Harmonic mean curvature flow in Riemannian manifolds and Ricci flow on noncompact manifolds

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Abstract

This thesis treats two topics about geometric flows. One topic concerns the deformation of hypersurfaces in negatively curved Riemannian manifolds using fully nonlinear parabolic equations defined in terms of the normal curvatures. It is shown that in a simply-connected negatively curved manifold, any strictly convex compact initial hypersurface flowing by Harmonic Mean Curvature Flow produces a solution which converges to a single point in finite time, and becomes spherical as the limit point is approached. We give several examples of hypersurfaces $\phi_t(M^n)$ evolving in time with speed determined by functions of the normal curvatures in an $(n + 1)$-dimensional hyperbolic manifold; we emphasize the case of flow by harmonic mean curvature. The examples converge to a totally geodesic submanifold of any dimension from 1 to n, and include cases which exist for infinite time.

The other topic concerns Ricci flow on noncompact manifolds. Following Deane Yang's sketch, we give the first detailed proof of the short-time existence of the local Ricci flow introduced by him. Using the local Ricci flow, we prove the short-time existence of the Ricci flow on noncompact manifolds whose Ricci curvature has a global lower bound and sectional curvature has only a local average integral bound. The short-time existence of the Ricci flow on noncompact manifolds was studied by Wan-Xiong Shi in 1989, who required a uniform bound of curvature tensors. As a corollary of our main theorem, we prove short-time existence in this more general context.

Contents

Chapter 1

Introduction

Geometric flow is the most active area in the subject of geometric analysis, its origin goes back to Eells and Sampson's paper [1] in 1964. There are several types of geometric flows, extrinsic flows on hypersurfaces in an ambient manifold and intrinsic flows on the manifold itself are the most important two types among them. One of the brilliant triumphs of extrinsic flows is Huisken and Ilmanen's proof of Penrose's inequality using inverse mean curvature flow in 1997, see [2]. On the other hand, Hamilton and Perelman's proof of Poincare's conjecture using Ricci flow is one of the most remarkable results in geometry and topology from 1982 to 2003, see [3].

This thesis treats some topics in Harmonic mean curvature flow (extrinsic flow) and Ricci flow (intrinsic flow). Both flows are parabolic flows, in PDE terminology. But harmonic mean curvature flow is a fully nonlinear parabolic equation (see [4]), and Ricci flow is a semilinear parabolic system after using DeTurck's trick (see [5]). Unless otherwise mentioned, all Riemannian manifolds in this thesis are complete.

1.1 Harmonic mean curvature flow

Since G. Huisken's paper [6], mean curvature flow of hypersurfaces in $\mathbb{R}^n$ has been an active research field. Other related curvature flows on hypersurfaces or curves in Riemannian manifolds were also studied intensively, see "curve shortening flow" in [7] and harmonic mean curvature flow in [8], also see [9].

Let M^n be a smooth, connected compact manifold of dimension $n \geq 2$, without boundary, and let (N^{n+1}, g^N) be a smooth connected Riemannian manifold. Write σ^N for any sectional curvature of N^{n+1}, R^N for the Riemann tensor on N^{n+1}, and ∇^N for the Levi-Civita connection corresponding to g^N. In this thesis, Einstein's summation notation is used, and the index set of summation is always $\{1, 2, \dots, n\}$ unless otherwise mentioned.

Suppose $\varphi_0 : M^n \to N^{n+1}$ is a smooth immersion of M^n. We choose a coordinate system $\{x_i\}_{i=1}^n$ of M, denote $e_i = \frac{\partial}{\partial x_i}$. For simplicity, it will sometimes be written as $e_i = \partial_{x_i}$. For $i = 1, 2, \dots, n$, adopt the frame $\{e_0, e_1, \dots, e_n\}$, where $\vec{v} = e_0$ is the ourward normal vector to $\varphi_0(M)$. The second fundamental form of M is a matrix A, where the entry $A_{ij} = h_{ij} = \langle \nabla^N_{e_i} \vec{v}, e_j \rangle_{g^N}$. The Weingarten matrix is $\mathscr{W} = AG^{-1}$, with entries $\omega_i^k = h_{ij} g^{jk}$ where g^{jk} is the entry of the inverse matrix of $\{g_{jk}\}$.

We seek a solution $\varphi : M^n \times [0, T) \to N^{n+1}$ to an equation

$$\frac{\partial}{\partial t}\varphi(x,t) = -f(\lambda(\mathscr{W}(x,t)))\vec{v}(x,t) \tag{1.1}$$

$$\varphi(x,0) = \varphi_0(x)$$

where $F(x,t) = f(\lambda(\mathscr{W}(x,t)))$ and f is a smooth symmetric function. Here $\vec{v}(x,t)$ is the outward unit normal vector to $\varphi(M^n, t)$. $\mathscr{W}(x,t)$ is the Weingarten matrix of $\varphi(M^n, t)$ in N^{n+1}, and $\lambda(\mathscr{W})$ is the set of eigenvalues $(\lambda_1, \dots, \lambda_n)$ of $\mathscr{W}$. Define $\varphi_t(x) = \varphi(x,t)$, then $(\lambda_1, \dots, \lambda_n)$ are the principal curvatures of the hypersurface $\varphi_t(M) \subset N$.

Such solution φ is called the **Harmonic Mean Curvature Flow** when $f(\lambda) = (\sum_i \lambda_i^{-1})^{-1}$.

So the solution $\varphi : M^n \times [0,T) \to N^{n+1}$ of the following equations:

$$\frac{\partial}{\partial t}\varphi(x,t) = -(\sum_i \lambda_i^{-1})^{-1}\vec{\nu}(x,t) \tag{1.2}$$

$$\varphi(x,0) = \varphi_0(x)$$

where $(\lambda_1, \ldots, \lambda_n)$ are the principal curvature of the hypersurface $\varphi_t(M) \subset N$, is Harmonic Mean Curvature Flow.

If we choose a normal coordinate system which diagonizes $\mathscr{W}$ at one point, then we call this coordinate system a curvature coordinate system. In [10] Huisken has considered the mean curvature flow in this setting, in his case $f(\lambda) = \sum_{i=1}^n \lambda_i$. The main theorem of [10] may be stated as follows:

Theorem 1 *(Huisken [10]) Let M^n and φ_0 be as above, and assume (N^{n+1}, g^N) satisfies the following conditions:*

$$-K_1 \le \sigma^N \le K_2, \ \left|\nabla^N R^N\right|_{g^N} \le L$$

for some nonnegative constants K_1, K_2 and L. Assume in addition that the injectivity radii $i_y(N)$ of N^{n+1} have a positive lower bound $i(N)$, and that the principal curvatures of φ_0 satisfy the inequality:

$$Hh_{ij} - nK_1 g_{ij} > \frac{n^2 L}{H} g_{ij}$$

where $H = f(\lambda_1, \cdots, \lambda_n) = \sum_{i=1}^n \lambda_i$, $(\lambda_1, \ldots, \lambda_n)$ are the principal curvatures of the hypersurface $\varphi_t(M) \subset N$. Then there exists a unique smooth solution to (1.1) on a maximal time interval $[0,T)$. The immersions φ_t converge uniformly to a constant $p \in N^{n+1}$ as t approaches T. The rescaled immersions $\tilde{\varphi}_\tau$ obtained by rescaling a neigborhood of p by a factor $(2n(T-t))^{-\frac{1}{2}}$ converge to the unit sphere $S_1^n(0)$ in Euclidean space, exponentially in C^∞ with respect to the natural time parameter $\tau = -\frac{1}{2}\ln(1-\frac{t}{T})$.

Later, Andrews proved the following theorem in [8]:

Theorem 2 *(Andrews [8]) Let M^n and φ_0 be assumed as at the beginning of this paper, (N^{n+1}, g^N) satisfies the following conditions:*

$$-K_1 \leq \sigma^N \leq K_2, \ \left|\nabla^N R^N\right|_{g^N} \leq L$$

for some nonnegative constants K_1, K_2 and L.
Assume that $f(\lambda) = (\frac{1}{n}\sum_{i=1}^n(\lambda_i - \sqrt{K_1})^{-1})^{-1}$, and every principal curvature λ_i of φ_0 satisfies the following condition:

$$\lambda_i > \sqrt{K_1}$$

Then there exists a unique smooth solution to (1.1) on a maximal time interval $[0, T)$, and the immersions φ_t converge uniformly to a point p in N^{n+1} as t approaches T. Expanding a neighborhood of p by a factor $(2(T-t))^{-\frac{1}{2}}$ gives rescaled immersions $\tilde{\varphi}_\tau$ which converge in C^∞ to the unit sphere about the origin in Euclidean space, exponentially with respect to the natural rescaled time parameter $\tau = -\frac{1}{2}\ln(1-\frac{t}{T})$.

Motivated by the above theorems, we proved the following theorem in Chapter 2:

Theorem 3 *Let M^n be a smooth, connected compact manifold of dimension $n \geq 2$, without boundary. Assume N^{n+1} is a negatively curved smooth simply connected manifold, and suppose $\varphi_0 : M^n \to N^{n+1}$ is a smooth immersion of M^n. Assume $f(\lambda) = (\sum_{i=1}^n(\lambda_i)^{-1})^{-1}$, and every principal curvature λ of φ_0 satisfies the following condition:*

$$\lambda_i > 0$$

Then there exists a unique smooth solution to (1.1) on a maximal time interval $[0, T)$, and the immersions φ_t converge uniformly to a point p in N^{n+1} as t approaches T. Expanding a *neighborhood of p by a factor* $(2n^{-1}(T-t))^{-\frac{1}{2}}$ *gives rescaled immersions* $\tilde{\varphi}_\tau$ *which converge in* C^∞ *to the unit sphere about the origin in Euclidean space, exponentially with respect to the natural rescaled time parameter* $\tau = -\frac{1}{2}\ln(1-\frac{t}{T})$.

It should be emphasized that the details of the rescaling process will be explained in the proof of the theorems of chapter 2, and it is not the same rescaling process as in Euclidean space. The hypotheses of this theorem differ from those proved by Andrews in three respects: no bound on the $\left|\nabla^N R^N\right|$ is required; we add the condition that N^{n+1} is simply connected; and the condition $-K_1 \le \sigma^N \le K_2$ is replaced by $\sigma^N \le 0$.

In Chapter 3, we give several examples of hypersurfaces $\varphi_t(M^n)$ evolving in time with speed determined by functions of the normal curvatures in an $(n+1)$-dimensional hyperbolic manifold (see also Robert Gulliver and Guoyi Xu [11]). We emphasize the case of flow by harmonic mean curvature. The examples converge to a totally geodesic submanifold of any dimension from 1 to n, and include cases which exist for infinite time.

1.2 Ricci flow

In his well-known paper [12], R. Hamilton introduced the Ricci flow on compact Riemannian manifolds which has proved to be very useful in the research of differential geometry. Let us recall the definition of the Ricci flow. It is the solution of the evolution equation deforming the metric on any n-dimensional Riemannian manifold (M, g_{ij}):

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.3}$$

where R_{ij} is the Ricci tensor of M at time t.

The first important thing in the study of the Ricci flow on Riemannian manifolds which we have to consider is the short-time existence of the solution. In the case where M is a compact manifold, it is well known that for any given initial metric g_{ij} on M, the evolution equation (1.3) always has a unique solution for a short time (see [5], [12]).

To prove Yau's **Uniformization Conjecture** (see Problem 34 on page 678 of [13]), Wan-Xiong Shi initiated studying the Ricci flow on complete noncompact Riemannian manifolds. The short-time existence problem of the evolution equation (1.3) is more difficult than the compact case. In his paper [14] in 1989, Shi proved the following theorem:

Theorem 4 (Shi, [14]) *Let (M, g_0) be an n-dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $Rm = \{R_{ijkl}\}$ satisfying*

$$|Rm| \le k_0 \qquad on\ M$$

where $0 < k_0 < +\infty$ is a constant. Then there is a constant $T(n, k_0) > 0$ depending only on n and k_0 such that the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij}(x,t) &= -2R_{ij}(x,t) \\ g(x,0) &= g_0(x) \end{aligned} \tag{1.4}$$

has a smooth solution $g_{ij}(x,t)$ on $[0, T(n, k_0)]$, and satisfies the following estimates. For any integer $m \ge 0$, there exist constants $C_m > 0$ depending only on n, m and k_0 such that the covariant derivatives satisfy

$$\sup_{x \in M} |\nabla^m Rm(x,t)| \le \frac{C_m}{t^{\frac{m}{2}}}, \quad 0 < t \le T(n, k_0)$$

Note in fact Shi's theorem requires that the initial manifold (M, g_0) have bounded geometry, i.e the curvature of M must have a point-wise estimate. In this paper, using Deane Yang's local Ricci flow, we prove the following theorem which requires only the integral conditions on the curvature tensor of the initial manifold and a local Sobolev inequality.

To use Deane Yang's local Ricci flow, in Chapter 4, we give the first detailed proof of the short-time existence of local Ricci flow by following Deane Yang's sketch in [15]. Then we give the estimates of curvatures and existence time of local Ricci flow in Chapter 5 by using Moser iteration and evolution equations of curvatures.

For a point $x \in M$, the open geodesic ball of radius R centered at x on (M, g_0) will be denoted as $\tilde{B}_x(R)$. And $⨍$ means the usual average of the integrand with respect to Riemannian volume. In Chapter 6, we prove the following theorem (see also Guoyi Xu [16]):

Theorem 5 *Assume (M, g_0) is a n-dimensional $(n \geq 3)$ complete noncompact Riemannian manifold, satisfying the following conditions:*

$$\begin{cases} \left(⨍_{\tilde{B}_x(4r)} h^{\frac{2n}{n-2}} dV_{g_0} \right)^{\frac{n-2}{n}} & \leq A_0 r^2 ⨍_{\tilde{B}_x(4r)} |\nabla h|^2 dV_{g_0} \,, \\ \left(⨍_{\tilde{B}_x(r)} |Rm(g_0)|^{p_0} dV_{g_0} \right)^{\frac{1}{p_0}} & \leq K_1 \end{cases} \tag{1.5}$$

for any $x \in M$ and $h \in C_0^\infty(\tilde{B}_x(4r))$, where A_0 and K_1 are some positive constants, r and p_0 are some constants such that $0 < r \leq 1$, $p_0 > \frac{n}{2}$.

Then the Ricci flow (1.4) has a smooth solution $g_{ij}(x, t)$ on $[0, T]$, and satisfies the following estimates. For any integer $m \geq 0$, there exists a positive constant $C(A_0, K_1, m, n, p_0, r)$ depending only on A_0, K_1, m, n, p_0 and r, such that

$$\sup_{x \in M} |\nabla^m Rm(x, t)| \leq \frac{C(A_0, K_1, m, n, p_0, r)}{t^{\frac{m}{2} + \frac{n}{2p_0}}}, \quad 0 < t \leq T \tag{1.6}$$

where $T = T(A_0, K_1, n, p_0, r)$ is some positive constant depending only on A_0, K_1, n, p_0 and r.

Remark 6 *Note the above theorem provides a method to smooth complete noncompact manifolds or orbifolds. For the compact case, see [17].*

We have the following corollary from Theorem 80:

Corollary 7 *Assume* (M, g_0) *is an n-dimensional* $(n \geq 3)$ *complete noncompact Riemannian manifold, satisfying the following conditions:*

$$\begin{cases} Rc(g_0) & \geq -Kg_0 \,, \\ \Big(\fint_{\tilde{B}_x(r)} |Rm(g_0)|^{p_0} dV_{g_0} \Big)^{\frac{1}{p_0}} & \leq K_1 \end{cases} \tag{1.7}$$

for any $x \in M$*, where* K *and* K_1 *are some positive constants,* r *and* p_0 *are some constants such that* $0 < r \leq 1$*,* $p_0 > \frac{n}{2}$*.*

Then the Ricci flow (1.4) has a smooth solution $g_{ij}(x,t)$ *on* $[0, T]$*, and satisfies the following estimates. For any integer* $m \geq 0$*, there exists a positive constant* $C(K, K_1, m, n, p_0, r)$ *depending only on* K*,* K_1*,* m*,* n*,* p_0 *and* r*, such that*

$$\sup_{x \in M} |\nabla^m Rm(x,t)| \leq \frac{C(K, K_1, m, n, p_0, r)}{t^{\frac{m}{2} + \frac{n}{2p_0}}}, \quad 0 < t \leq T$$

where $T = T(K, K_1, n, p_0, r)$ *is some positive constant depending only on* K*,* K_1*,* n*,* p_0 *and* r*.*

Proof: Because $Rc(g_0) \geq -Kg_0$, by theorem 3.1 in [18] and corollary 1.1 in [19], we get $A_0 \leq C(n, K)$, then apply Theorem 80 to get our conclusion. □

We also have the following corollary which is part of Shi's Theorem 4.

Corollary 8 *Let* (M, g_0) *be an n-dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor* $\{R_{ijkl}\}$ *satisfying* $|Rm| \leq k_0$ *on* M*, where* $0 < k_0 < +\infty$ *is a constant. Then there is a constant* $T(n, k_0) > 0$ *depending only on* n *and* k_0 *such that the evolution equation (1.4) has a smooth solution* $g_{ij}(x,t)$ *on* $[0,\ T(n, k_0)]$*.*

Proof: By $|Rm| \leq k_0$, there exists $K = nk_0$ such that

$$Rc(g_0) \geq -Kg_0,$$

and we also have

$$\Big(\fint_{\tilde{B}_x(1)} |Rm(g_0)|^n dV_{g_0} \Big)^{\frac{1}{n}} \le k_0$$

Now we choose

$$K = nk_0,\ K_1 = k_0,\ p_0 = n,\ r = 1.$$

Then hypothesis (1.7) in corollary 7 are satisfied. By Corollary 7 we get our corollary. □

And the other theorem is the case $p_0 = \frac{n}{2}$, where p_0 is the power of $|Rm|$:

Theorem 9 *Assume* (M, g_0) *is an n-dimensional* $(n \ge 3)$ *complete noncompact Riemannian manifold, satisfying the following conditions:*

$$\begin{cases} Vol_{g_0}(\tilde{B}_x(\rho)) & \ge N_0\Big(\frac{\rho}{4r}\Big)^n Vol_{g_0}(\tilde{B}_x(4r))\ , \\ \Big(\int_{\tilde{B}_x(r)} h^{\frac{2n}{n-2}} dV_{g_0} \Big)^{\frac{n-2}{n}} & \le A_0 \int_{\tilde{B}_x(r)} |\nabla h|^2 dV_{g_0}\ , \\ (\int_{\tilde{B}_x(r)} |Rm(g_0)|^{\frac{n}{2}} dV_{g_0})^{\frac{2}{n}} & \le (\tau n A_0)^{-1}\ , \\ \Big(\int_{\tilde{B}_x(r)} |Rc(g_0)|^{p_1} dV_{g_0} \Big)^{\frac{1}{p_1}} & < K_1 \end{cases} \tag{1.8}$$

for any $x \in M$, $0 < \rho \le 4r$, $h \in C_0^\infty(\tilde{B}_x(r))$, *where* $p_1 > \frac{n}{2}$, A_0, N_0 ,K_1 *and* $0 < r \le 1$ *are some positive constants. Here* τ *is some positive constant satisfying* $\tau > \max\Big\{36N, \frac{16p_1N}{\sqrt{n}}\Big\}$, *where* $N \triangleq 8^n N_0^{-2}$.

Then the Ricci flow (1.4) has a smooth solution $g_{ij}(x,t)$ *on* $[0, T]$, *and satisfies the following estimates. For any integer* $m \ge 0$, *there exist constants* $C(n, m, N_0, \tau, r) > 0$ *depending only on* n, m, N_0, τ *and* r, *such that*

$$\sup_{x\in M} |\nabla^m Rm(x,t)| \le \frac{C(n, m, N_0, \tau, r)}{t^{\frac{m}{2}+1}}, \quad 0 < t \le T \tag{1.9}$$

where T *is defined as the following:*

$$T = C(n, N_0, p_1, \tau, r) \min\Big\{1,\ A_0^{-\frac{n}{2p_1-n}} K_1^{-\frac{2p_1}{2p_1-n}}\Big\} \tag{1.10}$$

The typical examples of our theorem include manifolds with bounded curvature and the suitable perturbations of Euclidean space $\mathbb{R}^n$ which have unbounded curvatures but satisfy the assumptions (6.1).

Chapter 2

Harmonic Mean Curvature Flow in Riemannian manifolds

In the following C is always a constant, $C = C(M, N, \varphi_0)$ means that C is determined by M, N, φ_0. In other cases $C = C(\cdot)$ is understood in the same way. In this chapter, unless otherwise mentioned, we assume that $f(\lambda) = (\sum_i \lambda_i^{-1})^{-1}$, and recall $F(x,t) = f(\lambda(\mathscr{W}(x,t)))$ in Chapter 1.

Definition 10

$$\dot{F}^{kl} = \frac{\partial F}{\partial h_{kl}}\ ,\ \ddot{F}^{kl,pq} = \frac{\partial^2 F}{\partial h_{kl}\partial h_{pq}}$$

we define $|A|^2 = g^{ij}g^{kl}h_{ik}h_{jl}$.

$$\mathscr{R}_{ij} = R_{i0j0}$$

where 0 *appearing in curvature tensors represents the outward normal vector* $\vec{\nu}$ *of* $\varphi(M)$ *in* N. *Assume* W *is a function on* M^n, *then we define:*

$$\mathscr{L}(W) = \dot{F}^{kl}\nabla_k\nabla_l W$$

And define: $(\mathscr{W}^2)_{kl} = \omega_k^m h_{ml}$, $|\mathscr{W}| = \sqrt{\sum_{i,k}(\omega_i^k)^2}$

And H is the mean curvature, $\dot{H}^i_k = \frac{\partial H}{\partial \omega^k_i} = \delta^k_i$, where $\delta^k_i = 1$ if $i = k$ and $\delta^k_i = 0$ if $i \neq k$. When we use notation $< B, D >$ where $B = (b_{ij})$,$D = (d_{ij})$ are both matrices, we mean $< B, D >= \sum_{i,j} b_{ij} d_{ij}$ which is a real number.

Suppose $\psi_0 : \Sigma^n \to N^{n+1}$ is a smooth immersion of a compact manifold Σ (possible with a smooth boundary). We extend ψ to $\Sigma^n \times (-\epsilon, \epsilon)$ by the following equations:

$$\frac{\partial}{\partial s}\psi(\xi, s) = \hat{\nu}(\xi, s)$$

$$\psi(\xi, 0) = \psi_0(\xi)$$

for every ξ in Σ^n and every s in $(-\epsilon, \epsilon)$, where $\hat{\nu}(\xi, s)$ is a unit normal to $\psi(\Sigma^n, s)$ at $\psi(\xi, s)$, such that the maps $\psi^{(s)} = \psi(\cdot, s)$ are nondegenerate. The corresponding induced metric, connection and second fundamental form on Σ are denoted by $g^{(s)}$, $\nabla^{(s)}$, $A^{(s)}$, and the Weingarten matrix is $\Pi^{(s)} = A^{(s)}(g^{(s)})^{-1}$.

Lemma 11 *(Andrews [8]) For Σ and ψ_0 as above, there exists a map $\psi : \Sigma^n \times (-\epsilon, \epsilon) \to N^{n+1}$ satisfying*

$$\frac{\partial}{\partial s}\psi(\xi, s) = \hat{\nu}(\xi, s)$$

$$\psi(\xi, 0) = \psi_0(\xi)$$

for some sufficiently small positive ϵ, and also a constant $C = C(\psi_0, N)$ such that

$$C^{-1} g^{(0)}(u, u) \leq g^{(s)}(u, u) \leq C g^{(0)}(u, u)$$

$$\left|A^{(s)}\right|_{g^{(0)}} \leq C, \ \left|\nabla^{(s)}_u v - \nabla^{(0)}_u v\right|_{g^{(0)}} \leq C$$

for all unit vectors u, v in $T\Sigma^n$.

Let y_0 be a point in N^{n+1}, P is an n-dimensional subspace of $T_{y_0}N$, and e_0 is a unit vector normal to P in $T_{y_0}N$. Define a map $\psi_0 : P \to N$ according to the equation:

$$\psi_0(\xi) = Exp_{y_0}(\xi)$$

for every ξ in P. On a region Σ of P where ψ_0 is nondegenerate, we can get $\psi(\xi, s)$ such that

$$\frac{\partial}{\partial s}\psi(\xi, s) = \hat{v}(\xi, s)$$

$$\psi(\xi, 0) = \psi_0(\xi) = Exp_{y_0}(\xi)$$

where $\hat{v}(\xi, 0) = (T_\xi Exp_{y_0})(e_0)$.

The map ψ produced in this way is called the Graphical Coordinate System over P. The metric on $P \subset T_{y_0}N$ will be denoted by $\langle\cdot,\cdot\rangle$, and the corresponding norm by $|\cdot|$. The standard (flat) connection on P is denoted by d. Then we have the following lemma:

Lemma 12 *(Andrews [8]) Suppose N satisfies*

$$-K_1 \leq \sigma^N \leq K_2,\ |\nabla^N R^N|_{g^N} \leq L$$

with $K_1 = K_2 = L = 1$.

Then the graphical coordinate system ψ over any n-dimensional hyperplane P is nondegenerate on the domain $B_{\rho_0} \times (-\rho_0, \rho_0) \subset P \oplus \mathbb{R}e_0$ for some fixed $\rho_0 > 0$ depending only on n, where $\Sigma = B_{\rho_0}$ is the ball of radius ρ_0 in P. Furthermore, the following estimates hold for some fixed constant C:

$$C^{-1}|u|^2 \leq g^{(s)}(u,u) \leq C|u|^2, |\Pi^{(s)}(u,u)| \leq C$$

$$|\nabla^{(s)}_u v - d_u v| \leq C,\ |\nabla^{(s)}\Pi^{(s)}| \leq C$$

for all u and v in P, where $C = C(\phi_0, K_1, K_2, L) = C(\phi_0)$.

The following lemma connects the flow problem with parabolic equations, and is a key lemma.

Lemma 13 *(Andrews [8]) Let $\psi : \Sigma^n \times (-\epsilon, \epsilon) \to N^{n+1}$ be a nondegenerate map given by*

$$\frac{\partial}{\partial s}\psi(\xi, s) = \hat{v}(\xi, s)$$

$$\psi(\xi,0)=\psi_0(\xi)$$

and $\phi_0 : M^n \to N$ *is a smooth strictly convex immersion. Suppose there exists a nondegenerate map* $\chi_0 : \Sigma^n \to M^n$*, and a smooth function* $s_0 : \Sigma^n \to (-\epsilon,\epsilon)$ *such that*

$$\phi_0(\chi_0(\xi)) = \psi(\xi, s_0(\xi))$$

$$g^N(\nu(\chi_0(\xi)), \hat{\nu}^{(s_0)}(\xi)) > 0$$

for all ξ *in* Σ^n*, where* $\nu(\chi_0(\xi))$ *is the outward normal vector to* $\phi_0(M^n)$ *in* N^{n+1}*. If* $\phi : M^n \times [0,T) \to N$ *is a family of strictly convex immersions satisfying equation (1.1), then for sufficiently small* $t_0 > 0$ *there exist a smooth family of nondegenerate maps* $\chi : \Sigma^n \times [0,t_0) \to M^n$ *and a smooth family of functions* $s : \Sigma^n \times [0,t_0) \to (-\epsilon,\epsilon)$ *such that*

$$\phi_t(\chi_t(\xi)) = \psi(\xi, s_t(\xi))$$

for all (ξ,t) *in* $\Sigma^n \times [0,t_0)$*.*

The following strictly parabolic equation holds on $\Sigma \times [0,t_0)$*:*

$$\frac{\partial}{\partial t}s(\xi,t) = f(\lambda(\mathscr{A}))$$

$$s(\xi,0) = s_0(\xi)$$

where $\mathscr{A}$ *is a matrix, its entry* $\mathscr{A}_{ik} = s_{ik} - h^{(s)}_{ii}\delta^k_i - (h^{(s)}_{ii} + h^{(s)}_{kk})s_i s_k$*,* $s_i = \frac{\partial s}{\partial x_i}$*,* $s_{ik} = \frac{\partial^2 s}{\partial x_i \partial x_k}$*, and* $h^{(s)}_{ik} =< \nabla^{(s)}_{\partial x_i}\hat{\nu}^{(s)}, \partial x_k >$ *depends on* $s(\xi,t)$*, but is independent of* s_i*,* s_{ik}*.*

Conversely, if $s : \Sigma^n \times [0,t_0) \to (-\epsilon,\epsilon)$ *is smooth and satisfies the above equalities, then for every point* (ξ,t_1) *in* $\Sigma^n \times [0,t_0)$*, there exist a manifold* $\bar{M}$ *and a smooth family of diffeomorphism* $\bar{\chi}$ *of* $\bar{M} \times [t_1,t_2)$ *onto regions of* Σ^n *containing* ξ*, for some* $t_2 \in (t_1,t_0)$*, such that the map* $\bar{\phi} : \bar{M} \times [t_1,t_2) \to N$ *given by*

$$\bar{\phi}_t(\bar{x}) = \psi(\bar{\chi}_t(\bar{x}), s_t(\bar{\chi}_t(\bar{x})))$$

is a smooth family of strictly convex immersions satisfying equation (1.1). If s is produced from ϕ *as above, then there exists a nondegenerate map* $\Phi : \bar{M} \to M$ *such that*

$$\phi_t(\Phi(\bar{x})) = \bar{\phi}_t(\bar{x})$$

for all $(\bar{x}, t)$ *in* $\bar{M} \times [0, t_0)$.

2.1 Differentiating eigenvalues and eigenvectors

This section owes much to Andrews' [20].

Lemma 14 *(Andrews [4]) Fix i and j in* $\{1, \cdots, n\}$*. Suppose* f *is a symmetric convex (concave) function, and* f *is defined on* Γ_+ *or* Γ_-*, where* $\Gamma_+ = \{\lambda = (\lambda_1, \cdots, \lambda_n) : \lambda_i > 0, i = 1, \cdots, n\}$*,* $\Gamma_- = \{\lambda | -\lambda \in \Gamma_+\}$*, then if* $\lambda_i \neq \lambda_j$*, we have*

$$\frac{\frac{\partial f}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_j}}{\lambda_i - \lambda_j}$$

is non-negative (non-positive).

Proof: Assume $\lambda_i > \lambda_j$, f is defined on Γ_+. Let η be the vector $e_i - e_j$ in $\mathbb{R}^n$. Define a curve $\tilde{\lambda} : [0, 1] \to \Gamma_+$ by $\tilde{\lambda}(r) = \lambda + r\frac{\lambda_i - \lambda_j}{2}\eta$. Note $D_\eta f = \frac{\partial f}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_j}$.

$$D_\eta f(\tilde{\lambda}(1)) = \frac{\partial f}{\partial \lambda_i}|_{(\lambda_1, \cdots, \lambda_{i-1}, \frac{\lambda_i + \lambda_j}{2}, \lambda_{i+1}, \cdots, \lambda_{j-1}, \frac{\lambda_i + \lambda_j}{2}, \lambda_{j+1}, \cdots, \lambda_n)}$$
$$-\frac{\partial f}{\partial \lambda_j}|_{(\lambda_1, \cdots, \lambda_{i-1}, \frac{\lambda_i + \lambda_j}{2}, \lambda_{i+1}, \cdots, \lambda_{j-1}, \frac{\lambda_i + \lambda_j}{2}, \lambda_{j+1}, \cdots, \lambda_n)}$$

Now by the symmetric property of f, we know

$$\frac{\partial f}{\partial \lambda_i}|_{(\lambda_1, \cdots, \lambda_{i-1}, \frac{\lambda_i + \lambda_j}{2}, \lambda_{i+1}, \cdots, \lambda_{j-1}, \frac{\lambda_i + \lambda_j}{2}, \lambda_{j+1}, \cdots, \lambda_n)}$$
$$= \frac{\partial f}{\partial \lambda_j}|_{(\lambda_1, \cdots, \lambda_{i-1}, \frac{\lambda_i + \lambda_j}{2}, \lambda_{i+1}, \cdots, \lambda_{j-1}, \frac{\lambda_i + \lambda_j}{2}, \lambda_{j+1}, \cdots, \lambda_n)}$$

So $D_\eta f|_{\tilde\lambda(1)} = 0$.

then we have

$$D_\eta f|_{\tilde\lambda(1)} - D_\eta f|_{\tilde\lambda(0)} = \int_0^1 \frac{\partial}{\partial r}(D_\eta f(\tilde\lambda(r)))dr$$
$$= -\int_0^1 \frac{\lambda_i - \lambda_j}{2} D_\eta D_\eta f(\tilde\lambda(r))dr \le 0$$

which is non-positive by convex property of f and $\lambda_i > \lambda_j$. By the above we get $0-(\frac{\partial f}{\partial\lambda_i}-\frac{\partial f}{\partial\lambda_j}) \le 0$ that is $\frac{\partial f}{\partial\lambda_i} - \frac{\partial f}{\partial\lambda_j} \ge 0$, finally we get

$$\frac{\frac{\partial f}{\partial\lambda_i} - \frac{\partial f}{\partial\lambda_j}}{\lambda_i - \lambda_j} \ge 0$$

Similarly we can prove the other cases. □

Theorem 15 *Let f be a C^2 symmetric function defined on a symmetric region Ω in $\mathbb{R}^n$. Let Γ be a region in $\mathbb{R}^m$, and $\mathscr{A}(x)$ be a $n \times n$ matrix valued function defined on Γ. Define*

$$\tilde\Omega = \{\mathscr{A}(x) \in Sym(n) : \lambda(\mathscr{A}(x)) \in \Omega, x \in \Gamma\}$$

and define $F : \tilde\Omega \to \mathbb{R}$ by $F(\mathscr{A}) = f(\lambda(\mathscr{A}))$. Then at any $\mathscr{A} \in \tilde\Omega$, for any unit vector γ in $\mathbb{R}^m$, we have

$$F^{'} = D_\gamma F = \sum_{i,j,k} \dot f^k \mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}^{'}_{ij}$$

If $\mathscr{A}$ has distinct eigenvalues, we have

$$F^{''} = D_{\gamma\gamma}F = \sum_{k,l} \ddot f^{kl}(\mathscr{M}^T\mathscr{A}^{'}\mathscr{M})_{kk}(\mathscr{M}^T\mathscr{A}^{'}\mathscr{M})_{ll} + \sum_{i\ne j}[(\mathscr{M}^T\mathscr{A}^{'}\mathscr{M})_{ij}]^2(\frac{\dot f^j - \dot f^i}{\lambda_j - \lambda_i})$$
$$+ \sum_{i,j,k} \dot f^k \mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}^{''}_{ij}$$

where we assume $\mathscr{M}^T\mathscr{A}\mathscr{M} = diag(\lambda)$, $diag(\lambda)$ is the diagonal matrix whose ii-entry is λ_i, $\lambda = (\lambda_1, \cdots, \lambda_n)$, $\mathscr{M}\mathscr{M}^T = I$, $(\lambda_1, \cdots, \lambda_n)$ are eigenvalues of $\mathscr{A}$, $\mathscr{A}^{'} = D_\gamma\mathscr{A}$, $\mathscr{A}^{'}_{ij} = (D_\gamma\mathscr{A})_{ij}$, $\dot f^k = \frac{\partial f}{\partial\lambda_k}$ and $\ddot f^{kl} = \frac{\partial^2 f}{\partial\lambda_k\partial\lambda_l}$.

Proof: Assume $\mathscr{M}' = \mathscr{M}\Lambda$, then $(\mathscr{M}\mathscr{M}^T)' = (I)' = 0$

$$\Rightarrow \mathscr{M}'\mathscr{M}^T + \mathscr{M}(\mathscr{M}^T)' = 0$$

$$\mathscr{M}\Lambda\mathscr{M}^T + \mathscr{M}\Lambda^T\mathscr{M}^T = 0 \Rightarrow \Lambda + \Lambda^T = 0$$

Now $0 = \mathscr{M}^T\mathscr{A}\mathscr{M} - diag(\lambda_i)$, take the derivatives on both sides, we get, for any $i, j = 1, \cdots, n$,

$$0 = (\mathscr{M}^T\mathscr{A}'\mathscr{M} + (\mathscr{M}^T)'\mathscr{A}\mathscr{M} + \mathscr{M}^T\mathscr{A}\mathscr{M}')_{ij} - \lambda_i'\delta_i^j$$

$$= (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij} + ((\mathscr{M}\Lambda)^T\mathscr{A}\mathscr{M})_{ij} + (\mathscr{M}^T\mathscr{A}(\mathscr{M}\Lambda))_{ij} - \lambda_i'\delta_i^j$$

$$= (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij} + (\Lambda^T(\mathscr{M}^T\mathscr{A}\mathscr{M}))_{ij} + ((\mathscr{M}^T\mathscr{A}\mathscr{M})\Lambda)_{ij} - \lambda_i'\delta_i^j$$

$$= (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij} + \sum_k(\Lambda^T)_{ik}(\mathscr{M}^T\mathscr{A}\mathscr{M})_{kj} + \sum_k(\mathscr{M}^T\mathscr{A}\mathscr{M})_{ik}\Lambda_{kj} - \lambda_i'\delta_i^j$$

$$= (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij} - \sum_k\Lambda_{ik}(\mathscr{M}^T\mathscr{A}\mathscr{M})_{kj} + \sum_k\lambda_k\delta_i^k\Lambda_{kj} - \lambda_i'\delta_i^j$$

$$= (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij} - \sum_k\Lambda_{ik}\lambda_j\delta_j^k + \sum_k\lambda_k\delta_i^k\Lambda_{kj} - \lambda_i'\delta_i^j$$

$$= (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij} - \Lambda_{ij}\lambda_j + \lambda_i\Lambda_{ij} - \lambda_i'\delta_i^j$$

$$= (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij} + (\lambda_i - \lambda_j)\Lambda_{ij} - \lambda_i'\delta_i^j$$

let $i = j$ in the above,

$$\Rightarrow \lambda_i' = (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ii}$$

If $\mathscr{A}$ has distinct eignevalues, we get that for $i \neq j$

$$\Lambda_{ij} = -\frac{(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij}}{\lambda_i - \lambda_j}$$

So $F' = \sum_{k=1}^n \dot{f}^k\lambda_k' = \sum_{k=1}^n \dot{f}^k(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{kk} = \sum_{i,j,k}\dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}_{ij}'$, the first equality is proved.

Now we assume $\mathscr{A}$ has distinct eignevalues, take the derivatives on both sides of

$$0 = (\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij} + (\lambda_i - \lambda_j)\Lambda_{ij} - \lambda_i'\delta_i^j$$

we get

$$0 = ((\mathscr{M}^T)'\mathscr{A}'\mathscr{M} + \mathscr{M}^T\mathscr{A}''\mathscr{M} + \mathscr{M}^T\mathscr{A}'\mathscr{M}')_{ij} + (\lambda_i' - \lambda_j')\Lambda_{ij} + (\lambda_i - \lambda_j)\Lambda_{ij}' - \lambda_i''\delta_i^j$$

in the above take $i = j$, we get

$$\lambda_i'' = ((\mathscr{M}^T)'\mathscr{A}'\mathscr{M} + \mathscr{M}^T\mathscr{A}''\mathscr{M} + \mathscr{M}^T\mathscr{A}'\mathscr{M}')_{ii}$$

Now

$$\begin{aligned}
\sum_k \dot{f}^k\lambda_k'' &= \sum_{k,i,j} \dot{f}^k\mathscr{A}_{ij}'((\mathscr{M}_{ik})'\mathscr{M}_{jk} + \mathscr{M}_{ik}(\mathscr{M}_{jk})') + \sum_{i,j,k}\dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}_{ij}'' \\
&= tr(\mathscr{A}'\mathscr{M}(\Lambda(\dot{f}) + (\dot{f})\Lambda^T)\mathscr{M}^T) + \sum_{i,j,k}\dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}_{ij}'' \\
&= tr(\mathscr{M}^T\mathscr{A}'\mathscr{M}(\Lambda(\dot{f}) + (\dot{f})\Lambda^T)) + \sum_{i,j,k}\dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}_{ij}'' \\
&= \sum_{i,j}(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij}(\Lambda_{ij}\dot{f}^j + \dot{f}^i\Lambda_{ji}) + \sum_{i,j,k}\dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}_{ij}'' \\
&= \sum_{i,j}(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij}\Lambda_{ij}(\dot{f}^j - \dot{f}^i) + \sum_{i,j,k}\dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}_{ij}'' \\
&= \sum_{i\neq j}[(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij}]^2(\frac{\dot{f}^j - \dot{f}^i}{\lambda_j - \lambda_i}) + \sum_{i,j,k}\dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}_{ij}''
\end{aligned}$$

so by all the above we get

$$D_{\gamma\gamma}F = F'' = \sum_{k,l}\ddot{f}^{kl}\lambda_k'\lambda_l' + \sum_k \dot{f}^k\lambda_k''$$

$$= \sum_{k,l}\ddot{f}^{kl}(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{kk}(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ll} + \sum_{i\neq j}[(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij}]^2(\frac{\dot{f}^j - \dot{f}^i}{\lambda_j - \lambda_i}) + \sum_{i,j,k}\dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}\mathscr{A}_{ij}''$$

□

Corollary 16 *Taking the assumptions of Theorem 15, if f is concave (convex), that is $(\ddot{f}^{kl})$ is negative-definite (positive-definite); we get that $(\ddot{F}^{ij,kl}) = (\frac{\partial^2 F}{\partial\mathscr{A}_{ij}\mathscr{A}_{kl}})$ is negative-definite (positive-definite), that is $\sum_{i,j,k,l}\xi_{ij}\ddot{F}^{ij,kl}\xi_{kl} \leq 0 (\geq 0)$ for any matrix (ξ_{ij}).*

Proof: Using Lemma 14 and Theorem 15. □

2.2 Short and Finite Time Existence

In this section, we always assume the assumptions of Theorem 3.

Theorem 17 *(Short Time Existence) If $\mathscr{W}(x,0)$ is strictly positive definite, there exists a unique smooth solution to equations (1.2) on some time interval $[0,T_0)$, where $T_0 > 0$.*

Proof: In Lemma 3.2 of Andrews [8], if we choose $\Sigma^n = M^n$, $\psi_0 = \phi_0$, then we only need to show that there exists a solution for a short time to equations

$$\frac{\partial}{\partial t} s(\xi,t) = f(\lambda(\mathscr{A}))$$

$$s(\xi,0) = s_0(\xi)$$

where $\mathscr{A}$ is a matrix, its entry $\mathscr{A}_{ik} = s_{ik} - h^{(s)}_{ii}\delta^k_i - (h^{(s)}_{ii} + h^{(s)}_{kk})s_i s_k$, $s_i = \frac{\partial s}{\partial x_i}$, $s_{ik} = \frac{\partial^2 s}{\partial x_i \partial x_k}$, and $h^{(s)}_{ik} =< \nabla^{(s)}_{\partial x_i}\hat{\nu}^{(s)}, \partial x_k >$ depends on $s(\xi,t)$, but is independent of s_i, s_{ik}.

Let $\mathscr{F}(x,s,Ds,D^2s) = f(\lambda(\mathscr{A}))$, $\alpha \in (0,1)$, see Chapter IV in Lieberman's [21] for definition of $H_{1+\alpha}$, $H_{2+\alpha}$ etc, where $H_{1+\alpha}$ and $H_{2+\alpha}$ are the space of Holder continuous function in space and time, and 1 time derivative corresponds 2 space derivative. Set $C_0 = 1 + |s_0|_{H_{1+\alpha}}$, and for $\epsilon > 0$ to be choosen later, set $M^n_\epsilon = M^n \times [0,\epsilon)$,

$$\mathbb{S} = \{v \in H_{1+\alpha}(M^n_\epsilon) : |v|_{1+\alpha} \le C_0\}$$

We then define the map $J : \mathbb{S} \to H_{1+\alpha}$ by $s = Jv$, if

$$-s_t + \mathscr{F}(x,v,Dv,D^2s) = 0$$

in M^n_ϵ and $s(x,t) = s_0(x)$ on $M^n \times \{0\}$.

Let $G(x, D^2 s) = \mathscr{F}(x, v(x), Dv(x), D^2 s)$. Notice if $G(x, D^2 s)$ is uniformly parabolic at $D^2 s_0$, i.e.

$$c_1 tr(\eta) \leq G(x, D^2 s_0 + \eta) - G(x, D^2 s_0) \leq c_2 tr(\eta)$$

because M^n is compact, so there exists a constant $\delta > 0$, we can take $\tilde{G}$ to be uniformly parabolic at r such that

$$\frac{c_1}{2} tr(\eta) \leq \tilde{G}(x, r + \eta) - \tilde{G}(x, r) \leq 2c_2 tr(\eta)$$

and $\tilde{G}(x, D^2 s) = G(x, D^2 s)$ for $|D^2 s - D^2 s_0| < \delta$. If G is convex or concave in $D^2 s$, we can take $\tilde{G}$ to be convex or concave in $D^2 s$. If we know $|G(X, r) - G(Y, r)| \leq |X - Y|^\beta [b_1 + b_2 |r|]$ for some positive constants β, b_1, b_2, we can take $\tilde{G}$ such that

$$|\tilde{G}(X, r) - \tilde{G}(Y, r)| \leq |X - Y|^\beta [b_1 + b_2 |r|]$$

So if $G(x, D^2 s)$ satisfies the following three conditions:

(I): G is convex in r.

(II): $c_1 tr(\eta) \leq G(x, D^2 s_0 + \eta) - G(x, D^2 s_0) \leq c_2 tr(\eta)$

(III): $|G(X, r) - G(Y, r)| \leq |X - Y|^\beta [b_1 + b_2 |r|]$

where $c_1, c_2, \beta, b_1, b_2$ are positive constants, G is defined on $M^n \times (S^n)^+$, $(S^n)^+$ denotes the set of all positive real symmetric $n \times n$ matrices, and $G(x, r)$ is well defined for $(x, r) \in M^n \times (S^n)^+$. We can get $\tilde{G}$ satisfying the following three conditions:

($\tilde{I}$): $\tilde{G}$ is convex in r.

($\tilde{II}$): $\frac{c_1}{2} tr(\eta) \leq \tilde{G}(x, r + \eta) - \tilde{G}(x, r) \leq 2c_2 tr(\eta)$

$(I\tilde{I}I)$: $\tilde{G}(X,r) - \tilde{G}(Y,r)| \leq |X - Y|^{\beta}[b_1 + b_2|r|]$

Then define the map $\tilde{J} : \mathbb{S} \to H_{1+\alpha}$ by $s = \tilde{J}v$, if

$$-s_t + \tilde{G}(x, D^2 s) = 0$$

in M^n_ϵ and $s(x,t) = s_0(x)$ on $M^n \times \{0\}$. Because $\tilde{G}$ satisfies the conditions $(\tilde{I})$, $(\tilde{II})$ and $(I\tilde{I}I)$, we can use Corollary 14.9 and Theorem 14.10 in Lieberman's [21] to get

$$|s|_{2+\alpha} \leq CC_0$$

for some positive constant α where C is independent of ϵ and v, $C = C(n, \beta, c_1, c_2, b_1, b_2)$. So $|s|_2 \leq CC_0$, then we have

$$|s - s_0| \leq |s_t| \cdot \epsilon \leq |s|_2 \epsilon \leq CC_0 \epsilon$$

in M^n_ϵ. And we also know

$$|s - s_0|_{2+\alpha} \leq |s|_{2+\alpha} + |s_0|_{2+\alpha} \leq CC_0 + C$$

By the interpolation inequality, we get

$$|s - s_0|_{1+\alpha} \leq c \cdot |s - s_0|^{\frac{1}{2+\alpha}} |s - s_0|^{\frac{\alpha}{2+\alpha}}_{2+\alpha} \leq C\epsilon^{\frac{1}{2+\alpha}}$$

If we choose ϵ such that $C\epsilon^{\frac{1}{2+\alpha}} \leq 1$ and

$$|D^2 s - D^2 s_0| \leq \epsilon^{\alpha} |s - s_0|_{2+\alpha} \leq \epsilon^{\alpha}(CC_0 + C) < \delta,$$

then we get $|s|_{1+\alpha} \leq |s - s_0|_{1+\alpha} + |s_0|_{1+\alpha} \leq 1 + |s_0|_{1+\alpha} = C_0$, hence $\tilde{J}$ maps $\mathbb{S}$ into itself for such an ϵ. Also notice for such ϵ, we have $|D^2 s - D^2 s_0| < \delta$ in M^n_ϵ, so we have $G(x, D^2 s) = \tilde{G}(x, D^2 s)$ in M^n_ϵ. Since $\mathbb{S}$ is a convex compact subset of H_1, by Theorem 8.1 in Lieberman's [21] we know $\tilde{J}$ has a fixed point s. That is $s = \tilde{J}(s) = J(s)$ by the fact $G(x, D^2 s) = \tilde{G}(x, D^2 s)$ in M^n_ϵ.

$$-s_t + \mathscr{F}(x, s, Ds, D^2 s) = 0$$

in M^n_ϵ and $s = s_0$ on $M^n \times \{0\}$. By regularity theory, we have $s \in H_{2+\alpha}(M^n_\epsilon)$, so we get short time existence of the solution.

In the rest of the proof, we will prove the condition (I), (II) and (III) for G, then complete the whole proof.

Firstly we verify (I):

Notice $\mathscr{A} = -A\sqrt{1+|\nabla s|^2_{g(s)}}$, where A is the second fundamental form of $\phi(M^n)$ in N^{n+1}. So we could find orthonormal matrix P such that $PAP^t = (\lambda_i)$, where (λ_i) is the diagonal matrix whose entries are λ_i, $i = 1, 2, \ldots, n$. And by the initial condition we know $\lambda_i > 0$. Choose curvature coordinate system, we get $\mathscr{A}_{ik} = -\lambda_i \delta^k_i$, so

$$\frac{\partial^2 f}{\partial \mathscr{A}^2_{ii}} = \frac{2\mathscr{A}^{-3}_{ii}(-\sum_{k\neq i}\mathscr{A}^{-1}_{kk})}{(\sum_k \mathscr{A}^{-1}_{kk})^3} = \frac{2\lambda^{-3}_i(\sum_{k\neq i}\lambda^{-1}_k)}{(\sum_k \lambda^{-1}_k)^3} \geq 0$$

By Corollary 16, we get $\frac{\partial^2 G}{\partial \mathscr{A}_{ij}\partial \mathscr{A}_{kl}} \geq 0$, so G is convex in r.

Next we verify (II):

Define $\tilde{\lambda}_m = -\lambda_m\sqrt{1+|\nabla s|^2_{g(s)}}$,

$$\begin{aligned} G(x, r+\eta) - G(x, r) &= \sum_{i,j} \frac{\partial G}{\partial r_{ij}} \cdot \eta_{ij} \\ &= tr((\frac{\partial G}{\partial r_{ij}})\eta) \\ &= tr(P^t(\frac{\partial f}{\partial \tilde{\lambda}_m})P\eta) \\ &= tr((\frac{\partial f}{\partial \tilde{\lambda}_m})B) \end{aligned}$$

where $(\frac{\partial G}{\partial r_{ij}})$ is the matrix whose entry is $\frac{\partial G}{\partial r_{ij}}$, and $B = P\eta P^t$, b_{ij} is its entry. Then $G(x, D^2 s_0+\eta) - G(x, D^2 s_0) = \sum_i \frac{\partial f}{\partial \tilde{\lambda}_i} b_{ii}$, by the compactness of M^n we know that there exist positive constants c_1 and c_2 satisfying $c_1 \leq \frac{\partial f}{\partial \tilde{\lambda}_i} \leq c_2$.

Using the fact $tr(B) = tr(\eta)$, we get

$$c_1 tr(\eta) \leq \mathscr{F}(x, r+\eta) - \mathscr{F}(x, r) \leq c_2 tr(\eta)$$

Finally, we verify (III):

$$|G(x,r) - G(y,r)| = |\mathscr{F}(x, v(x), Dv(x), r) - \mathscr{F}(y, v(y), Dv(y), r)|$$

$$\leq |\mathscr{F}_x||x-y| + |\mathscr{F}_v||v|_\beta|x-y|^\beta + |\mathscr{F}_{Dv}||v|_{1+\beta}|x-y|^\beta$$

By Lemma 5, we know $\left|\frac{\partial h_{ii}^{(s)}}{\partial x_k}\right| \leq C$. Using the fact that $\sum_i \left|\frac{\partial f}{\partial \mathscr{A}_{ii}}\right| \leq 1$, we get

$$|\mathscr{F}(x,r) - \mathscr{F}(y,r)| \leq C\,|x-y|^\beta$$

(III) gets verified.

Uniqueness is obvious, by all the above, we prove the theorem. □

Theorem 18 *(Finite Time Extinction) For any strictly convex initial immersion φ_0, there exists a unique smooth solution φ on a finite time interval $[0,T)$ to equation (1.2), $[0,T)$ is the maximal time interval for existence, where $T > 0$.*

Proof: From Theorem 17, we get the short time existence and uniqueness. Firstly, consider some typical case. In $\mathbb{R}^{n+1}$ the flow $\psi(x,t) : S_R^n \times [0,\infty) \to \mathbb{R}^{n+1}$ satisfying the following equations:

$$\frac{\partial}{\partial t}\psi(x,t) = -f_1(\lambda(\mathscr{W}_\psi(x,t)))\vec{v}(x,t)$$

$$\psi(x,0) = S_R^n$$

where $S_R^n = \{x | x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \sum_{i=1}^n x_i^2 = R^2\}$, $f_1(\lambda(\mathscr{W}_\psi)) = f(\lambda(\mathscr{W}_\psi - \frac{1}{4R}Id))$, $\mathscr{W}_\psi$ is the Weingarten matrix of $\psi(S_R^n)$ in $\mathbb{R}^{n+1}$, and $\vec{v}$ is the outward normal vector of $\psi(S_R^n, t)$. Let x_0 be a point in N^{n+1}, because $\sigma^N \leq 0$ and N is simply connected, by Cartan-Hadamard theorem, we know

$$Exp_{x_0} : \mathbb{R}^{n+1} \to N^{n+1}$$

is a diffeomorphism.

Let $S^n_R|_{N^{n+1}} = Exp_{x_0}(S^n_R|_{\mathbb{R}^{n+1}})$, $\zeta : S^n_R \times [0,\infty) \to N^{n+1}$ is another flow satisfying

$$\frac{\partial}{\partial t}\zeta = -f(\lambda(\mathscr{W}_\zeta))\vec{v}$$

$$\zeta(x,0) = S^n_R|_{N^{n+1}}$$

where $\mathscr{W}_\zeta$ is the Weingarten matrix of ζ in N^{n+1}. Let $\rho(x) = d(x_0, x)$, by Hessian Comparision theorem we have

$$D^2\rho|_{N^{n+1}} \geq D^2\rho|_{\mathbb{R}^{n+1}}$$

We know the second fundamental form $A = D^2\rho$ as matrix and $\mathscr{W} = AG^{-1}$, choose curvature coordinate, we see that $\mathscr{W} = D^2\rho$, so we get

$$\mathscr{W}(S^n_R|_{N^{n+1}}) \geq \mathscr{W}(S^n_R|_{\mathbb{R}^{n+1}})$$

Then $S^n_R|_{N^{n+1}}$ is strictly $\frac{1}{2R}$-convex, which means that every eigenvalue of the Weingarten matrix of $S^n_R|_{N^{n+1}}$ is $> \frac{1}{2R}$. Let $\mathscr{W}_\zeta(0) = \mathscr{W}(S^n_R|_{N^{n+1}})$, $\mathscr{W}_\psi(0) = \mathscr{W}(S^n_R|_{\mathbb{R}^{n+1}})$, so $\mathscr{W}_\zeta(0) \geq \mathscr{W}_\psi(0)$. Using the fact f is increasing function in λ_i, we get

$$f(\lambda(\mathscr{W}_\zeta(0))) \geq f(\lambda(\mathscr{W}_\psi(0))) > f_1(\lambda(\mathscr{W}_\psi(0)))$$

So there exists $t_0 > 0$ such that for any $t \in (0, t_0)$, $\psi(t)$ is disjoint from $Exp^{-1}_{x_0}(\zeta(t))$. We claim that $\psi(t)$ is disjoint from $Exp^{-1}_{x_0}(\zeta(t))$ when they exist and $t > 0$. By contradicition, if at (x_1, t_1), they firstly touch each other, where $t_1 > 0$, then

$$\frac{\partial}{\partial t}d(\psi, Exp^{-1}_{x_0}(\zeta)) \geq f(\lambda(\mathscr{W}_\zeta)) - f_1(\lambda(\mathscr{W}_\psi))$$

$$\geq f(\lambda(\mathscr{W}_\psi)) - f(\lambda(\mathscr{W}_\psi - \frac{1}{4R}Id))$$

$$\geq \frac{1}{4R} < inf(\dot{F}), (Id) >$$

$$\geq \frac{1}{n} \cdot \frac{1}{4R}$$

in the last inequality we use the fact

$$< \dot{F}, (Id) >= \sum_i \frac{\partial f}{\partial \lambda_i} \cdot 1 = \frac{(\sum_i \lambda_i^{-2})}{(\sum_k \lambda_k^{-1})^2} \geq \frac{1}{n}$$

which is calculated by using curvature coordinate system. Because it is the first time when they touch, we have $\frac{\partial}{\partial t} d(\psi, Exp_{x_0}^{-1}(\zeta)) \leq 0$. Then we get $0 \geq \frac{1}{n} \cdot \frac{1}{4R}$, it is the contradiction!

Now $\psi(x,t)$ contracts to a single point at finite time, to keep disjoint from $\psi(x,t)$, $Exp_{x_0}^{-1}(\zeta(x,t))$ exists for only finite time. So $\zeta(x,t)$ exists for only finite time. Now there exists $R > 0$, such that $\varphi_0(M^n) \subset B_R^{n+1}$ in N^{n+1}, $\mathscr{W}(\varphi_0(M^n)) \geq \frac{1}{R} Id$, and $\varphi_0(M^n) \cap S_R^n = \emptyset$. Consider the flow:

$$\frac{\partial}{\partial t}\varphi = -f(\lambda(\mathscr{W}))\vec{v}$$

$$\varphi(x,0) = \varphi_0(M^n) \subset N^{n+1}$$

where $\mathscr{W}$ is the Weingarten matrix of $\varphi(M)$ in N. We claim that $\psi(t)$ is disjoint from $Exp_{x_0}^{-1}\varphi(t)$ for $t > 0$ when both flows exist. By contradiction argument like above, if at (x_1, t_1), $\psi(t)$ firstly touches $Exp_{x_0}^{-1}(\varphi(t))$, then becuase at (x_1, t_1), using curvature coordinate:

$$\mathscr{W} = D^2\rho|_{N^{n+1}} \geq D^2\rho|_{\mathbb{R}^{n+1}} = \mathscr{W}_\psi$$

then,

$$0 \geq \frac{\partial}{\partial t} d(\psi, Exp_{x_0}^{-1}(\varphi))$$

$$\geq f(\lambda(\mathscr{W})) - f_1(\lambda(\mathscr{W}_\psi))$$

$$\geq f(\lambda(\mathscr{W}_\psi)) - f(\lambda(\mathscr{W}_\psi - \frac{1}{4R} Id))$$

$$\geq \frac{1}{n} \cdot \frac{1}{4R}$$

Contradictions! So $\varphi(t)$ exists for only finite time. $\square$

2.3 Pinching Estimates

In this section, we always assume the assumptions of Theorem 3.

Proposition 19 *Let ϕ be a solution of (1.2) on the domain $M^n \times [0,T)$, then $\varphi_t(M^n)$ is always strictly convex.*

Proof: By Andrews' [8], we know

$$\frac{\partial F}{\partial t} = \mathscr{L}(F) + F < \dot{F}, (\mathscr{W}^2) > + F < \dot{F}, (\mathscr{R}_{0i0j}) >$$

By Theorem 18, $\varphi_t(M^n)$ is contained in some $S^n_R|_{N^{n+1}}$, we can assume the flow determined by φ_0 is contained in some compact set K, such that there exists some constant $K_1 > 0$, and $0 \geq \sigma^N|_K \geq -K_1$. We know that there exists $t_0 > 0$, when $t \in [0, t_0)$,$\varphi_t(M^n)$ is strictly convex in B^{n+1}_R, by initial strictly convex condition. And when $t \in [0, t_0)$,$F(\lambda) > 0$, and $\frac{\partial f}{\partial \lambda_i} \geq 0$, then using curvature coordinate system at one point to calculate:

$$\begin{aligned}
\frac{\partial F}{\partial t} &= \mathscr{L}(F) + F < \dot{F}, (\mathscr{W}^2) > + F < \dot{F}, (\mathscr{R}_{0i0j}) > \\
&= \mathscr{L}(F) + \sum_i F(\frac{\partial f}{\partial \lambda_i})(\lambda_i^2 + \mathscr{R}_{0i0i}) \\
&\geq \mathscr{L}(F) + F \sum_i (\frac{\partial f}{\partial \lambda_i})(\lambda_i^2 - K_1) \\
&\geq \mathscr{L}(F) + F \sum_i (\frac{\partial f}{\partial \lambda_i})(-K_1) \\
&= \mathscr{L}(F) - FK_1 \sum_i (\frac{\partial f}{\partial \lambda_i})
\end{aligned}$$

Because $\sum_i \frac{\partial f}{\partial \lambda_i} = \frac{\sum_i \lambda_i^{-2}}{(\sum_k \lambda_k^{-1})^2} \leq 1$, we have $\frac{\partial F}{\partial t} \geq \mathscr{L}(F) - FK_1$ when $t \in [0, t_0)$. Consider ODE

$$\frac{\partial \tilde{F}}{\partial t} = -\tilde{F} K_1$$

$$\tilde{F}(0) = \min_{x \in M^n} \tilde{F}(x, 0)$$

then $\tilde{F}(t) = \tilde{F}(0)e^{-K_1 t}$. By Maximum Principle, $F(x,t) \geq \tilde{F}(t) = \tilde{F}(0)e^{-K_1 t}$, $\forall x, t \in [0, t_0)$. Assume $[0, T)$ is the maximum time interval for flow φ_t by Theorem 1 above, and assume $[0, t_1)$ is the maximum time interval for strictly convex φ_t. Then if $t_1 = T$, we are done! Otherwise, if $t_1 < T$, because for $\forall t \in [0, t_1)$

$$F(x,t) \geq \tilde{F}(0)e^{-K_1 t_1} > 0$$

there exists t_2, such that $t_2 > t_1$, $[0, t_2)$ is the maximum time interval for strictly convex φ_t. Contradiction! So $t_1 = T$, we are done. □

Corollary 20 *There exists $C = C(M, \varphi_0, N) > 0$, such that $F(x,t) \geq C$.*

Theorem 21 *(Pinching Estimates) Let φ be a solution of (1.2) on the domain $M^n \times [0, T)$, there exist constants $C = C(M, N, \varphi_0) > 0$, $\beta = \beta(M, N, \varphi_0) > 0$, such that the following estimates hold:*

$$\lambda_i(x,t) \geq C\lambda_j(x,t)\ ,\ \lambda_i(x,t) \geq \beta$$

for all i and j, and all (x,t) in $M^n \times [0, T)$.

Proof: Firstly by Andrews' [8]:

$$\frac{\partial}{\partial t}\omega_i^r = \dot{F}^{kl}\nabla_k\nabla_l\omega_i^r + \ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr}$$

$$+\dot{F}^{kl}(h_{ml}\omega_k^m)\omega_i^r$$

$$+\dot{F}^{st}\mathscr{R}_{st}h_{ij}g^{jr}$$

$$+2\dot{F}^{pm}g^{tr}\omega_m^q\mathscr{R}_{piqt}$$

$$-\dot{F}^{pq}g^{tr}\omega_i^s\mathscr{R}_{psqt} - \dot{F}^{pq}g^{ts}\omega_s^r\mathscr{R}_{piqt}$$

$$+\dot{F}^{pq}g^{tr}(\nabla_i\mathscr{R}_{tpq0}-\nabla_p\mathscr{R}_{qit0})$$

Now define:

$$(I)=\dot{H}^i_r\dot{F}^{kl}(h_{ml}\omega^m_k)\omega^r_i$$

$$(II)=\dot{H}^i_r\dot{F}^{st}\mathscr{R}_{st}h_{ij}g^{jr}$$

$$(III)=2\dot{H}^i_r\dot{F}^{pm}g^{tr}\omega^q_m\mathscr{R}_{piqt}$$

$$(IV)=-\dot{H}^i_r(\dot{F}^{pq}g^{tr}\omega^s_i\mathscr{R}_{psqt}+\dot{F}^{pq}g^{ts}\omega^r_s\mathscr{R}_{piqt})$$

$$(V)=\dot{H}^i_r\dot{F}^{pq}g^{tr}(\nabla_i\mathscr{R}_{tpq0}-\nabla_p\mathscr{R}_{qit0})$$

then

$$\frac{\partial}{\partial t}H=\dot{H}^i_r(\frac{\partial}{\partial t}\omega^r_i)$$

$$=\dot{H}^i_r(\dot{F}^{kl}\nabla_k\nabla_l\omega^r_i)+\dot{H}^i_r\ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr}+(I)+\cdots+(V)$$

Notice

$$\dot{F}^{kl}\nabla_k\nabla_l H=\dot{F}^{kl}\nabla_k(\dot{H}^i_r\nabla_l\omega^r_i)$$

$$=\dot{F}^{kl}\ddot{H}^{i,\tilde{i}}_{r,\tilde{r}}(\nabla_k\omega^{\tilde{r}}_{\tilde{i}})(\nabla_l\omega^r_i)+\dot{F}^{kl}\dot{H}^i_r\nabla_k\nabla_l\omega^r_i$$

So we get:

$$\frac{\partial}{\partial t}H=\dot{F}^{kl}\nabla_k\nabla_l H$$

$$+\dot{H}^i_r\ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr}-\dot{F}^{kl}\ddot{H}^{i,\tilde{i}}_{r,\tilde{r}}(\nabla_k\omega^{\tilde{r}}_{\tilde{i}})(\nabla_l\omega^r_i)$$

$$+(I)+\cdots+(V)$$

Define

$$(\tilde{I})=\dot{H}^i_r\ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr}-\dot{F}^{kl}\ddot{H}^{i,\tilde{i}}_{r,\tilde{r}}(\nabla_k\omega^{\tilde{r}}_{\tilde{i}})(\nabla_l\omega^r_i)$$

$$\Rightarrow\frac{\partial}{\partial t}H=\mathscr{L}(H)+(\tilde{I})+(I)+\cdots+(V)$$

So

$$\frac{\partial}{\partial t}(\frac{H}{F}) = \frac{1}{F^2}(\frac{\partial}{\partial t}H \cdot F - H \cdot \frac{\partial}{\partial t}F)$$

$$= \frac{1}{F}[\mathscr{L}(H) + (\tilde{I}) + (I) + \cdots + (V)] - \frac{H}{F^2}[\mathscr{L}(F) + F < \dot{F}, (\mathscr{W}^2) > + F\dot{F}^{ij}\mathscr{R}_{i0j0}]$$

$$= (\frac{1}{F}\mathscr{L}(H) - \frac{1}{F^2}H \cdot \mathscr{L}(F)) + \frac{(\tilde{I})}{F}$$

$$+ \frac{(I) + (II) - H[< \dot{F}, (\mathscr{W}^2) > + \dot{F}^{ij}\mathscr{R}_{i0j0}]}{F}$$

$$+ \frac{1}{F}[(III) + (IV)] + \frac{1}{F}(V)$$

It's straightforward to verify that

$$(I) + (II) = H[< \dot{F}, (\mathscr{W}^2) > + \dot{F}^{ij}\mathscr{R}_{i0j0}]$$

And

$$\mathscr{L}(\frac{H}{F}) = \dot{F}^{kl}\nabla_k\nabla_l(\frac{H}{F})$$

$$= \dot{F}^{kl}\nabla_k(\frac{(\nabla_l H)F - H(\nabla_l F)}{F^2})$$

$$= \frac{1}{F}\mathscr{L}(H) - \frac{H}{F^2}\mathscr{L}(F) - \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l(\frac{H}{F})$$

Then

$$\frac{\partial}{\partial t}(\frac{H}{F}) = \mathscr{L}(\frac{H}{F}) + \frac{2}{F}\dot{F}^{kl}\nabla_k F\nabla_l(\frac{H}{F}) + \frac{1}{F}[(III) + (IV)] + \frac{(\tilde{I})}{F} + \frac{(V)}{F} \tag{2.1}$$

Choose curvature coordinate system around one point, then we could do the following simple calculation:

$$(\tilde{I}) = \delta^i_r\ddot{F}^{kk,pp}(\nabla_i h_{kk})(\nabla_j h_{pp})\delta^r_j - 0$$

$$= \ddot{F}^{kk,pp}(\nabla_i h_{kk})(\nabla_i h_{pp})$$

But by Corollary 16, we know F is concave from the fact that f is concave. So we get $(\tilde{I}) \leq 0$.

And

$$\left|\frac{(V)}{F}\right| = \left|\frac{1}{F}\frac{\partial f}{\partial \lambda_i}(\nabla_t\mathscr{R}_{tii0} - \nabla_i\mathscr{R}_{itt0})\right|$$

$$\leq \frac{1}{C}C_1(\sum_i \left|\frac{\partial f}{\partial \lambda_i}\right|)$$
$$\leq C_0$$

Where $C_0 = C(M, N, \varphi_0)$. For the first inequality, we use two facts: $F \geq C$ from Corollary 9 and $|\nabla \mathscr{R}| \leq C_1$, which is implied by the fact that the flow is contained in some compact subset of N^{n+1} determined by initial immersion. The second inequality comes from $(\sum_i \left|\frac{\partial f}{\partial \lambda_i}\right|) \leq 1$. Now

$$(III) + (IV) = 2\dot{H}^i_r \dot{F}^{pm} g^{tr} \omega^q_m \mathscr{R}_{piqt} - \dot{H}^i_r (\dot{F}^{pq} g^{tr} \omega^s_i \mathscr{R}_{psqt} + \dot{F}^{pq} g^{ts} \omega^r_s \mathscr{R}_{piqt})$$
$$= 2\delta^i_r \frac{\partial f}{\partial \lambda_p} \delta^m_p \delta^r_t \lambda_q \delta^m_q \mathscr{R}_{piqt} - \delta^i_r (\frac{\partial f}{\partial \lambda_p} \delta^q_p \delta^r_t \lambda_i \delta^s_i \mathscr{R}_{psqt} + \frac{\partial f}{\partial \lambda_p} \delta^q_p \delta^s_t \lambda_s \delta^r_s \mathscr{R}_{piqt})$$
$$= 2\mathscr{R}_{prpr} \frac{\partial f}{\partial \lambda_p}(\lambda_p - \lambda_r)$$
$$= 2\sum_{p<r} \mathscr{R}_{pr}(\frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_r})(\lambda_p - \lambda_r)$$
$$= \sum_{p,r} \mathscr{R}_{pr}(\frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_r})(\lambda_p - \lambda_r)$$
$$= (\sum_k \lambda_k^{-1})^{-2} \cdot \sum_{i,j}(-\mathscr{R}_{ij}) \cdot (\lambda_i - \lambda_j)^2 (\lambda_i + \lambda_j) \cdot \lambda_i^{-2} \lambda_j^{-2}$$
$$\leq K_1 \sum_{i,j}(\lambda_i + \lambda_j) \cdot (\frac{\lambda_i^{-1} - \lambda_j^{-1}}{\sum_k \lambda_k^{-1}})^2$$
$$\leq K_1 \sum_{i,j}(\lambda_i + \lambda_j) = 2K_1 H$$

The first inequality uses the fact: the flow is contained in some compact subset of N^{n+1} so that the curvature of $\varphi(M)$ has a lower bound $-K_1$, where $K_1 = K_1(M, N, \varphi_0)$

$$\Rightarrow \frac{(III) + (IV)}{F} \leq 2K_1 \frac{H}{F}$$

Consider $\psi = \frac{H}{F}$, by all the above, we get

$$\frac{\partial}{\partial t}\psi \leq \mathscr{L}(\psi) + \frac{2}{F}\dot{F}^{kl}(\nabla_k F)(\nabla_l \psi) + C + 2K_1\psi$$

Because

$$\psi = (\sum_i \lambda_i)(\sum_i \lambda_i^{-1}) \geq n$$

we get $C \leq Cn \leq C\psi$. Then,

$$\frac{\partial \psi}{\partial t} \leq \mathscr{L}(\psi) + \frac{2}{F}\dot{F}^{kl}(\nabla_k F)(\nabla_l \psi) + C_3\psi$$

where $C_3 = C + 2K_1$. Consider ODE:

$$\frac{\partial \tilde{\psi}}{\partial t} = C_0\tilde{\psi}$$

$$\tilde{\psi}(0) = \max_{x\in M^n} \psi(x, 0)$$

then

$$\tilde{\psi}(t) = e^{C_0 t}\tilde{\psi}(0) \leq e^{C_0 T}\tilde{\psi}(0) = C(M, N, \varphi_0)$$

By Maximum Principle, we get $\psi(x, t) \leq \tilde{\psi}(t) \leq C$. Then $\frac{\lambda_i}{\lambda_j} \leq \psi \leq C$ for any i, j, so $\lambda_i \geq \frac{1}{C}\lambda_j \geq C(M, N, \varphi_0)\lambda_j$. Assume $\lambda_i = \min\{\lambda_1, \cdots, \lambda_n\}$ and $\lambda_j = \max\{\lambda_1, \cdots, \lambda_n\}$, then

$$\lambda_i \geq C\lambda_j \geq C\frac{n}{\sum_k \lambda_k^{-1}} \geq CnF \geq \beta(M, N, \varphi_0)$$

For the last inequaility, we use the Corollary 20. So for any k, $\lambda_k \geq \beta(M, N, \varphi_0)$. □

2.4 Local estimates

Lemma 22 *Let $\phi : M^n \times [-1, 1] \to N^{n+1}$ be a smooth strictly convex solution to Harmonic Mean Curvature Flow with* $\sup_{M^n} |\mathscr{W}(x, 0)| = 1$*, and suppose $\phi(M^n \times [-1, 1]) \subset \tilde{N}^{n+1}$ which is a submanifold of N^{n+1} and $g^{\tilde{N}}$ is the restriction of g^N on $\tilde{N}^{n+1}$, we have*

$$-K_1 \leq \sigma^{\tilde{N}} \leq K_2, \left|\nabla^{\tilde{N}} R^{\tilde{N}}\right|_{g^{\tilde{N}}} \leq L$$

fro some positive constants K_1, K_2, L, *where* $\sigma^{\tilde{N}}$ *is the sectional curvature of* $\tilde{N}$ *and* $R^{\tilde{N}}$ *is the Riemannian curvature tensor of* $\tilde{N}$. *And assume* $\max\{K_1, K_2, L\} \leq 1$. *Choose* x_0 *in* M^n, *and let* $P = T_{x_0}\phi_0(T_{x_0}M^n) \subset T_{\phi_0(x_0)}N$. *Let* ψ *be the graphical coordinates over* P. *Then on a domain* $B_\delta(0)\times[-\tau,\tau] \subset P\times\mathbb{R}$, *there exists a smooth function* s *corresponding to* ϕ *by equation* $\phi(\chi(\xi,t),t) = \psi(\xi, s(\xi,t))$, *and we have*

$$\sup_{B_\delta\times[-\tau,\tau]} |s| \leq \epsilon$$

$$\sup_{B_\delta\times[-\tau,\tau]} |Ds| \leq 1$$

$$\sup_{(\xi,t)\in(B_\delta\times[-\tau,\tau])} |\mathscr{W}(\chi(\xi,t),t)| \leq 2$$

$$\sup_{B_\delta\times[-\tau,\tau]} |D^2 s| \leq C$$

where ϵ, δ, τ *and* C *are constants depending only on* n.

Proof: At the initial time $t = 0$, we can construct the required map $\chi_0(\cdot) = \chi(\cdot,0)$ and function $s_0(\cdot) = s(\cdot,0)$ giving the graphical parametrisation of ϕ_0. Set $s_0(0,\cdots,0) = 0$ and $\chi_0(0,\cdots,0) = x_0$, where $(0,\cdots,0)\in\mathbb{R}^n$, and extend s_0, χ_0 according to the following differential equations:

$$\nabla^{(s_0)} s_0(\xi) = (T_\xi\psi^{(s_0)})^{-1}(\hat{\upsilon}^{(s_0)}(\xi) - \frac{\upsilon(\chi_0(\xi))}{\langle\upsilon(\chi_0(\xi)), \hat{\upsilon}^{(s_0)}(\xi)\rangle_{g^N}})$$

$$T_\xi\chi_0(u) = (T_{\chi_0(\xi)}\phi_0)^{-1}(T_\xi\psi^{(s_0)}(u) + \nabla_u s_0(\xi)\hat{\upsilon}^{(s_0)}(\xi))$$

where $\psi^{(s_0)}(\xi) = \psi(\xi, s_0(\xi))$, $u\in P$, $\upsilon \perp \phi_0(M)$, $\hat{\upsilon}^{(s_0)} \perp \psi(\cdot, s_0)$. These expressions can be used to solve s_0 and χ_0 along radial curves from the origin in P.

Because $\langle\upsilon(\chi_0(\xi)), \hat{\upsilon}^{(s_0)}(\xi)\rangle_{g^N}^{-1} = \sqrt{1+|\nabla s_0|^2_{g^{(s_0)}}}$, the solution s_0 and χ_0 along such a curve can be extended within the region of definition of ψ as long as $|s_0| < \epsilon$ and $|\nabla s_0|_{g^{(s_0)}}$ remains bounded, for some positive constant ϵ which does not depend on x_0 or ϕ_0. Since $|\mathscr{W}(x,0)| \leq 1$, $\mathscr{A}_{ik} = s_{ik} - h^{(s)}_{ii}\delta^k_i - (h^{(s)}_{ii} + h^{(s)}_{kk})s_i s_k$, at time $t = 0$ we get

$$1 \geq |\mathscr{W}| \geq \left|(\sqrt{1 + |\nabla s_0|^2_{g^{(s_0)}}})^{-1}\mathscr{A}_{ik}\right|$$

$$|Ds|^2 + 1 \geq |\mathscr{A}_{ik}| \geq |s_{ik}| - C(|Ds|^2 + 1)$$

$$\Rightarrow |s_{ik}| \leq C(|Ds|^2 + 1) \leq C(|Ds|^2 + 1)^{\frac{3}{2}}$$

because the above inequality is valid at time $t = 0$, we get

$$|D^2 s_0| \leq C(|Ds_0|^2 + 1)^{\frac{3}{2}}$$

where $C = C(K_1, K_2, L)$. Because $\upsilon = \hat{\upsilon}$ at $((0, \cdots, 0), 0)$ by definition, we get

$$|Ds_0|(0, \cdots, 0) = |\nabla s_0|(0, \cdots, 0) = 0$$

Then by the inequality $|D^2 s| \leq C(|Ds|^2 + 1)^{\frac{3}{2}}$, we get a bound on $|Ds_0|$ on a ball of radius r_0 where $r_0 = r_0(K_1, K_2, L)$. We assume that we have taken r_0 sufficiently small to ensure that $|Ds_0| \leq \frac{1}{2}$. By taking r_0 smaller if necessary, this also ensures $|s_0| \leq \frac{\epsilon}{2}$, where $r_0 = r_0(K_1, K_2, L)$.

Note that this also implies a bound on $|\nabla s|_{g^{(s)}}$, since $g^{(s)}$ is uniformly equivalent to the metric on P in the region considered. That is

$$|Ds|_{g^{(s)}} \leq C(K_1, K_2, L)|Ds_0| \leq C(K_1, K_2, L)$$

(I) Step 1: estimate $|\mathscr{W}|$.

Let $Q = |\mathscr{W}| = \sqrt{\sum_{i,k}(\omega^k_i)^2}$, $\dot{Q}^i_r = \frac{\partial Q}{\partial \omega^r_i} = \frac{\omega^r_i}{Q}$ and $\ddot{Q}^{p,k}_{q,l} = \frac{\partial^2 Q}{\partial \omega^q_p \omega^l_k}$. Consider the equation as in the proof of Theorem 21:

$$\frac{\partial}{\partial t}Q = \dot{Q}^i_r(\frac{\partial}{\partial t}\omega^r_i)$$

$$= \dot{Q}^i_r \dot{F}^{kl}\nabla_k\nabla_l\omega^r_i + \dot{Q}^i_r \ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr}$$

$$+\dot{Q}^i_r\dot{F}^{kl}(h_{ml}\omega^m_k)\omega^r_i$$

$$+\dot{Q}^i_r\dot{F}^{st}\mathscr{R}_{st}h_{ij}g^{jr}$$

$$+2\dot{Q}^i_r\dot{F}^{pm}g^{tr}\omega^q_m\mathscr{R}_{piqt}$$

$$-\dot{Q}^i_r(\dot{F}^{pq}g^{tr}\omega^s_i\mathscr{R}_{psqt}+\dot{F}^{pq}g^{ts}\omega^r_s\mathscr{R}_{piqt})$$

$$+\dot{Q}^i_r\dot{F}^{pq}g^{tr}(\nabla_i\mathscr{R}_{tpq0}-\nabla_p\mathscr{R}_{qit0})$$

Now define:

$$(I)=\dot{Q}^i_r\dot{F}^{kl}(h_{ml}\omega^m_k)\omega^r_i$$

$$(II)=\dot{Q}^i_r\dot{F}^{st}\mathscr{R}_{st}h_{ij}g^{jr}$$

$$(III)=2\dot{Q}^i_r\dot{F}^{pm}g^{tr}\omega^q_m\mathscr{R}_{piqt}$$

$$(IV)=-\dot{Q}^i_r(\dot{F}^{pq}g^{tr}\omega^s_i\mathscr{R}_{psqt}+\dot{F}^{pq}g^{ts}\omega^r_s\mathscr{R}_{piqt})$$

$$(V)=\dot{Q}^i_r\dot{F}^{pq}g^{tr}(\nabla_i\mathscr{R}_{tpq0}-\nabla_p\mathscr{R}_{qit0})$$

using curvature coordinate system, we get

$$\frac{\partial}{\partial t}Q=\mathscr{L}(Q)+(\dot{Q}^i_i\ddot{F}^{kl,pq}-\dot{F}^{ii}\ddot{Q}^{p,k}_{q,l})\nabla_i(h_{kl})\nabla_i(h_{pq})+(I)+\cdots+(V)$$

Note $\dot{Q}^i_i\geq 0$, $\dot{F}^{ii}\geq 0$, $\sum_{kl,pq}\ddot{F}^{kl,pq}\nabla_i(h_{kl})\nabla_i(h_{pq})\leq 0$, and $\ddot{Q}^{p,k}_{q,l}\nabla_i(h_{kl})\nabla_i(h_{pq})\geq 0$, we have

$$\frac{\partial}{\partial t}Q\leq\mathscr{L}(Q)+(I)+\cdots+(V)$$

using Pinch Estimates in Theorem 21 and $\max\{K_1,K_2,L\}\leq 1$, it is easy to get

$$(II)+\cdots+(V)\leq CQ$$

also it is straightforward to get $(I)\leq CQ^3$, where C depends only on K_1,K_2,L and the pinching bound in Theorem 21. Finally we get the following

$$\frac{\partial}{\partial t}|\mathscr{W}|\leq\mathscr{L}(|\mathscr{W}|)+C|\mathscr{W}|^3+C|\mathscr{W}|$$

Let $W(t) = \sup_{x\in M^n} |\mathscr{W}(x,t)|$, then $W(0) = 1$. by

$$\frac{\partial}{\partial t}|\mathscr{W}| \le \mathscr{L}(|\mathscr{W}|) + C|\mathscr{W}|^3 + C|\mathscr{W}|$$

we get

$$\frac{\partial}{\partial t}W(t) \le C(W^3(t) + W(t))$$
$$\Rightarrow \frac{\partial}{\partial t}\ln(\frac{W^2(t)}{W^2(t)+1}) \le C$$
$$\Rightarrow \ln(\frac{W^2(t_1)}{W^2(t_1)+1}) - \ln(\frac{W^2(0)}{W^2(0)+1}) \le Ct_1$$

finally we get $W^2(t_1) \le \frac{e^{Ct_1}}{2-e^{Ct_1}}$.

So we can find a small interval $[0,\tau]$ on which

$$\sup_{B_{r_0}(0)\times[0,\tau]} |\mathscr{W}| \le 2$$

where $\tau = \tau(K_1, K_2, L, n, f)$ and $r_0 = r_0(K_1, K_2, L)$.

(II) Step 2: estimate $|s|$.

On this time interval $[0,\tau]$, we also have a bound $F \le \frac{|\mathscr{W}|}{\sqrt{n}} \le \frac{2}{\sqrt{n}}$. On $[0,\tau]$, the solution stays in a neighborhood of width $\frac{2\tau}{\sqrt{n}}$ of the initial immersion ϕ_0. Note $\frac{\partial}{\partial t}s = -\sqrt{1+|\nabla s|^2_{g(s)}}F$. By the above argument, we know that $|\nabla s|_{g(s)}$ and F are bounded on $B_{r_0}(0)\times[0,\tau]$ by some constant C if we choose τ sufficiently small. So we get $s_0 - C\tau \le s \le s_0 + C\tau$ on $B_{r_0}(0)\times[0,\tau]$. Because $|s_0| \le \frac{\epsilon}{2}$, we can choose τ small enough to ensure that $|s| \le \epsilon$ on $B_{\frac{r_0}{2}} \times [0,\tau]$, where $\tau = \tau(K_1, K_2, L, n, f)$.

(III) Step 3: estimate $|Ds|$.

Note

$$D_i|Ds| = D_i(\sqrt{\sum_j (D_j s)^2}) = \frac{\sum_j D_j s D_{ij} s}{\sqrt{\sum_j (D_j s)^2}} \le \sqrt{\sum_j (D_{ij} s)^2}$$

we have

$$|D|Ds|| = \sqrt{\sum_i (D_i|Ds|)^2} \leq \sqrt{\sum_{i,j} |D_{ij}s|^2} = |D^2 s|$$

by the former inequality about $|D^2 s|$, we get

$$|D|Ds|| \leq C(1 + |Ds|^2)^{\frac{3}{2}}$$

We want to estimate $|Ds|$ on $B_{\frac{r_0}{2}}(0) \times [0, \tau]$, choose $t \in [0, \tau]$, any point x in $B_{\frac{r_0}{3}}(0)$. Consider a curve $\gamma(\rho) : \mathbb{R} \to P$ which begins at point x and follows the direction of the steepest ascent of s and $|\gamma^{'}(\rho)| = 1$, which means that $(D_\rho s)(\gamma(\rho), t) \geq \frac{1}{n}|Ds(\gamma(\rho), t)|$. Then define $f_0(\rho) = |Ds|(\gamma(\rho), t)$, we have

$$f_0^{'}(\rho) \leq |D|Ds|| \leq C(1 + |Ds|^2)^{\frac{3}{2}} = C(1 + |f_0|^2)^{\frac{3}{2}}$$

$$\Rightarrow \frac{f_0^{'}(\rho)}{(\sqrt{1 + f_0^2})^3} \leq C$$

Let $f_0(\rho) = \tan(\theta(\rho))$, we get

$$(\cos\theta)^{'} \leq C$$

$$\Rightarrow \int_{\gamma(\rho)} \cos\theta d\theta \leq \int_{\gamma(\rho)} C$$

by $(-\sin\theta) = -\sqrt{\frac{f_0^2}{f_0^2+1}}$ we get

$$-\sqrt{\frac{f_0^2(r)}{f_0^2(r) + 1}} + \sqrt{\frac{f_0^2(0)}{f_0^2(0) + 1}} \leq Cr$$

where we assume γ is defined on $[0, r]$.

$$\Rightarrow f_0(r) \geq \frac{A - Cr}{\sqrt{1 - (A - Cr)^2}}$$

where $A = \frac{|Ds|(\gamma(0),t)}{\sqrt{1+|Ds(\gamma(0),t)|^2}} = \frac{f_0(0)}{\sqrt{1+f_0^2(0)}}$.

$$|Ds|(\gamma(r), t) \geq \frac{A - Cr}{\sqrt{1 - (A - Cr)^2}}$$

this holds for $A - Cr \geq 0$. Integrating the above inequality, we get

$$\int_{\gamma(\rho)} D_\rho s d\rho \geq \int_{\gamma(\rho)} \frac{1}{n}|Ds| \geq \frac{1}{n}\int_0^r \frac{A - C\rho}{\sqrt{1-(A-C\rho)^2}} d\rho$$

$$\Rightarrow s(\gamma(r),t) - s(\gamma(0),t) \geq (\frac{1}{nC}\sqrt{1-(A-C\rho)^2})|_0^r$$

$$= (nC)^{-1}(\sqrt{1-(A-Cr)^2} - \sqrt{1-A^2}) \geq (nC)^{-1}(\sqrt{1-(A-(nC)r)^2} - \sqrt{1-A^2})$$

$$= C_1^{-1}(\sqrt{1-(A-C_1 r)^2} - \sqrt{1-A^2})$$

in the last equlity we choose $C_1 = nC$. We want to show $|Ds|(x,t) = |Ds|(\gamma(0),t) \leq 1$. By contradiction, suppose $f_0(0) = |Ds|(\gamma(0),t) > 1$, then $A = \frac{f_0(0)}{\sqrt{1+f_0^2(0)}} > \frac{1}{\sqrt{2}}$. For $r \leq \frac{2}{3C_1}(\sqrt{2}-1)$, we have $r \leq \frac{2}{3C_1}(2A - \sqrt{2}\sqrt{1-A^2})$, which is equivalent to

$$C_1^{-1}(\sqrt{1-(A-C_1 r)^2} - \sqrt{1-A^2}) \geq \frac{r}{\sqrt{2}}$$

we get $s(\gamma(r),t) - s(\gamma(0),t) \geq \frac{r}{\sqrt{2}}$. But

$$s(\gamma(r),t) - s(\gamma(0),t) = [s(\gamma(r),t) - s(\gamma(r),0)]$$

$$+[s(\gamma(r),0) - s(\gamma(0),0)] + [s(\gamma(0),0) - s(\gamma(0),t)]$$

$$\leq Ct + |Ds_0|r + Ct \leq C_2 t + \frac{r}{2}$$

in the second inequality we use $\frac{\partial}{\partial t}s \leq C$ and in the last inequality we use $|Ds_0| \leq \frac{1}{2}$. Because $\gamma(0) = x \in B_{\frac{r_0}{3}}(0)$, consider paths γ of fixed length r, where

$$r \leq \min\{\frac{r_0}{6}, \frac{2}{3C_1}(\sqrt{2}-1)\}$$

then $\gamma(r) \in B_{\frac{r_0}{2}}(0)$. If $t \leq \tau < C_2^{-1}r(\frac{1}{\sqrt{2}} - \frac{1}{2})$, We have

$$s(\gamma(r),t) - s(\gamma(0),t) \geq \frac{r}{\sqrt{2}} > \frac{1}{2}r + C_2 t$$

It's the contradiction. We restrict $\tau < C_2^{-1} r(\frac{1}{\sqrt{2}} - \frac{1}{2})$ further, finally we get

$$\sup_{B_{\frac{r_0}{3}} \times [0,\tau]} |Ds| \leq 1$$

The same techniques show that the solution can be extended backward in time to $-\tau$, since we have assumed a curvature bound on $[-\tau, 0]$ too.

(IV) Step 4: estimate $|D^2 s|$ By (I) we know that on $B_{r_0} \times [0, \tau]$, $|\mathscr{W}| \leq 2$, we can get

$$|D^2 s| \leq C(|Ds|^2 + 1)^{\frac{3}{2}} \leq C(K_1, K_2, L) \leq C$$

where the last second inequality used (III) and the last inequality used the assumption $\max\{K_1, K_2, L\} \leq 1$ □

Theorem 23 *Let $\phi : M^n \times [-1, 1] \to N^{n+1}$ be a smooth strictly convex solution to harmonic mean curvature flow with* $\sup_{M^n} |\mathscr{W}(x, 0)| = |\mathscr{W}(x_0, 0)| = 1$, *and suppose $\phi(M^n \times [-1, 1]) \subset \tilde{N}^{n+1}$ which is a submanifold of N^{n+1} and $g^{\tilde{N}}$ is the restriction of g^N on $\tilde{N}^{n+1}$, we have*

$$-K_1 \leq \sigma^{\tilde{N}} \leq K_2, \left|\nabla^{\tilde{N}} R^{\tilde{N}}\right|_{g^{\tilde{N}}} \leq L$$

fro some positive constants K_1, K_2, L. And assume $\max\{K_1, K_2, L\} \leq 1$. *Let $P = T_{x_0}\phi_0(T_{x_0} M^n) \subset T_{\phi_0(x_0)} N$. Let ψ be the graphical coordinates over P. Then on a domain $B_\delta(0) \times [-\tau, \tau] \subset P \times \mathbb{R}$ there exists a smooth function s corresponding to ϕ by equation $\phi(\chi(\xi, t), t) = \psi(\xi, s(\xi, t))$, then*

$$||\mathscr{W}||_{C^{0,\beta(x)}(B_\delta(x) \times [\frac{3}{4}\tau, \tau])} \leq C(x)$$

where δ and τ is a constant depending only on n, $\beta(x)$ and $C(x)$ are functions of n and $d_0(x, x_0)$, d_0 being the distance in M^n with respect to g at time $t = 0$.

Proof: We choose curvature coordinate system at point (x, t), and we have

$$\frac{\partial}{\partial t} s(\xi, t) = f(\lambda(\mathscr{A}))$$

where $\mathscr{A}$ is a matrix, its entry $\mathscr{A}_{ik} = s_{ik} - h^{(s)}_{ii}\delta^k_i - (h^{(s)}_{ii} + h^{(s)}_{kk})s_i s_k$, $s_i = \frac{\partial s}{\partial x_i}$, $s_{ik} = \frac{\partial^2 s}{\partial x_i \partial x_k}$, $h^{(s)}_{ik} =< \nabla^{(s)}_{\partial x_i}\hat{\nu}^{(s)}, \partial x_k >$ is independent of $s(\xi, t)$, $f(\lambda) = (\sum_{k=1}^n \lambda_k^{-1})^{-1}$ and $\lambda = (\lambda_1, \cdots, \lambda_n)$. Let $\mathscr{F}(x, s, Ds, D^2 s) = f(\lambda(\mathscr{A}))$, and $x_0 = (0, \cdots, 0)$ for simplicity. Assume s is defined on $B_{\delta_1}(0) \times [-\tau_1, \tau_1]$ as in Lemma 22, choose $\sqrt{\tau} = \delta = \frac{1}{8}\min\{\delta_1, \sqrt{\tau_1}\}$. Firstly we want to estimate $\|s_t\|_{C^\alpha(B_\delta(0)\times[0,\tau])}$ which is written in $\|s_t\|_{C^\alpha}$ for simplicity in the following, where $s_t = \frac{\partial s}{\partial t}$, by Theorem 15, we get

$$\frac{\partial s_t}{\partial t} = \sum_{k=1}^n \dot{f}^k (\frac{\partial \lambda_k}{\partial t}) = \sum_{i,j,k} \dot{f}^k \mathscr{M}_{ik}\mathscr{M}_{jk}\frac{\partial \mathscr{A}_{ij}}{\partial t}$$

Let $g_{ij} = \sum_{k=1}^n \dot{f}^k \mathscr{M}_{ik}\mathscr{M}_{jk}$, then as a matrix,

$$(g_{ij}) = (\mathscr{M}(\dot{f})\mathscr{M}^T)_{ij}$$

, where $(\dot{f}) = diag\{\dot{f}^1, \cdots, \dot{f}^n\}$. But $\dot{f}^k = \frac{\lambda_k^{-2}}{(\sum_i \lambda_i^{-1})^2}$. By the fact $\lambda_k = -\sqrt{1 + |\nabla s|^2}\tilde{\lambda}_k$ and the pinching estimates of $\tilde{\lambda}_k$ in Theorem 21 (where $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n)$ are the principal curvatures of $\phi(M)$ in N^{n+1}), we know $C_1 \le (\dot{f}) = \frac{\tilde{\lambda}_k^{-2}}{(\sum_i \tilde{\lambda}_i^{-1})^2} \le C_2$ as matrix, where $C_i = C_i(M, N, \phi_0)$ $(i = 1, 2)$, and C_i is independent of rescaling metric of M^n, N^{n+1}. Because $\mathscr{M}$ is an orthogonal matrix, we get $C_1 \le (g_{ij}) \le C_2$ as matrix.

Let $v = s_t$, we get

$$-v_t + \sum_{i,j} g_{ij}(\mathscr{A}_{ij})_t = 0$$

where

$$(\mathscr{A}_{ij})_t = (s_{ij} - h^{(s)}_{ii}\delta^j_i - (h^{(s)}_{ii} + h^{(s)}_{jj})s_i s_j)_t$$

we get

$$-v_t + \sum_{i,j} g_{ij}v_{ij} + \sum_i b_i v_i + cv = 0$$

where $b_i = -2\sum_j g_{ij}s_j(h_{ii}^{(s)} + h_{jj}^{(s)})$ and $c = -\sum_{i,j} g_{ij}(\frac{\partial h_{ii}^{(s)}}{\partial s}\delta_i^j + \frac{\partial h_{ii}^{(s)}}{\partial s}s_i s_j + \frac{\partial h_{jj}^{(s)}}{\partial s}s_i s_j)$. Then b_i and c are all bounded because $|s|$, $|DS|$, $\left|D^2 s\right|$, $|s_t| = |f(\lambda(\mathscr{A}))|$ and the derivatives of $h_{ii}^{(s)}$ are all bounded, and we also have $C_1 \leq (g_{ij}) \leq C_2$ as matrix. By Corollary 7.26 of Lieberman's [21], there exists a positive constant α, such that

$$\|v\|_{C^\alpha(B_\delta(0)\times[0,\tau])} \leq C(M, N, \phi_0)$$

where $C(M, N, \phi_0)$ is independent of rescaling metric of M^n, N^{n+1}.

Secondly, we want to estimate $\left\|D^2 s\right\|_{C^\beta}$ for some positive constant β. Let γ be any unit vector in $\mathbb{R}^n$. By Theorem 15, if $\mathscr{A}$ has distinct eigenvalues, we have

$$\frac{\partial s_{\gamma\gamma}}{\partial t} = \sum_{k,l} \ddot{f}^{kl}(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{kk}(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ll} + \sum_{i,j,k} \dot{f}^k \mathscr{M}_{ik}\mathscr{M}_{jk}(\mathscr{A}_{ij})_{\gamma\gamma} + \sum_{i\neq j}[(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij}]^2(\frac{\dot{f}^j - \dot{f}^i}{\lambda_j - \lambda_i})$$

where $\mathscr{A}' = D_\gamma\mathscr{A}$, $(\mathscr{A}_{ij})_\gamma = D_\gamma(\mathscr{A}_{ij})$. Now We have

$$\ddot{f}^{kl} = \frac{2\lambda_k^{-2}\lambda_l^{-2} - 2\lambda_k^{-3}\delta_k^l(\sum_i \lambda_i^{-1})}{(\sum_i \lambda_i^{-1})^3}$$

choose any $\xi = (\xi^1, \cdots, \xi^n) \in \mathbb{R}^n$, we get

$$\sum_{k,l} \ddot{f}^{kl}\xi^k\xi^l = \frac{2(\sum_i \xi^i\lambda_i^{-2})^2 - 2(\sum_i(\xi^i)^2\lambda_i^{-3})(\sum_i \lambda_i^{-1})}{(\sum_i \lambda_i^{-1})^3}$$

Note $\lambda_i = -\sqrt{1 + |\nabla s|^2}\tilde{\lambda}_i \leq 0$ and using Schwartz's inequality we get

$$\sum_{k,l} \ddot{f}^{kl}\xi^k\xi^l \geq 0$$

We know $\ddot{f}^{kl} \geq 0$ as matrix, that is f is convex. By Lemma 14, we get $(\frac{\dot{f}^j - \dot{f}^i}{\lambda_j - \lambda_i}) \geq 0$, so

$$\sum_{i\neq j}[(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ij}]^2(\frac{\dot{f}^j - \dot{f}^i}{\lambda_j - \lambda_i}) \geq 0$$

$$\Rightarrow \frac{\partial s_{\gamma\gamma}}{\partial t} \geq \ddot{f}^{kl}(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{kk}(\mathscr{M}^T\mathscr{A}'\mathscr{M})_{ll} + \dot{f}^k\mathscr{M}_{ik}\mathscr{M}_{jk}(\mathscr{A}_{ij})_{\gamma\gamma}$$

$$\geq g_{ij}(\mathscr{A}_{ij})_{\gamma\gamma}$$

By continuity, we get $\frac{\partial s_{\gamma\gamma}}{\partial t} \geq g_{ij}(\mathscr{A}_{ij})_{\gamma\gamma}$ for any symmetric matrix $\mathscr{A}$.

$$\Rightarrow \frac{\partial s_{\gamma\gamma}}{\partial t} \geq g_{ji}(s_{ij} - h_{ii}^{(s)}\delta_i^j - (h_{ii}^{(s)} + h_{jj}^{(s)})s_i s_j)_{\gamma\gamma}$$

$$\geq g_{ij}(s_{\gamma\gamma})_{ij} - A_0 \left|D^3 s\right| - B_0$$

where

$$A_0 = C \times (\sup_{i,s} \left|h_{ii}^{(s)}\right|)(\sup_k |s_k|) = A_0(M, N, \phi_0)$$

$$B_0 = C \times (\sup_{s,i} \left|\frac{\partial^2 h_{ii}^{(s)}}{\partial \gamma^2}\right| + (\sup \left|D^2 s\right| + 1)^2 (\sup |Ds| + 1)^2 \sup_{s,i}(\left|h_{ii}^{(s)}\right| + \left|D_\gamma h_{ii}^{(s)}\right| + \left|D_{\gamma\gamma} h_{ii}^{(s)}\right|))$$

$$= B_0(M, N, \phi_0)$$

Let $M_2 = \sup_{B_{\delta_1}(0)\times[-\tau_1,\tau_1]} \left|D^2 s\right|$, $h_k = \frac{1}{2}(1 - \frac{s_{\gamma_k\gamma_k}}{1+M_2})$, γ_k will be chosen later, then we could get

$$-\frac{\partial h_k}{\partial t} + \sum_{i,j} g_{ij}(h_k)_{ij} \geq -C\frac{A_0 \left|D^3 s\right| + B_0}{1 + M_2}$$

We also have the following useful lemma,

Lemma 24

$$g_{ij}(Y)(s_{ij}(Y) - s_{ij}(X)) \geq F(Y, s(Y), Ds(Y), D^2 s(Y)) - F(Y, s(Y), Ds(Y), D^2 s(X))$$

Proof: Take $g_{ij}(Y) = g_{ij}(Y, s(Y), Ds(Y), D^2 s(Y))$, then

$$g_{ij}(Y)(s_{ij}(Y) - s_{ij}(X)) - [F(Y, s(Y), Ds(Y), D^2 s(Y)) - F(Y, s(Y), Ds(Y), D^2 s(X))]$$

$$= \int_0^1 (g_{ij}(Y)(s_{ij}(Y) - s_{ij}(X)) - \frac{\partial}{\partial t} F(Y, s(Y), Ds(Y), tD^2 s(Y) + (1-t)D^2 s(X))dt$$

$$= \int_0^1 (g_{ij}(Y, s(Y), Ds(Y), D^2 s(Y)) - g_{ij}(Y, s(Y), Ds(Y), tD^2 s(Y) + (1-t)D^2 s(X)))(s_{ij}(Y) - s_{ij}(X))dt$$

$$= \int_0^1 \int_0^1 \frac{\partial}{\partial \eta} g_{ij}(Y, s(Y), Ds(Y), \eta D^2 s(Y) + (1-\eta)(tD^2 s(Y) + (1-t)D^2 s(X)))d\eta dt \cdot (s_{ij}(Y) - s_{ij}(X))$$

$$= (\int_0^1 \int_0^1 (g_{ij,kl}) \cdot (1-t) d\eta dt) \cdot (s_{kl}(Y) - s_{kl}(X))(s_{ij}(Y) - s_{ij}(X))$$

but by the argument above we know that $(g_{ij,kl}) \geq \ddot{f}^{pq} \mathscr{M}_{ip} \mathscr{M}_{jp} \mathscr{M}_{kq} \mathscr{M}_{lq}$, so $(g_{ij,kl})$ is a nonnegative definite matrix.

So the last term is nonnegative. The lemma is proved. □

By Lemma 17.13 in [22], we can get a family of unit vectors $\{\gamma_k\}_{k=1}^{N_2}$ which include the coordinate directions e_i, $i = 1, \cdots, n$, together with the directions $\frac{1}{\sqrt{2}}(e_i \pm e_j)$, $i < j$, $i, j = 1, \cdots, n$, such that $(g_{ij}) = \sum_{k=1}^{N_2} \beta_k \gamma_k \otimes \gamma_k$. By our choice of γ_k, we can estimate

$$\left|D^3 s\right|^2 = \sum_{i,j,l} \left|D_{ijl} s\right|^2 \leq 4(1+M_2)^2 \sum_{k=1}^{N_2} |Dh_k|^2$$

Let $\omega_k = h_k + \epsilon \upsilon$, $k = 1, 2, \cdots, N_2$, $\upsilon = \sum_{i=1}^{N_2} (h_i)^2$, ϵ is some positive constant which will be determined later. We have

$$-\frac{\partial \omega_k}{\partial t} + \sum_{i,j} g_{ij} (\omega_k)_{ij} = -\frac{\partial h_k}{\partial t} + \sum_{i,j} g_{ij} (h_k)_{ij}$$

$$+2\epsilon \sum_{i,j,p} h_p g_{ij} (h_p)_{ij} - 2\epsilon \sum_p h_p \frac{\partial h_p}{\partial t}$$

$$+2\epsilon \sum_{i,j,p} g_{ij} (h_p)_i (h_p)_j$$

because for any $p = 1, \cdots, N_2$, we have

$$-\frac{\partial h_p}{\partial t} + \sum_{i,j} g_{ij} (h_p)_{ij} \geq -C \frac{A_0 \left|D^3 s\right| + B_0}{1 + M_2}$$

Times the above inequality by $2\epsilon h_p$, take the sum over p, note $0 \leq h_p \leq 1$, we get

$$2\epsilon \sum_{i,j,p} h_p g_{ij} (h_p)_{ij} - 2\epsilon \sum_p h_p \frac{\partial h_p}{\partial t} \geq -2\epsilon \sum_p h_p (C \frac{A_0 \left|D^3 s\right| + B_0}{1 + M_2})$$

$$\geq -C \frac{A_0 \left|D^3 s\right| + B_0}{1 + M_2}$$

So

$$-\frac{\partial\omega_k}{\partial t}+\sum_{i,j} g_{ij}(\omega_k)_{ij} \geq -C\frac{A_0\left|D^3 s\right|+B_0}{1+M_2}+2\epsilon\sum_{i,j,p} g_{ij}(h_p)_i(h_p)_j$$

$$\geq -\frac{CA_0}{1+M_2}\left|D^3 s\right|-\frac{CB_0}{1+M_2}+2\epsilon C_1\sum_p\left|Dh_p\right|^2$$

(where we use $C_1 \leq (g_{ij}) \leq C_2$ as the matrix)

$$\geq 2C_1\epsilon\sum_p\left|Dh_p\right|^2-\frac{CA_0}{1+M_2}\sqrt{4(1+M_2)^2\sum_{p=1}^{N_2}\left|Dh_p\right|^2}-\frac{CB_0}{1+M_2}$$

$$\geq 2C_1\epsilon\sum_p\left|Dh_p\right|^2-2CA_0\sqrt{\sum_{p=1}^{N_2}\left|Dh_p\right|^2}-\frac{CB_0}{1+M_2}$$

but

$$2CA_0\sqrt{\sum_{p=1}^{N_2}\left|Dh_p\right|^2}\leq 2C_1\epsilon\sum_p\left|Dh_p\right|^2+\frac{C^2}{2C_1\epsilon}A_0^2$$

$$\Rightarrow -\frac{\partial\omega_k}{\partial t}+\sum_{i,j} g_{ij}(\omega_k)_{ij}\geq -\frac{C^2}{2C_1\epsilon}A_0^2-\frac{CB_0}{1+M_2}=-C_1\bar{\mu}$$

which is valid on $Q(X_0,8\delta)\subset B_{\delta_1}(0)\times[-\tau_1,\tau_1]$, for any $k=1,\cdots,N_2$, where $\bar{\mu}=\frac{1}{C_1}(\frac{C^2}{2C_1\epsilon}A_0^2+\frac{CB_0}{1+M_2})$, define $|x|=\sqrt{\sum_{i=1}^n(x^i)^2}$, $|X|=\max\{|x|,|t|^{\frac{1}{2}}\}$, and $Q(X_0,R)=Q(R)=\{X\in\mathbb{R}^{n+1}:|X-X_0|<R, t<t_0\}$, where $X_0=(x_0,t_0)$, $x_0=(0,\cdots,0)$, $t_0=2\tau$.

Let $r=\sqrt{2\tau}=\sqrt{2}\delta$, define $\Theta(r)=Q((x_0,t_0-4r^2),r)$, and let $M_{k_l}=\sup_{Q(lr)}\omega_k$, $m_{k_l}=\inf_{Q(lr)}\omega_k$, where $l=1$ or 4. By Theorem 7.22 in Lieberman's [21], we get

$$\Phi_{p,r}(M_{k_4}-\omega_k(X))=(r^{-n-2}\int_{\Theta(r)}(M_{k_4}-\omega_k(X))^p dX)^{\frac{1}{p}}\leq C(\inf_{Q(r)}(M_{k_4}-\omega_k(X))+r^{\frac{n}{n+1}}|C_1\bar{\mu}|)$$

$$\leq C(M_{k_4}-M_{k_1}+\bar{\mu}r^2)$$

where p is a positive constant. Let $H_{k_l}=\sup_{Q(lr)}h_k(X)$, $\tilde{h}_{k_l}=\inf_{Q(lr)}h_k(X)$, and $\omega(lr)=\sum_{k=1}^{N_2}(H_{k_l}-\tilde{h}_{k_l})$. Now for $X\in\Theta(r)\subset Q(4r)$

$$M_{k_4}-\omega_k(X)\geq M_{k_4}-h_k(X)-\epsilon\upsilon(X)$$

$$\geq H_{k_4} + \epsilon\upsilon(X_1) - h_k(X) - \epsilon\upsilon(X)$$

$$\geq H_{k_4} - h_k - \epsilon\sum_{p=1}^{N_2}(h_p(X) - h_p(X_1))(h_p(X) + h_p(X_0))$$

$$\geq H_{k_4} - h_k - 2\epsilon\omega(4r)$$

we have

$$\Phi_{p,r}(H_{k_4} - h_k) \leq \Phi_{p,r}(M_{k_4} - \omega_k + 2\epsilon\omega(4r))$$

$$\leq (r^{-n-2}\int_{\Theta(r)} 2^p[(M_{k_4} - \omega_k)^p + (2\epsilon\omega(4r))^p])^{\frac{1}{p}}$$

$$\leq C[(\Phi_{p,r}(M_{k_4} - \omega_k))^p + (2\epsilon\omega(4r))^p]^{\frac{1}{p}}$$

$$\leq C[(M_{k_4} - M_{k_1} + \bar{\mu}r^2)^p + (\epsilon\omega(4r))^p]^{\frac{1}{p}}$$

$$\leq C[2(M_{k_4} - M_{k_1} + \bar{\mu}r^2 + \epsilon\omega(4r))^p]^{\frac{1}{p}}$$

$$\leq C(M_{k_4} - M_{k_1} + \epsilon\omega(4r) + \bar{\mu}r^2)$$

but $M_{k_4} - M_{k_1} \leq H_{k_4} - H_{k_1} + 2\epsilon\omega(4r)$, so

$$\Phi_{p,r}(H_{k_4} - h_k) \leq C(H_{k_4} - H_{k_1} + \epsilon\omega(4r) + \bar{\mu}r^2)$$

On the other side, by Lemma 24 and the choice of $\{\gamma_l\}_{l=1}^{N_2}$, we can get

$$\sum_{l=1}^{N_2}\beta_l(Y)(s_{\gamma_l\gamma_l}(X) - s_{\gamma_l\gamma_l}(Y)) = -g_{ij}(Y)(s_{ij}(Y) - s_{ij}(X))$$

$$\leq F(Y, s(Y), Ds(Y), D^2s(X)) - F(Y, s(Y), Ds(Y), D^2s(Y))$$

$$\leq F(Y, s(Y), Ds(Y), D^2s(X)) - F(X, s(X), Ds(X), D^2s(X)) + \frac{\partial s}{\partial t}(X) - \frac{\partial s}{\partial t}(Y)$$

$$\leq |F_x||Y - X| + |F_z||Ds||Y - X| + \left|F_p\right|\left|D^2s\right||Y - X| + |s_t(X) - s_t(Y)|$$

(where $F(x, z, p, r)$ is the expression used for derivatives above.)

$$\leq C|X - Y| + C|X - Y|^\alpha \leq C|X - Y|^\alpha$$

in the above we use the Holder estimate of s_t and the fact that F_x, F_s, F_{p_i}, Ds, $D^2 s$ are bounded which can be verified directly with pinching estimates. Choose any $Y \in \Theta(r)$ and $X \in Q(4r)$, for any $l = 1, \cdots, N_2$,

$$\beta_l(Y)(h_l(Y) - h_l(X)) = \beta_l(Y)(\frac{1}{2(1+M_2)})(s_{\gamma_l\gamma_l}(X) - s_{\gamma_l\gamma_l}(Y))$$

$$\leq -\frac{1}{2(1+M_2)}(\sum_{q\neq l}\beta_q(Y)(s_{\gamma_q\gamma_q}(X) - s_{\gamma_q\gamma_q}(Y))) + C\,|X-Y|^{\alpha}$$

$$\leq C(\sum_{q\neq l}(H_{q_4} - h_q(Y)) + r^{\alpha})$$

and

$$\beta_l(Y)(h_l(Y) - h_l(X)) \geq C_1(h_l(Y) - \tilde{h}_{l_4})$$

$$\Rightarrow h_l(Y) - \tilde{h}_{l_4} \leq C(\sum_{q\neq l}(H_{q_4} - h_q(Y)) + r^{\alpha})$$

$$\Phi_{p,r}(h_l(Y) - \tilde{h}_{l_4}) \leq C[\Phi_{p,r}(\sum_{q\neq l}(H_{q_4} - h_q(Y))) + r^{\alpha}]$$

Note

$$\Phi_{p,r}(\sum_{q\neq l}(H_{q_4} - h_q)) \leq C\sum_{q\neq l}\Phi_{p,r}(H_{q_4} - h_q)$$

$$\leq C(\sum_{q\neq l}(H_{q_4} - H_{q_1}) + (N_2-1)\epsilon\omega(4r) + (N_2-1)\bar{\mu}r^2)$$

$$\leq C((N_2-1)(\omega(4r) - \omega(r)) + (N_2-1)\epsilon\omega(4r) + (N_2-1)\bar{\mu}r^2)$$

$$\leq C((1+\epsilon)\omega(4r) - \omega(r) + \bar{\mu}r^2)$$

We get (II)

$$\Phi_{p,r}(h_l - \tilde{h}_{l_4}) \leq C((1+\epsilon)\omega(4r) - \omega(r) + \bar{\mu}r^2 + r^{\alpha})$$

for any $l = 1, \cdots, N_2$. But we have shown (I)

$$\Phi_{p,r}(H_{l_4} - h_l) \leq C(H_{l_4} - H_{l_1} + \epsilon\omega(4r) + \bar{\mu}r^2)$$

$$\leq C((1+\epsilon)\omega(4r) - \omega(r) + \bar{\mu}r^2)$$

for any $l = 1, \cdots, N_2$. Add (I) to (II),

$$H_{l_4} - \tilde{h}_{l_4} = \Phi_{p,r}(H_{l_4} - \tilde{h}_{l_4}) \leq C(\Phi_{p,r}(H_{l_4} - h_l) + \Phi_{p,r}(h_l - \tilde{h}_{l_4}))$$

$$C((1+\epsilon)\omega(4r) - \omega(r) + \bar{\mu}r^2 + r^\alpha)$$

Take the sum over l, we get

$$\omega(4r) \leq C((1+\epsilon)\omega(4r) - \omega(r) + \bar{\mu}r^2 + r^\alpha)$$

$$\Rightarrow \omega(r) \leq (1 + \epsilon - \frac{1}{C})\omega(4r) + \bar{\mu}r^2 + r^\alpha$$

Let $\epsilon = \frac{1}{2C}$, we get

$$\omega(r) \leq (1 - \frac{1}{2C})\omega(4r) + \bar{\mu}r^2 + r^\alpha$$

By the iteration method (like Lemma 8.23 in [22]) we have

$$\left\|D^2 s\right\|_{C^{0,\beta}(Q_r)} \leq C$$

for some constant $\beta = \beta(M, N, \phi_0, r)$ and $C = C(M, N, \phi_0, r)$. Notcie $Q(r) = Q(X_0, r) = B_\delta(0) \times (2\tau - r^2, 2\tau) = B_\delta(0) \times (0, 2\tau)$, so $B_\delta(0) \times [\frac{3}{4}\tau, \tau] \subset Q(r)$. Finally, we get

$$\left\|D^2 s\right\|_{C^{0,\beta}(B_\delta(0)\times[\frac{3}{4}\tau,\tau])} \leq C$$

From this inequality and the compactness of M^n, it is easy to get our conclusion. □

2.5 global results

Next we show the convergence to the sphere after an appropriate rescaling.

Proposition 25 *Suppose* $\phi : M^n \times [0, t_0) \to N^{n+1}$ *is a smooth strictly convex solution to (1.2) and* $\sup_{M^n \times [0,t_0)} F < \infty$*. Then* ϕ *extends uniquely to* $M^n \times [0, t_1)$ *for some* $t_1 > t_0$*.*

Proof: From $\sup_{M^n \times [0,t_0)} F < \infty$ and the pinching estimates proved before, it's easy to get $\sup_{M^n \times [0,t_0)} |A| \le C_0 < \infty$. Then by $C^{2,\alpha}$-estimates for the graphical parametrisation function s and the compactness of M^n, we could get uniform C^α-estimate for the curvature of ϕ. On the other side, from the proof of Finite Time Extinction Theorem we know the image of ϕ is contained in a compact set of N^{n+1} on this time interval. Consequently we have bounds on all the higher derivatives of the Riemann tensor of N^{n+1}. So standard Schauder estimates provide bounds on all the derivatives of the curvature of ϕ (see, for example, Chapter 14 of [21]).
Then when $0 \le t < t_0$, $|\nabla^m A| \le C_m$ for all $m \ge 1$, $C_m = C(m, n, \phi_0, M, C_0, N)$. Because of the pinching estimates, and the boundness of Riemannian tensor of N^{n+1} mentioned above, we get $|\nabla^m F| \le C_m$ by $|\nabla^m A| \le C_m$. Note

$$\frac{\partial g_{ij}}{\partial t} = -2F h_{ij}$$

The above ensures C^∞ convergence to an immersion ϕ_{t_0} (see, for example, section 8 in [6]). The short time existence result theorem now applies to extend the solution to a longer time interval.

□

Theorem 26 *(Convergence To A Sphere) Suppose* $\phi : M^n \times [0, T) \to N^{n+1}$ *is a smooth strictly convex solution to (1.2), assume* $[0, T)$ *is the maximal time interval for existence,* $\phi(M, t)$ *converges uniformly to a single point* $p \in N^{n+1}$ *as* $t \to T$*. If we take for* $t \to T$ *homothetic expansions of normal coordinates around* p *such that the total area of the expanded hypersurface* $\tilde{\phi}(M, t)$ *is fixed, then* $\tilde{\phi}(M, t)$ *converges to a round sphere of that area in the* C^∞*-topology.*

Proof: By the theorem above, we get

$$\lim_{t\to T}(\sup_{x\in M^n, 0\le t_1<T} |A|(x,t_1)) = \infty$$

Choose a point $x(t) \in M^n$ such that:

$$|A|(x(t),t) = \max_{x\in M^n} |A|(x,t)$$

and set $P(t) = \phi(x(t),t) \in N^{n+1}$. In the following we dilate the hypersurface $\phi(M,t)$ along the base point $P(t)$. Denote $|A|_{max}(t) = \max_{x\in M^n} |A|(x,t)$ and $\bar{g}^t_{\alpha\beta}(\bar{y}) = (|A|_{max}(t))^2 g^N_{\alpha\beta}(\bar{y})$, where $\bar{y} \in N^{n+1}$. Consider the one-parameter family of marked Riemannian manifolds $(N^{n+1}, \bar{g}^t_{\alpha\beta}, P(t))$. Since $(N^{n+1}, \bar{g}_{\alpha\beta})$ is negatively curved, its injectivity radius has uniformly bounded from below. It followed that the marked Riemannian manifolds $(N^{n+1}, \bar{g}^t_{\alpha\beta}, P(t))$ converge in the C^∞_{loc}-topology to the marked Euclidean space $(\mathbb{R}^{n+1}, \delta_{\alpha\beta}, O)$ in the usual sense of Gromov (see, for example, [23]). Choose

$$A_k = \sup_{t\in[0,\beta_k], x\in M^n} |A|(x,t)$$

where $\beta_k = max\{0, T - \frac{1}{k}\}$, and $A(x(t_k),t_k) = A_k$, $t_k \in [0,\beta_k]$. Let $x_k = x(t_k)$, define $\phi^{(k)}(x,\tau) = \phi(x, t_k + A_k^{-2}\tau)$ for $\tau \in [-A_k^2 t_k, A_k^2(\beta_k - t_k)]$. And by the theorem above, we know $t_k \to T$ as $k \to \infty$.

By Lemma 5.1 in Andrews' [8], we get that $\phi^{(k)}(x,\tau)$ satisfy the equation (1.2) in $(\bar{g}^{(t_k)}_{\alpha\beta}, N^{n+1})$. Similar with the proof of Proposition 25 we get that all covariant derivatives of $A_k(x,t)$ are also uniformly bounded for all $k \ge k_0$ and $\tau \in [-B, B]$, where B is a positive constant. (Note $\left|A^k(x,t)\right|^2 = A_k^{-2}|A|^2 \le 1$ when $t \in [-A_k^2 t_k, A_k^2(\beta_k - t_k)]$, where $A^{(k)}(x,t)$ is the second fundamental form of $\phi^{(k)}(x,t)$ in $(N^{n+1}, \bar{g}_{\alpha\beta}(t_k))$.) Then let

$$\tilde{\phi}^{(k)}(\xi,\tau) = (Exp_{x_k})^{-1}\Big(\psi(\chi_k^{-1}(\xi,\tau), s_k(\xi,\tau))\Big)$$

We know that for $\phi^{(k)}(x,\tau)$, $\sup_{x\in M^n}\left|A^{(k)}(x,0)\right| = 1$. On each $B_r \subset \mathbb{R}^{n+1} = T_{\phi(x_k)}N$, Theorem 23 gives $|s_k|_{C^{2+\beta}} \le C(n,f,r)$ and C is independent of k.

Note the definition χ_k is dependent on s_k, so for each positive integer r we can find a subsequence $\{t_{k_r}\}_{k=1}^{\infty}$ of $\{t_k\}_{k=1}^{\infty}$, for which the flow $\tilde{\phi}^{(k_r)}(\xi,\tau)$ converges to a flow $\phi_\infty(\xi,\tau)$ when $(\xi,\tau) \in B_r(0) \subset \mathbb{R}^{n+1}$. Furthermore we can arrange that $\{t_{k_{r+1}}\}_{k=1}^{\infty}$ is a subsequence of $\{t_{k_r}\}_{k=1}^{\infty}$ for each positive integer r. Taking a diagonal subsequence $\{t_{k_k}\}_{k=1}^{\infty}$, we obtain convergence to a limit flow in $\mathbb{R}^{n+1}$.

By Theorem 23 we know that the flow consists of strictly convex hypersurfaces, and the curvature of hypersurfaces satisfy C^α estimate which depends only on the distance from the origin. Also the curvature of the hypersurfaces in the limit flow is also bounded ($|A| \leq 2$), and the limit flow is a solution to equation (1.2). It follows (using the estimates mentioned above and Schauder theory) that the limit flow is smooth. And by the pinching estimate theorem, we could get the hypersurfaces in the limit flow has pinched principal curvatures.

We could now employ the following result due to Hamilton [24]:

Theorem 27 *A complete, smooth, strictly convex hypersurface with pinched principal curvatures in Euclidean space is compact.*

This implies that the solution converges for a subsequence of times to some point p of N^{n+1}, since the hypersurfaces approach a compact hypersurface after arbitrarily large rescaling, and so have diameter tending to zero. It follows that we have convergence to a point of the whole solution, since later hypersurfaces are contained by earlier hypersurfaces.

We have the following claim:

Claim 28 *The limit hypersurfaces in the limit flow must be in fact spheres.*

Note the limit flow is the solution to equation (1.2) in the ambient manifold $\mathbb{R}^{n+1}$. By the equation (2.1), in $\mathbb{R}^{n+1}$, we know that

$$\frac{1}{F}[(III) + (IV)] + \frac{(V)}{F} = 0$$

and $\frac{(\tilde{I})}{F} \leq 0$. Then we have

$$\frac{\partial}{\partial t}(\frac{H}{F}) = \mathscr{L}(\frac{H}{F}) + \frac{2}{F}\dot{F}^{kl}\nabla_k F \nabla_l(\frac{H}{F}) + \frac{(\tilde{I})}{F} \tag{2.2}$$

and

$$\frac{\partial}{\partial t}(\frac{H}{F}) \leq \mathscr{L}(\frac{H}{F}) + \frac{2}{F}\dot{F}^{kl}\nabla_k F \nabla_l(\frac{H}{F}) \tag{2.3}$$

it follows that in the limit flow the maximum of this quotient is nonincreasing. By the strong maximum principle, the maximum is strictly decreasing unless $\frac{H}{F}$ is a constant. But if the maximum decreases on the limit flow, we have a contradiction to the convergence (note that every hypersurface in the limit flow is in fact the limit surface of rescaling original surfaces in the original flow and the quantity is unaffected by the rescaling process). Hence $\frac{H}{F}$ is a constant in the limit flow, in the equation (2.2) $\frac{(\tilde{I})}{F} \equiv 0$, which implies that $\nabla_i h_{kk} \equiv 0$ for all i, k in the curvature coordinate system. Every limit hypersurface has constant curvature in $\mathbb{R}^{n+1}$ and is therefore sphere. The limit flow converges to a point O at finite time T_∞, and from

$$\lim_{k\to\infty}(t_k + A_k^{-2}T_\infty) = T_\infty$$

we get

$$T_\infty = \lim_{k\to\infty} A_k^{-2}(T - t_k)$$

The limit flow is harmonic mean curvature flow in Euclidean space $\mathbb{R}^{n+1}$, and the maximal time interval is $[0, T_\infty)$. Now proceeds exactly as in section 7 of [4] or refer Theorem 1.2 in [4], we get the following conclusion:

The rescaled immersions given by

$$\tilde{\phi}_\infty(\xi, \tilde{\tau}) = (2n^{-1}(T_\infty - \tau))^{-\frac{1}{2}}(\phi_\infty(\xi, \tau) - O)$$

converge to a smooth embedding $\tilde{\phi}_{\infty,\infty}$ with image equal to the unit sphere in $\mathbb{R}^{n+1}$, exponentially in C^∞ with respect to the natural rescaled time parameter $\tilde{\tau} = -\frac{1}{2}\ln(1 - \frac{\tau}{T_\infty})$, where O is the origin point of $\mathbb{R}^{n+1}$. □

Chapter 3

Examples of hypersurfaces flowing by curvature in Riemannian manifold

In this Chapter, except for Section 4, and unless otherwise mentioned, we consider harmonic mean curvature flow and let $f(\lambda) = (\sum_i \lambda_i^{-1})^{-1}$. We provide two specific examples of harmonic mean curvature flow: with dimension reduction in the limit case and the limit manifold of the same dimension as M case in section 1. Note these examples in section 1 provide barriers for harmonic mean curvature flow in Riemannian manifolds. We discuss the limit behavor of the harmonic mean curvature flow at infinite time in section 2. Then we treat the special consequences of the Gauss-Bonnet theorem for 2-dimensional surfaces in section 3, and turn to examples of more general flows by functions of normal curvatures in section 4.

3.1 The dimension reduction and non-reduction example

In this section, firstly, we give an example where ϕ_t converges to ϕ_∞ in the C^∞ topology but the dimension of $M_\infty = \phi_\infty(M)$ is less than the dimension of M_t; i.e. there is **dimension reduction**.

Theorem 29 *Let N^3 be a hyperbolic manifold containing an embedded closed geodesic M_∞. Then there is a flow $\phi_t : M^2 \to N^3$ by harmonic mean curvature, where M^2 is a torus, which converges to M_∞ as $t \to +\infty$. The flow consists of immersions ϕ_t, which become embedded for t sufficiently large.*

For example, we may let the ambient manifold N be $H^3/\mathbb{Z}$, where H^3 is hyperbolic space, represented as the Poincaré half space $(R^3)^+ = \{(x, y, z)|(x, y, z) \in \mathbb{R}^3, z > 0\}$ with the metric $g^N_{ij} = \frac{1}{z^2}\delta_{ij}$, and the $\mathbb{Z}$ action $f : \mathbb{Z} \times H^3 \to H^3$ is defined as:

$$f(k)(x, y, z) = 2^k(x, y, z).$$

Recall that $f(k)$ is an isometry of H^3 for each $k \in \mathbb{Z}$.

Now we let N be the quotient manifold of H^3 under the $\mathbb{Z}$-action, with fundamental domain $\{(x, y, z)|\, 1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2\}$. Then M_∞ = the positive z-axis, modulo $f(1)$, is a closed geodesic in N.

Proof: Let $\psi_0 : \mathbb{S}^1 \to N$ be an embedding as the given closed geodesic curve M_∞ in N. We choose a parallel unit vector field $w(x)$ in $(T_x\psi_0)^\perp$. Then for $r > 0$, we define

$$\psi(x, \theta, r) = \psi_r(x, \theta) : \mathbb{S}^1 \times \mathbb{S}^1 \to N^3$$

by

$$\psi(x, \theta, r) = \psi_r(x, \theta) = \gamma(x, \theta, r),$$

where $\gamma(x, \theta, \cdot)$ is the unit-speed geodesic in N with $\gamma(x, \theta, 0) = \psi_0(x)$ and $\frac{d}{dr}\gamma(x, \theta, r) = \vec{N}(x, \theta)$ at $r = 0$. Here $\vec{N}(x, \theta)$ is the unit tangent vector in $T_{\psi_0(x)}N^3$ which is perpendicular to $T_x\psi_0$ and makes the angle θ with $w(x)$. Then $\psi_r(\mathbb{S}^1 \times \mathbb{S}^1)$ has two principal curvatures:

$$\lambda_1(r) \equiv \tanh r, \quad \lambda_2(r) \equiv \coth r.$$

In fact, for $i = 1, 2$, $\lambda_i(r)$ is the logarithmic derivative of the length of a Jacobi field, and hence satisfies the Ricatti equation $\lambda_i'(r) + (\lambda_i(r))^2 = 1$.

We have constructed a one-parameter family of immersions $\psi_r : M \to N$, $-\infty < r < \infty$, with two principal curvatures: $\lambda_1(r) \equiv \tanh r$ and $\lambda_2(r) \equiv \coth r$. It may be observed that ψ_r is an embedding for r sufficiently small.

Now consider the harmonic mean curvature flow $\phi_t = \psi_{r(t)} : M \to N$, with initial conditions $\phi_0 = \psi_{r_0}$, $r(0) = r_0$, where r_0 is some fixed positive constant. The speed must satisfy:

$$\begin{aligned}\frac{\partial r}{\partial t} &= \Big\langle \frac{\partial \gamma}{\partial r}\frac{\partial r}{\partial t}, \vec{v}\Big\rangle = \Big\langle \frac{\partial \gamma(x,r)}{\partial t}, \vec{v}\Big\rangle \\ &= \Big\langle \frac{\partial \psi(x,r)}{\partial t}, \vec{v}\Big\rangle = \Big\langle \frac{\partial \phi(x,t)}{\partial t}, \vec{v}\Big\rangle \\ &= \Big\langle -F\vec{v}, \vec{v}\Big\rangle = -F(\lambda_1, \lambda_2) \\ &= -\frac{1}{\lambda_1^{-1} + \lambda_2^{-1}} = -\frac{\sinh r \cosh r}{(\sinh r)^2 + (\cosh r)^2}.\end{aligned}$$

In the first equality we use the fact $\frac{\partial \gamma}{\partial r} = \vec{v}$; in the third equality we use the definition of ψ_r, where $\vec{v} = \vec{N}(x,\theta)$ is the outward normal vector of $\psi_r(M)$ at $(x,\theta) \in \mathbb{S}^1 \times \mathbb{S}^1$.

Solving, we find

$$r(t) = \frac{1}{2}\sinh^{-1}\Big(e^{-t}\sinh 2r_0\Big).$$

Note that $r(t) \to 0$ as $t \to \infty$. $\square$

Next, we give one example in which M_t converges to M_∞ in the C^∞ topology and the dimension of M_∞ is the same as the dimension of M_t, i.e. there is **no dimension reduction.**

Theorem 30 *There is a compact surface M^2 of genus 2, a hyperbolic manifold N^3 diffeomorphic to $M \times \mathbb{R}$, a totally geodesic embedding $\psi_0 : M \to N$ and a flow by harmonic mean curvature $\phi_t : M \to N$ such that as $t \to +\infty$, $\phi_t(M) \to \psi_0(M)$ smoothly.*

Proof: Let Ω be a regular geodesic octagon in the hyperbolic plane H^2, with angles $\pi/2$, and thus area 4π. Label the edges as

$$\beta_1, \alpha_1', -\beta_1', -\alpha_1, \beta_2, \alpha_2', -\beta_2', -\alpha_2,$$

in that order, where the signs indicate orientation. Let A_1 be the orientation-preserving isometry of H^2 which maps the oriented geodesic segments α_1 to α_1'; A_2 maps α_2 to α_2'; B_1 maps β_1 to β_1'; and B_2 maps β_2 to β_2'. The group G of isometries of H^2 generated by A_1, A_2 and B_1 also includes B_2. G is isomorphic to the fundamental group of the compact surface of genus 2. (See pp. 95–98 in Katok [25] for the arithmetic properties of the group G.)

Let $\psi_0 : H^2 \to H^3$ be an embedding as a totally geodesic surface in H^3. The isometries in G extend in a well-known fashion to isometries of H^3, leaving the distance from $\psi_0(H^2)$ invariant.

Choose a unit normal vector field $\vec{N}$ to $\psi_0(H^2)$. Define $\psi(\cdot, r) : H^2 \to H^3$ by $\psi(x, r) = \psi_r(x) = \gamma(x, r)$ and $\psi(x, 0) = \psi_0(x)$, where $\gamma(x, \cdot)$ is the unit-speed geodesic in H^3 with $\gamma(x, 0) = x$ and $\frac{\partial}{\partial r}\gamma(x, 0) = \vec{N}(x)$.

Then $\psi_r(H^2)$ is totally umbilic, with normal curvatures $\lambda(r) \equiv \tanh r$. In fact, $\lambda(r)$ satisfies the Ricatti equation $\lambda'(r) + (\lambda(r))^2 = 1$, with the initial condition $\lambda(0) = 0$.

Now let the group G act by isometries on H^2 and on H^3. The quotient $H^2/G = M^2$ is a compact surface of genus 2, with fundamental domain Ω, and the quotient $H^3/G = N^3$ is a non-compact hyperbolic manifold diffeomorphic to $M \times \mathbb{R}$. The group G acting on N preserves each of the hypersurfaces $\psi_r(H^2)$. We have constructed a one-parameter family of totally umbilic embeddings $\psi_r : M \to N$, $-\infty < r < \infty$, with normal curvatures $\equiv \tanh r$.

Now consider the harmonic mean curvature flow $\phi_t : M \to N$, with initial conditions $\phi_0 = \psi_{r_0}$, where r_0 is some fixed positive constant. The speed must satisfy

$$\begin{aligned}\frac{\partial r}{\partial t} &= \Big\langle \frac{\partial \gamma}{\partial r}\frac{\partial r}{\partial t}, \vec{v}\Big\rangle = \Big\langle \frac{\partial \gamma(x,r)}{\partial t}, \vec{v}\Big\rangle = \Big\langle \frac{\partial \psi(x,r)}{\partial t}, \vec{v}\Big\rangle \\ &= \Big\langle \frac{\partial \phi(x,t)}{\partial t}, \vec{v}\Big\rangle = \Big\langle -F\,\vec{v}, \vec{v}\Big\rangle = -F(\lambda_1,\lambda_2) \\ &= -\frac{1}{\lambda_1^{-1}+\lambda_2^{-1}} = -\frac{1}{2}\tanh r.\end{aligned}$$

In the first equality we use the fact $\frac{\partial \gamma}{\partial r} = \vec{N}(x) = \vec{v}$. In the third equality we use the definition of ψ_r, where $\vec{v}$ is the outward normal vector of ψ_r.

Solving, we find

$$r(t) = \sinh^{-1}\Big(e^{-t/2}\sinh r_0\Big).$$

Note that $r(t) \to 0$ as $t \to \infty$. □

3.2 The limit behavior of harmonic mean curvature flow at infinite time

In this section, we will give a sufficient condition where harmonic mean curvature flow will exist forever, and discuss the limit behavior. Let $\varphi_t : M \to N$ be an immersion of M^n into a hyperbolic manifold N^{n+1}.

Definition 31 *We define the following notation:*

$$\dot{F}^{kl} = \frac{\partial F}{\partial h_{kl}}, \quad \ddot{F}^{kl,pq} = \frac{\partial^2 F}{\partial h_{kl}\partial h_{pq}}, \quad \dot{H}^i_k = \frac{\partial H}{\partial \omega^k_i}, \quad \ddot{H}^{s,i}_{r,k} = \frac{\partial^2 H}{\partial \omega^k_i \partial \omega^r_s}, \quad \mathscr{R}_{ij} = \mathscr{R}_{i0j0},$$

where 0 *appearing as a tensor index represents the normal vector $\vec{v}$ of $\varphi(M)$ in N. For any $W : M \to \mathbb{R}$, we define:*

$$\mathscr{L}(W) = \dot{F}^{kl}\nabla_k\nabla_l W.$$

Recall from Andrews [8] that $\mathscr{L}$ is elliptic as long as $\phi_t(M)$ remains locally strictly convex.

Lemma 32 *If N^{n+1} is a hyperbolic manifold, $F(x) < \frac{1}{n}$ for any $x \in M$, then $\phi_t(M)$ remains locally convex and $F(x,t) < \frac{1}{n}$ for any $x \in M$, $t \in [0, +\infty)$, $\lim_{t\to\infty} F(x,t) = 0$, and the harmonic mean curvature flow exists for all t in $[0, +\infty)$.*

Proof: By Andrews [8], using a curvature coordinate system at one point, we have the following formula:

$$\frac{\partial F}{\partial t} = \mathscr{L}(F) + F < \dot{F}, (\mathscr{W}^2) > +F < \dot{F}^{ij}, (\mathscr{R}_{ij}) >$$

$$= \mathscr{L}(F) + \sum_i F \frac{\partial f}{\partial \lambda_i} (\lambda_i^2 + \mathscr{R}_{ii}) \tag{3.1}$$

$$\leq \mathscr{L}(F) + F^3(n - \sum_i \lambda_i^{-2})$$

$$\leq \mathscr{L}(F) + F^3\Big(n - \frac{1}{n}F^{-2}\Big).$$

Consider the ODE

$$\frac{\partial \tilde{F}}{\partial t} = \tilde{F}^3(n - \frac{1}{n}\tilde{F}^{-2}),$$

$$\tilde{F}(0) = \max_{x\in M} F(x,0).$$

Solving the above ODE, we get $\widetilde{F}(t)^{-2} - n^2 = (\widetilde{F}(0)^{-2} - n^2)e^{2t/n}$. Because $0 < \widetilde{F}(0) = \max_{x\in M^n} F(x,0) < \frac{1}{n}$, we get $\lim_{t\to\infty} \tilde{F}(t) = 0$.

By the maximum principle, $F(x,t) \leq \tilde{F}(t) < \frac{1}{n}$, for all $x \in M$, $t \in [0, +\infty)$, and therefore $\lim_{t\to\infty} F(x,t) = 0$.

On the other hand, we have the following estimate by the above evolution equation of F:

$$\frac{\partial F}{\partial t} \geq \mathscr{L}(F) + F^3(-\sum_i \lambda_i^{-2}) \geq \mathscr{L}(F) - F.$$

Now consider the ODE

$$\frac{\partial \widehat{F}}{\partial t} = -\widehat{F},$$

$$\widehat{F}(0) = \min_{x\in M} F(x,0).$$

Then by the maximum principle again, we get for all $x \in M, t \in [0,+\infty)$:

$$F(x,t) \geq \widehat{F}(t) = \min_{x\in M} F(x,0)\, e^{-t} > 0$$

In particular, $\phi_t(M)$ remains convex for all t.

Finally, we have the following estimate of H. By Andrews' [8]:

$$\frac{\partial}{\partial t}\omega_i^r = \dot{F}^{kl}\nabla_k\nabla_l\omega_i^r + \ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr}$$

$$+\dot{F}^{kl}(h_{ml}\omega_k^m)\omega_i^r + \dot{F}^{st}\mathscr{R}_{st}h_{ij}g^{jr} + 2\dot{F}^{pm}g^{tr}\omega_m^q\mathscr{R}_{piqt}$$

$$-\dot{F}^{pq}(g^{tr}\omega_i^s\mathscr{R}_{psqt} + g^{ts}\omega_s^r\mathscr{R}_{piqt}) + \dot{F}^{pq}g^{tr}(\nabla_i\mathscr{R}_{tpq0} - \nabla_p\mathscr{R}_{qit0})$$

Now referring to the last five terms above, we define:

$$(I) = \dot{H}_r^i\dot{F}^{kl}(h_{ml}\omega_k^m)\omega_i^r\ ,\ (II) = \dot{H}_r^i\dot{F}^{st}\mathscr{R}_{st}h_{ij}g^{jr}$$

$$(III) = 2\dot{H}_r^i\dot{F}^{pm}g^{tr}\omega_m^q\mathscr{R}_{piqt}\ ,\ (IV) = -\dot{H}_r^i(\dot{F}^{pq}g^{tr}\omega_i^s\mathscr{R}_{psqt} + \dot{F}^{pq}g^{ts}\omega_s^r\mathscr{R}_{piqt})$$

$$(V) = \dot{H}_r^i\dot{F}^{pq}g^{tr}(\nabla_i\mathscr{R}_{tpq0} - \nabla_p\mathscr{R}_{qit0})$$

then

$$\frac{\partial}{\partial t}H = \dot{H}_r^i(\frac{\partial}{\partial t}\omega_i^r)$$

$$= \dot{H}_r^i(\dot{F}^{kl}\nabla_k\nabla_l\omega_i^r) + \dot{H}_r^i\ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr} + (I) + \cdots + (V)$$

Note

$$\dot{F}^{kl}\nabla_k\nabla_l H = \dot{F}^{kl}\nabla_k(\dot{H}^i_r\nabla_l\omega^r_i) = \dot{F}^{kl}\ddot{H}^{i,\tilde{i}}_{r,\tilde{r}}(\nabla_k\omega^{\tilde{r}}_{\tilde{i}})(\nabla_l\omega^r_i) + \dot{F}^{kl}\dot{H}^i_r\nabla_k\nabla_l\omega^r_i$$

Define

$$(J) = \dot{H}^i_r\ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr} - \dot{F}^{kl}\ddot{H}^{i,\tilde{i}}_{r,\tilde{r}}(\nabla_k\omega^{\tilde{r}}_{\tilde{i}})(\nabla_l\omega^r_i)$$

We get

$$\frac{\partial}{\partial t}H = \mathscr{L}(H) + (J) + (I) + \cdots + (V)$$

It's straightforward to get

$$(I) + (II) = H[< \dot{F}, (\mathscr{W}^2) > + \dot{F}^{ij}\mathscr{R}_{i0j0}] \leq nF^2 H \leq \frac{1}{n}H$$

and

$$(V) = \left|\frac{\partial f}{\partial\lambda_i}(\nabla_j\mathscr{R}_{jii0} - \nabla_i\mathscr{R}_{ijj0})\right| = 0$$

Choose curvature coordinate system around one point, then we could do the following simple calculation:

$$(J) = \ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_i h_{pq})$$

But by the Lemma 2.22 in [4], we know F is concave from the fact that f is concave. So we get $(J) \leq 0$.

Now

$$(III) + (IV) = 2\dot{H}^i_r\dot{F}^{pm}g^{tr}\omega^q_m\mathscr{R}_{piqt} - \dot{H}^i_r(\dot{F}^{pq}g^{tr}\omega^s_i\mathscr{R}_{psqt} + \dot{F}^{pq}g^{ts}\omega^r_s\mathscr{R}_{piqt})$$

$$= 2\delta^i_r\frac{\partial f}{\partial\lambda_p}\delta^m_p\delta^r_t\lambda_q\delta^m_q\mathscr{R}_{piqt} - \delta^i_r(\frac{\partial f}{\partial\lambda_p}\delta^q_p\delta^r_t\lambda_i\delta^s_i\mathscr{R}_{psqt} + \frac{\partial f}{\partial\lambda_p}\delta^q_p\delta^s_t\lambda_s\delta^r_s\mathscr{R}_{piqt})$$

$$= 2\mathscr{R}_{prpr}\frac{\partial f}{\partial\lambda_p}(\lambda_p - \lambda_r) = 2\sum_{p<r}\mathscr{R}_{prpr}(\frac{\partial f}{\partial\lambda_p} - \frac{\partial f}{\partial\lambda_r})(\lambda_p - \lambda_r)$$

$$= (\sum_k \lambda_k^{-1})^{-2} \cdot \sum_{i,j} (-\mathscr{R}_{ijij}) \cdot (\lambda_i - \lambda_j)^2 (\lambda_i + \lambda_j) \cdot \lambda_i^{-2} \lambda_j^{-2}$$

$$\leq \sum_{i,j} (\lambda_i + \lambda_j) \cdot \Big(\frac{\lambda_i^{-1} - \lambda_j^{-1}}{\sum_k \lambda_k^{-1}} \Big)^2 \leq \sum_{i,j} (\lambda_i + \lambda_j) = 2nH$$

We have the following estimate of H by the above estimates:

$$\frac{\partial H}{\partial t} \leq \mathscr{L}(H) + (2n + \frac{1}{n})H$$

Now consider the ODE

$$\frac{\partial \widehat{H}}{\partial t} = \Big(2n + \frac{1}{n}\Big)\widehat{H},$$

$$\widehat{H}(0) = \max_{x \in M} H(x, 0).$$

Then by the maximum principle again, we get for all $x \in M, t \in [0, +\infty)$:

$$H(x, t) \leq \widehat{H}(t) = \max_{x \in M} H(x, 0)\, e^{(2n + \frac{1}{n})t} < +\infty$$

This shows that the harmonic mean curvature flow exists on $[0, +\infty)$. □

In the rest of this section, we do not assume the ambient manifold N^{n+1} is a hyperbolic manifold.

Proposition 33 *Assume N^{n+1} is a smooth $n + 1 \geq 3$ dimensional manifold which is convex at infinity, the maximal existence time of the harmonic mean curvature flow $\phi : M \times [0, T) \to N$ is $T = +\infty$, and as $t \to +\infty$, $M_t = \phi(M, t)$ converges to a smooth n dimensional submanifold M_∞ of N in the C^∞-topology; then*

$$\max_{x \in M,\, t \in [0, +\infty)} \{|F(x, t)|, |\nabla F(x, t)|, |\nabla^2 F(x, t)|\} \leq C,$$

where C is a constant depending on M_0, N^{n+1} and M_∞.

Proof: Straightforward from the assumptions. □

Proposition 34 *Assume N and $M_t \to M_\infty$ are as in the hypotheses of Proposition 33. Then*

$$\lim_{t\to\infty} \int_{M_t} F^2 \, d\mu_t = 0.$$

Proof: By Theorem 1.1 in [9], we have the formula $\frac{\partial}{\partial t}(\int_{M_t} d\mu_t) = -\int_{M_t} FH d\mu_t$. Because $\int_{M_t} d\mu_t \to \mu(M_\infty)$ as $t \to \infty$, we could find an ϵ-dense set $\{t_k\}_{k=1}^{\infty}$ for any positive constant $\epsilon > 0$ such that

$$\lim_{k\to\infty} t_k = \infty$$

and

$$\lim_{k\to\infty} \int_{M_{t_k}} FH \, d\mu_{t_k} = 0.$$

Then using the inequality $H \geq n^2 F$, we get $\lim_{k\to\infty} \int_{M_{t_k}} F^2 \, d\mu_{t_k} = 0.$

Now to get our conclusion we only need to show $\frac{\partial}{\partial t} \int_{M_t} F^2 d\mu_t$ is uniformly bounded. First, we know from Proposition 33 that $|F|$, $|\nabla F|$ and $|\nabla^2 F|$ are uniformly bounded. So we have

$$\frac{\partial}{\partial t}\Big(\int_{M_t} F^2 \, d\mu_t \Big) = \int 2FF_t + F^2(-FH) \, d\mu_t$$

$$= \int 2F\Big(\mathscr{L}(F) + \sum_{i=1}^{n} F\Big(\frac{\partial f}{\partial \lambda_i}\Big)(\lambda_i^2 + \mathscr{R}_{ii})\Big) - F^3 H \, d\mu_t$$

(where we use equation (3.1))

$$= \int 2nF^4 + 2F^4\Big(\sum_{i=1}^{n} \lambda_i^{-2} \mathscr{R}_{ii}\Big) + 2F\mathscr{L}(F) - F^3 H \, d\mu_t$$

$$\leq \int 2F^4 K_2\Big(\sum_{i=1}^{n} \lambda_i^{-2}\Big) d\mu_t + \int 2F\mathscr{L}(F) \, d\mu_t$$

$$\leq C \int F^2 \, d\mu_t + 2 \int F \mathscr{L}(F) \, d\mu_t,$$

where the first inequality uses the following facts:

M_t is always contained in some compact set of N^{n+1}, since N^{n+1} is convex at infinity, so its sectional curvature is bounded above by some constant K_2; and $HF^{-1} = (\sum_{i=1}^{n} \lambda_i)(\sum_{i=1}^{n} \lambda_i^{-1}) \geq n^2 \geq 2n$.

Next, since we know the volume of M_t is always non-increasing and $|F|$ is uniformly bounded, we get

$$C \int_{M_t} F^2 \, d\mu_t \leq C_1,$$

where C_1 is some constant depending only on M_0, N and M_∞.

Since $|\nabla^2 F|$ is uniformly bounded, we get

$$2 \int F \mathscr{L}(F) \, d\mu_t \leq 2n^2 \int F |\nabla^2 F| \, d\mu_t \leq C_2,$$

where C_2 is some constant depending on M_0, N and M_∞.

By all the above we get

$$\frac{\partial}{\partial t}\Big(\int F^2 \, d\mu_t \Big) \leq C_3,$$

where C_3 is another constant depending on M_0, N and M_∞.

Therefore

$$\lim_{t \to \infty} \int_{M_t} F^2 \, d\mu_t = 0.$$

□

Corollary 35 *Assume N and $M_t \to M_\infty$ are as assumed for Proposition 33. Then we have*

$$\lim_{t \to \infty} \Big(\max_{x \in M} F(x, t) \Big) = 0.$$

Proof: By Proposition 34, we have

$$0 = \lim_{t\to\infty} \int_{M_t} F^2 \, d\mu_t = \int_{M_\infty} \lim_{t\to\infty} F^2(x,t) \, d\mu_\infty,$$

so the corollary follows. □

By the above results, we know that $F \equiv 0$ on the limit surface M_∞ if M_∞ is the smooth limit of the harmonic mean curvature flow, which implies that $\det \mathscr{W} = 0$ on M_∞.

3.3 Classification of harmonic mean curvature flow on surfaces

In this section, we consider harmonic mean curvature flow for $n = 2$, where M^2 is an orientable surface, N^3 is a hyperbolic manifold, and the harmonic mean $f(\lambda) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$. As before, we assume that $\phi_0(M)$ is locally strictly convex.

In the following we always assume $F(x,0) < \frac{1}{2}$, i.e. $\lambda_1^{-1} + \lambda_2^{-1} > 2$, which will guarantee the harmonic mean curvature flow exists forever by Lemma 32. Note that, for example, $f(\lambda_1, \lambda_2) < \frac{1}{2}$ for the examples of Theorems 29 and 30, and that the horospheres have $f(\lambda_1, \lambda_2) \equiv \frac{1}{2}$.

We define $C_0 = 2\pi\chi(M_0) = \int_{M_t} (K - 1) \, d\mu_t$, where the second equality is true for any M_t because of the Gauss-Bonnet theorem, where $\chi(M_0)$ is the Euler number of M_0; $K(x,t) = \lambda_1(x,t)\lambda_2(x,t)$, $\lambda_1(x,t)$ and $\lambda_2(x,t)$ are the principal curvatures at the point x on M_t in the ambient hyperbolic manifold N^3; and the Gauss equation, which implies the Gauss curvature $= K - 1$.

First, define $V(t) = \int_{M_t} 1 \, d\mu_t$, the area of M_t. Then using the formula

$$\frac{\partial}{\partial t} d\mu_t = -FH d\mu_t$$

we get

$$\frac{d}{dt}V(t) = \int_{M_t} \frac{\partial}{\partial t}\, d\mu_t = \int_{M_t} (-FH)\, d\mu_t = \int_{M_t} (-K)\, d\mu_t$$
$$= -\int_{M_t} (K-1)\, d\mu_t - \int_{M_t} 1\, d\mu_t = -C_0 - V(t).$$

Solving the above ODE, we get

$$V(t) = (V(0) + C_0)e^{-t} - C_0.$$

This shows that the area of M_t is determined by its genus and the area $V(0)$ of the initial surface M_0.

There are three cases: $C_0 < 0$, $C_0 = 0$, $C_0 > 0$, corresponding to the surfaces with genus $g > 1$ (Case I) , $g = 1$ (Case II) and $g = 0$ (Case III).

(I). Let us first consider the case $C_0 = 2\pi\chi(M_0) < 0$. In this case, we have

$$\lim_{t\to\infty} V(t) = -C_0 > 0$$

which means the limit surface has non-zero volume. We **conjecture** that in a hyperbolic manifold N^3, the limit surface will be the totally geodesic surface, if there is one in the homotopy class of M_0. This behavior is seen in Theorem 30.

(II). When $C_0 = 2\pi\chi(M_0) = 0$, we have

$$\lim_{t\to\infty} V(t) = -C_0 = 0$$

which means the limit surface has zero volume. In fact we could prove the following:

Proposition 36 *If N^3 is a hyperbolic manifold, $F(x, 0) < \frac{1}{2}$ for all $x \in M$ and the genus of M* $= 0$*, then*

$$\lim_{t\to\infty}(\max_{x\in M_t} H(x, t)) = +\infty.$$

Proof: Because $\int_{M_t}(K-1)\,d\mu_t = C_0 = 0$, we have $\max_{x\in M_t} K(x,t) \geq 1$. We also have $\lim_{t\to\infty}(\max_{x\in M_t} F(x,t)) = 0$, using the assumption $F(x,0) < \frac{1}{2}$, by Lemma 32. Then for any $x \in M_t$, $t > 0$, we have the following:

$$K(x,t) = H(x,t)F(x,t) \leq F(x,t)(\max_{x\in M_t} H(x,t)).$$

Taking the maximum on the both sides of the above inequality, we have

$$1 \leq \max_{x\in M_t} K(x,t) \leq (\max_{x\in M_t} F(x,t))(\max_{x\in M_t} H(x,t)).$$

So

$$\max_{x\in M_t} H(x,t) \geq \frac{1}{\max_{x\in M_t} F(x,t)}.$$

Taking the limit on both sides, we get

$$\lim_{t\to\infty}(\max_{x\in M_t} H(x,t)) \geq \frac{1}{\lim_{t\to\infty}(\max_{x\in M_t} F(x,t))} = +\infty.$$

□

The above proposition means that there exists at least one blowup point on the limit set; the example of Theorem 29 blows up at every point.

(III) Finally, when $C_0 = 2\pi\chi(M_0) > 0$, we have an interesting geometric result. In this case, because

$$V(t) = (V(0) + C_0)e^{-t} - C_0,$$

there exists some T_0, $0 < T_0 < +\infty$, such that $V(T_0) = 0$. That means the harmonic mean curvature flow stops in finite time. But we already proved that the flow will exist forever if $F < \frac{1}{2}$. So under the assumption $F < \frac{1}{2}$, this surface will not exist.

Remark 37 *Observe that the nonexistence of the initial surfaces in Case (III) above may also be proven by lifting the simply-connected surface* M_0 *to the universal cover* H^3 *of* N^3 *and applying the comparison principle with shrinking spheres centered at a point: the sphere of radius* r *has* $F = \frac{1}{2}\coth r > \frac{1}{2}$.

3.4 General geometric flows

In this section we give examples for a general geometric flow (1.1) in a hyperbolic manifold N^{n+1} which will exist forever or for a computable finite time, and converge to a given totally geodesic submanifold P^k of any codimension. In this section, we always assume the existence of a totally geodesic submanifold P^k in N^{n+1}.

Firstly, by similar methods to those of section 2 and section 3, we may prove a theorem for general dimensions and codimensions:

Theorem 38 *Assume* P^k *is a compact totally geodesic submanifold of the hyperbolic manifold* N^{n+1}*, where* $1 \le k \le n$*. Let* M *be diffeomorphic to the unit sphere bundle of the normal bundle* $\perp P$ *when* $k < n$ *; we choose* M *to be one of the two connected components of the unit sphere bundle of the normal bundle* $\perp P$ *when* $k = n$*. Then we have a flow by harmonic mean curvature* $\phi_t : M \to N$ *such that as* $t \to +\infty$*,* $\phi_t(M) \to P$*.*

Proof: We only sketch the proof. We find the second fundamental form matrix of $\psi_r(M)$ with respect to a basis of curvature directions is the following:

$$\mathcal{W} = \begin{pmatrix} I_k \tanh r & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & I_{n-k}\coth r \end{pmatrix}$$

Then we find

$$\frac{\partial r}{\partial t} = -F = -\frac{\tanh r}{k + (n-k)(\tanh r)^2} \tag{3.2}$$

Solving this ODE, we get

$$(\sinh r(t))^k (\cosh r(t))^{n-k} = Ce^{-t}$$

where $C = (\sinh r_0)^k (\cosh r_0)^{n-k}$ is a fixed positive constant. This shows that $\phi_t := \psi_{r(t)}$ is a solution of harmonic mean curvature flow.

Note that $r(t) \to 0$ as $t \to +\infty$. □

Now let M^n be diffeomorphic to (one connected component of) the unit sphere normal bundle of P^k in N^{n+1}, and let $\psi_r : M \to N$ define the hypersurface at distance $r > 0$ from P^k. We consider flow by an arbitrary symmetric function of the normal curvatures:

Theorem 39 *For the symmetric function $f(\lambda_1, \cdots, \lambda_n)$, define*

$$h(r) = f(\tanh r, \cdots, \coth r),$$

where $\tanh r$ *is repeated k times and* $\coth r$ *is repeated $n-k$ times. Choose $r_0 > 0$ and define*

$$T_0 = \int_0^{r_0} \frac{1}{h(r)} dr, \quad 0 < T_0 \le +\infty.$$

Then we may construct a flow

$$\frac{\partial}{\partial t}\phi(\cdot, t) = f(\lambda(\mathscr{W}(x,t)))\vec{\nu}(x,t) \tag{3.3}$$

with initial condition $\phi(\cdot, 0) = \psi_{r_0}$, which exists for time $0 \le t \le T_0 \le \infty$, and $\phi(\cdot, t)$ converges to the totally geodesic k-dimensional submanifold P^k as $t \to T_0$.

Proof: The hypersurface defined by $\phi(\cdot, t) := \psi_{r(t)}$ flows by (3.3) if

$$\frac{\partial r}{\partial t} = -F(x,t) \equiv -h(r) \tag{3.4}$$

$$\Longrightarrow \int_{r(0)}^{r(T_0)} \frac{1}{h(r)} dr = \int_0^{T_0} -1\, dt$$

$$\Longrightarrow T_0 = \int_0^{r_0} \frac{1}{h(r)} dr.$$

The conclusion now follows from the proof of Theorem 38, replacing equation (3.2) with equation (3.4). □

Remark 40 *Note that the flow (3.3) is parabolic if* $\frac{\partial f}{\partial \lambda_i} > 0$ $(1 \le i \le n)$*; parabolic for backwards time if* $\frac{\partial f}{\partial \lambda_i} < 0$ $(1 \le i \le n)$*; and is a first-order PDE if* f *is constant.*

The following corollary is a generalization of both mean curvature flow ($m = 1$, $\ell = 0$) and of harmonic mean curvature flow ($m = n$, $\ell = n - 1$).

Corollary 41 *Assume* P^k *is a compact totally geodesic submanifold of* N^{n+1}*, where* $1 \le k \le n$*. Let* M *be diffeomorphic to the unit sphere bundle of the normal bundle* $\perp P$ *when* $k < n$*;* M *is one of the two components of the unit sphere bundle of* $\perp P$ *when* $k = n$*.*

For integers $0 \le m, \ell \le n$*, let* S_m*,* S_ℓ *be the elementary symmetric functions of degree* m*,* ℓ *respectively, of the principal curvatures* $\lambda_1, \ldots, \lambda_n$ *of* M_t*. We have a flow by curvature function*

$$F(x,t) = \frac{S_m(\lambda_1, \cdots, \lambda_n)}{S_\ell(\lambda_1, \cdots, \lambda_n)}$$

for time $0 \le t < \infty$*, such that* $\phi(t) : M \to N$*, and* $\phi_t(M) \to P$ *as* $t \to +\infty$*; assuming that the integers* m*,* ℓ *satisfy* $|m - (n-k)| < |\ell - (n-k)|$*.*

Remark 42 *Theorem 39 also may be applied to prove a partial converse of Corollary 41: assuming* P^k *and* N^{n+1} *are as in Corollary 41, if the opposite condition* $|m-(n-k)| \ge |\ell-(n-k)|$ *holds, then the same construction yields a flow of hypersurfaces by the curvature function* $F = \frac{S_m}{S_\ell}$ *which converges to the totally geodesic submanifold* P^k *in* **finite** *time* T_0*.*

Remark 43 $\frac{S_m(\lambda_1,\cdots,\lambda_n)}{S_\ell(\lambda_1,\cdots,\lambda_n)}$ *is also called Q-Curvature in literature, see [26], [27].*

Proof: In the following, we fix an arbitrary positive constant $r(0) = r_0$. Firstly we have

$$S_m = \sum_{\substack{p+q=m \\ 0\le p\le k \\ 0\le q\le n-k}}^{n} C_k^p (\tanh r)^p C_{n-k}^q (\coth r)^q = \sum C_k^p C_{n-k}^q (\coth r)^{q-p},$$

where C_k^p is the combinatorial coefficient $\frac{k!}{p!(k-p)!}$.

Since $\coth r \ge 1$, it is easy to see

$$S_m \sim \begin{cases} (\coth r)^m & \text{if } m \le n-k \\ (\coth r)^{2(n-k)-m} & \text{if } m > n-k \end{cases}$$

where the notation $S_m \sim (\coth r)^j$ means that there exist positive constants C_1, C_2 such that $C_1(\coth r)^j \le S_m \le C_2(\coth r)^j$. Here C_1 and C_2 will depend only on m, n, k, ℓ and r_0.

Similarly, we have

$$S_\ell \sim \begin{cases} (\coth r)^\ell & \text{if } \ell \le n-k \\ (\coth r)^{2(n-k)-\ell} & \text{if } \ell > n-k. \end{cases}$$

Therefore

$$F = \frac{S_m}{S_\ell} \sim \begin{cases} (\coth r)^{m-\ell} & \text{if } m, \ell \le n-k \\ (\coth r)^{\ell-m} & \text{if } m, \ell > n-k \\ (\coth r)^{2(n-k)-m-\ell} & \text{if } \ell \le n-k < m \\ (\coth r)^{m+\ell-2(n-k)} & \text{if } m \le n-k < \ell. \end{cases}$$

By Theorem 39, we obtain that the flow exists forever if and only if the power of $\coth r$ is negative in the estimate for F above. That is, if and only if m, ℓ satisfy one of the following conditions:

$$\begin{cases} m < \ell & \text{if } m, \ell \leq n-k \\ \ell < m & \text{if } m, \ell > n-k \\ 2(n-k) < m+\ell & \text{if } \ell \leq n-k < m \\ m+\ell < 2(n-k) & \text{if } m \leq n-k < \ell. \end{cases}$$

It is straightforward to see the above inequalities are equivalent to the inequality $|m-(n-k)| < |\ell-(n-k)|$, which is our conclusion. □

Remark 44 *In particular, the case $k = n, m = 1, \ell = 0$ is the first example we are aware of in the literature of a locally* **convex** *compact hypersurface flowing by mean curvature and converging smoothly to a submanifold in infinite time. And the case $k = n-1, m = 0, \ell = 1$ gives an example of (backwards parabolic) inverse mean curvature flow existing forever and converging to a totally geodesic hypersurface.*

Chapter 4

Local Ricci flow and its Short-time existence

First, we recall the definition of local Ricci flow introduced by Deane Yang in [15]. Let M^n be a smooth n-dimensional manifold with Riemannian metric g_0, Ω an open bounded domain of M and $n \geq 3$. Let χ be a nonnegative smooth compactly supported function on Ω, and $0 \leq \chi \leq 1$. The local Ricci flow is the solution of the following evolution equation:

$$\frac{\partial g}{\partial t} = -2\chi^2 Rc(g), \quad g(0) = g_0, \quad \chi \in C_0^\infty(\Omega) \tag{4.1}$$

To prove the short-time existence of the local Ricci flow, we will following the brief sketch in Deane Yang's [15]. Firstly by DeTurck's trick, we can equivalently consider the following local Ricci-DeTurck flow:

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} &= -2\chi^2 R_{ij} + \nabla_i(\chi^2 W)_j + \nabla_j(\chi^2 W)_i \\ \frac{\partial}{\partial t}\phi_t(p) &= -\chi^2(p) W(\phi_t(p), t), \quad p \in M \\ \phi_0 &= Id_M, \quad g_{ij}(x,0) = \tilde{g}_{ij}(x) \triangleq (g_0)_{ij}(x) \end{cases} \tag{4.2}$$

$$W_j = g_{jk}g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq})$$

Note the second equation in (4.2) is a quasilinear ODE, the solution on $M \setminus \Omega$ is $Id_{M\setminus\Omega}$, so we only need to consider the first family of equations in (4.2).

Lemma 45 *The following modified evolution equations (4.3) is a parabolic system.*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} &= -2\chi^2 R_{ij} + \nabla_i(\chi^2 W)_j + \nabla_j(\chi^2 W)_i \\ g_{ij}(x,0) &= \tilde{g}_{ij}(x) \end{cases} \tag{4.3}$$

Proof: Note on $M \setminus \Omega$, $g_{ij}(x,t) \equiv \tilde{g}_{ij}(x)$, so we only need to consider the evolution equations on Ω. Calculate (4.3) directly, using lemma 2.1 in [14], we get

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} &= \chi^2 g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{ij} - \chi^2(g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{j\alpha q\beta} + g^{\alpha\beta}g_{jp}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta}) \\ & +\frac{1}{2}\chi^2 g^{\alpha\beta}g^{pq}(\tilde{\nabla}_i g_{p\alpha}\tilde{\nabla}_j g_{q\beta} + 2\tilde{\nabla}_\alpha g_{jp}\tilde{\nabla}_q g_{i\beta} - 2\tilde{\nabla}_\alpha g_{jp}\tilde{\nabla}_\beta g_{iq} \\ & \quad -2\tilde{\nabla}_j g_{p\alpha}\tilde{\nabla}_\beta g_{iq} - 2\tilde{\nabla}_i g_{p\alpha}\tilde{\nabla}_\beta g_{jq}) \\ & +\chi\chi_i g^{pq}(\tilde{\nabla}_p g_{jq} + \tilde{\nabla}_q g_{pj} - \tilde{\nabla}_j g_{pq}) + \chi\chi_j g^{pq}(\tilde{\nabla}_p g_{iq} + \tilde{\nabla}_q g_{pi} - \tilde{\nabla}_i g_{pq}) \\ g_{ij}(x,0) &= \tilde{g}_{ij}(x), \quad x \in \Omega. \\ g_{ij}(x,t) &= \tilde{g}_{ij}(x), \quad x \in \partial\Omega,\ t \in [0,T]. \end{cases} \tag{4.4}$$

where $\chi_i = \tilde{\nabla}_i\chi$. Then (4.4) is a degenerate parabolic system, so we are done. □

Now we consider the following parabolic system, where ϵ is some positive constant and $0 < \epsilon \leq 1$.

$$\begin{cases} \frac{\partial}{\partial t} v_{ij} &= (\chi^2 + \epsilon)u^{\alpha\beta}(v_{ij})_{\alpha\beta} + D_{ij}(U,V) + B_{ij}(U,\tilde{\nabla}U) \\ v_{ij}(x,0) &= \tilde{g}_{ij}(x) \quad x \in \Omega. \\ v_{ij}(x,t) &= \tilde{g}_{ij}(x) \quad x \in \partial\Omega,\ t \in [0,T]. \end{cases} \tag{4.5}$$

where $U = (u_{ij})_{i,j=1}^n$, $V = (v_{ij})_{i,j=1}^n$, $(v_{ij})_{\alpha\beta} = \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta (v_{ij})$,

$$D_{ij}(U, V) = -\chi^2[(u^{\alpha\beta}\tilde{g}^{pq}\tilde{R}_{j\alpha q\beta})v_{ip} + (u^{\alpha\beta}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta})v_{jp}]$$

$$B_{ij}(U, \tilde{\nabla}U) = E_{ij} + F_{ij}$$

$$\begin{aligned} E_{ij} &= \tfrac{1}{2}\chi^2 u^{\alpha\beta}u^{pq}(\tilde{\nabla}_i u_{p\alpha}\tilde{\nabla}_j u_{q\beta} + 2\tilde{\nabla}_\alpha u_{jp}\tilde{\nabla}_q u_{i\beta} - 2\tilde{\nabla}_\alpha u_{jp}\tilde{\nabla}_\beta u_{iq} \\ &\quad -2\tilde{\nabla}_j u_{p\alpha}\tilde{\nabla}_\beta u_{iq} - 2\tilde{\nabla}_i u_{p\alpha}\tilde{\nabla}_\beta u_{jq}) \end{aligned}$$

$$F_{ij} = \chi\chi_i u^{pq}(\tilde{\nabla}_p u_{jq} + \tilde{\nabla}_q u_{pj} - \tilde{\nabla}_j u_{pq}) + \chi\chi_j u^{pq}(\tilde{\nabla}_p u_{iq} + \tilde{\nabla}_q u_{pi} - \tilde{\nabla}_i u_{pq})$$

We define $\Phi_\epsilon(U) = V$, if V is the solution of (4.5), and $A_\epsilon^{\alpha\beta} \triangleq (\chi^2 + \epsilon)u^{\alpha\beta}$.

For simplicity reason, unless otherwise mentioned, in the rest of this section, we use the notation $\Phi(U) = V$, $A^{\alpha\beta}$ to replace $\Phi_\epsilon(U) = V$ and $A_\epsilon^{\alpha\beta}$, we also use ∇ to replace $\widetilde{\nabla}$. We get

$$\frac{\partial}{\partial t}v_{ij} = A^{\alpha\beta}(v_{ij})_{\alpha\beta} + D_{ij}(U, V) + B_{ij}(U, \nabla U) \tag{4.6}$$

We define the following function spaces and norms:

Definition 46

$$\mathfrak{A}_T = \{U\Big|\frac{1}{2}\tilde{g}_{ij} \le u_{ij} \le 2\tilde{g}_{ij};\ U(x, 0) = \tilde{g}(x), x \in \Omega;$$

$$U(x, t) = \tilde{g}, x \in \partial\Omega;\ U \in C^0(\bar{\Omega} \times [0, T], \otimes^2 T^*(\bar{\Omega}))\}$$

$$\big|U(x, t)\big|_N = \Big(\sum_{|\alpha|\le N}\int_\Omega |\nabla^\alpha U(x, t)|^2 d\mu_{\tilde{g}}\Big)^{\frac{1}{2}},\quad \big|U(x, t)\big|_{T,N} = \sup_{t\in[0,T]}(\big|U(x, t)\big|_N)$$

$$\|f\|_N \triangleq \sum_{|\alpha|\le N} |\nabla^\alpha f|_{L^\infty(\Omega)}$$

where $|\nabla^\alpha U(x, t)|^2 = \tilde{g}^{i_1 j_1} \cdots \tilde{g}^{i_m j_m}\tilde{g}^{pk}\tilde{g}^{ql}(\nabla_{i_1} \cdots \nabla_{i_m} u_{pq})(\nabla_{j_1} \cdots \nabla_{j_m} u_{kl})$, *and* $|\alpha| = m \ge 0$.

Choosing some fixed $\tilde{p} > \frac{n}{2}$*, then let*

$$\mathbb{B}_0(R)_T = \{U \big| \big|U\big|_{T,\tilde{p}+2} \leq R; U \in C^1(\bar{\Omega} \times [0,T], \otimes^2 T^*(\bar{\Omega}))\}$$

$$\mathbb{B}_N(R_N)_T = \{U \big| \big|U\big|_{T,N} \leq R_N; U \in C^1(\bar{\Omega} \times [0,T], \otimes^2 T^*(\bar{\Omega}))\}$$

where $N \geq \tilde{p} + 2$*.*

we choose $R = 2\big|\tilde{g}\big|_{\tilde{p}+2}$ and $R_N = 2\big|\tilde{g}\big|_N$ in the following.

Firstly we have the following lemma:

Lemma 47 *If* $|U^{-1}|_{L^\infty} \leq C_0$*, then*

$$\big|U^{-1}\big|_{k+1} \leq C(k, C_0, C_s(\Omega))(\big|U\big|_{\tilde{p}+1} + 1)^{k+1}(\big|U\big|_{k+1} + 1),$$

$\forall k \geq 0$*, where* $C_s(\Omega)$ *is the Sobolev constant of* Ω*.*

Proof:

$$\big|\nabla^{k+1} U^{-1}\big|_{L^2} = |\nabla^k (U^{-1} U^{-1} \nabla U)|_{L^2} = |\nabla^k (U^{-2} \nabla U)|_{L^2}$$

$$\leq C((|U^{-2}|_{L^\infty} + 1)|\nabla U|_k + |U^{-2}|_k (|\nabla U|_{L^\infty} + 1))$$

$$\leq C|\nabla U|_k + C(|U^{-1}|_k (|U^{-1}|_{L^\infty} + 1)(|U|_{\tilde{p}+1} + 1))$$

$$\leq C|U|_{k+1} + C|U^{-1}|_k (|U|_{\tilde{p}+1} + 1)$$

Let $a_k = \big|U^{-1}\big|_k$, $c_k = \big|U\big|_k$, $b = (\big|U\big|_{\tilde{p}+1} + 1)$, we get $a_{k+1} \leq C c_{k+1} + C b a_k$. By induction, it is easy to get

$$a_{k+1} \leq C(b+1)^{k+1}(c_{k+1} + 1)$$

We have

$$\big|U^{-1}\big|_{k+1} \leq C(\big|U\big|_{\tilde{p}+1} + 1)^{k+1}(\big|U\big|_{k+1} + 1)$$

□

From now on, we will denote all the uniformly bounded quantities on $[0, T]$ by C. C in different places may have different meanings, but it should be clear from the context what the C represents or depends on. If some quantity is assumed to be uniformly bounded on $[0, T]$ in a specific lemma, proposition or theorem, we will also denote it by C.

Then we have the following proposition:

Proposition 48 *There exists some $T > 0$, such that $\Phi : \mathfrak{A}_T \cap \mathbb{B}_0(R)_T \to \mathfrak{A}_T \cap \mathbb{B}_0(R)_T$.*

Proof: Firstly choose $U \in \mathfrak{A}_{t_1} \cap \mathbb{B}_0(R)_{t_1}$, t_1 is to be determined later, we want to show $V = \Phi(U) \in \mathfrak{A}_{t_1}$ for some $t_1 > 0$.

We choose a positive integer m such that $2^m > 2n$, then if we choose a normal coordinate system $\{x^i\}$, such that at one point

$$\{\tilde{g}_{ij}\} = \begin{pmatrix} 1 & & & \mathbf{0} \\ & 1 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{pmatrix}$$

$$\{v_{ij}\} = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{pmatrix}$$

then

$$\{v^{ij}\} = \begin{pmatrix} \lambda_1^{-1} & & & \mathbf{0} \\ & \lambda_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n^{-1} \end{pmatrix}$$

Now we can define:

$$\phi_m \triangleq v^{\alpha_1\beta_1}\tilde{g}_{\beta_1\alpha_2}\cdots v^{\alpha_m\beta_m}\tilde{g}_{\beta_m\alpha_1} = \sum_{i=1}^{n}\lambda_i^{-m}$$

By (4.6) and $\frac{\partial}{\partial t}v^{ij} = -v^{ik}v^{jl}\frac{\partial}{\partial t}v_{kl}$, we have

$$\frac{\partial}{\partial t}v^{ij} = -v^{ik}v^{jl}[A^{\alpha\beta}(v_{kl})_{\alpha\beta} + D_{kl}(U,V) + B_{kl}(U,\nabla U)] \tag{4.7}$$

By (61) on page 236 of [14], we get

$$(v^{ij})_{\alpha\beta} = \nabla_\alpha\nabla_\beta(v^{ij}) = -v^{ik}v^{jl}\nabla_\alpha\nabla_\beta(v_{kl}) - \nabla_\alpha(v^{ik}v^{jl})\nabla_\beta v_{kl} \tag{4.8}$$

By (4.7) and (4.8), we get

$$\frac{\partial}{\partial t}v^{ij} = A^{\alpha\beta}(v^{ij})_{\alpha\beta} + A^{\alpha\beta}(v^{ik}v^{jl})_\alpha(v_{kl})_\beta - v^{ik}v^{jl}[D_{kl}(U,V) + B_{kl}(U,\nabla U)]$$

$$\begin{aligned} (\phi_m)_t &= v^{\alpha_1\beta_1}\cdots(\tfrac{\partial}{\partial t}v^{\alpha_k\beta_k})\cdots\tilde{g}_{\beta_m\alpha_1} \\ &= v^{\alpha_1\beta_1}\cdots[A^{\alpha\beta}(v^{\alpha_k\beta_k})_{\alpha\beta} + A^{\alpha\beta}(v^{\alpha_k p}v^{\beta_k q})_\alpha(v_{pq})_\beta \\ &\quad -v^{\alpha_k p}v^{\beta_k q}[D_{pq}(U,V) + B_{pq}(U,\nabla U)]]\cdots\tilde{g}_{\beta_m\alpha_1} \\ &= A^{\alpha\beta}(\phi_m)_{\alpha\beta} + 2\lambda_i^{-m}\chi^2 u^{\alpha\beta}R_{i\alpha i\beta} - \lambda_i^{-m-1}B_{ii} \\ &\quad -2A^{\alpha\beta}\lambda_p^{-m-1}\lambda_q^{-1}(v_{pq})_\alpha(v_{pq})_\beta \\ &\quad -2\textstyle\sum_{k,l}\sum_{p<q}A^{\alpha\beta}(\lambda_k^{-1})^{m+p-q-3}(\lambda_l^{-1})^{q-p-3}(v_{kl})_\alpha(v_{kl})_\beta \end{aligned} \tag{4.9}$$

Note the last term in (4.9):

$$\begin{aligned} &-2\textstyle\sum_{k,l}\sum_{p<q}A^{\alpha\beta}(\lambda_k^{-1})^{m+p-q-3}(\lambda_l^{-1})^{q-p-3}(v_{kl})_\alpha(v_{kl})_\beta \\ =& -m\textstyle\sum_{k,l}A^{\alpha\beta}\sum_{\sigma=-2}^{m-4}(\lambda_k^{-1})^{m-\sigma-6}(\lambda_l^{-1})^{\sigma}(v_{kl})_\alpha(v_{kl})_\beta \end{aligned} \tag{4.10}$$

By (4.9) and (4.10), we get

$$\begin{aligned}(\phi_m)_t \quad &= (\chi^2+\epsilon)(u^{\alpha\beta})(\phi_m)_{\alpha\beta} - m(\chi^2+\epsilon)\textstyle\sum_{\sigma=-2}^{m-4} u^{\alpha\beta}(v_{kl})_\alpha(v_{kl})_\beta(\lambda_k^{-1})^{m-\sigma-6}(\lambda_l^{-1})^{\sigma}\\ &-2(\lambda_p^{-1})^{m+1}\lambda_q^{-1}(\chi^2+\epsilon)u^{\alpha\beta}(v_{pq})_\alpha(v_{pq})_\beta\\ &+2\lambda_i^{-m}\chi^2u^{\alpha\beta}\tilde{R}_{i\alpha i\beta} - \lambda_i^{-m-1}B_{ii}\end{aligned}$$

then we get

$$(\phi_m)_t \le (\chi^2+\epsilon)u^{\alpha\beta}(\phi_m)_{\alpha\beta} + C(n)\chi^2\left|\tilde{Rm}\right|\phi_m - \lambda_i^{-m-1}B_{ii} \tag{4.11}$$

Because $U \in \mathfrak{A}_{t_1} \cap \mathbb{B}_0(R)_{t_1}$, then

$$-\lambda_i^{-m-1}B_{ii} \le C\cdot(R+1)^2\phi_m^{1+\frac{1}{m}} \tag{4.12}$$

From (4.11), (4.12) and Maximum Principle, we have

$$\big(\max_{x\in\Omega}\phi_m(x,t)\big)_t \le C\big((\max_{x\in\Omega}\phi_m(x,t))+1\big)^2$$

$$\phi_m(x,t) \le \frac{1}{C_0 - Ct} - 1, \quad C_0 = \frac{1}{n+1}$$

Note $\phi_m(0) = 2n$, so we can choose $t_{11} > 0$, such that when $t \in [0, t_{11}]$, $\phi_m(x,t) \le 2^m$,

$$\lambda_i \ge \frac{1}{2}, \quad i = 1, 2, \cdots, n$$

Now we define

$$\psi_m = v_{\alpha_1\beta_1}\tilde{g}^{\beta_1\alpha_2}\cdots v_{\alpha_m\beta_m}\tilde{g}^{\beta_m\alpha_1} = \sum_{i=1}^{n}\lambda_i^m$$

$$\begin{aligned}(\psi_m)_t \quad &= (\chi^2+\epsilon)u^{\alpha\beta}(\psi_m)_{\alpha\beta} - 2(\chi^2+\epsilon)\textstyle\sum_{j<k}\lambda_q^{m+j-k-1}\lambda_r^{k-j-1}u^{\alpha\beta}(v_{qr})_\alpha(v_{qr})_\beta\\ &\quad -2\chi^2\lambda_i^m u^{\alpha\beta}\tilde{R}_{i\alpha i\beta} + \lambda_i^{m-1}B_{ii}\\ &\le (\chi^2+\epsilon)u^{\alpha\beta}(\psi_m)_{\alpha\beta} + C(\psi_m+1)^2\end{aligned} \tag{4.13}$$

Similarly, by (4.13) and Maximum Principle, we have

$$\phi_m(x,t) \le \frac{1}{C_0 - Ct} - 1, \quad C_0 = \frac{1}{n+1}$$

Choose $t_{12} > 0$, such that when $t \in [0, t_{12}]$, $\psi_m(x,t) \le 2^m$,

$$\lambda_i \le 2 \quad i = 1, 2, \cdots, n$$

We set $t_1 = \min\{t_{11}, t_{12}\}$,then when $t \in [0, t_1]$, $\frac{1}{2} \le \lambda_i \le 2$, we get $V \in \mathfrak{A}_{t_1}$. Note in fact we prove that $V \in \mathfrak{A}_{t_1}$ if $U \in \mathfrak{A}_{t_1} \cap \mathbb{B}_0(R)_{t_1}$.

By (4.6), for $N \ge \tilde{p} + 2$, we have

$$\frac{\partial}{\partial t}\left|V\right|_N^2 = 2\sum_\alpha \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha v_{kl}\nabla^\alpha\left(\frac{\partial}{\partial t}v_{ij}\right) = I_1 + I_2 + I_3 + I_4 \tag{4.14}$$

where

$$I_1 = 2\sum_\alpha \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha v_{kl}\nabla^\alpha (v_{ij})_{pq}A^{pq}$$

$$I_2 = 2\sum_\alpha \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha v_{kl}\left[\nabla^\alpha(A^{pq}(v_{ij})_{pq}) - A^{pq}\nabla^\alpha(v_{ij})_{pq}\right]$$

$$I_3 = 2\sum_\alpha \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha v_{kl}\nabla^\alpha D_{ij}, \quad I_4 = 2\sum_\alpha \int_\Omega (\tilde{g}^{-1})^{|\alpha+2}\nabla^\alpha v_{kl}\nabla^\alpha B_{ij}$$

$$\begin{aligned} I_1 &\le -2\sum_\alpha \int (\tilde{g}^{-1})^{|\alpha|+2}(\nabla^\alpha v_{kl})_p\nabla^\alpha(v_{ij})_q A^{pq} - 2\sum_\alpha \int (\tilde{g}^{-1})^{|\alpha|+2}(\nabla^\alpha v_{kl})\nabla^\alpha(v_{ij})_q(A^{pq})_p \\ &\le -\frac{1}{2}\sum_{|\alpha|=N}\int (\chi^2+\epsilon)\left|\nabla^{\alpha+1}V\right|^2 + C\left|V\right|_N^2 \end{aligned} \tag{4.15}$$

where we use the fact $|\nabla U^{-1}| \le |U^{-1}|^2|\nabla U| \le C(R)$.

$$I_2 \le 2\sum_\alpha \left|\nabla^\alpha V\right|_{L^2} \cdot \left[C\left(\int (\chi^2+\epsilon)\left|\nabla^{\alpha+1}V\right|^2\right)^{\frac{1}{2}} + \left|\nabla^{\alpha-2}[\nabla^2 A^{pq}(V)_{pq}]\right|_{L^2}\right] \tag{4.16}$$

and it is easy to get

$$\left|\nabla^{\alpha-2}[\nabla^2 A^{pq}(V)_{pq}]\right|_{L^2} \leq C|V|_N + C(|V|_{\tilde{p}+2}+1)(|A^{pq}|_N+1)$$

but we know from lemma 47:

$$\left|U^{-1}\right|_{k+1} \leq C(\left|U\right|_{\tilde{p}+1}+1)^{k+1}(\left|U\right|_{k+1}+1)$$

we have

$$\left|\nabla^{\alpha-2}[\nabla^2 A^{pq}(V)_{pq}]\right|_{L^2} \leq C\left|V\right|_N + C(\left|V\right|_{\tilde{p}+2}+1)(\left|U\right|_N+1) \tag{4.17}$$

By (4.16) and (4.17), we get

$$I_2 \leq \frac{1}{16}\sum_{|\alpha|=N}\int(\chi^2+\epsilon)\left|\nabla^{\alpha+1}V\right|^2 + C\left|V\right|_N^2 + C\left|V\right|_N(\left|V\right|_{\tilde{p}+2}+1)(\left|U\right|_N+1) \tag{4.18}$$

$$I_3 \leq 2\left|V\right|_N \textstyle\sum_\alpha |\nabla^\alpha D|_{L^2} \leq C|V|_N^2 + C|V|_N(\left|V\right|_{\tilde{p}}+1)(|U|_N+1) \tag{4.19}$$

$$I_4 = I_5 + I_6 \tag{4.20}$$

where

$$I_5 = 2\sum_\alpha\int(\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha v_{kl}\nabla^\alpha E_{ij}$$

$$I_6 = 2\sum_\alpha\int(\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha v_{kl}\nabla^\alpha F_{ij}$$

for I_5, we only estimate one term of it as the following, the rest are similar.

$$
\begin{aligned}
&2\sum_\alpha\int(\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha v_{kl}\nabla^\alpha[\tfrac{1}{2}\chi^2u^{rs}u^{pq}(u_{rp})_i(u_{sq})_j]\\
&\quad\le C(n)\sum_\alpha|\nabla^\alpha V|_{L^2}\cdot|\nabla^{\alpha-1}[\nabla(\chi^2U^{-2})\cdot(\nabla U)^2]|_{L^2}\\
&\qquad+C(n)\sum_\alpha\int\nabla^\alpha V\cdot(\chi^2U^{-2})\nabla^\alpha(\nabla U\nabla U)\\
&\quad\le C|V|_N\cdot[|U^{-2}|_N|((\nabla U)^2|_{\tilde{p}}+1)+(|U^{-2}|_{\tilde{p}+1}|+1)(\nabla U)^2|_{N-1}]\\
&\qquad+C(n)\sum_\alpha\int\left|\nabla^{\alpha+1}V\cdot(\chi^2U^{-2})\nabla^{\alpha-1}(\nabla U\nabla U)\right|\\
&\qquad+C(n)\sum_\alpha\int\left|\nabla^\alpha V\cdot\nabla(\chi^2U^{-2})\nabla^{\alpha-1}(\nabla U\nabla U)\right|\\
&\le C|V|_N(|U|_N+1)+\tfrac{1}{64}\sum_{|\alpha|=N}\int\chi^2|\nabla^{\alpha+1}V|^2+C|U|_N^2+C|V|_N|U|_N
\end{aligned}
$$

we get

$$
I_5\le\frac{9}{64}\sum_{|\alpha|=N}\int\chi^2|\nabla^{\alpha+1}V|^2+C|V|_N(|U|_N+1)+C|U|_N^2
$$

for I_6, we only estimate one term of it as the following, the rest are similar.

$$
\begin{aligned}
&2\sum_\alpha\int(\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha v_{kl}\nabla^\alpha[\chi\chi_iu^{pq}(u_{jq})_p]\\
&\le C\sum_\alpha\nabla^\alpha V\cdot\nabla^{\alpha-1}[\nabla(\chi\nabla\chi U^{-1})\cdot(\nabla U)]+2\sum_\alpha\int\nabla^\alpha V\cdot(\chi\nabla\chi U^{-1})\nabla^{\alpha+1}U\\
&\le C|V|_N(|U|_N+1)+\tfrac{1}{64}\sum_{|\alpha|=N}\int\chi^2|\nabla^{\alpha+1}V|^2+C|U|_N^2+C|V|_N|U|_N
\end{aligned}
$$

we have

$$
I_6\le\frac{6}{64}\sum_{|\alpha|=N}\int\chi^2|\nabla^{\alpha+1}V|^2+C|U|_N^2+C|V|_N(|U|_N+1)
$$

Then we get

$$
I_4\le\frac{15}{64}\sum_{|\alpha|=N}\int\chi^2|\nabla^{\alpha+1}V|^2+C|U|_N^2+C|V|_N(|U|_N+1) \tag{4.21}
$$

By (4.14), (4.15), (4.18), (4.19) and (4.21), we get

$$
\frac{\partial}{\partial t}|V|_N^2\le C|V|_N^2+C|V|_N(|V|_{\tilde{p}+2}+1)(|U|_N+1)+C|U|_N^2 \tag{4.22}
$$

Let $\phi(t)=|V|_N^2(t)$, then

$$
\phi_t\le C\phi+C\phi^{\frac{1}{2}}Q_1+Q_2
$$

where

$$Q_1 = (|V|_{\tilde{p}+2} + 1)(|U|_N + 1), \quad Q_2 = C|U|_N^2$$

we can get

$$\phi_t \leq C(\phi + 1)(Q_1^2 + Q_2 + 1) \leq CQ_4(\phi + 1) \tag{4.23}$$

where

$$Q_4 = (|V|_{\tilde{p}+2}^2 + 1)(|U|_N + 1)^2 \tag{4.24}$$

When we choose $N = \tilde{p} + 2$, then $\phi(t) = |V|_{\tilde{p}+2}^2(t)$, define $\psi(t) = |U|_{\tilde{p}+2}(t)$, then

$$Q_4 = (\phi + 1)(\psi + 1)^2$$

$$\phi_t \leq C(\phi + 1)^2(\psi + 1)^2 \leq C(R + 1)^2(\phi + 1)^2 = C(\phi + 1)^2$$

where $\psi \leq R$ because $U \in \mathbb{B}_0(R)_{t_1}$ and $C = C(n, N, R, ||\chi||_N, ||\tilde{Rm}||_N, C_s(\Omega))$.

$$\phi(t) \leq \frac{1}{C_0 - Ct} - 1, \quad C_0 = (\phi(0) + 1)^{-1}$$

Note $\phi(0) = (\frac{R}{2})^2$, and there exists $t_2 > 0$, such that if $t \in [0, t_2]$, $\phi(t) \leq R^2$

$$\Longrightarrow |V|_{\tilde{p}+2} \leq R$$

We choose

$$T = t_3 = \min\{t_1, t_2\} \tag{4.25}$$

we get $\phi(U) = V \in \mathfrak{A}_T \cap \mathbb{B}_0(R)_T$ if $U \in \mathfrak{A}_T \cap \mathbb{B}_0(R)_T$. We have

$$\phi : \mathfrak{A}_T \cap \mathbb{B}_0(R)_T \longrightarrow \mathfrak{A}_T \cap \mathbb{B}_0(R)_T$$

□

Proposition 49 *For any $N \geq \tilde{p} + 2$, there exists $T > 0$, such that*

$$\Phi : \mathfrak{A}_T \cap \mathbb{B}_0(R)_T \cap \mathbb{B}_N(R_N)_T \longrightarrow \mathfrak{A}_T \cap \mathbb{B}_0(R)_T \cap \mathbb{B}_N(R_N)_T$$

Proof: Choose $U \in \mathfrak{A}_{t_3} \cap \mathbb{B}_0(R)_{t_3} \cap \mathbb{B}_N(R_N)_{t_3}$, where t_3 is defined in (4.25). Then by Proposition 48, we get $V = \Phi(U) \in \mathfrak{A}_{t_3} \cap \mathbb{B}_0(R)_{t_3}$, so $|V|_{\tilde{p}+2} \leq R$. And we also have $|U|_N \leq R_N$, so by (4.23) and (4.24), we get

$$\phi_t \leq C(R^2 + 1)(R_N + 1)^2(\phi + 1) \leq C(\phi + 1)$$

where $\phi(t) = |V|_N^2(t)$ and $C = C(n, N, R, R_N, \|\chi\|_N, \|\tilde{R}m\|_N, C_s(\Omega))$.

$$\phi(t) \leq (C_0 - Ct)^{-1} - 1, \quad C_0 = (\phi(0) + 1)^{-1}$$

Note $\phi(0) = (\frac{R_N}{2})^2$.

Then there exists $t_4 > 0$, such that if $t \in [0, t_4]$,

$$\phi(t) \leq (R_N)^2$$

which implies $|V|_N \leq R_N$.

Now we choose

$$T = t_5 = \min\{t_3, t_4\}$$

Then when $t \in [0, T]$, we have

$$|V(x, t)|_N \leq R_N$$

$\Longrightarrow V \in \mathfrak{A}_T \cap \mathbb{B}_0(R)_T \cap \mathbb{B}_N(R_N)_T$ if $U \in \mathfrak{A}_T \cap \mathbb{B}_0(R)_T \cap \mathbb{B}_N(R_N)_T$. □

Next we show that Φ is a contraction map.

Proposition 50 *There exists $T > 0$, such that*

$$|\Phi(U) - \Phi(\tilde{U})|_{T,\tilde{p}} \leq \delta |U - \tilde{U}|_{T,\tilde{p}}$$

for some $0 < \delta < 1$, *any* $U, \tilde{U} \in \mathfrak{A}_T \cap \mathbb{B}_0(R)_T$.

Proof: By (4.6) we have the following

$$\begin{cases} V_t &= (\chi^2 + \epsilon) u^{\alpha\beta} (V)_{\alpha\beta} + P(U)V + B(U, \nabla U) \\ \tilde{V}_t &= (\chi^2 + \epsilon) \tilde{u}^{\alpha\beta} (\tilde{V})_{\alpha\beta} + P(\tilde{U})\tilde{V} + B(\tilde{U}, \nabla \tilde{U}) \end{cases}$$

where

$$V = \Phi(U);\ (P(U)V)_{ij} = D_{ij}(U, V);\ (B(U, \nabla U))_{ij} = B_{ij}(U, \nabla U);\ (V)_{\alpha\beta} = \nabla_\alpha \nabla_\beta V$$

similar notation for $\tilde{V}$ etc.

Define $W \triangleq V - \tilde{V} = \Phi(U) - \Phi(\tilde{U})$, then

$$W_t = (\chi^2 + \epsilon) u^{\alpha\beta} w_{\alpha\beta} + P(U)W + H$$

where

$$H = (\chi^2 + \epsilon)(u^{\alpha\beta} - \tilde{u}^{\alpha\beta})(\tilde{V})_{\alpha\beta} + (P(U) - P(\tilde{U}))\tilde{V} + \left(B(U, \nabla U) - B(\tilde{U}, \nabla \tilde{U})\right)$$

We consider $|W|^2_{\tilde{p}}$:

$$\frac{\partial}{\partial t} |W|^2_{\tilde{p}} = J_1 + J_2 + J_3 + J_4 \tag{4.26}$$

where

$$J_1 = 2 \sum_{|\alpha| \leq \tilde{p}} \int (\tilde{g}^{-1})^{|\alpha|+2} \nabla^\alpha w_{kl} \nabla^\alpha (w_{ij})_{pq} A^{pq}$$

$$J_2 = 2 \sum \int (\tilde{g}^{-1})^{|\alpha|+2} \nabla^\alpha w_{kl} [\nabla^\alpha (A^{pq} (w_{ij})_{pq}) - A^{pq} \nabla^\alpha (w_{ij})_{pq}]$$

$$J_3 = 2\sum_{|\alpha|\le\tilde{p}}\int(\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha w_{kl}\nabla^\alpha(P(U)W)_{ij}, \quad J_4 = 2\sum_{|\alpha|\le\tilde{p}}\int(\tilde{g}^{-1})^{|\alpha|+2}\nabla^\alpha w_{kl}\nabla^\alpha H_{ij}$$

Recall $A^{pq} = (\chi^2+\epsilon)u^{pq}$. Similar to (4.15), we have

$$J_1 \le -\frac{1}{2}\sum_{|\alpha|=\tilde{p}}\int(\chi^2+\epsilon)|\nabla^{\alpha+1}W|^2 + C|W|^2_{\tilde{p}} \tag{4.27}$$

Similar to (4.16), we get

$$J_2 \le 2\sum_{|\alpha|\le\tilde{p}}|\nabla^\alpha W|_{L^2}\cdot[C(\int(\chi^2+\epsilon)|\nabla^{\alpha+1}W|^2)^{\frac{1}{2}} + |\nabla^{\alpha-2}[\nabla^2 A^{pq}(W)_{pq}]|_{L^2}]$$

and we have

$$\begin{aligned}|\nabla^{\alpha-2}[\nabla^2 A^{pq}(W)_{pq}]|_{L^2} &\le C\textstyle\sum_{\beta+\gamma=\alpha}|\nabla^\beta(\nabla^2 A^{pq})\nabla^\gamma W|_{L^2}\\ &\le C((|\nabla^2 A^{pq}|_{L^\infty}+1)|\nabla^\alpha W|_{L^2} + |W|_{L^\infty}(|\nabla^\alpha(\nabla^2 A^{pq})|_{L^2}+1))\\ &\le C[(|U|_{\tilde{p}+2}+1)|W|_{\tilde{p}} + |W|_{\tilde{p}}(|U|_{\tilde{p}+2}+1)] \le C|W|_{\tilde{p}}\end{aligned}$$

so

$$J_2 \le \frac{1}{16}\sum_{|\alpha|=\tilde{p}}\int(\chi^2+\epsilon)|\nabla^{\alpha+1}W|^2 + C|W|^2_{\tilde{p}} \tag{4.28}$$

Similar to (4.19), we get

$$J_3 \le C|W|^2_{\tilde{p}} \tag{4.29}$$

$$J_4 = J_5 + J_6 + J_7 \tag{4.30}$$

where

$$J_5 = 2\sum_\alpha\int(\tilde{g}^{-1})^{\alpha+2}\nabla^\alpha W\cdot\nabla^\alpha[(\chi^2+\epsilon)(U^{-1}-\tilde{U}^{-1})\nabla^2\tilde{V}]$$

$$J_6 = 2\sum_\alpha\int(\tilde{g}^{-1})^{\alpha+2}\nabla^\alpha W\cdot\nabla^\alpha[(P(U)-P(\tilde{U}))\tilde{V}]$$

$$J_7 = 2\sum_{\alpha}\int(\tilde{g}^{-1})^{\alpha+2}\nabla^{\alpha}W\cdot\nabla^{\alpha}[B(U,\nabla U) - B(\tilde{U} - \nabla\tilde{U})]$$

$$\begin{aligned} J_5 &\leq C|W|_{\tilde{p}}\Big(\textstyle\sum_{\alpha}|\nabla^{\alpha}[(\chi^2+\epsilon)\nabla^2\tilde{V}\cdot(U^{-1}-\tilde{U}^{-1})]|_{L^2}\Big) \\ &\leq C|W|_{\tilde{p}}(|\tilde{V}|_{\tilde{p}+2}+1)|U^{-1}-\tilde{U}^{-1}|_{\tilde{p}} \leq C|W|_{\tilde{p}}|\tilde{U}-U|_{\tilde{p}} \end{aligned} \tag{4.31}$$

It is easy to get

$$J_6 \leq C|W|_{\tilde{p}}[|P(U) - P(\tilde{U})|_{\tilde{p}}(|\tilde{V}|_{\tilde{p}}+1)] \leq C|W|_{\tilde{p}}|\tilde{U}-U|_{\tilde{p}} \tag{4.32}$$

We only need to estimate two terms of J_7 like the following J_8 and J_9, the others are similar.

$$J_8 = \sum_{\alpha}\int(\tilde{g}^{-1})^{|\alpha|+2}\cdot\nabla^{\alpha}W\cdot\nabla^{\alpha}[\chi^2(U^{-2}(\nabla U)^2 - \tilde{U}^{-2}(\nabla\tilde{U})^2)]$$

$$J_9 = \sum_{\alpha}\int(\tilde{g}^{-1})^{|\alpha|+2}\cdot\nabla^{\alpha}W\cdot\nabla^{\alpha}[\chi\nabla\chi(U^{-1}\nabla U - \tilde{U}^{-1}\nabla\tilde{U})]$$

We first estimate J_8

$$J_8 \leq J_{10} + J_{11} + J_{12} + J_{13} \tag{4.33}$$

where

$$J_{10} = C\sum_{\alpha}\int(\tilde{g}^{-1})^{|\alpha|+2}\cdot\nabla^{\alpha}W\cdot\nabla^{\alpha}[(U^{-1}-\tilde{U}^{-1})U^{-1}(\nabla U)^2\chi^2]$$

$$J_{11} = C\sum_{\alpha}\int(\tilde{g}^{-1})^{|\alpha|+2}\cdot\nabla^{\alpha}W\cdot\nabla^{\alpha}[(U^{-1}-\tilde{U}^{-1})\tilde{U}^{-1}(\nabla U)^2\chi^2]$$

$$J_{12} = C\sum_{\alpha}\int(\tilde{g}^{-1})^{|\alpha|+2}\cdot\nabla^{\alpha}W\cdot\nabla^{\alpha}[(\tilde{U}^{-2}\nabla U\chi^2)\nabla(U-\tilde{U})]$$

$$J_{13} = C\sum_{\alpha}\int(\tilde{g}^{-1})^{|\alpha|+2}\cdot\nabla^{\alpha}W\cdot\nabla^{\alpha}[(\tilde{U}^{-2}\nabla\tilde{U}\chi^2)\nabla(U-\tilde{U})]$$

Firstly, it is easy to get

$$J_{10} \leq C|W|_{\tilde{p}}\Big(|U^{-1} - \tilde{U}^{-1}|_{\tilde{p}}(|U^{-1}(\nabla U)^2|_{\tilde{p}} + 1)\Big) \leq C|W|_{\tilde{p}}|\tilde{U} - U|_{\tilde{p}} \tag{4.34}$$

Similarly, we get

$$J_{11} \leq C|W|_{\tilde{p}}|\tilde{U} - U|_{\tilde{p}} \tag{4.35}$$

We have $J_{12} \leq J_{12.1} + J_{12.2}$, where

$$J_{12.1} = C\sum_{\alpha}\int(\tilde{g}^{-1})^{|\alpha|+2} \cdot \nabla^{\alpha} W \cdot \nabla^{\alpha-1}[\nabla(\tilde{U}^{-2}\nabla U\chi^2)\nabla(U - \tilde{U})]$$
$$J_{12.2} = C\sum_{\alpha}\int(\tilde{g}^{-1})^{|\alpha|+2} \cdot \nabla^{\alpha} W \cdot (\tilde{U}^{-2}\nabla U\chi^2) \cdot \nabla^{\alpha+1}(U - \tilde{U})$$

$$J_{12.1} \leq C|W|_{\tilde{p}}\sum_{\alpha}\left|\nabla^{\alpha-1}[f\nabla g]\right|_{L^2} \tag{4.36}$$

where $f = \nabla(\tilde{U}^{-2}\chi^2\nabla U)$, $g = U - \tilde{U}$, and note $|f|_{\tilde{p}} \leq C(|u|_{\tilde{p}+2} + |\tilde{u}|_{\tilde{p}+2} + 1) \leq C$.

Then

$$\begin{aligned} |\nabla^{\alpha-1}[f\nabla g]|_{L^2} &\leq C\textstyle\sum_{\beta+\gamma=\alpha}|\nabla^{\beta} f\nabla^{\gamma} g|_{L^2} \\ &\leq C((|f|_{L^\infty} + 1)|\nabla^{\alpha} g|_{L^2} + |g|_{L^\infty}(|\nabla^{\alpha} f|_{L^2} + 1)) \\ &\leq C|g|_{\tilde{p}} = C|U - \tilde{U}|_{\tilde{p}} \end{aligned} \tag{4.37}$$

With (4.36) and (4.37), we get

$$J_{12.1} \leq C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}} \tag{4.38}$$

it is easy to get

$$J_{12.2} \leq \tfrac{1}{1000}\textstyle\sum_{|\alpha|=\tilde{p}}\int|\nabla^{\alpha+1}W|^2\chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}} \tag{4.39}$$

by (4.38) and (4.39), we get

$$J_{12} \leq \frac{1}{1000}\sum_{|\alpha|=\tilde{p}}\int|\nabla^{\alpha+1}W|^2\chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}} \tag{4.40}$$

Similarly,

$$J_{13} \leq \frac{1}{1000} \sum_{|\alpha|=\tilde{p}} \int |\nabla^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}} |U - \tilde{U}|_{\tilde{p}} \tag{4.41}$$

By (4.34), (4.35), (4.40), (4.41) and (4.33), we get

$$J_8 \leq \frac{1}{500} \sum_{|\alpha|=\tilde{p}} \int |\nabla^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}} |U - \tilde{U}|_{\tilde{p}} \tag{4.42}$$

$$J_9 \leq J_{14} + J_{15} \tag{4.43}$$

where

$$J_{14} = \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \nabla^\alpha W \cdot \nabla^\alpha [(\chi \nabla \chi \nabla U)(U^{-1} - \tilde{U}^{-1})]$$

$$J_{15} = \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \nabla^\alpha W \cdot \nabla^\alpha [(\chi \nabla \chi \tilde{U}^{-1}) \nabla (U - \tilde{U})]$$

It is easy to get

$$J_{14} \leq C|W|_{\tilde{p}} |U - \tilde{U}|_{\tilde{p}} \tag{4.44}$$

Similar with the argument for J_{12}, we could get

$$J_{15} \leq \frac{1}{1000} \sum_{|\alpha|=\tilde{p}} \int |\nabla^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}} |U - \tilde{U}|_{\tilde{p}} \tag{4.45}$$

By (4.44), (4.45) and (4.43), we get

$$J_9 \leq \frac{1}{1000} \sum_{|\alpha|=\tilde{p}} \int |\nabla^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}} |U - \tilde{U}|_{\tilde{p}} \tag{4.46}$$

By (4.42) and (4.46), we get

$$J_7 \leq \Big(\frac{9}{500} + \frac{6}{1000}\Big) \sum_{|\alpha|=\tilde{p}} \int |\nabla^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}} |U - \tilde{U}|_{\tilde{p}} \tag{4.47}$$

By (4.31), (4.32), (4.47) and (4.30), we get

$$J_4 \le \frac{3}{100} \sum_{|\alpha|=\tilde{p}} \int |\nabla^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|^2_{\tilde{p}} + C|W|_{\tilde{p}} |U - \tilde{U}|_{\tilde{p}} \tag{4.48}$$

By (4.27), (4.28), (4.29), (4.48), and (4.26) we get

$$\frac{\partial}{\partial t}(W|^2_{\tilde{p}}) \le C_1 |W|^2_{\tilde{p}} + C_2 |U - \tilde{U}|^2_{\tilde{p}} \tag{4.49}$$

where $C_1 = C_1(n, R, ||\chi||_{\tilde{p}}, ||\tilde{Rm}||_{\tilde{p}}, C_s(\Omega)) > 0$ and C_2 depends on the same parameter like C_1.

By Gronwall's inequality and (4.49), note $|W(x,0)|_{\tilde{p}} = 0$, we get

$$|W|^2_{\tilde{p}}(t) \le e^{C_1 t} \cdot \int_0^t C_2 |U - \tilde{U}|^2_{\tilde{p}}(s) ds \le C_2 t e^{C_1 t} |U - \tilde{U}|^2_{t,\tilde{p}} \tag{4.50}$$

We can choose $t_6 > 0$, such that $C_2 t_6 e^{C_1 t_6} = \delta^2 < 1$, then we choose

$$T = t_7 = \min\{t_5, t_6\} \tag{4.51}$$

then if $t \in [0, T]$, we get $|W|^2_{\tilde{p}}(t) \le \delta^2 |U - \tilde{U}|^2_{T,\tilde{p}}$. We have

$$|W|_{T,\tilde{p}} \le \delta |U - \tilde{U}|_{T,\tilde{p}}$$

that is

$$|\Phi(U) - \Phi(\tilde{U})|_{T,\tilde{p}} \le \delta |U - \tilde{U}|_{T,\tilde{p}}$$

□

Theorem 51 *There exists $T > 0$, such that on $\Omega \times [0, T]$, degenerate parabolic system (4.4) has a smooth solution.*

Proof: We choose $T = t_7$, and t_7 is defined in (4.51). Now we set $U^0(x,t) = \tilde{g}(x)$, and $U^{i+1} = \Phi(U^i)$ where U^i are all $(2,0)$-tensor on $\Omega \times [0, T]$, $i = 0, 1, 2 \cdots$. Note $U^0 \in \mathfrak{A}_T \cap$

$\mathbb{B}_0(R)_T \cap \mathbb{B}_N(R_N)_T$ for any $N \geq \tilde{p}+2$, by Proposition 49, we get $U^i \in \mathfrak{A}_T \cap \mathbb{B}_0(R)_T \cap \mathbb{B}_N(R_N)_T$ for any i. Then we have the following:

$$\sup_{t\in[0,T]} \sup_{x\in\Omega} \sup_{|\alpha|\leq N} |\nabla^\alpha U^i(x,t)| \leq C|U^i|_{T,\tilde{p}+N} \leq CR_{\tilde{p}+N} \tag{4.52}$$

for any $i, N \geq 0$, where $C = C(C_s(\Omega), N)$, and note C is independent of i.

Look at (4.5), we get

$$\sup_{t\in[0,T]} \sup_{x\in\Omega} |D_t^N U^i(x,t)| \leq C \tag{4.53}$$

for any $i, N \geq 0$, where $C = C(C_s, N, R, R_{\tilde{p}+3}, \cdots, R_{\tilde{p}+2N}, \|\chi\|_{\tilde{p}+2N}, \|\tilde{Rm}\|_{\tilde{p}+2N})$, note C is independent of i.

So $\{U^i\}_{i=1}^\infty$ are uniformly bounded in $C^k(\Omega\times[0,T])$ for any positive integer k, we can choose subsequence $U^{k_i} \to U^\infty$ in $C^{\tilde{p}}(\Omega\times[0,T])$. For $k_i \geq m > 0$, we get

$$|U^m - U^{k_i}|_{T,\tilde{p}} = |\Phi^m(U^0) - \Phi^{k_i}(U^0)|_{T,\tilde{p}} \leq \delta^m|U^0 - \Phi^{k_i-m}(U^0)|_{T,\tilde{p}} \leq 2\delta^m R$$

So we have

$$\lim_{m\to\infty} |U^m - U^\infty|_{T,\tilde{p}} \leq \lim_{m\to\infty} (|U^m - U^{k_i}|_{T,\tilde{p}} + |U^{k_i} - U^\infty|_{T,\tilde{p}}) = 0$$

Now

$$\begin{aligned} |U^\infty - \Phi(U^\infty)|_{T,\tilde{p}} &= \lim_{m\to\infty} |U^m - \Phi(U^\infty)|_{T,\tilde{p}} = \lim_{m\to\infty} |\Phi(U^{m-1}) - \Phi(U^\infty)|_{T,\tilde{p}} \\ &\leq \delta \lim_{m\to\infty} |U^{m-1} - U^\infty|_{T,\tilde{p}} = \delta\cdot 0 = 0 \end{aligned}$$

then we have

$$|U^\infty - \Phi(U^\infty)|_{C^0} \leq C_s(\Omega)|U^\infty - \Phi(U^\infty)|_{T,\tilde{p}} = 0$$

so

$$U^\infty = \Phi(U^\infty) \tag{4.54}$$

Now we come back to the notation with ϵ, then (4.54) is

$$U_\epsilon^\infty = \Phi_\epsilon(U_\epsilon^\infty) \tag{4.55}$$

with respect to (4.5) with ϵ.

By (4.52) and (4.53), we get $\{U_\epsilon^\infty\}_{0<\epsilon\leq 1}$ is uniformly bounded in $C^k(\Omega \times [0,T])$ for any k. There exists subsequence $\{U_{\epsilon_i}^\infty\}_{i=1}^\infty$, such that

$$\lim_{i\to\infty} U_{\epsilon_i}^\infty = \tilde{U}^\infty \quad in \quad C^{\tilde{p}}(\Omega \times [0,T])$$

where $\lim_{i\to\infty} \epsilon_i = 0$.

On the other hand, by (4.55) and (4.5), we have

$$(U_{\epsilon_i}^\infty)_t = (\chi^2 + \epsilon_i)(U_{\epsilon_i}^\infty)^{\alpha\beta}\nabla_\alpha\nabla_\beta(U_{\epsilon_i}^\infty) + D(U_{\epsilon_i}^\infty) + B(U_{\epsilon_i}^\infty, \nabla U_{\epsilon_i}^\infty) \tag{4.56}$$

Let $i \to \infty$ in (4.56), we get

$$(\tilde{U}^\infty)_t = \chi^2(\tilde{U}^\infty)^{\alpha\beta}\nabla_\alpha\nabla_\beta(\tilde{U}^\infty) + D(\tilde{U}^\infty) + B(\tilde{U}^\infty, \nabla\tilde{U}^\infty)$$

Then $\tilde{U}^\infty$ is a solution of (4.4) on $\Omega \times [0,T]$, and $\tilde{U}^\infty \in C^{\tilde{p}}(\Omega \times [0,T])$.

Because $\{U_\epsilon^\infty\}_{0<\epsilon\leq 1}$ is uniformly bounded in $C^k(\Omega \times [0,T])$ for any k, then for any k, we have subsequence of $\{U_{\epsilon_i}^\infty\}_{i=1}^\infty$ denoted as $\{U_j^\infty\}$ such that $\lim_{j\to\infty} U_j^\infty = \tilde{U}^{k,\infty}$ in $C^k(\Omega \times [0,T])$. We get that $\tilde{U}^\infty = \tilde{U}^{k,\infty} \in C^k(\Omega \times [0,T])$ for any k. Then we have

$$\tilde{U}^\infty \in C^\infty(\Omega \times [0,T])$$

So $\tilde{U}^\infty$ is a smooth solution of (4.4) on $\Omega \times [0,T]$. □

Chapter 5

Curvature and time estimates of local Ricci flow

The organization of this Chapter is as the following. In section 1, we discuss the evolution equations and inequalities of curvature tensors under local Ricci flow, which are almost parallel to Ricci flow case. In order to use our uniform local assumption in 6.1 or 6.19, we use the partition of unity principle to choose some suitable cut-off functions ξ_i on the whole manifold. Then we multiply $\chi^2 Rm$ with ξ_i to do estimates on them, that is so called curvature's "local local" estimates in section 2. The main technical tools are energy estimate and Moser iteration in this special situation. Finally we combine these results with the extension criteria results of local Ricci flow (see [28]) to prove the time estimate of the local Ricci flow in section 3.

5.1 Evolution equations of curvature tensors

We want to find the evolution equation of the curvature tensor Rm under local Ricci flow. We use the following notations: $R_{ijk}^{l} = g^{lp}R_{ijkp}$, $R_i^r = g^{rp}R_{ip}$, and R is the scalar curvature.

Lemma 52 *Under the local Ricci flow (4.1), we have*

$$\frac{\partial}{\partial t}R_{ijkl} = \chi^2 \Delta R_{ijkl} + Q + I_1 + I_2 + I_3 \tag{5.1}$$

where Q, I_1, I_2, I_3 are defined in (5.8), (5.4),(5.5) and (5.6) below.

Proof: Firstly, from [29], we know that if $\frac{\partial}{\partial t}g_{ij} = h_{ij}$, we have (see Remark 3.4 in [29])

$$\frac{\partial}{\partial t}R^l_{ijk} = \frac{1}{2}g^{lp}[\nabla_i\nabla_k h_{jp} + \nabla_j\nabla_p h_{ik} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_k h_{ip} - R^q_{ijk}h_{qp} - R^q_{ijp}h_{kq}]$$

In our case, we have

$$h_{ij} = -2\chi^2 R_{ij}.$$

Now

$$\frac{\partial}{\partial t}R_{ijkl} = (-2\chi^2 R_{lm})R^m_{ijk} + g_{lm}(\frac{\partial}{\partial t}R^m_{ijk}). \tag{5.2}$$

We calculate $g_{lm}\frac{\partial}{\partial t}R^m_{ijk}$:

$$g_{lm}\tfrac{\partial}{\partial t}R^m_{ijk} = \tfrac{1}{2}[\nabla_i\nabla_k h_{jl} + \nabla_j\nabla_l h_{ik} - \nabla_i\nabla_l h_{jk} - \nabla_j\nabla_k h_{il} - R^q_{ijk}h_{ql} - R^q_{ijl}h_{kq}].$$

We calculate $\nabla_i\nabla_k h_{jl}$ as a sample, using the notation $\chi_i = \nabla_i\chi$ and $\chi_{ij} = \nabla_i\nabla_j\chi$:

$$\nabla_i\nabla_k h_{jl} = -2(\chi^2\nabla_i\nabla_k R_{jl} + 2\chi\chi_i\nabla_k R_{jl} + 2\chi\chi_k\nabla_i R_{jl} + 2\chi_i\chi_k R_{jl} + 2\chi\chi_{ik}R_{jl})$$

After a straightforward computation we get

$$\begin{aligned} g_{lm}\tfrac{\partial}{\partial t}R^m_{ijk} &= \chi^2(\nabla_i\nabla_l R_{jk} + \nabla_j\nabla_k R_{il} - \nabla_i\nabla_k R_{jl} - \nabla_j\nabla_l R_{ik}) \\ &\quad + \chi^2(R^q_{ijk}R_{ql} + R^q_{ijl}R_{kq}) + I_1 + I_2 + I_3 \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} I_1 &= 2\chi[(\chi_k\nabla_j R_{il} + \chi_j\nabla_k R_{il} + \chi_l\nabla_i R_{jk} \\ &\quad + \chi_i\nabla_l R_{jk}) - (\chi_k\nabla_i R_{jl} + \chi_i\nabla_k R_{jl} + \chi_l\nabla_j R_{ik} + \chi_j\nabla_l R_{ik})] \end{aligned} \tag{5.4}$$

$$I_2 = 2[(\chi_i\chi_l R_{jk} + \chi_j\chi_k R_{il}) - (\chi_i\chi_k R_{jl} + \chi_j\chi_l R_{ik})] \tag{5.5}$$

$$I_3 = 2\chi[(\chi_{il}R_{jk} + \chi_{jk}R_{il}) - (\chi_{ik}R_{jl} + \chi_{jl}R_{ik})]. \tag{5.6}$$

Now we want to compute ΔR_{ijkl}. From the formula for ΔR^l_{ijk} on page 178 of [29], we know that

$$\begin{aligned}
\Delta R_{ijkl} &= \Delta(g_{lm}R^m_{ijk}) = g_{lm}\Delta R^m_{ijk} \\
&= (\nabla_i\nabla_l R_{jk} + \nabla_j\nabla_k R_{il} - \nabla_i\nabla_k R_{jl} - \nabla_j\nabla_l R_{ik}) \\
&\quad +(R^r_j R_{irkl} - R^r_i R_{jrkl}) - g^{pq}(R^r_{ijp}R_{rqkl} - 2R^r_{pik}R_{jqrl} + 2R^r_{jqk}R_{pirl})
\end{aligned} \tag{5.7}$$

By (5.2), (5.3) and (5.7), we get

$$\frac{\partial}{\partial t}R_{ijkl} = \chi^2\Delta R_{ijkl} + Q + I_1 + I_2 + I_3,$$

where

$$Q = Q_1 + Q_2 \tag{5.8}$$

$$Q_1 = \chi^2 g^{pq}(R^r_{ijp}R_{rqkl} - 2R^r_{pik}R_{jqrl} + 2R^r_{jqk}R_{pirl}) \tag{5.9}$$

$$Q_2 = -\chi^2(R_{mjkl}R^m_i + R_{imkl}R^m_j + R_{ijml}R^m_k + R_{ijkm}R^m_l). \tag{5.10}$$

This is our conclusion. □

Next we calculate $\frac{\partial}{\partial t}|Rm|^2$, we have the following lemma:

Lemma 53 *Under the local Ricci flow (4.1), we have*

$$\begin{aligned}
\frac{\partial}{\partial t}|Rm| &\le \chi^2\Delta|Rm| + \Big(\frac{\chi^2\big|\nabla|Rm|\big|^2}{|Rm|}\Big) - (1-\epsilon)\Big(\frac{\chi^2|\nabla Rm|^2}{|Rm|}\Big) \\
&\quad + 10\chi^2|Rm|^2 + \Big(\frac{10^5}{\epsilon}\Big)n|\nabla\chi|^2|Rm| + \Big(\frac{8\chi g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\chi_{il}R_{jk}}{|Rm|}\Big)
\end{aligned} \tag{5.11}$$

where ϵ is any positive constant satisfying $0 < \epsilon < 1$.

Proof:

$$\frac{\partial}{\partial t}|Rm|^2 = 2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\Big(\frac{\partial}{\partial t}R_{ijkl}\Big) + \frac{\partial}{\partial t}(g^{ri}g^{sj}g^{pk}g^{ql})R_{rspq}R_{ijkl}$$

Note

$$\frac{\partial}{\partial t}(g^{ri})g^{sj}g^{pk}g^{ql}R_{rspq}R_{ijkl} = 2\chi^2 g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}R_{mjkl}R_i^m$$

Similarly, we get

$$\frac{\partial}{\partial t}(g^{ri}g^{sj}g^{pk}g^{ql})R_{rspq}R_{ijkl} = -2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}Q_2$$

We have

$$\frac{\partial}{\partial t}|Rm|^2 = \chi^2[\Delta(|Rm|^2) - 2|\nabla Rm|^2] + 2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}[Q_1 + I_1 + I_2 + I_3]$$

Let $f = |Rm|$, we get

$$\frac{\partial}{\partial t}(f^2) = \chi^2\Delta(f^2) - 2\chi^2|\nabla Rm|^2 + 2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}[Q_1 + I_1 + I_2 + I_3]$$

Firstly we have the following estimate:

$$2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}Q_1 \leq 10\chi^2 f^3.$$

Next it is easy to get

$$2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}I_1 \leq 32\sqrt{n}\chi|\nabla\chi||\nabla Rm|f \leq 2\epsilon(\chi|\nabla Rm|)^2 + (\frac{10^4}{\epsilon})n|\nabla\chi|^2 f^2;$$

also

$$2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}I_2 \leq 16\sqrt{n}\,|\nabla\chi|^2 f^2$$

We can use the normal coordinate system to simplify:

$$2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}I_3 = 16\chi\sum_{ijkl}(\chi_{il}R_{jk}R_{ijkl}) = 16\chi g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\chi_{il}R_{jk}$$

Using normal coordinates it is also easy to see the following

$$\Big|\nabla|Rm|\Big| \leq |\nabla Rm|.$$

Then we get

$$\frac{\partial}{\partial t}(f^2) \leq 2f\chi^2\Delta f + 2\chi^2|\nabla f|^2 - (2-2\epsilon)\chi^2|\nabla Rm|^2 + 10\chi^2 f^3 + (\tfrac{10^5}{\epsilon})n|\nabla\chi|^2 f^2 + 16\chi g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\chi_{il}R_{jk},$$

we have

$$\frac{\partial}{\partial t}f \leq \chi^2\Delta f + (\tfrac{\chi^2|\nabla f|^2}{f}) - (1-\epsilon)(\tfrac{\chi^2|\nabla Rm|^2}{f}) + 10\chi^2 f^2 + (\tfrac{10^5}{\epsilon})n|\nabla\chi|^2 f + \Big(\tfrac{8\chi g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\chi_{il}R_{jk}}{f}\Big).$$

Replace f with $|Rm|$, we get our conclusion. □

Using the formula in section 1 of Chapter 3 of [29], we can get the following formula (because the calculation is very similar with the reduction of Lemma 52, we omit the calculation here, just give the formula):

Lemma 54 *Under the local Ricci flow (4.1), we have*

$$\frac{\partial}{\partial t}R_{ij} = \chi^2\Delta(R_{ij}) + J_1 + J_2 + J_3 + J_4, \tag{5.12}$$

$$\begin{aligned} \textit{where} \quad J_1 &= 2\chi g^{pq}\chi_p\nabla_q R_{ij} + \chi(\chi_i\nabla_j R + \chi_j\nabla_i R) \\ &\quad - 2\chi\chi_p g^{pq}(\nabla_i R_{qj} + \nabla_j R_{qi}), \\ J_2 &= -2g^{pq}\chi(\chi_{ip}R_{qj} + \chi_{jp}R_{qi}) + 2\chi\chi_{ij}R, \\ J_3 &= -2g^{pq}(\chi_i\chi_p R_{qj} + \chi_j\chi_p R_{qi}) + 2\chi_i\chi_j R, \\ J_4 &= 2\chi^2 g^{pq}(R^r_{qij}R_{rp} - R_{ip}R_{qj}). \end{aligned} \tag{5.13}$$

Imitating the reduction of Lemma 53, we can also get the following theorem about $|Rc|$:

Lemma 55 *Under the local Ricci flow (4.1), we have*

$$\begin{aligned}\frac{\partial}{\partial t}|Rc| \le \chi^2\Delta|Rc| &+ \Big(\frac{\chi^2\big|\nabla|Rc|\big|^2}{|Rc|}\Big) - (1-\epsilon)\Big(\frac{\chi^2|\nabla Rc|^2}{|Rc|}\Big) \\ &+ (\frac{10^5}{\epsilon})n|\nabla\chi|^2|Rc| + (2+\sqrt{n})\chi^2|Rc||Rm| \\ &+ \Big(\frac{g^{ik}g^{jl}R_{kl}J_2}{|Rc|}\Big)\end{aligned} \tag{5.14}$$

where J_2 in the last line comes from (5.13), and ϵ is any positive constant satisfying $0 < \epsilon < 1$.

Proof: Similar to the proof of Lemma 53. □

5.2 "Local local" curvature estimates

In this section, we will use the product of two different cut-off functions to do curvature estimates, so we call them "Local local" curvature estimates. We assume (M^n, g_0) satisfy the hypothesis of Theorem 80 (First case) or of Theorem 81 (second case).

5.2.1 The First Case: $p_0 > \frac{n}{2}$

We state some lemmas about Sobolev constants. Let us recall that $\tilde{B}_x(r)$ is the open geodesic ball of radius r centered at x on (M, g_0). There is well-known Sobolev inequality

$$\Big(\fint_{\tilde{B}_x(r)} f^{\frac{2n}{n-2}} dV_{g(t)}\Big)^{\frac{n-2}{n}} \le A(\tilde{B}_x(r), t) r^2 \fint_{\tilde{B}_x(r)} |\nabla f|^2 dV_{g(t)}, \quad \forall f \in C_0^\infty(\tilde{B}_x(r)) \tag{5.15}$$

with respect to each metric $g(t)$, note $A(\tilde{B}_x(r), t)$ depends on both t and $\tilde{B}_x(r)$. When r is fixed, we use the notation $A(t)$ to denote a uniform Sobolev constant for every $\tilde{B}_x(r)$, $x \in M$.

Lemma 56 *If $a^{-1}g_0 \le g(t) \le ag_0$, then we have the following inequality:*

$$a^{-(n+1)}A_0 \le A(t) \le a^{n+1}A_0$$

where $A_0 = A(0)$ is the uniform Sobolev constant with respect to $\tilde{B}_x(r)$ and g_0, and $a \geq 1$ is some positive constant.

Proof: It is straightforward from scaling argument and (5.15). $\square$

In the rest of this subsection, we assume (M, g_0) satisfies the assumption of Theorem 80, namely inequalities (6.1), with some fixed constant $0 < r \leq 1$.

By Theorem 5.2.1 in [30] and the Sobolev inequality in (6.1), we get that there exists $N_1 = N_1(A_0, r)$ such that

$$Vol_{g_0}(\tilde{B}_x(\rho)) \geq N_1\Big(\frac{\rho}{4r}\Big)^n Vol(\tilde{B}_x(4r)), \quad 0 < \rho \leq 4r$$

Then by the argument of Lemma 1.1 in [31], we have a sequence of points $\{x_i\}_{i=1}^{\infty}$ in M such that $M = \cup_{i=1}^{\infty}\tilde{B}_{x_i}(r)$ and $\tilde{B}_{x_j}(r)$ intersects at most $N = N(A_0, r, n)$ balls $\tilde{B}_{x_i}(r)$, $i \neq j$, and $\tilde{B}_{x_i}(\frac{r}{2}) \cap \tilde{B}_{x_j}(\frac{r}{2}) = \emptyset$ if $i \neq j$. Then by partition of unity principle, we have $\xi_i \in C_0^{\infty}(\tilde{B}_{x_i}(r))$, $\sum_i \xi_i^2 = 1$, $|\tilde{\nabla}\xi_i|^2 \leq C(r, N)$ (using the argument on page 60 of [31]) and $0 \leq \xi_i \leq 1$.

For simplicity reason, we use notation B_i to replace $\tilde{B}_{x_i}(r)$ in the rest of this section and section 5.3. And we also use ξ to replace ξ_i in some proofs of this section, it will be clear from the context when we do such replacement.

We use the following notations which are a little different from the usual, note the denominator of the right sides of the following definitions are the same volume. In fact, the right sides of the following are all the same, the choice of definition depends on containment relations between B_i and Ω:

$$\fint_{B_i\cap\Omega} h dV_{g(t)} \triangleq \frac{1}{V_{g(t)}(B_i)}\int_{B_i\cap\Omega} h dV_{g(t)}$$

$$\fint_{B_i} h dV_{g(t)} \triangleq \frac{1}{V_{g(t)}(B_i)}\int_{B_i\cap\Omega} h dV_{g(t)}$$

$$\fint_{\Omega} h dV_{g(t)} \triangleq \frac{1}{V_{g(t)}(B_i)} \int_{B_i \cap \Omega} h dV_{g(t)}$$

for any $h \in C_0^\infty(B_i)$.

By Theorem 51 and DeTurck's trick, (4.1) has a smooth solution on a sufficiently small time interval starting at $t = 0$. Let $[0, T_{\max})$ be a maximal time interval on which (4.1) has a smooth solution such that the following hold for each metric $g(t)$ (where $g(0) = g_0$):

$$\Big(\fint_{\tilde{B}_x(r)} h^{\frac{2n}{n-2}} dV_{g(t)} \Big)^{\frac{n-2}{n}} \leq 4A_0 r^2 \fint_{\tilde{B}_x(r)} |\nabla h|^2 dV_{g(t)} \ ; \tag{5.16}$$

$$\frac{1}{2} g_0 \leq g(t) \leq 2g_0 \ ; \tag{5.17}$$

$$\Big(\fint_{B_i \cap \Omega} |Rm(g(t))|^{p_0} dV_{g(t)} \Big)^{\frac{1}{p_0}} \leq (2N)^{\frac{1}{p_0}} K_1 \tag{5.18}$$

for any $x \in M$, $h \in C_0^\infty(\tilde{B}_x(r))$ and $i = 1, 2, \cdots$.

We always assume $\|\nabla \chi\|_\infty \triangleq \|\nabla \chi\|_{(g_0, \infty)} = \sup_{x \in \Omega} (g_0^{ij} \nabla_i \chi \nabla_j \chi)^{\frac{1}{2}} \leq 1$ in this section.

Firstly, we have the following kind of "energy estimate" for $|Rm|$.

Lemma 57 *If $0 \leq t < T_{\max}$, then for any i, $q \geq 1$, $p \geq \frac{n}{2}$, $p' \geq 0$ and $p \geq p'$, there exists some positive constant $C_1(A_0, K_1, n, p_0, r)$ such that we have the following:*

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\fint_{\Omega} \xi_i^{2q} \chi^{2p'} |Rm|^p dV_{g(t)} \Big) + \frac{1}{10}\Big(\fint_{\Omega} \big|\nabla(\xi_i^q \chi^{p'+1} |Rm|^{\frac{p}{2}})\big|^2 dV_{g(t)} \Big) \\ &\leq C_1(A_0, K_1, n, p_0, r) p^{\frac{n}{2p_0 - n} + 3} q^2 \Big(\fint_{B_i \cap \Omega} \xi_i^{2q-2} \chi^{2p'} |Rm|^p dV_{g(t)} \Big) \end{aligned} \tag{5.19}$$

Proof: Set $f(x,t) = |Rm(x)|_{g(t)}$. Using Lemma 53 and $\frac{\partial}{\partial t}(dV_{g(t)}) = \frac{1}{2}[g^{ij} \frac{\partial}{\partial t} g_{ij}] = -\chi^2 R$, we get the following:

$$\frac{\partial}{\partial t}\Big(\fint_{\Omega} \xi^{2q} \chi^{2p'} f^p \Big) \leq \sum_{k=1}^{7} I_k \tag{5.20}$$

where

$$I_1 = p \fint \xi^{2q}\chi^{2p'+2} f^{p-1}\Delta f, \quad I_2 = p \fint \xi^{2q}\chi^{2p'+2} f^{p-2}|\nabla f|^2$$

$$I_3 = -(1-\epsilon)p \fint \xi^{2q}\chi^{2p'+2} f^{p-2}|\nabla Rm|^2, \quad I_4 = 10p \fint \xi^{2q}\chi^{2p'+2} f^{p+1} - \fint \xi^{2q}\chi^{2p'+2} f^{p} R$$

$$I_5 = \Big(\frac{10^5}{\epsilon}\Big)np \fint \xi^{2q}\chi^{2p'}|\nabla\chi|^2 f^p, \quad I_6 = 8p \fint \xi^{2q}\chi^{2p'+1} f^{p-2} g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}R_{jk}\chi_{il}$$

$$I_7 = n(n-1)\Big(\fint_\Omega \xi^{2q}\chi^{2p'} f^p\Big)\cdot\Big(\fint_{B_i}\chi^2 f\Big)$$

For simplification, we set the following notations:

$$\tilde{a} = \xi^q\chi^{p'} f^{\frac{p}{2}}|\nabla\chi|; \quad \tilde{b} = \xi^q\chi^{p'+1} f^{\frac{p}{2}-1}|\nabla f|;$$

$$\tilde{c} = \xi^q\chi^{p'+1} f^{\frac{p}{2}-1}|\nabla Rm|; \quad \tilde{d} = \fint \Big|\nabla(\xi^q\chi^{p'+1} f^{\frac{p}{2}})\Big|^2; \quad \tilde{e} = \xi^{q-1}\chi^{p'+1} f^{\frac{p}{2}}|\nabla\xi|$$

Then we get

$$I_1 \le -p(p-1)\fint \tilde{b}^2 + 2\epsilon p \fint \tilde{b}^2 + \tfrac{pq^2}{\epsilon}\fint \tilde{e}^2 + \tfrac{p(p'+1)^2}{\epsilon}\fint \tilde{a}^2$$

$$I_2 \le p\fint \tilde{b}^2, \quad I_3 = -(1-\epsilon)p\fint \tilde{c}^2, \quad I_5 \le \Big(\frac{10^5}{\epsilon}\Big)np\fint \tilde{a}^2$$

where $0 < \epsilon < \frac{1}{8}$ is some constant to be determined later.

For I_6, we get

$$\begin{aligned} I_6 &= -8p\fint [\nabla_l(\xi^{21}\chi^{2p'+1} f^{p-2} g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}R_{jk})]\chi_i \\ &\le \epsilon p \fint \tilde{c}^2 + \tfrac{1}{\epsilon}10^6 n^3p^3q^2\Big(\fint \tilde{a}^2 + \fint \tilde{e}^2\Big) \end{aligned}$$

And we also have

$$I_7 \le n(n-1)\fint_\Omega \xi^{2q}\chi^{2p'} f^p \cdot \Big(\fint_{B_i\cap\Omega} f^{p_0}\Big)^{\frac{1}{p_0}} \le n(n-1)(2N)^{\frac{1}{p_0}} K_1\Big(\fint_{B_i\cap\Omega}\xi_i^{2q-2}\chi^{2p'} f^p dV_{g(t)}\Big)$$

Note that $|R| \leq n|Rm| \leq 2pf$, $p_0^* < \frac{n}{n-2}$ where $\frac{1}{p_0} + \frac{1}{p_0^*} = 1$, we have

$$\begin{aligned} I_4 &\leq 12p \fint \xi^{2q}\chi^{2p'+2} f^{p+1} \leq 12p \nparallel f \|_{p_0, B_i\cap\Omega} \cdot \Big[\fint (\xi^{2q}\chi^{2p'+2} f^p)^{p_0^*} \Big]^{\frac{1}{p_0^*}} \\ &\leq 12p(2N)^{\frac{1}{p_0}} K_1 \Big[C(\epsilon, p_0, n)(2NpK_1A_0)^{\frac{n}{2p_0-n}} \fint \xi^{2q}\chi^{2p'+2} f^p + \frac{\epsilon}{2NpK_1A_0} \Big(\fint (\xi^{2q}\chi^{2p'+2} f^p)^{\frac{n}{n-2}} \Big)^{\frac{n-2}{n}} \Big] \\ &\leq C(\epsilon, A_0, K_1, n, p_0) p^{\frac{n}{2p_0-n}+1} q^2 \fint_{B_i\cap\Omega} \xi^{2q-2}\chi^{2p'+2} f^p + 100\epsilon\tilde{d} \end{aligned} \tag{5.21}$$

where $\nparallel f \|_{p_0, B_i\cap\Omega} = \Big(\fint_{B_i\cap\Omega} f^{p_0} \Big)^{\frac{1}{p_0}}$.

Combining the above estimates together, note $\tilde{b} \leq \tilde{c}$, we have

$$\begin{aligned} \frac{\partial}{\partial t}\Big(\fint_{\Omega} \xi^{2q}\chi^{2p'} f^p \Big) &\leq p(1 + 4\epsilon - p) \fint \tilde{b}^2 + 100\epsilon\tilde{d} + \frac{2}{\epsilon} 10^6 n^3 p^3 q^2 \Big(\fint \tilde{a}^2 + \fint \tilde{e}^2 \Big) \\ &\quad + C(\epsilon, A_0, K_1, n, p_0) p^{\frac{n}{2p_0-n}+1} q^2 \fint_{B_i\cap\Omega} \xi^{2q-2}\chi^{2p'} f^p \end{aligned} \tag{5.22}$$

On the other hand, we have

$$\begin{aligned} \Big| \nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}}) \Big|^2 &\leq (q\tilde{e}^2 + (p'+1)\tilde{a} + \tfrac{p}{2}\tilde{b})^2 \\ &\leq (2 + \tfrac{1}{2\epsilon}) q^2 \tilde{e}^2 + (2 + \tfrac{1}{2\epsilon})(p'+1)^2 \tilde{a}^2 + (\tfrac{1}{4} + \epsilon) p^2 \tilde{b}^2 \end{aligned}$$

Then we get

$$\fint \tilde{b}^2 \geq \Big[(\frac{1}{4} + \epsilon) p^2 \Big]^{-1} \tilde{d} - \Big[(\frac{1}{4} + \epsilon) p^2 \Big]^{-1} \Big(2 + \frac{1}{2\epsilon} \Big) (q + p' + 1)^2 \fint (\tilde{a}^2 + \tilde{e}^2) \tag{5.23}$$

By (5.22), (5.23), using $|\tilde{\nabla}\xi_i|^2 \leq C(r, N)$, $0 \leq \xi_i \leq 1$, $0 \leq \chi \leq 1$, and note $1 + 4\epsilon - p < 0$, we get

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\fint \xi^{2q}\chi^{2p'} f^p \Big) + \Big(\frac{4(p-1-4\epsilon)}{(1+4\epsilon)p} - 100\epsilon \Big) \tilde{d} \\ &\leq C(\epsilon, A_0, K_1, n, p_0, r) p^{\frac{n}{2p_0-n}+3} q^2 \fint_{B_i\cap\Omega} \xi^{2q-2}\chi^{2p'} f^p \end{aligned} \tag{5.24}$$

Choose $\epsilon = \frac{1}{1000}$ in (5.24), we get

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\fint \xi^{2q}\chi^{2p'} f^p \Big) + \frac{1}{10}\Big(\fint |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2 \Big) \\ &\leq C(A_0, K_1, n, p_0, r) p^{\frac{n}{2p_0-n}+3} q^2 \fint_{B_i\cap\Omega} \xi^{2q-2}\chi^{2p'} f^p \end{aligned} \tag{5.25}$$

Replace f with $|Rm|$, the lemma is proved.

□

For simplicity reason, we use C_1 intead of $C_1(A_0, K_1, n, p_0, r)$ in the rest of this section, similar for C_2 etc.

In the next lemma, we use Moser iteration to get local C^0-estimate of $|\chi^2 Rm(g(t))|$.

Lemma 58 *If* $0 < t < T_{\max} < 1$*, then there exists some positive constant* $C_2(A_0, K_1, n, p_0, r)$*, such that*

$$|\chi^2 Rm(g(t))| \leq C_2 t^{-\frac{n}{2p_0}} \tag{5.26}$$

Proof: Let $f(x,t) = |Rm(x)|_{g(t)}$. Given any t_1 and t_2 such that $0 < t_1 < t_2 < T_{\max}$, set

$$\psi(t) = \begin{cases} 0 & 0 \leq t \leq t_1 \\ \frac{t-t_1}{t_2-t_1} & t_1 \leq t \leq t_2 \\ 1 & t_2 \leq t < T_{\max} \end{cases} \tag{5.27}$$

By Lemma 57, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(⨍_\Omega \xi_i^{2q} \chi^{2p'} f^p dV_{g(t)}\Big) + \frac{1}{10}\Big(⨍_\Omega \big|\nabla(\xi_i^q \chi^{p'+1} f^{\frac{p}{2}})\big|^2 dV_{g(t)}\Big) \\ &\leq C_2 p^{\frac{n}{2p_0-n}+3} q^2 \Big(⨍_{B_i\cap\Omega} \xi_i^{2q-2} \chi^{2p'} f^p dV_{g(t)}\Big) \end{aligned} \tag{5.28}$$

In the rest of the proof, we use $C = C(A_0, K_1, n, p_0, r)$ for simplification. We multiply (5.28) by $\psi(t)$, integrate it from 0 to t_0 with respect to t, where $t_2 \leq t_0 < T_{\max}$. Then we get

$$\begin{aligned} &⨍_\Omega (\xi^{2q} \chi^{2p'} f^p) dV_{g(t_0)} + \frac{1}{10}\int_{t_2}^{t_0} ⨍_\Omega |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2 \\ &\leq C p^{\frac{n}{2p_0-n}+3} q^2 \Big(1 + \frac{1}{t_2-t_1}\Big) \int_{t_1}^{t_0} ⨍_{B_i\cap\Omega} \xi^{2(q-1)} \chi^{2p'} f^p \end{aligned} \tag{5.29}$$

Note (5.29) is in fact valid for any $0 < t_1 < t_2 \leq t_0 < T_{\max}$.

For $0 < t \leq t_0$, we denote

$$H(p, p', q, t) = \int_t^{t_0} ⨍_{B_i\cap\Omega} \xi^{2q} \chi^{2p'} f^p$$

then for any $0 < \iota < \iota' \le t_0 < T_{\max}$, using (5.29), we have

$$\begin{aligned}
&H(p(1+\tfrac{2}{n}), p'(1+\tfrac{2}{n})+1, q(1+\tfrac{2}{n}), \iota') = \int_{\iota'}^{t_0} \fint_{B_i\cap\Omega} (\xi^{2q}\chi^{2p'} f^p)^{\frac{2}{n}} (\xi^q \chi^{p'+1} f^{\frac{p}{2}})^2 \\
&\le \int_{\iota'}^{t_0} \Big[\fint (\xi^{2q}\chi^{2p'} f^p) \Big]^{\frac{2}{n}} \cdot 4A_0 r^2 \Big[\fint |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2 \Big] dt \\
&\le C p^{\frac{4(3p_0-n)}{2p_0-n}} q^{2(1+\frac{2}{n})} \Big(1 + \tfrac{1}{\iota'-\iota}\Big)^{1+\frac{2}{n}} H(p, p', q-1, \iota)^{1+\frac{2}{n}}
\end{aligned}$$

Denote $\beta = \frac{4(3p_0-n)}{2p_0-n}$, $\nu = 1 + \frac{2}{n}$, $\eta = \nu^{n+2}$, fixed $t \in (0, t_0)$, set

$$\begin{aligned}
&p_k = p_0 \nu^k\,, \quad p'_k = p_k - \tfrac{n}{2}\,, \quad p'_0 = p_0 - \tfrac{n}{2}\,, \\
&q_k = n\nu^k - (\tfrac{n}{2}+1)\,, \quad q_0 = \tfrac{n}{2} - 1\,, \quad \iota_k = t(1-\eta^{-k})\,, \\
&H_k = H(p_k, p'_k, q_k, \iota_k)\,, \quad \Phi_k = H_k^{\frac{1}{p_k}}
\end{aligned}$$

so

$$\Phi_0 = \Big(\int_0^{t_0} \fint \xi^{n-2} \chi^{2p_0-n} f^{p_0} \Big)^{\frac{1}{p_0}}$$

Then for any $k \ge 0$, we get the following:

$$\begin{aligned}
H_{k+1} &= H(p_{k+1}, p'_{k+1}, q_{k+1}, \iota_{k+1}) \\
&= H\Big(p_k(1+\tfrac{2}{n}), p'_k(1+\tfrac{2}{n})+1, (q_k+1)(1+\tfrac{2}{n}), \iota_{k+1}\Big) \\
&\le C p_k^{\beta} (q_k+1)^{2(1+\frac{2}{n})} \Big(1 + \tfrac{1}{\iota_{k+1}-\iota_k}\Big)^{1+\frac{2}{n}} H(p_k, p'_k, q_k, \iota_k)^{1+\frac{2}{n}} \\
&\le C p_0^{\beta} \eta^{k\nu\left(\frac{\beta+2}{n+2}+1\right)} \Big(1 + t^{-1}\tfrac{\eta}{\eta-1}\Big)^{\nu} H_k^{\nu}
\end{aligned}$$

Taking p_{k+1}-root on both sides of the above inequality, we have:

$$\Phi_{k+1} \le C^{\frac{\nu^{-(k+1)}}{p_0}} p_0^{\beta\frac{\nu^{-(k+1)}}{p_0}} \eta^{\left(\frac{\beta+2}{n+2}+1\right)\frac{k\nu^{-k}}{p_0}} \Big(1 + t^{-1}\frac{\eta}{\eta-1}\Big)^{\frac{\nu^{-k}}{p_0}} \Phi_k$$

By induction, we get

$$\Phi_{k+1} \le C^{\frac{\sigma_{k+1}-1}{p_0}} p_0^{\beta\frac{(\sigma_{k+1}-1)}{p_0}} \eta^{\left(\frac{\beta+2}{n+2}+1\right)\frac{\sigma'_k}{p_0}} \Big(1 + t^{-1}\frac{\eta}{\eta-1}\Big)^{\frac{\sigma_k}{p_0}} \Phi_0$$

where $\sigma_k = \sum_{i=0}^k \nu^{-i}$ and $\sigma'_k = \sum_{i=0}^k i\nu^{-i}$.

Then we have

$$|\xi_i^{\frac{n}{p_0}}\chi^2 f(x,t)| \le \lim_{k\to\infty}\Phi_{k+1} \le C(1+t^{-1})^{\frac{n+2}{2p_0}}\Big(\int_0^{t_0} \fint_{B_i\cap\Omega} \xi_i^{n-2}\chi^{2p_0-n} f^{p_0}\Big)^{\frac{1}{p_0}} \tag{5.30}$$

Let $t_0 \to t$ in (5.30) and using (5.18), we get

$$|\xi_i^{\frac{n}{p_0}}\chi^2 f(x,t)| \le C(1+t^{-1})^{\frac{n+2}{2p_0}}\Big(\int_0^{t} \fint_{B_i\cap\Omega} \xi_i^{n-2}\chi^{2p_0-n} f^{p_0}\Big)^{\frac{1}{p_0}} \le Ct^{-\frac{n}{2p_0}}$$

So we get

$$|\xi_i^2\chi^2 f| \le |\xi_i^{\frac{n}{p_0}}\chi^2 f| \le Ct^{-\frac{n}{2p_0}}$$

Now for any $x \in \Omega$, we assume $x \in B_{i_0} \cap \Omega$, then we have

$$|\chi^2 f|(x,t) \quad = |\sum_{j=1}^{N} \xi_j^2\chi^2 f(x,t)| \le NCt^{-\frac{n}{2p_0}} \le C_2 t^{-\frac{n}{2p_0}}$$

Replace f with $|Rm|$, this is our conclusion. □

Straightforward from Lemma 58, we have the following two corollaries.

Corollary 59 *If* $0 < t < T_{\max} < 1$*, then there exists some positive constant* $C_3(A_0, K_1, n, p_0, r)$*, such that*

$$|\chi^2 Rc(g(t))| \le C_3 t^{-\frac{n}{2p_0}}$$

Corollary 60 *If* $0 \le t < T_{\max} < 1$*, then there exists some positive constant C, which is independent of t, such that*

$$|\chi^2 Rm(g(t))| \le C$$

5.2.2 The Second Case: Scale-invariant exponent $p_0 = \frac{n}{2}$

In this subsection, we assume (M, g_0) satisfies the assumption of Theorem 81, namely inequalities (6.19), and $N = 8^n N_0^{-2}$.

Similar argument like the beginning of the subsection 5.2.1, we assume $[0, T_{\max})$ is the maximal time interval on which (4.1) has a smooth solution and such that the following hold for each metric $g(t)$ (where $g(0) = g_0$):

$$\Big(\int_{\tilde{B}_x(r)} h^{\frac{2n}{n-2}} dV_{g(t)} \Big)^{\frac{n-2}{n}} \leq 4A_0 \int_{\tilde{B}_x(r)} |\nabla h|^2 dV_{g(t)} \ ; \tag{5.31}$$

$$\frac{1}{2} g_0 \leq g(t) \leq 2 g_0 \ ; \tag{5.32}$$

$$\Big(\int_{B_i \cap \Omega} |Rm(g(t))|^{\frac{n}{2}} dV_{g(t)} \Big)^{\frac{2}{n}} \leq (2N)^{\frac{2}{n}} (\tau n A_0)^{-1}. \tag{5.33}$$

for any $x \in M$, $h \in C_0^\infty(\tilde{B}_x(r))$ and $i = 1, 2, \cdots$.

We have the following similar estimate as Lemma 57, but the coefficents of the gradient integral are quite different. The difference is partially from the different estimates of I_4 in these two lemmas.

Lemma 61 *If $0 \leq t < T_{\max}$, then for any i, $q \geq 1$, $p \geq \frac{n}{2}$, $p' \geq 0$ and $p \geq p'$, there exists $\tilde{C}_1(n, N_0, r) > 0$ such that we have the following:*

$$\begin{aligned} &\tfrac{\partial}{\partial t}\Big(\int_\Omega \xi_i^{2q} \chi^{2p'} |Rm|^p dV_{g(t)} \Big) + \theta \Big(\int_\Omega \Big| \nabla (\xi_i^q \chi^{p'+1} |Rm|^{\frac{p}{2}}) \Big|^2 dV_{g(t)} \Big) \\ &\leq \tfrac{1}{\epsilon} \tilde{C}_1(n, N_0, r) p^3 q^2 \Big(\int_{B_i \cap \Omega} \xi_i^{2q-2} \chi^{2p'+2} |Rm|^p dV_{g(t)} \Big) \end{aligned} \tag{5.34}$$

where

$$\theta = \Big(\frac{4(p - 1 - 4\epsilon)}{(1 + 4\epsilon)p} \Big) - 48 p A_0 ||Rm||_{\frac{n}{2}, B_i \cap \Omega} \tag{5.35}$$

and $||Rm||_{\frac{n}{2}, B_i \cap \Omega} = \Big(\int_{B_i \cap \Omega} |Rm(x,t)|^{\frac{n}{2}} dV_{g(t)} \Big)^{\frac{2}{n}}$, ϵ is any positive constant satisfying $0 < \epsilon < \frac{1}{8}$.

Proof: Set $f(x,t) = |Rm(x)|_{g(t)}$. Using Lemma 53 and $\frac{\partial}{\partial t}(dV_{g(t)}) = \frac{1}{2}[g^{ij}\frac{\partial}{\partial t}g_{ij}] = -\chi^2 R$, we can get the following:

$$\frac{\partial}{\partial t}\Big(\int_\Omega \xi^{2q}\chi^{2p'}f^p\Big) \le \sum_{k=1}^{6} I_k \tag{5.36}$$

where

$$I_1 = p\int \xi^{2q}\chi^{2p'+2}f^{p-1}\Delta f, \quad I_2 = p\int \xi^{2q}\chi^{2p'+2}f^{p-2}|\nabla f|^2$$

$$I_3 = -(1-\epsilon)p\int \xi^{2q}\chi^{2p'+2}f^{p-2}|\nabla Rm|^2, \quad I_4 = 10p\int \xi^{2q}\chi^{2p'+2}f^{p+1} - \int \xi^{2q}\chi^{2p'+2}f^p R$$

$$I_5 = (\frac{10^5}{\epsilon})np\int \xi^{2q}\chi^{2p'}|\nabla\chi|^2 f^p, \quad I_6 = 8p\int \xi^{2q}\chi^{2p'+1}f^{p-2}g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\chi_{il}R_{jk}$$

For simplification, we set the following notations:

$$\tilde{a} = \xi^q\chi^{p'}f^{\frac{p}{2}}|\nabla\chi|; \quad \tilde{b} = \xi^q\chi^{p'+1}f^{\frac{p}{2}-1}|\nabla f|;$$
$$\tilde{c} = \xi^q\chi^{p'+1}f^{\frac{p}{2}-1}|\nabla Rm|; \quad \tilde{d} = \int\Big|\nabla(\xi^q\chi^{p'+1}f^{\frac{p}{2}})\Big|^2; \quad \tilde{e} = \xi^{q-1}\chi^{p'+1}f^{\frac{p}{2}}|\nabla\xi|$$

Note that $|R| \le n|Rm| = 2pf$, we have

$$\begin{aligned} I_4 &\le 12p\int \xi^{2q}\chi^{2p'+2}f^{p+1} \le 12p\|f\|_{\frac{n}{2},B_i\cap\Omega}\cdot 4A_0\cdot\int\Big|\nabla(\xi^q\chi^{p'+1}f^{\frac{p}{2}})\Big|^2 \\ &\le 48pA_0\|f\|_{\frac{n}{2},B_i\cap\Omega}\cdot\tilde{d}\end{aligned}$$

Similar to the proof of Lemma 57, we have

$$\begin{aligned}\tfrac{\partial}{\partial t}\Big(\int_\Omega \xi^{2q}\chi^{2p'}f^p\Big) &\le p(1+4\epsilon-p)\int\tilde{b}^2 + 48pA_0\|f\|_{\frac{n}{2},B_i\cap\Omega}\cdot\tilde{d} \\ &\quad + \tfrac{2}{\epsilon}10^6n^3p^3q^2\Big(\int\tilde{a}^2 + \int\tilde{e}^2\Big)\end{aligned} \tag{5.37}$$

Then we get

$$\begin{aligned}&\tfrac{\partial}{\partial t}\Big(\int \xi^{2q}\chi^{2p'}f^p\Big) + \theta\tilde{d} \le \tfrac{4}{\epsilon}10^6n^3p^3q^2\int(\tilde{a}^2+\tilde{e}^2) \\ &\le \tfrac{8}{\epsilon}10^6n^3p^3q^2\Big(\int_\Omega \xi^{2q}\chi^{2p'}|Rm|^p dV_{g(t)} + \int_\Omega \xi^{2q-2}\chi^{2p'+2}|Rm|^p|\nabla\xi|^2 dV_{g(t)}\Big)\end{aligned}$$

where θ is defined in (5.35). Replace f with $|Rm|$, using $|\tilde{\nabla}\xi_i|^2 \le C(r,N)$, $0 \le \xi_i \le 1$ and $0 \le \chi \le 1$, the lemma is proved.

□

We also have another technical lemma about $|Rm|$ to be used in the proof of Lemma 63:

Lemma 62 *If* $0 < t < T_{\max} < 1$*, then we have the following inequality:*

$$\int \xi_i^4 \chi^2 |Rm(x)|^{1+\frac{n}{2}} dV_{g(t)} \leq C(n, N_0, \tau, r) A_0^{-\frac{n}{2}} t^{-1}, \quad \forall i \tag{5.38}$$

Proof: Set $f(x,t) = |Rm(x)|_{g(t)}$. Given any t_1 and t_2 such that $0 < t_1 < t_2 < T_{\max}$, set

$$\psi(t) = \begin{cases} 0 & 0 \leq t \leq t_1 \\ \frac{t-t_1}{t_2-t_1} & t_1 \leq t \leq t_2 \\ 1 & t_2 \leq t < T_{\max} \end{cases} \tag{5.39}$$

In Lemma 61, let $q = 1$, $p' = 0$, $p = \frac{n}{2}$, $\epsilon = \frac{\epsilon_0}{2}$, where ϵ_0 is defined in (5.56), plug the above values into (5.35) we get $\theta = \theta(n, N_0, \tau) > 0$, then we have

$$\frac{\partial}{\partial t}\Big(\int_\Omega \xi^2 f^{\frac{n}{2}} dV_{g(t)}\Big) + \theta \int_\Omega |\nabla(\xi\chi f^{\frac{n}{4}})|^2 dV_{g(t)} \leq \frac{1}{\epsilon_0} C(n, N_0, r)\Big(\int_{B_i\cap\Omega} f^{\frac{n}{2}} dV_{g(t)}\Big)$$

Multiplying the above by ψ, integrate it from 0 to t_0 with respect to t, where t_0 satisfies $t_2 \leq t_0 < T_{\max}$, then let $t_1 = \frac{t_0}{4}$, $t_2 = \frac{t_0}{2}$, we get

$$\begin{aligned} \int_{\frac{t_0}{2}}^{t_0} \int_\Omega |\nabla(\xi\chi f^{\frac{n}{4}})|^2 &\leq C(n, N_0, r, \tau)\Big(1 + \tfrac{1}{t_0}\Big) \int_{\frac{t_0}{4}}^{t_0} \int_{B_i\cap\Omega} f^{\frac{n}{2}} \\ &\leq C(n, N_0, r, \tau) A_0^{-\frac{n}{2}} \Big(1 + \tfrac{1}{t_0}\Big) t_0 \end{aligned}$$

Again in Lemma 61, let $q = 2$, $p' = 1$, $p = 1 + \frac{n}{2}$, choose suitable $\epsilon = \epsilon(n, N_0, \tau)$ such that $\theta \geq 0$, then we have the following:

$$\frac{\partial}{\partial t}\Big(\int_\Omega \xi^4 \chi^2 f^{1+\frac{n}{2}} dV_{g(t)}\Big) \leq C(n, N_0, r, \tau)\Big(\int_\Omega \xi^2 \chi^2 f^{1+\frac{n}{2}} dV_{g(t)}\Big)$$

Multiplying the above by ψ, integrate it from 0 to t_0 with respect to t, where t_0 satisfies $t_2 \le t_0 < T_{\max}$, then let $t_1 = \frac{t_0}{2}$, $t_2 = t_0$, we get

$$\begin{aligned}
&\int_\Omega \xi^4\chi^2 f^{1+\frac{n}{2}} dV_{g(t_0)} \le C(n, N_0, r, \tau)(1 + \tfrac{1}{t_0})\Big(\int_{\frac{t_0}{2}}^{t_0} \int_\Omega \xi^2\chi^2 f^{1+\frac{n}{2}}\Big) \\
&\le C(n, N_0, r, \tau)(1 + \tfrac{1}{t_0})\Big(\sup_{\frac{t_0}{2}\le t\le t_0} |f(t)|_{\frac{n}{2}, B_i\cap\Omega}\Big) A_0\Big(\int_{\frac{t_0}{2}}^{t_0} \int_\Omega |\nabla(\xi\chi f^{\frac{n}{4}})|^2\Big) \\
&\le C(n, N_0, r, \tau)\Big(1 + \tfrac{1}{t_0}\Big)^2 A_0^{-\frac{n}{2}} t_0
\end{aligned}$$

Because we choose t_0 freely, in fact we get the following inequality for any $0 < t < T_{\max} < 1$:

$$\int_\Omega \xi^4\chi^2 f^{1+\frac{n}{2}} dV_{g(t)} \le C(n, N_0, \tau, r) A_0^{-\frac{n}{2}} t^{-1}$$

Replace f with $|Rm|$, we get our conclusion. □

Next we have the following lemma about $|Rc|$, and the key estimate is for I_4 where we use Lemma 62.

Lemma 63 *If* $0 < t < T_{\max} < 1$, $p \ge \frac{n}{2}$, $p' \ge 0$ *and* $p \ge p'$, *we have the following:*

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\Big(\int \xi_i^{2q}\chi^{2p'}|Rc|^p\Big) + \tfrac{1}{9}\Big(\int |\nabla(\xi_i^q\chi^{p'+1}|Rc|^{\frac{p}{2}})|^2\Big) \\
&\le C(n, N_0, \tau, r)p^{2+\frac{n}{2}} q^2(1 + t^{-1})\Big(\int_{B_i\cap\Omega} \xi_i^{2(q-2)}\chi^{2p'}|Rc|^p\Big)
\end{aligned} \tag{5.40}$$

for any $q \ge 2$. *There exists* $\tilde{C}_2(n, N_0, r) > 0$ *such that we have the following:*

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\Big(\int_\Omega \xi_i^{2q}\chi^{2p'}|Rc|^p dV_{g(t)}\Big) + \theta\Big(\int_\Omega \Big|\nabla(\xi_i^q\chi^{p'+1}|Rc|^{\frac{p}{2}})\Big|^2 dV_{g(t)}\Big) \\
&\le \tfrac{1}{\epsilon}\tilde{C}_2(n, N_0, r)p^3 q^2\Big(1 + \tfrac{1}{r^2}\Big)\Big(\int_{B_i\cap\Omega} \xi_i^{2(q-1)}\chi^{2p'+2}|Rc|^p\Big)
\end{aligned} \tag{5.41}$$

for any $q \ge 1$, *where* $\theta = \Big[\frac{4(p-1-4\epsilon)}{(1+4\epsilon)p} - \frac{32pN}{\tau\sqrt{n}}\Big]$.

Proof: Let $f(x, t) = |Rc(x)|_{g(t)}$. By Lemma 55, we get the following:

$$\frac{\partial}{\partial t}\Big(\int_\Omega \xi^{2q}\chi^{2p'} f^p\Big) \le \sum_{k=1}^{6} I_k \tag{5.42}$$

where

$$I_1 = p \int \xi^{2q} \chi^{2p'+2} f^{p-1} \Delta f, \quad I_2 = p \int \xi^{2q} \chi^{2p'+2} f^{p-2} |\nabla f|^2$$

$$I_3 = -(1-\epsilon) p \int \xi^{2q} \chi^{2p'+2} f^{p-2} |\nabla Rc|^2, \quad I_4 = 2(1+\sqrt{n}) p \int \xi^{2q} \chi^{2p'+2} f^p |Rm|$$

$$I_5 = (\frac{10^5}{\epsilon}) np \int \xi^{2q} \chi^{2p'} |\nabla \chi|^2 f^p, \quad I_6 = p \int \xi^{2q} \chi^{2p'} f^{p-2} g^{ik} g^{jl} R_{kl} J_2$$

where J_2 in I_6 comes from (5.13).

Similar to the proof of Lemma 61, we set

$$\tilde{a} = \xi^q \chi^{p'} f^{\frac{p}{2}} |\nabla \chi| \, ; \quad \tilde{b} = \xi^q \chi^{p'+1} f^{\frac{p}{2}-1} |\nabla f|$$
$$\tilde{c} = \xi^q \chi^{p'+1} f^{\frac{p}{2}-1} |\nabla Rc| \, ; \quad \tilde{d} = \int \left| \nabla (\xi^q \chi^{p'+1} f^{\frac{p}{2}}) \right|^2 ; \quad \tilde{e} = \xi^{q-1} \chi^{p'+1} f^{\frac{p}{2}} |\nabla \xi|.$$

Then we get the following

$$I_1 + I_2 + I_3 + I_5 + I_6 \leq \frac{4(1+4\epsilon-p)}{(1+4\epsilon)p} \tilde{d} + \frac{4}{\epsilon} 10^6 n^3 p^3 q^2 \Big(\int \tilde{a}^2 + \int \tilde{e}^2 \Big) \tag{5.43}$$

Finally, for I_4, we have:

$$\begin{aligned}
I_4 &\leq 2(1+\sqrt{n}) p \Big[\int_\Omega (\xi^{\frac{8}{n+2}} \chi^{\frac{4}{n+2}} |Rm|)^{\frac{n+2}{2}} \Big]^{\frac{2}{n+2}} \\
&\quad \cdot \Big[\int_{B_i \cap \Omega} (\xi^{\frac{4(q-2)}{n+2}} \chi^{\frac{4p'}{n+2}} f^{\frac{2p}{n+2}})^{\frac{n+2}{2}} \Big]^{\frac{2}{n+2}} \cdot \Big[\int_\Omega (\xi^{\frac{2qn}{n+2}} \chi^{\frac{(2p'+2)n}{n+2}} f^{\frac{pn}{n+2}})^{\frac{n+2}{n-2}} \Big]^{\frac{n-2}{n+2}} \\
&= 2(1+\sqrt{n}) p \Big(\int_\Omega \xi^4 \chi^2 |Rm|^{1+\frac{n}{2}} \Big)^{\frac{2}{n+2}} \cdot \Big(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \Big)^{\frac{2}{n+2}} \\
&\quad \cdot \Big[\int_\Omega (\xi^q \chi^{p'+1} f^{\frac{p}{2}})^{\frac{2n}{n-2}} \Big]^{\frac{n-2}{n+2}} \\
&\leq \hat{a} \cdot \hat{b}
\end{aligned}$$

where we use Lemma 62 and the following notations:

$$\begin{aligned}
\hat{a} &= \Big[C(n, N_0, \tau, r) p t^{-\frac{2}{n+2}} \Big(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \Big)^{\frac{2}{n+2}} \Big] \\
\hat{b} &= \Big(\int |\nabla (\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2 \Big)^{\frac{n}{n+2}} \\
s &= \tfrac{n+2}{2}, \quad s' = \tfrac{n+2}{n}, \quad \tfrac{1}{s} + \tfrac{1}{s'} = 1
\end{aligned}$$

For any $\delta > 0$, we have

$$\hat{a}\hat{b} = (\hat{a}\delta^{-\frac{1}{ss'}})(\hat{b}\delta^{\frac{1}{ss'}}) \le \frac{1}{s}(\hat{a}\delta^{-\frac{1}{ss'}})^{s} + \frac{1}{s'}(\hat{b}\delta^{\frac{1}{ss'}})^{s'}$$
$$\le (\hat{a}\delta^{-\frac{1}{ss'}})^{s} + (\hat{b}\delta^{\frac{1}{ss'}})^{s'} = \hat{a}^{s}\delta^{-\frac{1}{s'}} + \hat{b}^{s'}\delta^{\frac{1}{s}}$$

Let $\delta = k_0^{\frac{n+2}{2}}$, where k_0 is some positive constant which will be determined later.

Then we get

$$I_4 \le \hat{a}\hat{b} \le k_0^{-\frac{n}{2}} \cdot C(n, N_0, \tau, r) p^{\frac{n+2}{2}} t^{-1} \Big(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \Big) + k_0 \tilde{d} \tag{5.44}$$

Now we choose $\epsilon = \frac{1}{16}$, $k_0 = \frac{1}{9}$ in (5.43) and (5.44), by (5.42), we get

$$\frac{\partial}{\partial t}\Big(\int \xi^{2q} \chi^{2p'} f^p \Big) + \frac{1}{9} \int |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2$$
$$\le C(n, N_0, \tau, r)\Big[p^3 q^2 + p^{\frac{n+2}{2}} t^{-1} \Big] \cdot \Big(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \Big)$$
$$\le C(n, N_0, \tau, r) p^{2+\frac{n}{2}} q^2 (1 + t^{-1}) \cdot \Big(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \Big)$$

So (5.40) is proved.

For I_4, we have

$$I_4 \le 2(1 + \sqrt{n}) p \int |Rm| \xi^{2q} \chi^{2p'+2} f^p \le 8(1 + \sqrt{n}) p A_0 \|Rm\|_{\frac{n}{2}, B_i \cap \Omega} \cdot \tilde{d} \le \frac{32pN}{\tau\sqrt{n}} \tilde{d} \tag{5.45}$$

By (5.42), (5.43) and (5.45), we get

$$\frac{\partial}{\partial t}\Big(\int_\Omega \xi_i^{2q} \chi^{2p'} f^p dV_{g(t)} \Big) + \theta \Big(\int_\Omega \Big| \nabla(\xi_i^q \chi^{p'+1} f^{\frac{p}{2}}) \Big|^2 dV_{g(t)} \Big)$$
$$\le \frac{4}{\epsilon} 10^6 n^3 p^3 q^2 \Big(|\nabla\chi|_\infty^2 \int_\Omega \xi_i^{2q} \chi^{2p'} f^p + \int_{B_i \cap \Omega} \xi_i^{2(q-1)} \chi^{2p'+2} f^p |\nabla \xi_i|^2 \Big)$$

Replace f by $|Rc|$ and simplify it, we get our second conclusion. □

We set

$$\epsilon_1 = \frac{(p_1 - 1)\tau\sqrt{n} - 8p_1^2 N}{32 p_1^2 N + 4\tau\sqrt{n}} > 0$$

where we use $\tau > \frac{16 p_1 N}{\sqrt{n}}$.

Choose $\tilde{T}_1 > 0$ such that

$$\exp\left(\frac{1}{\epsilon_1}\tilde{C}_2 p_1^3 N \tilde{T}_1\right) = \frac{3}{2}$$

Now we set $\tilde{T}_2$ as the following:

$$\tilde{T}_2 = \min\{\tilde{T}_1, 1\}$$

Corollary 64 *If* $0 \leq t < T_{\max} < \tilde{T}_2$*, then*

$$\Big(\int_{B_i \cap \Omega} |Rc|^{p_1} dV_{g(t)}\Big)^{\frac{1}{p_1}} \leq \Big(\frac{3}{2}N\Big)^{\frac{1}{p_1}} K_1\,, \quad \forall i \tag{5.46}$$

Proof: Set $f = |Rc|$, choose $q = 1$, $p' = 0$, $p = p_1$ and $\epsilon = \epsilon_1$ in (5.41), then it is easy to check that $\theta \geq 0$ by the definition of θ in Lemma 63. Then we get

$$\frac{\partial}{\partial t}\int_{\Omega} \xi_i^2 |Rc|^{p_1} \leq \frac{1}{\epsilon_1}\tilde{C}_2 p_1^3 \Big(\int_{B_i \cap \Omega} |Rc|^{p_1}\Big)$$

Define $\phi(t) = \sup_{B_i \in Cov(\Omega)} \int_{B_i \cap \Omega} \xi_i^2 |Rc|^{p_1} dV_{g(t)}$, then we get

$$\frac{\partial}{\partial t}\phi(t) \leq \frac{1}{\epsilon_1}\tilde{C}_2 p_1^3 N \phi(t)$$

Then

$$\phi(t) \leq \exp\left(\tfrac{1}{\epsilon_1}\tilde{C}_2 p_1^3 N t\right)\phi(0) \leq \exp\left(\tfrac{1}{\epsilon_1}\tilde{C}_2 p_1^3 N T_2\right) K_1^{p_1} \leq \tfrac{3}{2} K_1^{p_1}$$

Now we have

$$\int_{B_i \cap \Omega} |Rc|^{p_1} dV_{g(t)} \leq N\phi(t) \leq \Big(\frac{3}{2}N\Big) K_1^{p_1}$$

We get our conclusion. □

Now we use Moser iteration to get local C^0-estimate $|\chi^2 Rc(g(t))|$.

Lemma 65 *If* $0 < t < T_{\max} < \tilde{T}_2$*, then there exists some constant* $\tilde{C}_3(n, N_0, \tau, p_1, r) > 0$*, such that*

$$|\chi^2 Rc(g(t))| \leq \tilde{C}_3 A_0^{\frac{n}{2p_1}} K_1 t^{-\frac{n}{2p_1}} \tag{5.47}$$

Proof: Let $f(x,t) = |Rc(x)|_{g(t)}$. By (5.40), similar argument as Lemma 58, but we denote

$$H(p,p',q,t) = \int_t^{t_0}\int_{B_i\cap\Omega} \xi^{2q}\chi^{2p'}f^p$$

And denote $\nu = 1+\frac{2}{n}$, $\eta = \nu^{n+2}$, fixed $t\in(0,t_0)$, set

$$\begin{aligned} &s_k = p_1\nu^k\ , s_0 = p_1 \quad p'_k = s_k - \tfrac{n}{2}\ , \quad p'_0 = p_1 - \tfrac{n}{2}\ , \\ &q_k = 2n\nu^k - (n+2)\ , \quad q_0 = n-2\ , \quad \iota_k = t(1-\eta^{-k})\ , \\ &H_k = H(s_k, p'_k, q_k, \iota_k)\ , \quad \Phi_k = H_k^{\frac{1}{s_k}} \end{aligned}$$

Then

$$\Phi_0 = \Big(\int_0^{t_0}\int \xi^{n-2}\chi^{2p_1-n}f^{p_1}\Big)^{\frac{1}{p_1}}$$

we get

$$\begin{aligned} |\xi_i^{\frac{2n}{p_1}}\chi^2 f(x,t)| &\le \lim_{k\to\infty}\Phi_{k+1} \\ &\le C(n,N,\tau,r,p_1)A_0^{\frac{n}{2p_1}}(1+t^{-1})^{\frac{n+2}{2p_1}}\Big(\int_0^{t_0}\int_{B_i\cap\Omega}\xi_i^{2(n-2)}\chi^{2p_1-n}f^{p_1}\Big)^{\frac{1}{p_1}} \end{aligned} \tag{5.48}$$

Let $t_0 \to t$ in (5.48) and using Corollary 64, we get

$$\begin{aligned} |\xi_i^{\frac{2n}{p_1}}\chi^2 f(x,t)| &\le C(n,N,\tau,r,p_1)A_0^{\frac{n}{2p_1}}(1+t^{-1})^{\frac{n+2}{2p_1}}\Big(\int_0^{t}\int_{B_i\cap\Omega}\xi_i^{2(n-2)}\chi^{2p_1-n}f^{p_1}\Big)^{\frac{1}{p_1}} \\ &\le C(n,N,\tau,r,p_1)A_0^{\frac{n}{2p_1}}K_1 t^{-\frac{n}{2p_1}} \end{aligned}$$

Then we get

$$|\xi_i^4\chi^2 f| \le |\xi_i^{\frac{2n}{p_1}}\chi^2 f| \le C(n,N,\tau,r,p_1)A_0^{\frac{n}{2p_1}}K_1 t^{-\frac{n}{2p_1}}$$

Now for any $x\in\Omega$, we assume $x \in B_{i_0}\cap\Omega$, then we have

$$\begin{aligned} |\chi^2 f|(x,t) &= |\textstyle\sum_{j=1}^N \xi_j^2\chi^2 f(x,t)| = |(\sum_{j=1}^N \xi_j^2)^2\chi^2 f| \\ &\le |N\textstyle\sum_{j=1}^N \xi_j^4\chi^2 f| \le N^2 C(n,N,\tau,r,p_1)A_0^{\frac{n}{2p_1}}K_1 t^{-\frac{n}{2p_1}} \\ &\le \tilde{C}_3(n,N_0,\tau,p_1,r)A_0^{\frac{n}{2p_1}}K_1 t^{-\frac{n}{2p_1}} \end{aligned}$$

Replace f with $|Rc|$, this is our conclusion. □

Lemma 66 *If* $0 \leq t < T_{\max} < 1$*, then for* $q \geq 2$*,* $p \geq \frac{n}{2}$*,* $p' \geq 0$ *and* $p \geq p'$*, we have the following:*

$$\frac{\partial}{\partial t}\Big(\int \xi_i^{2q}\chi^{2p'}|Rm|^p\Big) + \frac{1}{9}\Big(\int |\nabla(\xi_i^q\chi^{p'+1}|Rm|^{\frac{p}{2}})|^2\Big)$$
$$\leq C(n, N_0, \tau, r)p^{2+\frac{n}{2}}q^2(1+t^{-1})\Big(\int_{B_i\cap\Omega}\xi_i^{2(q-2)}\chi^{2p'}|Rm|^p\Big)$$

Proof: Similar to the proof of (5.40) in Lemma 63. □

Lemma 67 *If* $0 < t < T_{\max} < \tilde{T}_2$*, there exists some positive constant* $\tilde{C}_4(n, N_0, \tau, r)$*, which is independent of t, such that*

$$|\chi^2 Rm(g(t))| \leq \tilde{C}_4 t^{-1}$$

Proof: Similar to the proof of Lemma 65, but replacing p_1 with $\frac{n}{2}$ and using Lemma 66, (5.33) instead of Lemma 63, Corollary 64. □

Corollary 68 *If* $0 \leq t < T_{\max} < \tilde{T}_2$*, there exists some positive constant C, which is independent of t, such that*

$$|\chi^2 Rm(g(t))| \leq C$$

Proof: Straightforward from Lemma 67. □

5.3 Time estimates of the local Ricci flow

As in section 5.2, we consider two cases and assume that (M^n, g_0) satisfies the hypothesis of theorem 80 or theorem 81 respectively.

5.3.1 The First Case: $p_0 > \frac{n}{2}$

In Propositions 69, 70, 71 and 72, we assume (M, g_0) satisfies (6.1). Choose $T_1 > 0$ such that

$$\exp^{\left(C_1 p_0^{\frac{n}{2p_0-n}+3} N T_1\right)} = \frac{3}{2}$$

where C_1 is from Lemma 57.

Proposition 69 *If* $0 \le t < T_{\max} < T_1$, *we have*

$$\Big(\fint_{B_i\cap\Omega} |Rm(g(t))|^{p_0} dV_{g(t)}\Big)^{\frac{1}{p_0}} \le \Big(\frac{3}{2}N\Big)^{\frac{1}{p_0}} K_1, \qquad \forall i$$

Proof: Set $f(x,t) = |Rm(x)|_{g(t)}$. Let $q = 1$, $p' = 0$, $p = p_0$ in Lemma 57, we have the following:

$$\frac{\partial}{\partial t}\Big(\fint_{\Omega} \xi_i^2 f^{p_0} dV_{g(t)}\Big) \le C_1 p_0^{\frac{n}{2p_0-n}+3} \fint_{B_i\cap\Omega} f^{p_0} dV_{g(t)}$$

For Ω, define $Cov(\Omega) = \{B_i | B_i \cap \bar{\Omega} \neq \emptyset\}$, note that $Cov(\Omega)$ has only a finite number of elements. Now we define

$$\phi(t) = \sup_{B_i\in Cov(\Omega)} \Big[\fint_{B_i\cap\Omega} \xi_i^2 |Rm|^{p_0} dV_{g(t)}\Big]$$

Assume $\phi(t_0) = \fint_{B_{i_0}\cap\Omega} \xi_{i_0}^2 |Rm|^{p_0} dV_{g(t_0)}$, then

$$\begin{aligned} \tfrac{\partial}{\partial t}\phi(t_0) &\le \tfrac{\partial}{\partial t}\Big(\fint_{\Omega} \xi_{i_0}^2 f^{p_0}\Big)(t_0) \le C_1 p_0^{\frac{n}{2p_0-n}+3} \fint_{B_{i_0}\cap\Omega} f^{p_0} dV_{g(t_0)} \\ &\le C_1 p_0^{\frac{n}{2p_0-n}+3} \textstyle\sum_{j=1}^N \Big(\int_{B_{i_0}\cap\Omega} \xi_j^2 f^{p_0} dV_{g(t_0)}\Big) \le C_1 p_0^{\frac{n}{2p_0-n}+3} N\phi(t_0) \end{aligned} \tag{5.49}$$

Because t_0 is choosed freely, we have

$$\tfrac{\partial}{\partial t}\phi(t) \le \Big[C_1 p_0^{\frac{n}{2p_0-n}+3} N\Big]\phi(t) \tag{5.50}$$

Note by (1.7), $\phi(0) \le \sup_{B_i} \fint_{B_i\cap\Omega} |Rm|^{p_0} dV_{g_0} \le K_1^{p_0}$, then by the definition of T_1 and (5.50), we get

$$\phi(t) \le \frac{3}{2} K_1^{p_0}$$

Now we have

$$\fint_{B_i\cap\Omega} |Rm|^{p_0} dV_{g(t)} = \sum_{j=1}^{N} \fint_{B_i\cap\Omega} \xi_j^2 f^{p_0} \le N\phi(t) \le \tfrac{3}{2} N K_1^{p_0}$$

We get our conclusion. □

Set

$$T_2 = \min\Big\{T_1,\ 1,\ \Big(\frac{\ln 2}{2(n+1)C_3}\Big)^{\frac{2p_0}{2p_0-n}}\Big\}$$

where C_3 is from Corollary 59.

Proposition 70 *If* $0 \le t < T_{\max} < T_2$*, we have*

$$2^{-\frac{1}{n+1}} g_0 \le g(t) \le 2^{\frac{1}{n+1}} g_0 \tag{5.51}$$

Proof: Choose $0 \le t_0 < T_{\max} < T_2$. By Corollary 59, we have

$$\int_0^{t_0} |\tfrac{\partial}{\partial t} g| dt = \int_0^{t_0} |2\chi^2 Rc(g(t))| dt \le \int_0^{t_0} 2C_3 t^{-\frac{n}{2p_0}} dt \le 2C_3 T_2^{1-\frac{n}{2p_0}} \le \tfrac{\ln 2}{n+1}$$

By Lemma 6.49 in [29], we get our conclusion. □

Proposition 71 *If* $0 \le t < T_{\max} < T_2$*, we have*

$$\Big(\fint_{\tilde{B}_x(r)} h^{\frac{2n}{n-2}} dV_{g(t)}\Big)^{\frac{n-2}{n}} \le 2A_0 r^2 \fint_{\tilde{B}_x(r)} |\nabla h|^2 dV_{g(t)}, \quad \forall h \in C_0^\infty(\tilde{B}_x(r)) \tag{5.52}$$

Proof: By Lemma 56 and Proposition 70, we get our conclusion. □

Proposition 72 *If $T_{\max} < T_2$, then on $[0, T_{\max})$, we have*

$$\begin{cases} \Big(⨍_{\tilde{B}_x(r)} h^{\frac{2n}{n-2}} dV_{g(t)} \Big)^{\frac{n-2}{n}} & \leq 2A_0 r^2 ⨍_{\tilde{B}_x(r)} |\nabla h|^2 dV_{g(t)}, \\ 2^{-\frac{1}{4}} g_0 & \leq g(t) \leq 2^{\frac{1}{4}} g_0 \\ \Big(⨍_{\tilde{B}_x(r)} |Rm(g(t))|^{p_0} dV_{g(t)} \Big)^{\frac{1}{p_0}} & \leq \Big(\frac{3}{2}N\Big)^{\frac{1}{p_0}} K_1 \\ |\chi^2 Rm(g(t))| & \leq C \end{cases} \tag{5.53}$$

for any $x \in M$ and $h \in C_0^\infty(\tilde{B}_x(r))$, where C is independent of t.

Proof: Combining Propositions 69, 70, 71 and Corollary 60, we get our conclusion. □

Now we need to use the following two propositions due to Yuanqi Wang. Firstly, we define another Sobolev constant $C_s(\Omega, t)$ for $\Omega \subset (M, g(t))$ by the following

$$|h|_{C^0(\Omega)} \leq C_s(\Omega, t) \Big(\int_\Omega |\nabla h|^2 dV_{g(t)} \Big)^{\frac{1}{2}}, \quad \forall h \in C_0^\infty(\Omega)$$

Remark 73 *Because Ω is bounded domain in (M, g_0), by the proof of Theorem 3.5 in [31], we get $C_s(\Omega, 0)$ is finite. If $0 \leq t < T_{\max} < T_2$, by Proposition 72, we get $2^{-\frac{1}{4}} g_0 \leq g(t) \leq 2^{\frac{1}{4}} g_0$, so $C_s(\Omega, t)$ is uniformly bounded on $[0, T_{\max})$.*

Proposition 74 (Yuanqi Wang, [28]) *If $|\chi^2 Rm|_{(g(t),\infty)}$, $|\nabla \chi|_{(g(t),\infty)}$, $C_s(\Omega, t)$ are uniformly bounded on $[0, T)$, then for any $m \geq 0$, $|\nabla^m(\chi^2 Rm)|_{(g(t),\infty)}$ is uniformly bounded on $[0, T)$.*

Proof: Combining Lemmas 4.7–4.13 and Propositions 4.15–4.16 in [28], we get that for any $m \geq 0$, $|\nabla^m(\chi^2 Rm)|_{(g(t),\infty)}$ is uniformly bounded on $[0, T)$. □

Proposition 75 (Yuanqi Wang, [28]) *If $g(t)$, $t \in [0, T)$ is a solution to the local Ricci flow (4.1) on M, and for any $m \geq 0$, $|\nabla^m(\chi^2 Rm)|_{(g(t),\infty)}$ is uniformly bounded on $[0, T)$, then the local Ricci flow could extend smoothly through T.*

Proof: Using the similar argument in section 7 of chapter 6 of [29], the difference is that $\nabla^m(\chi^2 Rm)$ plays the key role in our local Ricci flow case instead of $\nabla^m Rm$ in Ricci flow case. □

Now we have the following theorem which has its root in Theorem 9.2 of [15].

Theorem 76 *Assume* (M, g_0) *is a n-dimensional* $(n \geq 3)$ *complete noncompact Riemannian manifold, which satisfies (6.1). Then there exists some positive constant* $T = T(A_0, K_1, n, p_0, r)$*, such that (4.1) has a smooth solution on* $[0, T]$*. Moreover, for* $t \in [0, T]$*, the metric satisfies (5.17) and the curvature tensors satisfy the following bounds:*

$$|\chi^2 Rc(g(t))| \leq C(A_0, K_1, n, p_0, r)t^{-\frac{n}{2p_0}} \tag{5.54}$$

$$|\chi^2 Rm(g(t))| \leq C(A_0, K_1, n, p_0, r)t^{-\frac{n}{2p_0}} \tag{5.55}$$

Proof: Firstly, by Theorem 51 and DeTurck's trick, the equation (4.1) has a smooth solution on a sufficiently small time interval starting at $t = 0$. Let $[0, T_{\max})$ be a maximal time interval on which (4.1) has a smooth solution and such that (5.16), (5.17), (5.18) hold for each metric $g(t)$. Choose $T = T(A_0, K_1, n, p_0, r) = \frac{T_2}{2} > 0$. We claim that $T_{\max} \geq T$, we prove it by contradiction.

If $T_{\max} < T$, then $T_{\max} < T_2$. By Proposition 72 and Remark 73, we get $|\chi^2 Rm|_{(g(t),\infty)}$, $|\nabla\chi|_{(g(t),\infty)}$, $C_s(\Omega, t)$ are uniformly bounded on $[0, T_{\max})$.

By Proposition 74, we get that for any $m \geq 0$, $|\nabla^m(\chi^2 Rm)|_{(g(t),\infty)}$ is uniformly bounded on $[0, T)$. Now using Proposition 75, we can extend the solution of (4.1) to $[0, T_{\max} + \delta)$, where $\delta > 0$ is some constant.

By (5.53), we can furthermore assume that (5.16), (5.17), (5.18) hold on $[0, T_{\max} + \delta)$, and it is the contradiction with the definition of $T_{\max}$. Then we have $T_{\max} \geq T$, and (4.1) has a smooth solution on $[0, T]$. We prove the first conclusion and $g(t)$ satisfies (5.17).

Note $T < T_2$, by Corollary 59, Lemma 58 and $T_2 \leq 1$, we get (5.54), (5.55). $\square$

5.3.2 The Second Case: $p_0 = \frac{n}{2}$

In this subsection, we assume that (M, g_0) satisfies the inequalities (6.19). Now we set

$$\epsilon_0 = \frac{\tau(n-2) - 12Nn}{8(\tau + 6Nn)} > 0 \tag{5.56}$$

where we use $\tau > 36N$ and N is the same as in Theorem 81.

Choose $\tilde{T}_3 > 0$ such that

$$\exp^{\left(\frac{1}{\epsilon_0}\tilde{C}_1 n^3 N \tilde{T}_3\right)} = \frac{3}{2}$$

where $\tilde{C}_1$ is from Lemma 61.

$$\tilde{T}_4 = \min\left\{\tilde{T}_2,\ \tilde{T}_3,\ \left(\frac{\ln 2}{2(n+1)\tilde{C}_3(n, N_0, \tau, p_1, r)}\right)^{\frac{2p_1}{2p_1-n}} A_0^{-\frac{n}{2p_1-n}} K_1^{-\frac{2p_1}{2p_1-n}}\right\}$$

where $\tilde{C}_3$ is from Lemma 65.

By similar argument as in Propositions 70, 71 and 72, we get:

Proposition 77 *If $T_{\max} < \tilde{T}_4$, then on $[0, T_{\max})$, we have*

$$\begin{cases} \left(\int_{\tilde{B}_x(r)} h^{\frac{2n}{n-2}} dV_{g(t)}\right)^{\frac{n-2}{n}} & \leq 2A_0 \int_{\tilde{B}_x(r)} |\nabla h|^2 dV_{g(t)}, \\ 2^{-\frac{1}{4}} g_0 & \leq g(t) \leq 2^{\frac{1}{4}} g_0 \\ \left(\int_{\tilde{B}_x(r)} |Rm(g(t))|^{\frac{n}{2}} dV_{g(t)}\right)^{\frac{2}{n}} & \leq \left(\frac{3}{2}N\right)^{\frac{2}{n}} (\tau n A_0)^{-1} \\ |\chi^2 Rm(g(t))| & \leq C \end{cases} \tag{5.57}$$

for any $x \in M$ and $h \in C_0^\infty(\tilde{B}_x(r))$, where C is independent of t.

Similar to the proof of Theorem 76, we have the following theorem:

Theorem 78 *Assume* (M, g_0) *is a n-dimensional* ($n \geq 3$) *complete noncompact Riemannian manifold, which satisfies (6.19). Then there exists positive constant* $C(n, N_0, \tau, p_1, r)$*, such that (4.1) has a smooth solution on* $[0, T]$*, where*

$$T = C(n, N_0, \tau, p_1, r) \min\left\{1,\ A_0^{-\frac{n}{2p_1 - n}} K_1^{-\frac{2p_1}{2p_1 - n}}\right\} \tag{5.58}$$

Moreover, for $t \in [0, T]$*, the metric satisfies (5.17) and the curvature tensors satisfy the following bounds:*

$$|\chi^2 Rc(g(t))| \leq \tilde{C}(n, N_0, \tau, p_1, r) A_0^{\frac{n}{2p_1}} K_1 t^{-\frac{n}{2p_1}} \tag{5.59}$$

$$|\chi^2 Rm(g(t))| \leq C(n, N_0, \tau, r) t^{-1} \tag{5.60}$$

Proof: Similar argument to the proof of Theorem 76. □

Remark 79 *Note the difference of the power of t between (5.54), (5.55) and (5.59), (5.60).*

Chapter 6

Short-time existence of Ricci flow on noncompact manifolds

In this chapter, we use the family of the local Ricci flows to converge to the Ricci flow on a noncompact manifold M, then we get the short time existence of the Ricci flow on M. At many points in this section, we will take a subsequence. To simplify notation, at each stage a sequence such as $\{\chi_k\}$ will be re-indexed to continue to be $\{\chi_k\}$. Hamilton's Compactness theorem for Ricci flow and uniform estimate of local Ricci flow from last chapter will be the key point of the proof.

Theorem 80 *Assume* (M, g_0) *is a n-dimensional* $(n \geq 3)$ *complete noncompact Riemannian manifold, satisfying the following conditions:*

$$\begin{cases} \left(⨍_{\tilde{B}_x(4r)} h^{\frac{2n}{n-2}} dV_{g_0} \right)^{\frac{n-2}{n}} & \leq A_0 r^2 ⨍_{\tilde{B}_x(4r)} |\nabla h|^2 dV_{g_0} \,, \\ \left(⨍_{\tilde{B}_x(r)} |Rm(g_0)|^{p_0} dV_{g_0} \right)^{\frac{1}{p_0}} & \leq K_1 \end{cases} \tag{6.1}$$

for any $x \in M$ *and* $h \in C_0^\infty(\tilde{B}_x(4r))$*, where* A_0 *and* K_1 *are some positive constants,* r *and* p_0 *are some constants such that* $0 < r \leq 1$*,* $p_0 > \frac{n}{2}$*.*

Then the Ricci flow (1.4) has a smooth solution $g_{ij}(x,t)$ on $[0,T]$, and satisfies the following estimates. For any integer $m \geq 0$, there exists a positive constant $C(A_0, K_1, m, n, p_0, r)$ depending only on A_0, K_1, m, n, p_0 and r, such that

$$\sup_{x\in M} |\nabla^m Rm(x,t)| \leq \frac{C(A_0, K_1, m, n, p_0, r)}{t^{\frac{m}{2}+\frac{n}{2p_0}}}, \quad 0 < t \leq T \tag{6.2}$$

where $T = T(A_0, K_1, n, p_0, r)$ is some positive constant depending only on A_0, K_1, n, p_0 and r.

Proof: Choose

$$\Omega_i = \{x | d_{g_0}(x, \mathbf{O}) \leq 4^i,\ x \in M\} \quad i = 1, 2, \cdots$$

where $\mathbf{O}$ is some fixed point in M. Constructing χ_i, which is a smooth cutoff function on M, such that

$$\chi_{i+1}(x) = \begin{cases} 1 & x \in \Omega_i \\ 0 & x \in M \backslash \Omega_{i+1} \end{cases}$$

and

$$0 \leq \chi_i \leq 1, \quad \|\nabla \chi_i\|_\infty \leq 1, \quad \lim_{i\to\infty} \chi_i(x) = 1 \tag{6.3}$$

Now by (6.3), we can choose T as in Theorem 76 and such T is independent of χ_i. For each k, $g_k(t)$ be the solution of the following local Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t} g_k &= -2\chi_k^2 Rc, \quad x \in M \\ g_k(x,0) &= g_0(x), \quad x \in M \end{cases} \tag{6.4}$$

For any $0 < t_1 < T$, any i, we have

$$\|\nabla_k^m Rm_k(x,t_1)\|_{(g_k(t_1),\infty)} \leq C(m, \Omega_i, t_1), \quad x \in \Omega_i, \quad k \leq i \tag{6.5}$$

where ∇_k and Rm_k are with respect to $g_k(t)$. For simplicity reason, we will use the notation $|\nabla_k^m Rm_k(x,t_1)|_\infty$ instead of $\|\nabla_k^m Rm_k(x,t_1)\|_{(g_k(t_1),\infty)}$. If $k > i$, then on Ω_i, g_k satisfies (1.4) which is the Ricci flow. From (5.55) we know

$$|\chi_k^2 Rm_k(x,t_1)|_\infty \le C(A_0,K_1,n,p_0,r)t_1^{-\frac{n}{2p_0}}\ , \quad x\in M, \quad \forall k \tag{6.6}$$

Note $C(A_0,K_1,n,p_0,r)$ in (6.6) is independent of k. We get

$$|Rm_k(x,t_1)|_\infty \le C(A_0,K_1,n,p_0,r)t_1^{-\frac{n}{2p_0}}\ , \quad x\in \Omega_i\ ,\ k>i \tag{6.7}$$

Then by Theorem 14.14 in [32], we have

$$|\nabla_k^m Rm_k(x,t_1)|_\infty \le C(A_0,K_1,n,m,p_0,r,t_1,\Omega_i)t_1^{-\frac{m}{2}} = C(A_0,K_1,n,m,p_0,r,t_1,\Omega_i) \tag{6.8}$$

for any $x\in\Omega_i$ and $k>i$.

By (6.5) and (6.8), we get

$$|\nabla_k^m Rm_k(x,t_1)|_\infty \le C(A_0,K_1,n,m,p_0,r,t_1,\Omega_i) \quad x\in\Omega_i,\ \forall k\ge 0 \tag{6.9}$$

Note $|\nabla_k^m Rm_k(x,t_1)|_\infty$ are bounded independent of k.

When $t\in[0,T]$, by Theorem 76 we know

$$\frac{1}{2}g_0 \le g_k(t) \le 2g_0, \tag{6.10}$$

we get

$$C_1\sqrt{\det g_0} \le \sqrt{\det g_k(t_1)} \le C_2\sqrt{\det g_0}$$

By Lemma 5.2 of [15], we get that there exists some $\tau_0>0$, such that

$$inj_{g_k(t_1)}(\mathbf{O}) \ge \tau_0 \tag{6.11}$$

By (6.9) and (6.11), Theorem 3.9 in [33] applies for $(\Omega_i, g_k(t_1), \mathbf{O})_{k\in\mathbb{N}}$ too, we get

$$\lim_{k\to\infty} g_k(x,t_1) = g_\infty(x,t_1)$$

where the limit is in C^∞ sense on Ω_i. We have

$$|\widehat{\nabla}^m g_k(x,t_1)|_\infty \triangleq |\nabla^m_{g_0} g_k(x,t_1)|_\infty \le C(m,\Omega_i) \quad x \in \Omega_i \tag{6.12}$$

where $C(m,\Omega_i)$ is independent of k.

And we also know from Theorem 14.14 in [32], if $k > i$,

$$|\nabla^m_k Rm_k(x,t)|_\infty \le C(A_0,K_1,n,m,p_0,r,t_1,\Omega_i)\ , \quad (x,t) \in \Omega_i \times [t_1,T] \tag{6.13}$$

If $k \le i$, it is easy to get (6.13) is also true. So we get for any k, (6.13) is true, where $C(A_0,K_1,n,m,p_0,r,t_1,\Omega_i)$ is independent of k.

Then by (6.10), (6.12), (6.13) and Lemma 3.11 in [33], we get

$$\left|\frac{\partial^q}{\partial t^q}\widehat{\nabla}^m g_k(x,t)\right|_\infty \le C(A_0,K_1,n,m,q,p_0,r,t_1,\Omega_i), \quad (x,t) \in \Omega_i \times [t_1,T], \quad \forall k \ge 0$$

Now by Corollary 3.15 in [33], we get

$$\lim_{k\to\infty} g_k(x,t) = g_\infty(x,t) \quad (x,t) \in \Omega_i \times [t_1,T]$$

where the limit is in C^∞ sense. Because i and t_1 are arbitrary, by the diagonal method, we get

$$g(x,t) \triangleq g_\infty(x,t) = \lim_{k\to\infty} g_k(x,t) \quad (x,t) \in M \times (0,T] \tag{6.14}$$

Then we have

$$\begin{aligned}
|\lim_{t\to 0}(g(x,t) - g_0(x))| &= |\lim_{t\to 0}\lim_{k\to\infty}(g_k(x,t) - g_k(x,0))| \\
&= |\lim_{t\to 0}\lim_{k\to\infty}\int_0^t \tfrac{\partial}{\partial s} g_k(x,s)ds| \\
&\le \lim_{t\to 0}\lim_{k\to\infty}\int_0^t |2\chi_k^2 Rc_k|_\infty \le C\lim_{t\to 0} t^{1-\frac{n}{2p_0}} \\
&= 0
\end{aligned}$$

In the last inequality above, we used (5.54).

We can define

$$g(x,0) = g_0(x) \quad x \in M \tag{6.15}$$

Then by (6.4), (6.14) and (6.15), we get $g(t)$ is the solution of the Ricci flow (1.4) on $M \times [0,T]$. This is the first conclusion of theorem 80; it remains to prove the estimate (6.2).

Now by (6.7) and (6.14), we get

$$|Rm_g(x,t_1)|_\infty \le \lim_{k\to\infty} |Rm_k(x,t_1)|_\infty \le C(A_0, K_1, n, p_0, r) t_1^{-\frac{n}{2p_0}}$$

for any $x \in M$ and $t_1 \in (0,T]$.

If we look at $(M, g(\frac{t_1}{2}))$ as the initial conditions of the Ricci flow, we have the Ricci flow on $M \times [\frac{t_1}{2}, T]$ and

$$\left| Rm_{g(\frac{t_1}{2})} \right|_\infty \le C(A_0, K_1, n, p_0, r) t_1^{-\frac{n}{2p_0}} \tag{6.16}$$

Now we define

$$\tilde{g}(x,t) = \left[C(A_0,K_1,n,p_0,r)t_1^{-\frac{n}{2p_0}}\right] g\Big(x, \frac{t_1}{2} + t\Big[C(A_0,K_1,n,p_0,r)t_1^{-\frac{n}{2p_0}}\Big]^{-1}\Big) \tag{6.17}$$

where $C(A_0, K_1, n, p_0, r)$ in (6.17) is the same as in (6.16). Then from scaling argument we can get

$$\begin{aligned} \tfrac{\partial}{\partial t}\tilde{g}_{ij}(x,t) &= -2\tilde{R}_{ij}(x,t) \\ \tilde{g}(x,0) &= \Big[C(A_0,K_1,n,p_0,r)t_1^{-\frac{n}{2p_0}}\Big] g(x, \tfrac{t_1}{2}) \end{aligned}$$

We already know that g_{ij} and its Ricci flow exist on $[\frac{t_1}{2}, T]$, so $\tilde{g}_{ij}$ and its Ricci flow exist on $\Big[0, (T - \frac{t_1}{2}) C(A_0, K_1, n, p_0, r) t_1^{-\frac{n}{2p_0}}\Big]$. We use $\{\tilde{R}_{ijkl}(x)\}$ and $\tilde{\nabla}$ to denote, repectively, the Riemannian curvature tensor and the covariant derivative with respect to $\tilde{g}_{ij}(x,0)$. Note $|\widetilde{Rm}| \le 1$ on M, by Lemma 7.1 of [14], there exists a positive constant $C(n,m)$, such that

$$|\tilde{\nabla}^m \widetilde{Rm}(x,t)|^2 \le \frac{C(n,m)}{t^m}; \quad 0 < t \le \Big[0, (T - \frac{t_1}{2}) C(A_0, K_1, n, p_0, r) t_1^{-\frac{n}{2p_0}}\Big],\ m \ge 0 \tag{6.18}$$

by (6.17) and (6.18), we get

$$|\nabla^m Rm(x,t)|^2 \le \frac{C(n,m)\Big[C(A_0,K_1,n,p_0,r)t_1^{-\frac{n}{2p_0}}\Big]^{m+2}}{\Big[C(A_0,K_1,n,p_0,r)t_1^{-\frac{n}{2p_0}}(t-\frac{t_1}{2})\Big]^m} \le \frac{C(A_0,K_1,m,n,p_0,r)}{t_1^{\frac{n}{p_0}}(t-t_1)^m}; \qquad \frac{t_1}{2} < t \le T$$

So we have

$$|\nabla^m Rm(x,t_1)|^2 \le \frac{C(A_0,K_1,m,n,p_0,r)}{t_1^{m+\frac{n}{p_0}}}, \quad m \ge 0$$

Because we choose t_1 freely, finally we get

$$|\nabla^m Rm(x,t)|^2 \le \frac{C(A_0,K_1,m,n,p_0,r)}{t^{m+\frac{n}{p_0}}}, \quad for\ \ 0 < t \le T, \quad m \ge 0$$

So we get our conclusion. □

Theorem 81 *Assume (M, g_0) is an n-dimensional ($n \ge 3$) complete noncompact Riemannian manifold, satisfying the following conditions:*

$$\begin{cases} Vol_{g_0}(\tilde{B}_x(\rho)) & \ge N_0\Big(\frac{\rho}{4r}\Big)^n Vol_{g_0}(\tilde{B}_x(4r)) , \\ \Big(\int_{\tilde{B}_x(r)} h^{\frac{2n}{n-2}} dV_{g_0}\Big)^{\frac{n-2}{n}} & \le A_0 \int_{\tilde{B}_x(r)} |\nabla h|^2 dV_{g_0} , \\ (\int_{\tilde{B}_x(r)} |Rm(g_0)|^{\frac{n}{2}} dV_{g_0})^{\frac{2}{n}} & \le (\tau n A_0)^{-1} , \\ \Big(\int_{\tilde{B}_x(r)} |Rc(g_0)|^{p_1} dV_{g_0}\Big)^{\frac{1}{p_1}} & < K_1 \end{cases} \tag{6.19}$$

for any $x \in M$, $0 < \rho \le 4r$, $h \in C_0^\infty(\tilde{B}_x(r))$, where $p_1 > \frac{n}{2}$, A_0, N_0 ,K_1 and $0 < r \le 1$ are some positive constants. Here τ is some positive constant satisfying $\tau > \max\Big\{36N, \frac{16p_1N}{\sqrt{n}}\Big\}$, where $N \triangleq 8^n N_0^{-2}$.

Then the Ricci flow (1.4) has a smooth solution $g_{ij}(x,t)$ on $[0,T]$, and satisfies the following estimates. For any integer $m \ge 0$, there exist constants $C(n,m,N_0,\tau,r) > 0$ depending only on n, m, N_0, τ and r, such that

$$\sup_{x\in M} |\nabla^m Rm(x,t)| \le \frac{C(n,m,N_0,\tau,r)}{t^{\frac{m}{2}+1}}, \quad 0 < t \le T \tag{6.20}$$

where T is defined as the following:

$$T = C(n, N_0, p_1, \tau, r) \min\left\{1,\ A_0^{-\frac{n}{2p_1-n}} K_1^{-\frac{2p_1}{2p_1-n}}\right\} \tag{6.21}$$

Proof: Similar argument as the proof of Theorem 80, except using the results in subsections 5.2.2, 5.3.2 instead of subsections 5.2.1, 5.3.1. □

References

[1] Jr. Eells James and J. H. Sampson. Harmonic mappings of riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.

[2] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the riemannian penrose inequality. *J. Differential Geom.*, 59, no. 3:353–437, 2001.

[3] John Morgan & Gang Tian. Ricci flow and the poincar conjecture. *Clay Mathematics Monographs, 3. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA,*, 2007.

[4] Ben Andrews. Contraction of convex hypersurfaces in euclidean space. *Calc.Var.*, 2:151–171, 1994.

[5] D. DeTurck. Deforming metrics in the direction of their ricci tensors. *J. Differential Geom.*, 18, no.1:157–162, 1983.

[6] Gerhard Huisken. Flow by mean curvature of convex hypersurfaces into spheres. *J.Differential Geometry*, 20:237–268, 1984.

[7] Matt Grayson. Shortening embedded curves. *Ann. of Math. (2)*, 129:71–111, 1989.

[8] Ben Andrews. Contraction of convex hypersurfaces in riemannian spaces. *J.Differential Geometry.*, 39:407–431, 1994.

[9] Gerhard Huisken. Evolution of hypersurfaces by their curvature in riemannian manifolds. *Proceedings of the International Congress of Mathematicians*, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II:349–360.

[10] Gerhard Huisken. Contracting convex hypersurfaces in riemannian manifolds by their mean curvature. *Invent. Math.*, 84:463–480, 1986.

[11] Robert Gulliver & Guoyi Xu. Examples of hypersurfaces flowing by curvature in a riemannian manifold. *Comm. Anal. Geom.*, 17, no.4:701–720, 2009.

[12] Richard Hamilton. Three-manifolds with positive ricci curvature. *J.Differential Geometry*, 17:255–306, 1982.

[13] Yau Shing-Tung. Problem section. *Seminar on Differential Geometry, Ann. of Math. Stud. , Princeton Univ. Press, Princeton, N.J.*, 102:669–706, 1982.

[14] Wan-Xiong Shi. Deforming the metric on complete riemannian manifolds. *J.Differential Geometry*, 30:223–301, 1989.

[15] Deane Yang. Convergence of riemannian manifolds with integral bounds on curvature.i. *Ann Sci.E.N.S*, pages 77–105, 1992.

[16] Guoyi Xu. Short-time existence of the ricci flow on noncompact riemannian manifolds. *arXiv: 0907.5604 [math.DG].*

[17] Min-Oo J. Bemelmans and E. A. Ruh. Smoothing riemannian metrics. *Math. Zeitschr.*, 188:69–74, 1984.

[18] L. Saloff-Coste. Uniformly elliptic operators on riemannian manifolds. *J.Differential Geometry*, 36:417–450, 1992.

[19] Peter Li & Richard Schoen. l^p and mean value properties of subharmonic functions on riemannian manifolds. *Acta Math.*, 153:279–301, 1984.

[20] Ben Andrews. Pinching estimates and motion of hypersurfaces by curvature functions. *J. reine angew. Math*, 608:17–33, 2007.

[21] G. Lieberman. Second order parabolic differential equations. *World Scientific*, 1996.

[22] D. Gilbarg and N. S. Trudinger. Elliptic differential equations of second order. Springer-Verlag:1998, Berlin.

[23] Richard Hamilton. A compactness property for solution of the ricci flow. *Amer. J. Math.*, 117:545–572, 1995.

[24] Richard Hamilton. Convex hypersurfaces with pinched second fundamental form. *Comm. Anal. Geom.*, 2:167–172, 1994.

[25] Svetlana Katok. Fuchsian groups. *University of Chicago Press*, 1992.

[26] T. P. Branson. Q-curvature and spectral invariants. *Proceedings of the 24thWinter School Geometry and Physics Srn, 2004. Suppl. Rend. Circ. Mat. Palermo*, 75:11–55, 2005.

[27] Orsted S.-Y. Alice Chang; Michael, Eastwood; Bent and Paul C. Yang. What is *q*-curvature? *Acta Appl. Math*, 102, no. 2-3:119–125, 2008.

[28] Yuanqi Wang. Pseudolocality of ricci flow under integral bound of curvature. *arXiv: 0903.2913v1 [math.DG].*

[29] B. Chow and D. Knopf. The ricci flow: An introduction. *Mathematical Surveys and Monographs*, 110:AMS, Providence, RI, 2004.

[30] L. Saloff-Coste. Aspects of sobolev-type inequalities. *London Math Society Lecture Note Series*, 289:Cambridge University Press, 2002.

[31] E. Hebey. Nonlinear analysis on manifolds: Sobolev spaces and inequalities. *AMS and Courant Institute of Mathematics Sciences*, 2000.

[32] D. Glickenstein C. Guenther J. Isenberg T. Ivey D. Knopf P. Lu F. Luo B. Chow, S. Chu and L. Ni. The ricci flow: Techniques and applications part ii: Analytic aspects. *Mathematical Surveys and Monographs*, 144:AMS, Providence, RI, 2008.

[33] D. Glickenstein C. Guenther J. Isenberg T. Ivey D. Knopf P. Lu F. Luo B. Chow, S. Chu and L. Ni. The ricci flow: Techniques and applications part i: Geometric aspects. *Mathematical Surveys and Monographs*, 135:AMS, Providence, RI, 2007.