APPLICATIONS OF FINAL POLYNOMIALS AND FINAL SYZYGIES

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APPLICATIONS OF FINAL POLYNOMIALS AND FINAL SYZYGIES

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Abstract. Final polynomials and final syzygies provide an explicit representation of polynomial identities promised by Hilbert's Nullstellensatz. Such representations can be obtained from a suitable Gröbner bases computation with slack variables. The resulting algorithm is not restricted to the classical complex case but can be used over the reals or the rationals as well.

In this paper we shall discuss applications of syzygy computations to invariant theory, real projective geometry and matrix theory. Our theoretical results include a topological criterion for the existence of final syzygies based on elimination theory, and upper bounds for the degrees of final polynomials.

1. Introduction. Consider a finite set $\mathcal{F} := \{ f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}) \} \subset \mathbb{C}[\mathbf{x}]$ of polynomials in n complex variables $\mathbf{x} := (x_1, x_2, \ldots, x_n)$. Hilbert's Nullstellensatz states that, either the equations

$$(1) \qquad f_1(\mathbf{x}) \quad = \quad f_2(\mathbf{x}) \quad = \ldots = \quad f_m(\mathbf{x}) \quad = \quad 0$$

have a solution in complex affine n-space $\mathbb{C}^n$, or the constant 1 is contained in the ideal $< \mathcal{F} >$ generated by $\mathcal{F}$. As is well known, the solvability of (1) over $\mathbb{C}$ can be decided quite efficiently by Buchberger's Gröbner basis method: (1) is inconsistent if and only if a Gröbner basis $\mathcal{G}$ of $\mathcal{F}$ contains a non-zero constant [4].

For certain applications of computer algebra, however, the information "$1 \in < \mathcal{F} >$" (which is provided by most Gröbner bases implementations) is insufficient. In geometry, for example, one is often interested in a refutational condition, i.e. an explicit representation

$$(2) \qquad 1 \quad = \quad \sum_{i=1}^{m} g_i(\mathbf{x}) \cdot f_i(\mathbf{x}) \qquad \text{in} \quad \mathbb{C}[\mathbf{x}].$$

A *final polynomial* for $\mathcal{F}$ is an element $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$, where $\mathbf{y} := (y_1, \ldots, y_m)$, such that

$$(3) \qquad p(\,\mathbf{x}\,,\, f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})\,) \;=\; 0 \quad \text{and} \quad p(\,\mathbf{x}\,, 0, \ldots, 0\,) \;=\; 1 \quad \text{in } \mathbb{C}[\mathbf{x}].$$

By the Nullstellensatz, either (1) has a solution, or $\mathcal{F}$ admits a final polynomial of the form $p(\mathbf{x}, \mathbf{y}) = -\sum_{i=1}^{m} g_i(\mathbf{x}) y_i + 1$ (also called *Hilbert equation* in the logic literature [17]). A final polynomial $p \in \mathbb{C}[\mathbf{y}]$ (not depending on $\mathbf{x}$ at all !) will be called a *final syzygy.*

We shall see in Section 3 that there are final syzygies if and only if (1) is inconsistent for all sufficiently small pertubations. We give a simple algorithm based on Gröbner bases to compute final syzygies or final polynomials whenever possible. Although most theoretical results are restricted to algebraically closed fields, it should be emphasized that the computation works for any field: in fact, the present investigation was motivated by J. Bokowski's final polynomial method [2] for realizability problems in $\mathbb{R}^d$.

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Applications of final polynomials and final syzygies to problems from invariant theory, real projective geometry and matrix theory will be discussed in Sections 4–6. All our examples have been computed with the Gröbner basis implementation in MAPLE.

2. Variants of the Nullstellensatz.

To begin with, we give a brief exposition of some recent results related to the Nullstellensatz. The classical proofs of the Nullstellensatz are non-constructive and do not give any a priori upper bound for the degrees of final polynomials. The first constructive proof due to G. Herrmann [8] implies that there is a final polynomial p with $\deg(p) \le (2D)^{2^n}$ where D bounds the degrees of the f_i. This doubly exponential bound has been replaced very recently by a singly exponential bound due to W.D. Brownawell.

THEOREM 2.1. (Brownawell [3]) *Let $\mathcal{F}$ as above with $\deg(f_i) \le D$, and assume that (1) has no solution in $\mathbb{C}^n$. Then there exists a final polynomial $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ with $\deg(p) \le \mu n D^\mu + \mu D$, where $\mu := \min\{m, n\}$.*

Brownawell's paper uses very powerful estimates from several complex variables, and it is substantially more difficult than the usual non-constructive proofs for the Nullstellensatz. An example family given in [3] shows that the bound in Theorem 2.1 is asymptotically optimal. If the f_i are homogeneous, then we have the following linear bound of D. Lazard.

THEOREM 2.2. (Lazard [11]) *Suppose that the polynomials f_i are homogeneous with $\deg(f_i) \le D$, and that (1) has no solution in $\mathbb{C}^n$ except 0. Then the ideal $< x_1, \ldots, x_m >^{nD}$ is contained in the ideal $< f_1, \ldots, f_n >$. Hence every affine chart $\{x_i = 1\}$ of projective $(n-1)$-space $\mathbb{P}^{n-1}$ admits a final polynomial p_i with $\deg(p_i) \le nD$.*

First Nullstellensätze for the real numbers $\mathbb{R}$ were proved by J.L. Krivine in 1964 and D.W. Dubois in 1969. For the history and details we refer to the survey articles on real algebraic geometry by E. Becker [1] and M.A. Dickmann [5]. In their expositions one finds also several *semi-algebraic Nullstellensätze* which deal with sharp and weak inequalities as well as equations. Here we state only the following basic version.

THEOREM 2.3. (Real Nullstellensatz) *Let K be a subfield of the real numbers $\mathbb{R}$, and let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Then either there exists an $\mathbf{x} \in \mathbb{R}^n$ such that $f_1(\mathbf{x}) = \ldots = f_m(\mathbf{x}) = 0$, or there exist $g_1, \ldots, g_m, h_1, \ldots, h_r \in K[x_1, \ldots, x_n]$ and positive numbers $\alpha_1, \ldots, \alpha_r \in K$ such that $g_1 f_1 + \ldots + g_m f_m + \alpha_1 {h_1}^2 + \ldots + \alpha_r {h_r}^2 = -1$.*

PROBLEM 2.4. *Is there an efficient version of the real Nullstellensatz which gives asymptotically the same bound as Brownawell's Theorem 2.1 ?*

Such a bound seems possible in view of the recent results by Grigor'ev & Vorobjov [7].

3. Computing final polynomials and final syzygies.

In this section we describe an algorithm based on Buchberger's Gröbner bases method [4] for the construction of final polynomials. Applying this method to a variety of problems, we found that in many instances the resulting final polynomial p does not depend on $\mathbf{x}$ at all. We shall prove below that the suggested algorithm produces such a final polynomial (called final syzygy) whenever one exists. In this situation, which can be characterized by an interesting topological condition, a linear bound on the degree is obtained from Lazard's Theorem 2.2.

Let $\mathcal{F}$ as in Section 1, and assume that (1) has no solutions in $\mathbb{C}^n$. In order to compute a final polynomial for $\mathcal{F}$, we introduce a slack variable y_i for each element of $\mathcal{F}$. Define

$$(4) \qquad \widehat{\mathcal{F}} := \{\widehat{f}_1, \dots, \widehat{f}_m\} \subset \mathbb{C}[\mathbf{x}, \mathbf{y}] \quad \text{where} \quad \widehat{f}_i(\mathbf{x}, \mathbf{y}) \quad := \quad f_i(\mathbf{x}) - y_i.$$

Applying an appropriate Gröbner basis computation to the set $\widehat{\mathcal{F}}$ of m polynomials in $m+n$ variables yields the desired result.

THEOREM 3.1. *Given $\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathbb{C}[\mathbf{x}]$, let $\widehat{\mathcal{G}} \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ be a Gröbner basis with respect to purely lexicographic order induced from $y_1 < \dots < y_m < x_1 < \dots < x_n$ for the set $\widehat{\mathcal{F}}$ in (4). Then $\widehat{\mathcal{G}}$ contains a final polynomial for $\mathcal{F}$ if and only if $\mathcal{F}$ has no zeros in $\mathbb{C}^n$. Moreover, if $\mathcal{F}$ admits a final syzygy, then there is a final syzygy in $\widehat{\mathcal{G}}$.*

Proof. Consider a final polynomial for $\mathcal{F}$ of the form $p(\mathbf{x}, \mathbf{y}) = -\sum_{i=1}^{m} g_i(\mathbf{x}) y_i + 1$. Since p is contained in the ideal generated by $\widehat{\mathcal{F}}$, its normal form modulo the Gröbner basis $\widehat{\mathcal{G}}$ must be zero. On the other hand, it is not difficult to see that, given the above purely lexicographical order, the polynomial p cannot reduce to zero modulo a set $\widehat{\mathcal{G}}$ unless $\widehat{\mathcal{G}}$ contains a final polynomial p'.

By a well known property of Gröbner bases, $\widehat{\mathcal{G}}$ contains a Gröbner basis for the *elimination ideal* $< \widehat{\mathcal{F}} > \cap \mathbb{C}[\mathbf{y}]$. This implies that, if $\mathcal{F}$ admits final syzygies at all, then there must a final syzygy $p(\mathbf{y})$ contained in $\widehat{\mathcal{G}}$. □

Example 3.2.

a) Let $\mathcal{F} := \{f_1, f_2\} \subset \mathbb{C}[x_1, x_2]$ where $f_1(\mathbf{x}) := x_1 x_2 + 1$ and $f_2(\mathbf{x}) = x_2$. Then $\widehat{\mathcal{G}} = \{x_2 - y_2, \underline{1 - y_1 + x_1 y_2}\}$; the final polynomial for $\mathcal{F}$ is underlined.

b) The line $f_1(\mathbf{x}) := x_1 - x_2 - 1$, the hyperbola $f_2(\mathbf{x}) := 3x_1 x_2 - 1$, and the unit circle $f_3(\mathbf{x}) := 1 - x_1^2 - x_2^2$ do not intersect. A final syzygy for $\mathcal{F} = \{f_1, f_2, f_3\}$ is given by

$$\frac{3}{2} y_1^2 + 3y_1 + y_2 + \frac{3}{2} y_3 + 1 \quad \in \quad \widehat{\mathcal{G}}.$$

Observe that the identity $3f_1^2 + 6f_1 + 2f_2 + 3f_3 + 2 = 0$ is also a "semi-algebraic final syzygy" for the inequality system $f_1 > 0, f_2 > 0, f_3 > 0$ (cf. Theorem 2.3 and [2]).

Given $\mathcal{F}$ as before, we consider the polynomial mapping

$$(5) \qquad \mathbf{f} : \mathbb{C}^n \to \mathbb{C}^m, \quad \mathbf{x} \mapsto (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})).$$

Denoting the zero vector in $\mathbb{C}^m$ with 0, we have that $\mathcal{F}$ admits a final polynomial if and only if 0 is not contained in the image $\mathrm{Im}(\mathbf{f})$ of $\mathbf{f}$.

LEMMA 3.3. *$\mathcal{F} \subset \mathbb{C}[\mathbf{x}]$ has a final syzygy if and only if 0 is not contained in the closure $\overline{\mathrm{Im}(\mathbf{f})}$ of $\mathrm{Im}(\mathbf{f})$ with respect to the Zariski topology on $\mathbb{C}^m$.*

Proof. Assume that 0 is contained in $\overline{\mathrm{Im}(\mathbf{f})}$. By definition of the Zariski topology, we have $p(0) = 0$ for every polynomial $p \in \mathbb{C}[\mathbf{y}]$ which vanishes on $\mathrm{Im}(\mathbf{f})$. $\mathbb{C}$ being an infinite field, p vanishes on $\mathrm{Im}(\mathbf{f})$ if and only if $p(f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) = 0$ in $\mathbb{C}[\mathbf{x}]$. Hence there is no final syzygy for $\mathcal{F}$. On the other hand, if 0 is not contained in $\overline{\mathrm{Im}(\mathbf{f})}$, then there exists $p \in \mathbb{C}[\mathbf{y}]$ which vanishes on $\mathrm{Im}(\mathbf{f})$ but not at 0. After scaling, p is a final syzygy for $\mathcal{F}$. □

Note that Lemma 3.3 remains valid if $\mathbb{C}$ is replaced by any infinite field. For the special case of the complex numbers, however, one has the following much stronger statement.

THEOREM 3.4. *$\mathcal{F} \subset \mathbb{C}[\mathbf{x}]$ has a final syzygy if and only if 0 is not contained in the closure of $\mathrm{Im}(\mathbf{f})$ with respect to the classical real topology on $\mathbb{C}^m$.*

Proof. By Lemma 3.3, it suffices to show that the closure of $\mathrm{Im}(\mathbf{f})$ with respect to the classical real topology is Zariski-closed in $\mathbb{C}^m$. We embed the affine spaces $\mathbb{C}^n$ and $\mathbb{C}^m$ into complex projective spaces $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, and we consider the induced mapping $\widetilde{\mathbf{f}} : \mathbb{P}^n \to \mathbb{P}^m$ (see Mumford [13, Chapter 2]). Given a set $A \subset \mathbb{P}^m$, we write $\mathrm{zcl}(A)$ for its Zariski-closure in $\mathbb{P}^m$ and $\mathrm{rcl}(A)$ for its closure in the classical real topology on $\mathbb{P}^m$, and similarly for subsets of $\mathbb{P}^n$.

By the *Main Theorem of Elimination Theory* [13, Theorem 2.23], we have that the image of $\widetilde{\mathbf{f}}$ is Zariski-closed (and hence classically closed) in $\mathbb{P}^m$. In other words,

$$\mathrm{zcl}\left(\widetilde{\mathbf{f}}(\mathrm{zcl}(\mathbb{C}^n))\right) = \mathrm{zcl}\left(\widetilde{\mathbf{f}}(\mathbb{P}^n)\right) = \widetilde{\mathbf{f}}(\mathbb{P}^n) = \mathrm{rcl}\left(\widetilde{\mathbf{f}}(\mathbb{P}^n)\right) = \mathrm{rcl}\left(\widetilde{\mathbf{f}}(\mathrm{rcl}(\mathbb{C}^n))\right).$$

$\widetilde{\mathbf{f}}$ being continuous in both topologies, the inner closure operators can be dropped:

$$\mathrm{zcl}\left(\widetilde{\mathbf{f}}(\mathbb{C}^n)\right) = \mathrm{rcl}\left(\widetilde{\mathbf{f}}(\mathbb{C}^n)\right).$$

Finally, we need the fact that both the classical topology and the Zariski topology on $\mathbb{C}^m$ are induced from the respective topologies on $\mathbb{P}^m$ [13, Proposition 2.5]. Intersecting both sides of the last equation with $\mathbb{C}^n$ shows that $\mathrm{Im}(\mathbf{f})$ has the same closure $\overline{\mathrm{Im}(\mathbf{f})}$ in $\mathbb{C}^m$ relative to both topologies:

$$\overline{\mathrm{Im}(\mathbf{f})} = \mathrm{zcl}(\widetilde{\mathbf{f}}(\mathbb{C}^n)) \cap \mathbb{C}^m = \mathrm{rcl}(\widetilde{\mathbf{f}}(\mathbb{C}^n)) \cap \mathbb{C}^m.$$

This completes the proof of Theorem 3.4. □

This result allows the following nice interpretation: there exists a final syzygy whenever $\mathcal{F} = \{f_1(\mathbf{x}), \dots, f_m(\mathbf{x})\}$ has no zeros, and it is stable with this property.

COROLLARY 3.5. *$\mathcal{F}$ has a final syzygy if and only if the system (1) is stably inconsistent, i.e., there exists an $\epsilon > 0$ such that $\mathcal{F}' := \{ f_1(\mathbf{x}) + \delta_1, \ldots, f_m(\mathbf{x}) + \delta_m \}$ has no zeros in $\mathbb{C}^n$ for all $\delta_1, \ldots, \delta_m \in \mathbb{C}$ with $|\delta_i| \leq \epsilon$.*

Thus the computation of $\widehat{\mathcal{G}}$ in Theorem 3.1 furnishes us with a decision procedure for stable inconsistency of polynomial systems. Moreover, applying Lazard's Theorem 2.2 to the homogenized mapping $\widetilde{\mathbf{f}}$ in the proof of Theorem 3.4 gives the following bound.

COROLLARY 3.6. *Let $\mathcal{F}$ be stably inconsistent as above, and suppose $\deg(f_i) \leq D$. Then there exists a final polynomial p with $\deg(p) \leq n \cdot D$.*

4. An application to projective geometry in the real plane.

While most of the above theoretical results are restricted to algebraically closed fields, the computation of $\widehat{\mathcal{G}}$ as suggested in Theorem 3.1 makes sense for any field K. In this section we shall apply our method of computing final polynomials to a non-trivial example from real projective geometry. As the defining equations have coefficients in the field $K = \mathbb{Q}$ of rational numbers, all computations will be carried out in $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$.

It is a belief frequently expressed in the computer algebra literature, that all "interesting" elementary geometry theorems in the real plane are also valid over $\mathbb{C}$, and therefore the consistency tests provided by ideal membership methods such as Gröbner bases [4,9] or Wu's method [18] are sufficient for automated theorem-proving in elementary geometry. Let us discuss a "counterexample" to that belief which is (from a geometer's point of view) both interesting and elementary, namely the famous Sylvester-Gallai-theorem: *Among any n points in the real Euclidean plane, not all on a line, there exist two points such that no other point is collinear with them.*

Consider the Sylvester-Gallai configuration [10] on nine *points* $1, 2, \ldots, 9$ and twelve *lines*

$$123 \quad 147 \quad 159 \quad 168 \quad 249 \quad 258 \quad 267 \quad 348 \quad 357 \quad 369 \quad 456 \quad 789. \tag{6}$$

If this combinatorial scheme could be realized by points and lines in the projective plane over a field K, this would imply that the Sylvester-Gallai-theorem fails for K. For, given any two points i and j, there exists a third point k which is collinear with i and j.

Suppose such a realization exists over $\mathbb{C}$, i.e., there exists a complex 3×9-matrix M (of homogeneous coodinates) whose vanishing 3×3-determinants are given precisely by (6). After a projective transformation every such matrix is of the form

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & x_1 & 1 & 0 & 1 & x_4 & 0 & x_6 & 1 \\ 0 & 0 & 0 & x_2 & x_3 & x_5 & 1 & x_7 & 1 \end{pmatrix} \tag{7}$$

where the indeterminants $x_1, x_2, \ldots, x_7$ are specialized to suitable complex numbers. We write $[ijk] \in \mathbb{C}[x_1, \ldots, x_7]$ for the 3×3-subdeterminant of M with column indices i, j and

k. (For example, $[159] = 1 - x_3$.) By construction of M, we have $[123] = [147] = [357] = 0$, and hence the realizations of (6) (modulo projective transformations) correspond to a Zariski-dense subset of the zero set in $\mathbb{C}^7$ of $\mathcal{F} = \{[159], [168], [249], [258], [267], [348], [369], [456], [789]\}$. We introduce nine slack variables $y_1, y_2, \ldots, y_9$, and we compute a Gröbner basis $\widehat{\mathcal{G}}$ with respect to lexicographic order induced from $y_1 < \ldots < y_9 < x_1 < \ldots < x_7$ of the set $\widehat{\mathcal{F}} =$

$$\{ [159]-y_1, [168]-y_2, [249]-y_3, [258]-y_4, [267]-y_5, [348]-y_6, [369]-y_7, [456]-y_8, [789]-y_9 \}.$$

We find that the polynomial

$$\begin{aligned} p(\mathbf{x}, \mathbf{y}) \;=\; & \underline{x_1^2 - x_1 + 1} - y_5 x_1 y_1 - x_1 y_9 y_1 - y_1 + y_2 + y_4 + y_6 + y_8 - 2y_9 + x_1 y_1 \\ & + 2y_9 y_1 - x_1 y_4 - x_1^2 y_1 + x_1 y_9 - y_9 y_8 - y_9 y_4 - y_9 y_6 + y_9^2 - y_9^2 y_1 - y_5 y_4 + y_5 x_1 \end{aligned}$$

is contained in $\widehat{\mathcal{G}}$. This proves that $x_1^2 - x_1 + 1 = (x_1 - \frac{1}{2})^2 + \frac{3}{4}$ is in the ideal $< \mathcal{F} > \subset \mathbb{C}[x_1, \ldots, x_7]$, and hence the configuration (6) is not realizable over $\mathbb{R}$ (compare [10, Theorem 3.4]). A realization over $\mathbb{C}$ is obtained by substituting $x_1 := \frac{1}{2} + \frac{1}{2}\sqrt{-3}$, $x_2 := \frac{1}{2} - \frac{1}{2}\sqrt{-3}$, $x_3 := 1$, $x_4 := \frac{1}{2} + \frac{1}{2}\sqrt{-3}$, $x_5 := 1$, $x_6 := 1$ and $x_7 := \frac{1}{2} - \frac{1}{2}\sqrt{-3}$ in (7). It turns out that the Gröbner basis $\widehat{\mathcal{G}}$ also contains a polynomial $q(\mathbf{y})$ entirely in the y_i's. Careful inspection of q, which is of degree 6 in 230 summands, shows that the configuration (6) expresses an incidence theorem over $\mathbb{C}$: *If in a set of nine points in the projective complex plane* 11 *of the* 12 *triples in (6) are collinear, then so is the twelfth.*

5. Applications to invariant theory.

Let $\mathcal{F} = \{f_1, \ldots, f_m\} \subset \mathbb{C}[\mathbf{x}]$ and $\widehat{\mathcal{F}}, \widehat{\mathcal{G}} \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ as in Section 3. A polynomial $f \in \mathbb{C}[\mathbf{x}]$ is contained in the <u>ideal</u> $< \mathcal{F} >$ generated by the f_i if and only if the normal form of f with respect to a Gröbner basis $\mathcal{G}$ of $\mathcal{F}$ is zero. Furthermore, we get a representation $f = \sum_i g_i f_i$ by computing the normal form of f with respect to $\widehat{\mathcal{G}}$.

This method of testing membership and computing representations generalizes to the <u>subring</u> $\mathbb{C}[\mathcal{F}]$ generated by the f_i in $\mathbb{C}[\mathbf{x}]$. The proof of this observation is straightforward using the methods from Section 3.

PROPOSITION 5.1. *A polynomial $f \in \mathbb{C}[\mathbf{x}]$ is contained in the subring $\mathbb{C}[\mathcal{F}]$ generated by $\mathcal{F}$ if and only if $f(\mathbf{x})$ reduces to a polynomial $p(\mathbf{y})$ modulo $\widehat{\mathcal{G}}$ in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. In that case, we have an explicit representation $f(\mathbf{x}) = p(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ in $\mathbb{C}[\mathbf{x}]$.*

A problem frequently arising in invariant theory is to express a given invariant f as a polynomial function in certain *fundamental invariants* $f_1, \ldots, f_m$ [6,16,17]. Before commenting on the general situtation, let us consider two important classical cases.

Example 5.2. An algorithmic version of the Main Theorem on Symmetric Functions.

A polynomial $f \in \mathbb{C}[\mathbf{x}]$ is *symmetric* if it is invariant under any permutation of the variables $x_1, \ldots, x_n$. The subset $\mathbb{C}[\mathbf{x}]^{\Sigma_n}$ of symmetric polynomials is closed under addition and multiplication and hence forms a subring of $\mathbb{C}[\mathbf{x}]$. Observe that the *elementary symmetric functions*

$$\begin{aligned}
\sigma_1(\mathbf{x}) &= x_1 + x_2 + \ldots + x_n, \\
\sigma_2(\mathbf{x}) &= x_1x_2 + x_1x_3 + \ldots + x_2x_3 + \ldots + x_{n-1}x_n, \\
\sigma_3(\mathbf{x}) &= x_1x_2x_3 + x_1x_2x_4 + \ldots + x_{n-2}x_{n-1}x_n, \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\sigma_n(\mathbf{x}) &= x_1x_2 \ldots x_n,
\end{aligned}$$

are contained in $\mathbb{C}[\mathbf{x}]^{\Sigma_n}$.

The *Main Theorem on Symmetric Functions* states that $\mathbb{C}[\mathbf{x}]^{\Sigma_n} = \mathbb{C}[\sigma_1, \sigma_2, \ldots, \sigma_n]$, i.e., every symmetric function can be written as a polynomial in the elementary symmetric functions. It is a very interesting fact that the classical proof for this result (see e.g. van der Waerden [15]) is not only constructive but uses a Gröbner-bases-type argument analogous to Proposition 5.1: Given any symmetric polynomial f with lexicographically leading term $\beta \cdot x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, then the polynomial $\sigma := \beta \cdot \sigma_1^{\alpha_1 - \alpha_2} \sigma_2^{\alpha_2 - \alpha_3} \ldots \sigma_n^{\alpha_n}$ has the same leading term. This proves the Main Theorem by induction because the leading term of the symmetric polynomial $f - \sigma$ is lexicographically earlier than the leading term of f.

As an example for Proposition 5.1 consider the case $n = 3$: the (reduced) Gröbner basis for $\widehat{\mathcal{F}} = \{\sigma_1(\mathbf{x}) - y_1, \sigma_2(\mathbf{x}) - y_2, \sigma_3(\mathbf{x}) - y_3\}$ with respect to purely lexicographic order induced from $y_1 < y_2 < y_3 < x_1 < x_2 < x_3$ is given by

$$\widehat{\mathcal{G}} = \{ -x_1^3 + x_1^2 y_1 - x_1 y_2 + y_3 \, , \ -x_2^2 - x_1 x_2 + x_2 y_1 - x_1^2 + x_1 y_1 - y_2 \, , \ x_3 + x_2 + x_1 - y_1 \}.$$

The symmetric function $D(\mathbf{x}) := (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2$ has the normal form

$$d(y_1, y_2, y_3) \quad = \quad -4\,y_2^3 + y_1^2\,y_2^2 + 18\,y_1 y_2 y_3 - 4\,y_1^3 y_3 - 27\,y_3^2$$

modulo $\widehat{\mathcal{G}}$. The resulting identity $D = d(\sigma_1, \sigma_2, \sigma_3)$ is the usual formula for the *discriminant* of a univariate cubic polynomial $z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_3$.

Example 5.3. An algorithmic version of the First Fundamental Theorem.

Let $\mathbf{X} := (x_{kl})$ be a variable $n \times d$-matrix, and let $\mathbb{C}[\mathbf{X}]$ denote the polynomial ring in the nd indeterminants x_{kl}. The group $SL(d, \mathbb{C})$ of complex $d \times d$-matrices with determinant 1 acts on $\mathbb{C}[\mathbf{X}]$ by right multiplication. The $d \times d$-subdeterminants of $\mathbf{X}$ are clearly invariant under that action, and so are successive sums and products thereof.

The *First Fundamental Theorem of Invariant Theory* (see e.g. Weyl [16] or Dieudonné & Carrell [6]) states that all invariants can be written as polynomials in the $d \times d$-minors of $\mathbf{X}$. In other words, the invariant ring $\mathbb{C}[\mathbf{X}]^{SL(d,\mathbb{C})}$ is generated as a $\mathbb{C}$-algebra by the fundamental invariants

$$\det \begin{pmatrix} x_{i_1,1} & \dots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_d,1} & \dots & x_{i_d,d} \end{pmatrix}$$

where $1 \leq i_1 < \ldots < i_d \leq n$.

In addition to the nd variables x_{kl} we introduce $\binom{n}{d}$ slack variables $[i_1, \ldots, i_d]$, and we compute a Gröbner basis $\widehat{\mathcal{G}}$ of

$$\widehat{\mathcal{F}} \quad := \quad \{ \det \begin{pmatrix} x_{i_1,1} & \dots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_d,1} & \dots & x_{i_d,d} \end{pmatrix} - [i_1, \ldots, i_d] \quad | \quad 1 \leq i_1 < \ldots < i_d \leq n \ \}$$

with respect to purely lexicographic order (assuming $\ldots < [i_1 \ldots i_d] < \ldots < x_{kl} < \ldots$). For any invariant polynomial $p(x_{kl}) \in \mathbb{C}[\mathbf{X}]^{SL(d,\mathbb{C})}$ an equivalent bracket polynomial is given by the normal form of p modulo $\widehat{\mathcal{G}}$. Let $n = 4, d = 2$, and consider the polynomial

$$\begin{aligned} p(\mathbf{X}) \quad = \quad & x_{11}x_{22}x_{31}x_{42} + x_{11}x_{22}x_{32}x_{41} + x_{12}x_{21}x_{31}x_{42} + x_{12}x_{21}x_{32}x_{41} \\ & - 2\,x_{11}x_{32}x_{21}x_{42} - 2\,x_{12}x_{31}x_{22}x_{41}. \end{aligned}$$

which is invariant under the action of $SL(2, \mathbb{C})$ on $\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix}$. Let us find an equivalent bracket expression (and a geometric interpretation) for p.

A Gröbner basis with respect to lexicographic order induced from $x_{11} > x_{12} > \ldots > x_{42} > [12] > [13] > \ldots > [34]$ for the set $\widehat{\mathcal{F}} = \{ x_{i1}x_{j2} - x_{i2}x_{j1} - [ij] \mid 1 \leq i < j \leq 4 \}$ is given by

$$\begin{aligned} \widehat{\mathcal{G}} \quad = \quad \widehat{\mathcal{F}} \ \cup \ \{ \, & x_{41}[12][23] - x_{31}[12][24] - x_{21}[14][23] + x_{21}[13][24], \\ & x_{41}[12] - x_{21}[14] + x_{11}[24], \quad x_{41}[23] - x_{31}[24] + x_{21}[34], \quad [13]x_{41} - [14]x_{31} + x_{11}[34], \\ & x_{22}[13] - x_{12}[23] - [12]x_{32}, \ -[24]x_{12} - x_{42}[12] + x_{22}[14], \ -x_{12}[34] - x_{42}[13] + x_{32}[14], \\ & - x_{22}[34] - x_{42}[23] + x_{32}[24], \quad -[12]x_{42}[23] + [12]x_{32}[24] + x_{22}[14][23] - x_{22}[13][24], \\ & x_{31}[12] - x_{21}[13] + x_{11}[23], \quad \underline{[14][23] - [13][24] + [12][34]} \ \} \end{aligned}$$

The normal form of $p(\mathbf{X})$ modulo $\widehat{\mathcal{G}}$ equals $-[14][23] - [13][24]$, and we have the following geometric interpretation: vanishing of the polynomial $p(\mathbf{X})$ is equivalent to $(1, 2 ; 3, 4) = 2$ where $(1, 2 ; 3, 4)$ denotes the projective cross ratio of the column vectors of $\mathbf{X}$.

Note that the underlined expression in the Gröbner basis $\widehat{\mathcal{G}}$ is a polynomial only in the brackets. It is a (Grassmann-Plücker) syzygy which, by construction, vanishes for the 2×2-minors of any 4×2-matrix. This observation generalizes to arbitrary n and d, and hence we have a combined algorithm for both the First and the Second Fundamental Theorem of Invariant Theory. (The latter describes the ideal of all syzygies among the maximal minors of an $n \times d$-matrix.) Our Gröbner basis $\widehat{\mathcal{G}}$ necessarily contains a Gröbner basis for the syzygy ideal; see [14] for details.

Finding geometric interpretations for algebraic expressions is a basic problem in computer-aided geometric reasoning. As a first step in algorithms dealing with this question, polynomial expressions must be rewritten in terms of geometric (= invariant) magnitudes. Example 5.3 suggests a solution to this subproblem which generalizes to all classical geometric groups. Recent progress on the substantially more difficult problem of rewriting invariant polynomials as synthetic constructions has been made by T. McMillan & N. White [12], who gave an algorithm for multilinear invariants of $SL(d, \mathbb{C})$. An improved version of Example 5.3 could serve as "preprocessor" to their *Cayley factorization* procedure which applies to a large number of projective geometry problems.

It has been suggested by Dieudonné & Carrell [6] that both classical and post-Hilbert invariant theory deserve further study from the computational point of view. We close this section by posing a specific problem in this direction. As motivation let us briefly recall the basic ideas in the proof of Hilbert's famous finiteness theorem. For simplicity we restrict ourselves to finite groups; the reader is referred to [6, Chapter 3] for the general case considered by Nagata (including the classical infinite groups).

THEOREM 5.5. (Hilbert's Finiteness Theorem for Finite Groups)
Let Γ be a finite group of algebra automorphisms of $\mathbb{C}[\mathbf{x}]$. Then the invariant ring

$$\mathbb{C}[\mathbf{x}]^\Gamma \quad = \quad \{ f \in \mathbb{C}[\mathbf{x}] \quad | \quad \forall \sigma \in \Gamma : \sigma(f) = f \}$$

is finitely generated, i.e., there exist invariants $f_1, \ldots, f_m$ such that $\mathbb{C}[\mathbf{x}]^\Gamma = \mathbb{C}[f_1, \ldots, f_m]$.

Sketch of the proof. Consider the ideal I_Γ generated by the (infinite) set $\mathbb{C}[\mathbf{x}]^\Gamma_+$ of homogeneous invariants of degree ≥ 1. $\mathbb{C}[\mathbf{x}]$ is noetherian by Hilbert's basis theorem, and we can find[1] $f_1, \ldots, f_m \in \mathbb{C}[\mathbf{x}]^\Gamma_+$ such that $I_\Gamma = < f_1, \ldots, f_m >$. Clearly, $\mathbb{C}[f_1, \ldots, f_m] \subset \mathbb{C}[\mathbf{x}]^\Gamma$.

Let $f \in \mathbb{C}[\mathbf{x}]^\Gamma$, and assume w.l.o.g. that f is homogeneous of degree ≥ 1. We will show $f \in \mathbb{C}[f_1, \ldots, f_m]$ by induction on the degree of f. Abbreviate $f^* := \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \sigma(f)$ for any $f \in \mathbb{C}[\mathbf{x}]$. We have $f = \sum_i g_i f_i \in I_\Gamma$. Applying the operator $*$ on both sides of that equation yields $f = f^* = \sum_i g_i^* f_i$. Clearly, the g_i^* are invariant, and, by homogeneity,

[1] This nonconstructive step in Hilbert's proof was meant when Gordan, the "king of invariants", exclaimed: "That is theology, not mathematics".

we have $\deg(g_i^*) < \deg(f)$. Hence $g_i^* \in \mathbb{C}[f_1, \ldots, f_m]$ by induction, and consequently $f \in \mathbb{C}[f_1, \ldots, f_m]$. □

PROBLEM 5.4. *Give an (efficient) algorithm to compute fundamental invariants f_i for the action of a finite group Γ on $\mathbb{C}[\mathbf{x}]$. Or, more generally, determine the computational complexity of Hilbert's finiteness theorem.*

6. An application to matrix theory.

An easy modification of the Second Fundamental Theorem (mentioned above) yields a complete description for the ideal of syzygies among all subdeterminants of an arbitrary matrix. However, in linear algebra one is frequently interested in restricted families of matrices and/or restricted families of subdeterminants. Syzygy computations can be helpful to gain more insight in such special situations.

As an example we discuss the following question which has been suggested to the author by C.R. Johnson [private communication]:

PROBLEM 6.1. (C.R. Johnson) *Which values can be attained by the $2^n - 1$ principal (= symmetric with respect to the main diagonal) minors of a symmetric $n \times n$-matrix M.*

With our final syzygy method it is an easy excercise to give a complete answer for $n = 3$, the first non-trivial case.

PROPOSITION 6.2. *A sequence of numbers $D_{123}, D_{12}, D_{13}, D_{23}, D_1, D_2, D_3 \in \mathbb{C}$ can be obtained as principle minors of a symmetric complex 3×3-matrix (e.g. D_{13} is the minor with row and column indices $1, 3$) if and only if*

$$\begin{aligned} 0 \quad = \quad & D_{123}^2 - 2\,D_{123}D_{23}D_1 - 2\,D_{123}D_{13}D_2 - 2\,D_{123}D_{12}D_3 \\ & + 4\,D_{123}D_1D_2D_3 - 2\,D_{12}D_{23}D_1D_3 - 2\,D_{13}D_{23}D_1D_2 - 2\,D_{12}D_{13}D_2D_3 \\ & + D_{12}^2D_3^2 + D_{13}^2D_2^2 + D_{23}^2D_1^2 + 4\,D_{23}D_{13}D_{12}. \end{aligned} \tag{8}$$

Proof. Let $\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix}$, and let $\mathcal{F} \subset \mathbb{C}[x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_{33}]$ be the set of principal subdeterminants of $\mathbf{X}$. We introduce seven slack variables $D_{123}, D_{12}, D_{13}, D_{23}, D_1, D_2, D_3$, and, as before, we compute a lexicographical Gröbner basis $\widehat{\mathcal{G}}$ for $\widehat{\mathcal{F}} \subset \mathbb{C}[x_{11}, \ldots, x_{33}, D_{123}, \ldots, D_3]$.

The set $\widehat{\mathcal{G}} \cap \mathbb{C}[D_{123}, \ldots, D_3]$ is a singleton; it consists of the polynomial in (8). This implies that the polynomial in (8) generates the elimination ideal $< \widehat{\mathcal{F}} > \cap\, \mathbb{C}[D_{123}, \ldots, D_1]$. The claim follows now from the Main Theorem of Elimination Theory [13, Theorem 2.23] (compare Section 3) because all polynomials in $\mathcal{F}$ are homogeneous. □

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