Essays on Economic Growth, Education, and the Distribution of Income: A Structural Analysis for the Case of South Africa

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Dedication

To my wife Annie and to my daughter Mercy-Grace

Abstract

Since the fall of the Apartheid regime in South Africa in 1994, the democratically elected post-Apartheid governments have engaged in social and economic reforms aimed at improving the welfare of millions of the left outs during that regime, and at enhancing economic growth. Among these reforms, the most important have consisted of achieving structural transformation of the South African economy through technological advancements, improving skills by investing massively in human capital through education and training, reducing income inequality across racial groups through access to programs that facilitate the acquisition of skills by the left outs of the Apartheid regime rather than through massive income redistribution, liberalizing the labor markets and the formation of labor unions, privatizing statal and para-statal corporations and abolishing monopolies in public services, liberalizing trade and capital movements, promoting new investment through tax incentives, promoting private initiative, reducing or at least freezing government expenditures, reducing poverty, and so on.

In this dissertation, I analyze the relationship between growth and some aspects of the economy targeted by the aforementioned reforms through two separate essays. In the first essay (Essay 1), I analyze the growth and welfare effects of public spending on education in post-Apartheid South Africa while in the second essay (Essay 2) I investigate the dynamics of income inequality, poverty, and growth in the post-Apartheid South Africa.

Starting with Essay 1, since the abolition of its Apartheid regime in 1994, South Africa has launched a massive program of education and training, which has been financed through resources representing annually on average 21% of the national budget,

or 7% of GDP. Today, the GDP share of public spending on education is 1.3 times the average of industrialized countries (5.4%) and almost twice that of developing countries (3.9%).

In this essay, I simulate fiscal policy experiments to analyze the growth and welfare effects of a reduction in or an elimination of spending on education in a model of endogenous growth with human capital accumulation and policies for the Post Apartheid South African economy. The first and second experiments consist of reducing the GDP share of educational spending to the averages of industrialized and developing countries, respectively; while the third experiment consists of eliminating government spending on the educational sector (a 100% tax reduction).

The results of the simulations demonstrate that a reduction in or an elimination of educational spending reduces the long run rates as well as the transition rates of growth, in per capita GDP, the wages of skilled workers, and overall welfare. The effects on the other variables in the economy (physical capital, human capital, labor, consumption, and the interest rate) vary across experiments. However, these growth and welfare effects are small.

Turning to Essay 2, I construct a model of growth with heterogeneity in asset holdings and skills and calibrate it to the Post-Apartheid South African economy in order to analyze the dynamics of income distribution, income inequality, poverty, and growth. I find that growth is achieved at all levels of incomes and poverty is totally eliminated by the end of the process, which lasts 73 years (from 1993 to 2065). Furthermore, the economy achieves the overall convergence of incomes to the income of the average consumer in the distribution. Indeed, poor consumers improve their relative positions of

in the distribution of wealth as well as in that of income while rich consumers worsen theirs. Next, I combine the results of the heterogeneous model with the microeconomic data (the South Africa's 1996 October Household Survey) to estimate the distribution of income, and to analyze thoroughly the interaction between growth, income inequality, and poverty. I find that a one percent increase in the rate of growth of income causes on average poverty to drop by 3.7%, using the $2 per day poverty-line, and by 1.3% using the $1 per day poverty-line. Moreover, growth causes overall decline in income inequality but the effect is very small. A one percent-increase in the rate of growth of income results on average in a decline in income inequality of only 0.056% by the Gini coefficient and of 0.11% by the Global Theil Index.

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I1. Introduction

The centrality of human capital in driving economic development has been the subject of an abundant literature in recent years. Whether human capital is accumulated through on job learning, learning by doing, or formal education, its benefits to society are enormous. These benefits have been assessed in a number of studies in the case of education. For instance, high levels of education are found to be associated with higher likelihood of participation in the political process (Mulligan et al, 2004), lower likelihood of criminal activities (Lochner and Moretti, 2004), improved health (Currie and Moretti, 2003), higher rates of own productivity and that of coworkers (Moretti, 2004).

Education is the most studied channel through which human capital accumulates, and the impact of its changes on productivity is analyzed empirically at the microeconomic level as well as at the macroeconomic level. At the microeconomic level, there exists strong and robust empirical evidence that investment in education is a key determinant of growth of earnings of workers (Psacharopoulos, 1994; Duflo, 2004).[1,2] At the macroeconomic level, however, empirical studies were not consistent regarding the role of human capital in the process of development until recently. For instance, Benhabib and Spiegel (1994) and Barro and Sala-i-Marin (1995) find no effect of changes in schooling on GDP growth in cross-country regressions. In subsequent papers, Topel (1999) and Krueger and Lindhal (2001) revisit the above studies to trace the reasons why the empirical microeconomic correlation between growth of earnings and

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[1] These studies use the real wage as a proxy for productivity because they assume that labor market is perfectly competitive. This assumption implies that labor is paid its marginal product.

[2] In this study, we adopt the view that education improves worker skills (Becker, 1964). There exists an opposing view to this one, that is, education is viewed as a signal of worker ability (Spence, 1973). However, the signaling hypothesis does not have data evidence for African countries (see Boissiere et al., 1985)

change in schooling is not evident at the macroeconomic level. They find that the schooling function used was misspecified and schooling adjusting data were measured with error. After correcting for these problems, they found that investment in education affects growth positively.

Regardless of its role in economic development, the levels of investment in education resulting from households' decisions may fall short of their optimal levels because of market failures. For instance, missing markets for financing education in many developing countries may prevent people from going to school, and thus slow the accumulation of human capital and, consequently the process of growth. Some parents can still afford to provide education to their children regardless of these missing markets, but they may not be willing to reduce their own consumption in order to finance their children's education. Even in countries where such markets exist, parents may not be able to borrow against their children future incomes because of lack of collateral.

The above reasons may explain why we observe today the large involvement of the public sector in human capital investment through schooling across countries. The most recent and striking public involvement in the educational sector has been that of the government of the Republic of South Africa. Since the abolition of its Apartheid regime in 1994, South Africa has launched a massive program of education aimed at improving its quality and facilitating its access to those of its citizens deprived of it during the Apartheid regime. According to Schmidt (2004), the resources allocated to education for the period 1995-2001 represented on average 21% of the South Africa's national budget, or 7% of its GDP. Today, education is the largest single item in South Africa's national budget, and one of the highest proportions worldwide. This proportion is 1.3 times the average of industrialized countries (5.4%) and almost twice that of less developed countries (3.9%). The question addressed by this essay is whether or not this large public involvement in the educational sector helps to sustain the development process.

The impact of public involvement in the educational sector on the growth process of an economy has been an issue of growing interest in recent years. This issue has been analyzed within two distinct strands in literature. The first strand is based on the view that government expenditures are unproductive consumption of economic resources, and that tax rates are distortionary. More specifically, a tax rate induces distortions, which negatively affect the growth rate of the economy and generate welfare loss (King and Rebello, 1990; Lucas, 1990; Jones *et al.*, 1993; Stokey and Rebello, 1995; Razin and Yuen, 1996; and Ortigueira, 1998). The second strand in literature, however, is based on the view that not all government expenditures are unproductive. Indeed, a tax rate does induce distortions on the allocation decisions of households, but at the same time, the public good it finances generate positive externalities, which may enhance growth and welfare. For instance, public spending allocated to education and/or health enhances the productivity of labor. Early contributions to this topic include Barro (1990) and Barro and Sala-i-Matin (1992).

In this paper, I build on the second strand in literature to analyze the growth and welfare effects of public spending on education in a model of endogenous growth with human capital accumulation, focusing on the South African economy. More specifically, I formulate a simple model of endogenous growth with human capital accumulation and policies of Lucas' type[3] and solve it numerically under the parameters estimated from the data. Then, I simulate three fiscal policy experiments and compare their solutions to that of the baseline case to analyze their growth and welfare effects. The first and second policy experiments consist of reducing the share of GDP spent on education to the averages of industrialized and developing countries, respectively. The third consists of eliminating all government involvement in the educational sector. In both the baseline case and the simulated experiments, I restrict the analysis to services from government spending flows (exogenously given) rather than stocks of public expenditures.

In all three cases, the results indicate that spending on education does have positive growth and welfare effects. These results, however, do not indicate the optimal size of government involvement in the educational sector.

This study shares many features with several works in the literature of endogenous growth with policies, such as Barro (1990), Jones *et al.* (1993), Alesina and Rodrik (1994), Barro and Sala-i-Martin (1992, 2004), Corsetti and Roubini (1996), Agenor (2005) and Greiner (2006). As in all of these works, I model public spending as an input of production in order to analyze its effects on growth and welfare. However, my study deviates from them in several aspects. For instance, in all these works except Grenier (2006), the tax rate is endogenously determined (optimal taxation), while the tax rate is exogenously given in this study. Grenier (2006) estimates the tax rates from data, but unlike this essay, he focuses on the allocation shift between public spending on education and on public capital. The growth and welfare effects of such a shift, unlike this study,

[3] For a model of endogenous growth with human capital accumulation, see Lucas (1988), Becker et al. (1990), Mulligan and Sala-i-Martin (1992), Caballe and Santos (1993), Ortigueira (1998), Boucekkine and Tamarit (2004), Barro and Sala-i-Martin (1992, 2004), and Boucekkine et al (2007).

are not separate , and thus do not indicate the direction and magnitude of the impact attributable to public spending on education alone. Another aspect and probably the most important is the choice of the sector that uses public spending as an input. In Barro (1990), Alesina and Rodrik (1994), Barro and Sala-i-Martin (1992, 2004), public spending is an input for the final goods sector, while it is an input of the physical capital accumulation sector in Jones *et al.* (1993). In Corsetti and Roubini (1996) and Agenor (2005), it is either an input of the final good sector or of the human capital accumulation sector. In Grenier (2006), as well as in this paper, it is an input only for the human capital accumulation sector. Nonetheless, the human capital technology adopted here allows public spending to exert its external effects to the accumulation of human capital in a linear fashion. This specification generates a series of real per capita GDP that mimics the data very closely. This feature is missing in Grenier (2006) and in most of the aforementioned studies. As far as the author knows, this study is the first to emphasize this dimension, and this its our contribution.

The remainder of this study is organized as follows. Section 2 describes the rationale for government intervention in the educational sector. In Section 3, I formulate a simple model of endogenous growth with human capital and characterize its equilibrium. In section 4, I solve numerically the model. Section 5 includes the comparison of the predictions of the model to the data. In Section 6, I simulate three fiscal policy experiments as well as their solutions. In Section 7, I analyze growth and welfare effects of fiscal policy experiments. Section 8 provides concluding remarks.

I2. Rationale for Government Involvement in the Education Sector

Education is an impure public good that can be provided privately. In a privately financed education model, education is as a normal good whose demand varies across households based on their income levels. As such, high income households would purchase more education for their children than would low income households. Furthermore, low income households can still increase the education of their children to the level they want by borrowing against their children's future labor earnings in the credit markets. If households can choose any amount of education for their children, then what justifies government involvement in the education sector? If the government chooses to intervene, what is the optimal size of its involvement in this sector?

Education is associated with public benefits or externalities that justify government involvement. These externalities lead to welfare improvements and higher economic growth.[4] On welfare grounds, public spending on education allows first a redistribution of resources from high income households (who bear the tax burden of education) to low income households. This redistribution allows for income mobility by providing children of low income households the opportunity to get an education so that they can increase their future incomes. Next, education may lower the likelihood of criminal activities. A Lower likelihood of criminal activities implies greater safety and lower costs of associated government programs. Education can also lead to the improvement in the quality of the democratic process since it allows those who have accumulated it to disseminate the information on the political process to all citizens of a country.

[4] For a complete discussion on public benefits of education see Gruber (2007) and all references cited in.

Moving to economic growth, education induces that growth in a number of ways. First, education improves individuals' productivity, which in turn sustains growth. The benefits from improved productivity can be private or social. Higher productivity from a worker implies a higher wage rate for that worker (private benefit), which increases his/her standard of living. High productivity from a worker can generate spillovers at 2 levels (social benefits). A more productive worker can increase coworkers' productivity, which raises their wages and standards of living. At the same time, increased wages generate more tax revenues. This implies high public saving, which is then allocated to financing investment projects for higher growth. Next, high quality education attracts more productive workers from other countries who can then establish in the host country after their studies (immigrants) to advance the growth process. Finally, public funding for education can overcome inefficiencies due to credit market failures (missing or inefficient credit markets for funding education) since it allows potential productive workers to get education, which they cannot get otherwise because of lack of collateral or low availability of credit. It also increases the amount of education when parents are unwilling to decrease their consumption to provide their children the optimal amount of education.

Next, the optimal size of government involvement in the education sector has to be determined by the extent of market failures as well as the return to public education. Determining the extent of market failures has proved difficult. Moreover, determining the public return to education requires a cost-benefit analysis. The costs of education are easy to assess. However, measuring benefits from education is very difficult due to the lack of appropriate procedures. These difficulties probably explain why most governments choose an arbitrary spending on education, which may or may not be optimal.

I.3. Model of Endogenous Growth with Human Capital and Policy

I.3.1 Setup

Consider a decentralized Ramsey model a la Lucas with an unbounded horizon and continuous time. The economy is closed and populated by many infinitely-lived, rational, and identical agents with homothetic preferences, many competitive firms with identical technology, and a government. A single consumption good $y(t)$ is produced in this economy in each period t from a technology that combines physical capital $(k(t))$ and effective labor $(u(t)h(t))$- a combination of raw labor $(u(t))$ with human capital $(h(t))$, where t is a time variable which takes on non-negative values, i.e., $t \geq 0$. Population is assumed constant over time.

A representative-agent derives her utility from consuming $c(t)$ units of the consumption good in each period (leisure does not enter the utility function). Assume that agent's preferences are characterized by a twice continuously-differentiable utility function $U(c(t))$ with $U' > 0$ and $U'' < 0$, for all $c(t) > 0$, and satisfies the Inada conditions, that is, $\lim_{c(t)\to 0} U'(c(t)) = \infty$ and $\lim_{c(t)\to \infty} U'(c(t)) = 0$, where U' and U'' are the first and the second derivatives of the utility function, respectively. The discounted sum of future utilities of the representative agent is given by:

$$\int_0^{\infty} U(c(t))e^{-\rho t}dt\,^{5}, \tag{1.1}$$

where c is measured in units of final output and ρ is the constant parameter of time preference. Assume that the representative agent supplies raw labor inelastically, where her supply of labor in each period is normalized to one. She has to decide on the fraction of labor to allocate to production (u) and the fraction to allocate to the accumulation of human capital $(1-u)$ in each period t. She receives a real wage rate w in exchange for supplying one unit of effective labor (uh) and a real rate of return on capital r for renting one unit of physical capital k in each period t, where w and r are measured in units of consumption good y[6]. She also receives a lump sum transfer T from the government in each period t. Raw labor u and human capital h are perfect substitutes. In each period t, she allocates her total income to spending on consumption c and to saving $\dot{k}$. Her budget constraint is given by

$$\dot{k} = (1-\tau_u)wuh + (1-\tau_k)rk - \delta_k k - c + \pi + T\,, \tag{1.2}$$

where π is profit earned by the agent who holds $1/N$ fraction of the firm's shares, N is the size of population, δ_k is the rate of depreciation of physical capital, τ_k is a flat rate of tax on capital income, τ_u is a flat rate of tax on labor income, and T represents the lump sum transfers. Human capital h accumulates according to

$$\dot{h} = \varphi(1-u)(h + f(g_h)) - \delta_h h, \tag{1.3}$$

where g_h is a stream of exogenously given government spending on education, $\varphi(.)$ is a

[5] From now on, I do not explicitly indicate the time dependence of variables if no ambiguity arises.
[6] Since the economy is closed, it is understood that physical capital is exactly equal to assets.

decreasing function in u, $f(.)$ is an increasing function in g_h, and δ_h is the rate of depreciation of human capital. We restrict the analysis to the educational services from government spending flows rather than a stock. Equation (1.3) implies that the rate of accumulation of h in each period t is a function of the time spent on the learning field $(1-u)$, the existing stock of h, and the public spending on education in that period. Lastly, the representative agent starts with some positive endowments of physical and human capital

$$k(0) = k_0,\ h(0) = h_0,\ k_0 \text{ and } h_0 \text{ are given,} \tag{1.4}$$

and all decision variables take only non-negative values:

$$c \geq 0,\ k \geq 0,\ h \geq 0,\ 0 \leq u \leq 1. \tag{1.5}$$

The supply side of the model consists of a representative competitive firm producing the consumption good y. The profit function of this firm in each period t is

$$\pi = y - rk^f - wz, \tag{1.6}$$

where k^f and z are the firm's demands of physical capital and effective labor, respectively, and y is its output produced according to

$$y = F\left(k^f, z\right)^7, \tag{1.7}$$

where F is a Constant Returns to Scale (CRS) technology in k^f and z. This function is assumed to be twice continuously differentiable in each argument with $F' > 0$, $F'' < 0$. It is also assumed to satisfy the Inada Conditions:

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[7] The production function in the Lucas' model has an external effect from human capital. I omit it here to ease the derivation of the Balanced Growth Path conditions.

$$\lim_{k^f\to\infty} F'(.,z)=0,\qquad \lim_{k^f\to 0} F'(.,z)=\infty,\qquad \lim_{z\to\infty} F'(k^f,.)=0,\qquad \text{and}\qquad \lim_{z\to 0} F'(k^f,.)=\infty.$$

Assume also that

$$y\geq 0,\ k^f\geq 0,\ z\geq 0. \tag{1.8}$$

A government intervenes in this economy through a fiscal policy, that is, it collects taxes on incomes, and uses the proceeds to make lump sum transfers to consumers (T) and to finance education (g_h). Its budget constraint (which by assumption must be balanced in each period) and the boundary conditions on the tax rates are, respectively:

$$g=g_h+T=\eta_h g+(1-\eta_h)g=\tau_u wuh+\tau_k rk,\qquad \forall t \tag{1.9}$$

$$0\leq\tau_u\leq 1,\ 0\leq\tau_k\leq 1, \tag{1.10}$$

where η_h is the constant budget shares of public spending on education. The assumption of a balanced budget for the government is intended to prevent it from running a deficit that it would finance by issuing debt (which it would pay by increasing the tax rates) or a surplus by accumulating wealth.

I.3.2 Equilibrium and its Characterization

A competitive equilibrium for this economy is a sequence of allocations of the representative agent $\{c(t),k(t),h(t),u(t)\}_{t=0}^{\infty}$, a sequence of allocations of the representative firm $\{y(t),k^f(t),z(t)\}_{t=0}^{\infty}$, a sequence of the rental rates of $\{r(t),w(t)\}_{t=0}^{\infty}$, and a sequence of policies $\{\tau_u,\tau_k,g_h,T\}_{t=0}^{\infty}$ such that:

i) Given $\{r(t),w(t)\}_{t=0}^{\infty}$ and $\{\tau_u,\tau_k,g_h,T\}_{t=0}^{\infty}$, $\{c(t),k(t),h(t),u(t)\}_{t=0}^{\infty}$ maximizes (1.1) subject to $(1.2)-(1.5)$, *ii*) The rental rates of physical capital and effective labor in each period t are given by:

$$r = \frac{\partial F\left(k^f, z\right)}{\partial k^f},\ w = \frac{\partial F\left(k^f, z\right)}{\partial z}, \tag{1.11}$$

iii) The government budget constraint (1.9) and the boundary conditions on the tax rates (1.10) hold in each period t,

iii) The following feasibility conditions hold in each period t:

$$c + g_h + \dot{k} = y\,, k^f = k\,,\ z = uh. \tag{1.12}$$

Assume that the utility function, the production function, and the functions $\varphi(.)$ and $f(.)$ take the following functional forms, respectively:

$$U(c) = c^{1-\sigma}/1-\sigma\,,\ \ F\left(k^f, z\right) = A\left(k^f\right)^{\alpha} z^{1-\alpha},\ \ \varphi(1-u) = \phi(1-u),\ f(g) = \xi g, \tag{1.13}$$

where σ is the inverse of the inter-temporal elasticity of substitution, A is the efficiency parameter, α is the output's share of physical capital, ϕ is the human capital technology parameter, and ξ is a constant parameter. The Current Value Hamiltonian is:

$$J = \frac{c^{1-\sigma}}{1-\sigma} + \lambda_k\left[(1-\tau_u)wuh + (1-\tau_k)rk - \delta_k k - c + (1-\eta_h)g\right] +$$

$$\lambda_h\left[\phi(1-u)(h + \xi\eta_h g) - \delta_h h\right]^{89},$$

where c and u are the control variables, k and h are the states variables, λ_k and λ_h are the co-state variables or the shadow prices of physical and human capital, respectively.

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[8] We omit profit in J since it is zero in each period due to CRS specification of the production function.

[9] J is derived from the Present Value Hamiltonian, that is, $J = J^P e^{\rho t}$, where J^P is given by

$$J^P = \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} + m_k \dot{k} + m_h \dot{h}$$

λ_k and λ_h are derived from the co-state variables m_k and m_h of k and h in J^P, that is, $\lambda_k = m_k e^{\rho t}$, $\lambda_h = m_h e^{\rho t}$. The first-order conditions from maximizing J are:

$$c^{-\sigma} - \lambda_k = 0, \qquad \forall t \tag{1.14}$$

$$\lambda_k w(1-\tau_u) - \lambda_h \phi(1 + \xi\eta_h (g/h)) = 0, \qquad \forall t \tag{1.15}$$

$$\dot{k} = (1-\tau_u)wuh + (1-\tau_k)rk - \delta_k k - c + (1-\eta_h)g, \tag{1.16}$$

$$\dot{h} = \phi(1-u)(h + \xi\eta_h g) - \delta_h h, \tag{1.17}$$

$$\dot{\lambda}_k = -\lambda_k [(1-\tau_k)r - \rho - \delta_k], \tag{1.18}$$

$$\dot{\lambda}_h = -\lambda_k (1-\tau_u)wu - \lambda_h [\phi(1-u) - \rho - \delta_h]. \tag{1.19}$$

The boundary conditions are the initial conditions (1.4) and the following transversality conditions (TVC):

$$\lim_{t\to\infty} \lambda_k(t) e^{-\rho t} k(t) = 0, \ \lim_{t\to\infty} \lambda_h(t) e^{-\rho t} h(t) = 0. \tag{1.20}$$

Taking the log and time-derivative of λ_k in (1.14) and substituting the resulting expression into (1.18) yields the following equation of motion of c:

$$\dot{c} = \sigma^{-1} c[(1-\tau_k)r - \rho - \delta_k]. \tag{1.21}$$

Manipulating $(1.14)-(1.19)$ we get the following equations of motion of u:[10]

$$\dot{u} = u\omega\left[\phi + \phi\xi\eta_h\left(\frac{g}{h}\right)u - (1-\tau_k)\alpha A k^{\alpha-1}(uh)^{1-\alpha} + \alpha\left(1 - \xi\eta_h\left(\frac{g}{h}\right)\right)\left(\frac{\dot{k}}{k} - \frac{\dot{h}}{h}\right)\right], \tag{1.22}$$

where $\omega = [\alpha + (1-\alpha)\xi\eta_h (g/h)]^{-1}$.

[10] See Appendix A for the derivation of the equations of motion of u.

Given the initial conditions (1.4), differential equations $(1.16),(1.17),(1.21),(1.22)$, and the *TVC* (1.20) form the dynamic system describing the evolution of this economy over time.[11]

I.3.3 Steady State

Let construct from equations $(1.16),(1.17),(1.21)$, and (1.22) the system' balanced growth paths (BGP), the system to which the equilibrium paths will converge. In the BGP, all variables grow at constant (possibly zero) rates. Let define these rates by:

$$\gamma_c = \frac{\dot{c}}{c},\ \gamma_k = \frac{\dot{k}}{k},\ \gamma_h = \frac{\dot{h}}{h},\ \gamma_g = \frac{\dot{g}}{g}, \gamma_w = \frac{\dot{w}}{w},\ \ \gamma_r = \frac{\dot{r}}{r},\ \gamma_u = \frac{\dot{u}}{u}. \tag{1.23}$$

From equation (1.21) and using (1.11)[12] and (1.23), we derive the following marginal product of k:

$$(1-\tau_k)\alpha A k^{\alpha-1}(uh)^{1-\alpha} = \rho + \sigma\gamma_c + \delta_k, \tag{1.24}$$

which is constant along the *BPG*. Substitute (1.11) for r and w into (1.16), divide by k, and use (1.23) for $\dot{k}/k$ to obtain

$$\frac{c}{k} = (1-\eta_h[\tau_u(1-\alpha)+\tau_k\alpha])\left(\frac{\rho+\sigma\gamma_c}{\alpha(1-\tau_k)}\right) - \gamma_k - \delta_k, \tag{1.25}$$

which is constant along the *BPG*. Taking the log and time-derivative of (1.25) yields $\dot{c}/c - \dot{k}/k = 0$, that is, c and k grow at the same rate $(\gamma_c = \gamma_k = \gamma)$. Moreover, divide the feasibility condition (1.12) by k to get:

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[11] See Mulligan and Sala-i-Martin (1992), Caballe and Santos (1993), Ortigueira (1998), Boucekkine and Tamarit (2004), Barro and Sala-i-Martin (1995, 2004), and Boucekkine et al (2007), among others for the description of the transition behavior of this system .

[12] The rental rates r and w are given by

$$r = \alpha k^{\alpha-1}(uh)^{1-\alpha},\ w = (1-\alpha)k^{\alpha}(uh)^{-\alpha}.$$

$$\frac{\dot{k} + c + g_h}{k} = Ak^{\alpha-1}(uh)^{1-\alpha}. \qquad (1.26)$$

Multiply the LHS of (1.26) by $\dot{k}/\dot{k}$, use (1.23) for $\dot{k}/k$, take the inverse the resulting expression, and rearrange to obtain the savings rate (s):

$$s = \frac{\alpha(1-\tau_k)\gamma}{\rho + \sigma\gamma + \delta_k}. \qquad (1.27)$$

Divide the government budget constraint (1.9) by k and substitute (1.11) for r and w into the result to get

$$\frac{g}{k} = [\tau_u(1-\alpha) + \tau_k\alpha]\left(\frac{\rho + \sigma\gamma + \delta_k}{\alpha(1-\tau_k)}\right). \qquad (1.28)$$

Differentiating (1.28) we get $\dot{g}/g - \dot{k}/k = 0,$ that is, g and k grow at the same rate $(\gamma_g = \gamma_k = \gamma)$. Furthermore, taking the log and time-derivative of (1.24)[13] yields $\dot{h}/h - \dot{k}/k = 0,$ that is, h and k grow at the same rate $(\gamma_h = \gamma_k = \gamma)$. Next, multiply both sides of (1.28) by k/h to obtain:

$$\frac{g}{h} = [\tau_u(1-\alpha) + \tau_k\alpha]\left(\frac{\rho + \sigma\gamma + \delta_k}{\alpha(1-\tau_k)}\right)\frac{k}{h}. \qquad (1.29)$$

Recovering k/h from (1.24)[14] and substituting its expression into (1.29) yields

[13] To get this result we have assumed that $\dot{u}/u = \gamma_u = 0$. We will show shortly that this is indeed the case.

[14] $\dot{k}/h = [(\rho + \sigma\gamma)/\alpha A(1-\tau_k)]^{1/(\alpha-1)}\, u$

$$\frac{g}{h} = \psi u, \tag{1.30}$$

where $\psi = A^{\frac{1}{1-\alpha}}\left[\tau_u(1-\alpha)+\tau_k\alpha\right]\left(\frac{\rho+\sigma\gamma}{\alpha(1-\tau_k)}\right)^{\frac{\alpha}{\alpha-1}}$. Taking the log and time-derivative of (1.14) and of (1.15) yields

$$\dot{\lambda}_k/\lambda_k = \dot{\lambda}_h/\lambda_h = -\sigma\gamma. \tag{1.31}$$

Divide (1.19) by λ_h, substitute (1.15) into the resulting expression, and rearrange to get

$$\dot{\lambda}_k/\lambda_k = \rho + \delta_h - \phi - \phi\xi\eta_h(g/h)u. \tag{1.32}$$

Set (1.31) and (1.32) equal to obtain

$$\left(\frac{g}{h}\right)u = \frac{\rho + \delta_h - \phi + \sigma\gamma}{\phi\xi\eta_h}. \tag{1.33}$$

Divide (1.17) by h, use (1.23), and rearrange the resulting expression to get

$$u - \xi\eta_h\left(\frac{g}{h}\right) + \xi\eta_h\left(\frac{g}{h}\right)u = 1 - \left(\frac{\gamma - \delta_h}{\phi}\right). \tag{1.34}$$

Substitute (1.30) and (1.33) into (1.34) and rearrange to obtain

$$u = (1 - \xi\eta_h\psi_1)^{-1}\left[1 - \left(\frac{\gamma(1+\sigma) + \rho + \delta_h - \phi}{\phi}\right)\right]. \tag{1.35}$$

Taking the log and time-derivative of (1.35) yields $\dot{u}/u = \gamma_u = 0$. Furthermore, it is obvious from (1.11) that the rates of growth of r and w are zeros in the BGP $(\gamma_r = \gamma_w = 0)$. However, the growth rate of w augmented for skill growth is $\gamma_w^* = \gamma_w + \gamma_h = \gamma$. This is also the rate of growth GDP as can be verified by differentiating the expression of GDP in (1.13). From all of the above, it is obvious that in the BGP; $c, k, h, g,$ and w augmented for skill growth grow at the same constant rate $(\gamma_c = \gamma_k = \gamma_h = \gamma_g = \gamma_w^* = \gamma)$; and u, r, w are constant $(\gamma_u = \gamma_r = \gamma_w = 0)$. The common rate of growth γ is recovered from (1.27). It is given by

$$\gamma = \frac{s(\rho + \delta_k)}{\alpha(1-\tau_k) - s\sigma}. \tag{1.36}$$

We use the above common rate of growth γ to normalize variables as follows:

$$\hat{c} = ce^{-\gamma t},\ \hat{k} = ke^{-\gamma t},\ \hat{h} = he^{-\gamma t},\ \hat{g} = ge^{-\gamma t}.$$

The dynamic system of the normalized variables is formed by the following:

$$\dot{\hat{k}} = [1 - \eta_h(\tau_u(1-\alpha) + \tau_k\alpha)]A(\hat{k})^\alpha (u\hat{h})^{1-\alpha} - \delta_k\hat{k} - \gamma\hat{k} - \hat{c},$$

$$\dot{\hat{h}} = \phi(1-u)(\hat{h} + \xi\eta_h\hat{g}) - (\gamma + \delta_h)\hat{h},$$

$$\dot{\hat{c}} = \sigma^{-1}\hat{c}\left[(1-\tau_k)\alpha A(\hat{k})^{\alpha-1}(u\hat{h})^{1-\alpha} - \rho - \delta_k - \gamma\sigma\right],$$

$$\dot{u} = u\omega\left[\phi + \phi\xi\eta_h\left(\frac{\hat{g}}{\hat{h}}\right)u - (1-\tau_k)\alpha A\hat{k}^{\alpha-1}(u\hat{h})^{1-\alpha} + \alpha\left(1 - \xi\eta_h\left(\frac{\hat{g}}{\hat{h}}\right)\right)\omega_1\right],$$

and the TVC as given in (1.20), (1.37)

where $\omega_1 = \left\{(1 - \eta_h(\alpha\tau_k + (1-\alpha)(1-\tau_u)))A\left(\frac{\hat{k}}{u\hat{h}}\right)^{\alpha-1} - \frac{\hat{c}}{\hat{k}} - \delta_k + \delta_h - \phi(1-u)\left(1 + \xi\eta_h\left(\frac{\hat{g}}{\hat{h}}\right)\right)\right\}$.

The TVC in (1.37) imply that the vector $x(t)$ approaches its steady state, that is:

$$\lim_{t\to\infty} x(t) = x_{ss} < \infty \leftrightarrow \lim_{t\to\infty}(\dot{x}(t)/x(t)) = 0,$$

where $x(t) = (\hat{c}, \hat{k}, \hat{h}, u)$. The steady state conditions are obtained by dividing the four equations in (1.37) by $\hat{c}$, $\hat{k}$, $\hat{h}$, and u, respectively, and by manipulating the resulting expressions. They are

$$u_{ss} = (1-\xi\eta_h\psi)^{-1}\left[1-\left(\frac{\gamma(1+\sigma)+\rho+\delta_h-\phi}{\phi}\right)\right],$$

$$\left(\frac{\hat{c}}{\hat{k}}\right)_{ss} = [1-\eta_h(\tau_u(1-\alpha)+\tau_k\alpha)]\left(\frac{\rho+\sigma\gamma_c}{\alpha(1-\tau_k)}\right)-\gamma_k-\delta_k,$$

$$\left(\frac{\hat{k}}{\hat{h}}\right)_{ss} = \left[\frac{(\rho+\delta_k+\gamma\sigma)}{\alpha A(1-\tau_k)}\right]^{1/(\alpha-1)} u_{ss},$$

$$\psi = A^{\frac{1}{1-\alpha}}[\tau_u(1-\alpha)+\tau_k\alpha]\left(\frac{\rho+\sigma\gamma}{\alpha(1-\tau_k)}\right)^{\frac{\alpha}{\alpha-1}},$$

$$\gamma = \frac{s(\rho+\delta_k)}{\alpha(1-\tau_k)-s\sigma}. \qquad (1.38)$$

The steady state values of the static equations are:

$$r_{ss} = (\rho+\delta_k+\gamma\sigma)/(1-\tau_k),$$

$$w_{ss} = (1-\alpha)A[(\rho+\gamma\sigma)/\alpha A(1-\tau_k)]^{\alpha/(\alpha-1)},$$

$$g_{ss} = [\tau_u(1-\alpha)+\tau_k\alpha][(\rho+\gamma\sigma)/\alpha(1-\tau_k)]. \qquad (1.39)$$

I.4. Numerical Solution

The dynamic system described in (1.37) does not admit a closed-form solution. Therefore we resort to the numerical solution. We use the relaxation algorithm[15] to solve numerically this dynamic system given the boundary conditions (initial and terminal conditions). The initial conditions include those on the state variables (k,h) and some

[15] See Timborn, Koch, and Steger (2004) for the description and the implementation of the algorithm.

Table 1: Parameter Estimates[16]

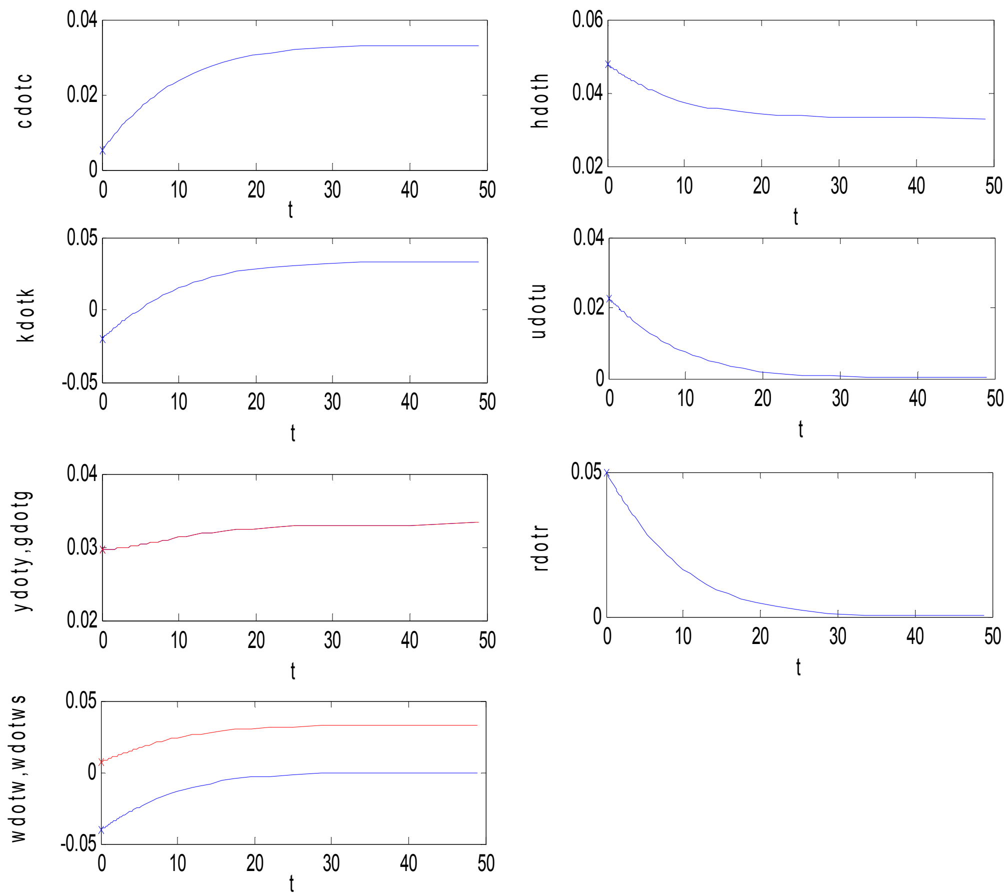
Parameter	Estimated Value
Efficiency Parameter (A)	1.000
Output Share of Physical Capital (α)	0.451
Initial Physical Capital (k_0)	11.600
Initial Human Capital (h_0)	6.030
Human Capital Technology Parameter (ϕ)	0.116
Government Spending Parameter (ξ)	0.034
Tax Rate $(\tau_k = \tau_u)$	0.250
Budget Share of Public Spending on Education (η_h)	0.270
Parameter of Time Preference(ρ)	0.068
Savings Rate (s)	0.100
Parameter of Substitution (σ)	1.460
Rate of Depreciation of Physical Capital (δ_k)	0.040
Rate of Depreciation of Physical Capital (δ_h)	0.000[17]

arbitrary guess on the control variables(c,u). The terminal conditions are the steady state conditions on the state and control variables. The relaxation algorithm transforms an infinite time variable into a time scale to facilitate the solution to the problem. It tries an arbitrary solution to both the state and control variables, assesses the deviation of the arbitrary solution to the true path by a multi-dimensional error function, and then uses the derivative of this function to boost the guess in an iteration of a Newton procedure type. At each point of the path, the adjustment is related to the incorrectness in slope and in the

[16] The estimation procedure is explained in Appendix B.
[17] We use the rate of depreciation of human capital of zero because of the difficulty to obtain an estimated value for this parameter. Also, any positive value we tries for this parameter causes the rate of growth of human capital to become negative.

static equations' solution. The algorithm keeps adjusting the trial until it reaches an optimal solution, that is, the one for which the error becomes sufficiently small.

Figure 1: Time-Paths of Rates of Growth of c, h, k, u, y, r, w, and w* (1995-2040)



I use the relaxation algorithm as well as the estimated values of parameters summarize summarized in Table 1 to solve numerically the dynamic system described in (1.37). The numerical solution to the model[18] is given by the time paths of the rates of growth of variables as depicted in Figure 1. Following the initial period as it is noticeable from this figure, the solution paths monotonically approach the steady state. In fact, the

[18] We solve the dynamic system of the normalized variables (1.37), and uses the solution to recover the dynamic system of the levels of variables, that is, the one formed by (1.16), (1.17), (1.21), and (1.22).

rates of growth of c, h, k, y, g, r, u, w and w^* converge to their steady state values, which are reached after 46 years or in 2040. In the BGP, c, h, k, y, g and w^* grow at the same constant rate of 3.47%; while u, r and w do not grow. Furthermore, in the steady state, a representative agent spends 67.70% of his/her time working and 32.30% accumulating human capital. Also, the interest and wage rates in the steady state are 15.84% and 1.297, respectively.

In the transition, on the other hand, c, k, y, g, w and w^* grow at increasing rates, while h, u, and r grow at decreasing rates.

The behaviors of the rates of growth of variables are consistent with the predictions of the Lucas model. In fact, in the long run, c, h, k, y, g and w^* grow at the same constant rate; while u, r and w do not grow. The transition behaviors of the rates of growth, on the other hand, are determined by the relative size of the parameter of substitution and the elasticity of physical capital as well as by the relationship between the initial and steady state ratios of physical to human capital. If the parameter of substitution is greater than the elasticity of physical capital $(\sigma > \alpha)$, an economy starting with higher (lower) physical-human capital ratio than that of the long run equilibrium will observe higher (lower) transition rates of growth of human capital than that of the long run equilibrium, and lower (higher) transition rates of growth of physical capital than that of the long run equilibrium. The transition for the case where the parameter of substitution is less than the elasticity of physical capital $(\sigma < \alpha)$, is obtained by applying symmetrically the above results.

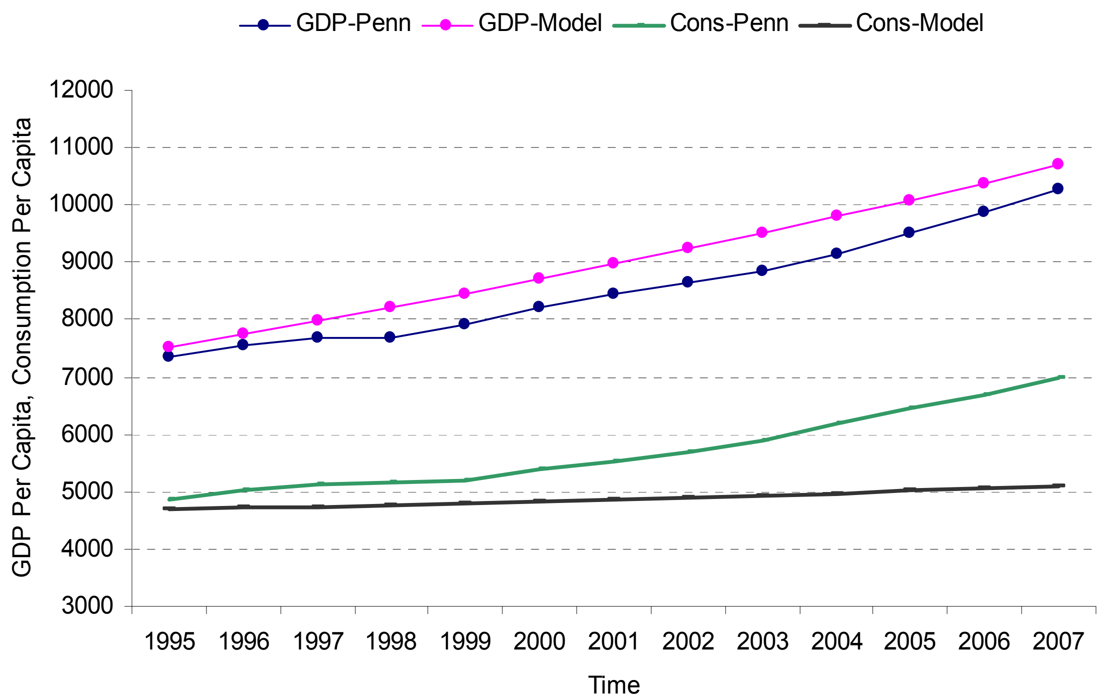
As indicated in table 1, the parameter of substitution is greater than the elasticity of physical capital. Further, the initial ratio of physical to human capital (not shown in table 1) is greater than its corresponding long run equilibrium ratio. This indicates that there exist some imbalances between physical and human capital, that is, human capital falls short of its transition equilibrium levels. Moreover, a low steady state ratio of physical-human capital ratio relative to its corresponding initial ratio suggests that human capital is increasing faster relative to physical capital in transition. This last result combined with the steady state behavior of the common rate of growth of both human and physical capital implies that the transition rate of growth of human capital will be above the common rate of growth, while the transition rate of growth of physical capital will be below the common rate of growth. This implies that the transition rate of growth of human capital is decreasing, whereas the transition rate of growth of physical capital is to increasing.

The other rates of growth in transition are determined by those of human and physical capital. Starting with the rental rates, a close look at their formulas (see footnote 10) shows that their rates of growth in transition (not showed but can be easily derived from their formulas) will depend on those of physical and human capital. In fact, the rate of growth of the interest rate is positively related to the former and negatively related to latter. The reverse is true for the wage rate. Since the rate of growth of human capital dominates that of physical capital during the transition, its follows that the rate of growth of the interest rate is decreasing, while those of wage and wage augmented for skill growth are increasing. Next, the rate of growth of labor is hard to predict since it depends not only on the difference between the rate of growth of physical capital and that of human capital $\left(\dot{k}/k - \dot{h}/h\right)$ but also on the sign of $\left(1 - \xi\eta_h\left(g/h\right)\right)$. If $\left(1 - \xi\eta_h\left(g/h\right)\right) < 0$, this rate is decreasing but positive since $\dot{h}/h$ dominates $\dot{k}/k$ and it is decreasing. But if $\left(1 - \xi\eta_h\left(g/h\right)\right) > 0$, this rate is negative but increasing since $\left(\dot{k}/k - \dot{h}/h\right)$ is negative. Our results are consistent with the case where $\left(1 - \xi\eta_h\left(g/h\right)\right) < 0$, that is, the rate of growth of labor is positive and increasing. Finally, the transition rate of growth of the output is increasing because it is positively related to both the transition rates of growth of physical and human capital.

I.5. Comparison of the model's predictions to the data

In this section, I compare the model's solution to data to see whether or not the model is capable of describing well the growth process of South Africa's economy from 1995 to 2007. I limit this comparison to variables whose data exist on the per year basis. Two variables meet this requirement, namely, the Real Consumption per capita and Real GDP per capita. The data I use for this purpose are those on the levels of the two variables from Penn World Tables 6.1. Each of the two variables is measured in the 2000 constant US \$. Starting with the Real GDP per capita, we can see from Figure 2 that the levels of the GDP per capita generated from the model are close to data on this variable over the period 1995-2007. As it is obvious from this figure, the two series are almost the same in the first three years of the process (1995-1997), period after which they start to differ slightly. However, after the 9^{th} year (2003); the differences between the two series seem to start vanishing, indicating thus that they are converging to the same values after the year 2007. On the other hand, the model series on the Real Consumption per capita seems to mimic data on consumption but not very closely. From Figure 2, it is clear that

Figure 2: Real Per capita GDP and Real Per Capita Consumption (1995-2007)



the consumption series from data grows faster than the consumption series from the model. Further, the gap between the two series is small in the first 5 years of the process (1995-1999) but is widening substantially thereafter. This widening gap between the two series indicates that either the predictive performance of the model with respect to the levels of consumption after the year 1999 is not very good or the time series data on consumption are measured with increasing error. If these patterns in the consumption data persist, then the two series will be totally different in years following 2007. It is noteworthy mentioning that time series data on consumption have been criticized for their high level of inaccuracy. Indeed, consumption data are not observed but calculated as residuals, that is, the difference between GDP and all of its other components. This implies that consumption may carry on all possible measurement errors which may probably be present in each of the components of the GDP.

I now supplement the above graphical analysis with a statistical analysis by using the Theil Inequality Coefficient (U). This statistics measures the predictive performance of a model and is bounded below by zero and above by one. Its expression is as follows:

$$U = \frac{\sqrt{(1/T)\sum_{t=1}^{T}\left(Y_t^M - Y_t^D\right)^2}}{\sqrt{(1/T)\sum_{t=1}^{T}\left(Y_t^M\right)^2} + \sqrt{(1/T)\sum_{t=1}^{T}\left(Y_t^D\right)^2}} \in [0,1],$$

where Y_t^M and Y_t^D are the values of the variable from the model and the data, respectively, and T is the sample size. A value of U of zero indicates a perfect fit $\left(Y_t^M = Y_t^D\right)$ and a value of U of one indicates a bad predictive performance of the model. Applying the above formula to the Real GDP per capita and the Real Consumption per capita's series generated by the model and those from the data over 1995-2007; we obtain values of U of 0.0012 and 0.0272 for the former and the latter, respectively. The value of U for the Real GDP per capita is very close to zero, and thus reflects a very good predictive performance of the model with respect to the levels of this variable. This value in the case of the Real Consumption per capita is 23 times that of the Real GDP per capita but it is still close to zero. This indicates that the Real Consumption per capita from the model tracks the data very well over 1995-2007. However, the patterns in data reveal that the value of U will increase as data on the Real Consumption per capita beyond 2007 become available and are included in the calculation of the U-statistics..

I.6. Simulation of Fiscal Policy Experiments

As mentioned above, South Africa has allocated on average 7% of its GDP or 21% of its national budget to spending on education since 1995. This share is 1.3 times the average of industrialized countries (5.4%) and almost twice that of developing countries (3.9%). Does allocating more resources to education accelerate economic growth and improve welfare? To answer this question, I simulate three different fiscal policy experiments. Further, I compare the results of these experiments to the baseline case to analyze the growth and welfare effects of public spending on education. The baseline case is the one associated with the parameters used to solve the basic model (see section *I.3.*) but with some modifications introduced to isolate the impact of change in spending on education from the composite impact of change in spending on education and change in transfers. In other words, I set transfers equal to zero so that the tax revenue is used exclusively to finance educational spending. This implies that the tax rate is set to equal the constant GDP share of spending on education (7%). This tax rate of 7% of GDP is far lower than the one that prevails when tax revenue is collected to finance both educational spending and transfers, that is, 25% of GDP. The change resulting from the elimination of transfers implies an increase in the constant budget share of public spending on education to one. With respect to the basic solution, these modifications imply that in the baseline case, all parameters maintain their values these modifications imply that in the baseline case, all parameters maintain their values except for the tax rate $(\tau_k = \tau_u = 7\%)$, the budget share of spending on education $(\eta_h = 1)$, and the budget share of transfers $(\eta_T = 0)$.

The experiments consist of maintaining all the parameters to their baseline values and of changing the tax rate in each case. Specifically, we consider the cases of reducing public spending on education to the average share of industrialized countries (5.4% of

Table 2: BGP's Results of Simulation of Fiscal Policy Experiments

Variables	Baseline	Policy-Exp 1	Policy-Exp 2	Policy-Exp 3
	Common Rates of Growth (in %)			
y, k, h, c, g, w^*	3.390	3.300	3.220	3.040
u, r, w	0.000	0.000	0.000	0.000
	Levels of Un-Normalized Variables			
u	0.621	0.642	0.660	0.703
r	0.132	0.128	0.125	0.117
w	1.508	1.543	1.577	1.664

Table 3: Transition Dynamics of Simulation of Policy Experiments

Variables	Baseline	Policy-Exp 1	Policy-Exp 2	Policy-Exp 3
	Median Rates of Growth (in %)			
y	3.257	3.245	3.211	3.245
h	4.227	4.281	4.023	4.501
k	0.195	0.056	0.631	-0.675
u	1.488	1.575	1.478	1.939
c	1.623	1.517	1.799	1.088
r	3.006	3.184	2.670	3.906
w	-2.469	-2.616	-2.193	-3.208
w^*	1.758	1.665	1.829	1.026

GDP or a decrease in the tax rate of 22.86%) for the first experiment (Experiment 1), to the average share of developing countries (3.9% of GDP or a decrease in the tax rate of 44.29%) for the second experiment (Experiment 2), and to the case of eliminating government from the model (no public spending on education or a 100% tax cut) for the last experiment (Experiment 3).

The solutions to the model under the 3 experiments (not shown) are qualitatively similar to that of the baseline case. In fact, the economy under each experiment converges after 46 years. Also, in the BGP, c, h, k, y, g and w^* grow at the same constant rate; while u, r and w do not grow in each of the three experiments. Plus, the transition rates of growth of c, k, y, w and w^* are increasing, while those of $u, r,$ and h are decreasing. The results of the baseline and of the three experiments are displayed in Tables 2 and 3.

Starting with the BGP results, we can see from Table 2 that a decrease in the GDP share of public spending on education from 7% to 5.4% (Experiment 1) or to 3.9% (Experiment 2) or to 0% (Experiment 3) does not affect the BGP common rate of growth of the rental rates and of the supply of labor. However, it does affect the common rate of growth of $y, k, h, c, g,$ and w^* as well as the levels of $r, w,$ and u in the three experiments. Also, Table 2 indicates that spending on education is positively related to the common rate of growth of $y, k, h, c, g,$ and w^* and to the interest rate, but negatively related to the wage rate and the supply of labor.

Turning to the effects in transition, Table 3 indicates that spending on education is positively related to the median rate of growth of GDP. However, the patterns of the relationship between spending on education and the median rates of growth of other variables are hard to predict. For instance, a decrease in spending on education causes the median rates of growth of human capital, the supply of labor, and the interest rate to increase in Experiments 1 and 3, but to decrease in Experiment 2. By contrast, a decrease in spending on education causes the median rates of growth of physical capital, consumption, wage, and wage augmented for skill growth to decrease in Experiments 1 and 3, but to increase in Experiment 2.

I.7. Growth and Welfare Effects of Fiscal Policy Experiments

In this section I use the results of simulations of fiscal policy experiments summarized in tables 2 and 3 to analyze the long run and transition growth effects and the welfare effects in each of the three experiments.

First, a decrease in spending on education affects the allocation decisions in the long run. A decrease in spending on education of 22.9% (Experiment 1), of 44.3% (Experiment 2), and of 100% (Experiment 3) translates into a decrease in the common growth rate (of physical capital, human capital, consumption, GDP, and wage augmented for skill growth), and in the interest rate in the long run. It also translates into an increase in the labor supply and the wage rate. In Experiment 1, the growth effects of a decrease of 22.9% in spending on education are -2.7%, 3.4%, -2.8%, and 2.4% for the common rate of growth, the supply of labor, the interest and wage rates, respectively. In Experiment 2, the growth effects of a decrease of 44.3% in spending on education on are -5.0%, 6.2%, -5.4%, and 4.6% for the common rate of growth, the supply of labor, the interest and wage rates, respectively. In Experiment 3, the growth effects of a 100% decrease in spending on education are -10.3, 13.2%, -11.3%, and 10.3% for the common rate of growth, the supply of labor, the interest and wage rates, respectively.

A close inspection of the growth effects in each of the three experiments reveals two important characteristics of their solutions. First, in each of these experiments, the effects

Table 4: BGP's Growth Effects of Fiscal Policy Experiments

Variables	Baseline	Policy-Exp 1	Policy-Exp2	Policy-Exp 3
	Growth Rate	**Growth Effects on Rates of Growth (in %)**		
y, k, h, c, g, w^*	3.390	-2.655	-5.015	-10.320
u, r, w	0.000	0.000	0.000	0.000
	Levels	**Growth Effect on Un-normalized variables**		
u	0.621	3.368	6.181	13.150
r	0.132	-2.805	-5.383	-11.296
w	1.508	2.375	4.600	10.341

Table 5: Transition Growth Effects of Fiscal Policy Experiments[19]

Variables	Baseline	Policy-Exp 1	Policy-Exp2	Policy-Exp 3
	Growth Rate	**Growth Effects on Cumulative Growth Rates (%)**		
y	1.499	-0.431	-3.960	-0.287
h	1.906	1.177	-3.415	5.798
k	0.222	1.280	-9.337	8.794
u	0.619	5.705	-5.239	29.154
c	0.823	-5.679	2.476	-26.193
r	1.247	6.253	-3.719	28.792
w	-1.025	6.253	-3.372	28.799
w^*	0.881	-4.474	-3.061	-20.955

[19] To assess growth effects in transition, we use the cumulative rate of growth. We could have used either the average rate of growth or the median rate of growth. The former is biased upward for all rates of growth except for human capital, which is biased downward. The latter is relatively more accurate compared to the former. However, it does not take into account the patterns of each variable in the earlier and late periods of the process.

the effects are small. Second, the magnitudes of these effects are related to the size of change in spending on education. Indeed, relatively large effects result from a large decrease in spending on education (Experiment 3), and relatively small effects result from a small decrease in this spending (Experiment 1). Between these two, relatively moderate effects result from a moderate decrease in this spending (Experiment 2).

The intuition behind these results follows from the productive nature of spending on education. Spending on education is financed exclusively through the collection of tax on income. As we know, an income tax rate creates distortions on the households' allocation decisions in the long run. At the same time, the public good it finances (education) generates long run externalities on all other variables in the economy. A tax rate generates externalities that create incentives for agents to choose high common rate of growth of physical capital, human capital, consumption, GDP, and wage augmented for skill growth. Thus, decreasing spending on education or decreasing the tax rate causes a decrease in incentives and thus induces agents to choose a lower common rate of growth of the mentioned variables. Also, a government involvement in the educational sector sends a strong signal to agents regarding the improvement in the quality of education. The converse is also true. The results in Table 4 (negative effects on the supply of labor) indicate that a reduction in spending on education in the three experiments may be interpreted as a decrease in the quality of education to which agents respond by reducing time they spend at school and by increasing their work effort in the long run. The increase in the work effort is also sustained by more attractive wage rates for unskilled workers. A decrease in the tax rate results in a wage differential between the baseline and each of the three experiments. It also makes working more attractive than learning. So agents respond to the high wage with high work effort. Further, a decrease in the tax rate creates incentives for the accumulation of physical capital in the long run which in turn increases the supply of physical capital, causing thus the interest rate to fall.

Turning now to the transition, it can be seen from Table 5 that a reduction in spending on education to the average of industrialized or the average of developing countries (Experiments 1 and 2) or its elimination (Experiment 3) does affect the cumulative rates of growth of all variables. In fact, the cumulative rate of growth of GDP decreases by 0.4%, 4.0%, and by 0.3% in Experiments 1, 2, and 3, respectively. As these results show, the effect is relatively larger when spending on education is reduced to the average of developing countries (Experiment 2). This may imply that spending is too small to generate enough externalities that would outweigh the tax distortions.

The effects on the cumulative rates of growth of GDP are the results of the combined effects of spending on education on human capital, on physical capital, and on the supply of labor in each of the three experiments.

Starting with Experiment 2, Table 5 shows that a reduction in spending on education to the average level among developing countries causes the cumulative rate of growth of the supply of labor to decrease by 5.2% in this experiment with respect to the baseline case. Indeed, agents perceive a decrease in spending on education as a signal of a decrease in the quality of education to which they would respond by increasing their work effort. However, they also look at levels of the wage rates of skilled and unskilled workers before they decide. For the present experiment, the wage rates of unskilled and skilled workers grow at a lower cumulative rate compared to the baseline case, that is, it is lower by 3.4% and 3.1% for the former and the latter, respectively. However, the wage rate of skilled workers grows at relatively higher rate than that of unskilled workers in this experiment. This implies that working is less attractive than building skills. Thus, agents react to a decrease in spending on education by spending more time at school in order to compensate for the decrease in the quality of education. Also, a decrease in the working time causes the accumulation of human capital to increase. But this negative effect of a decrease in the supply of labor on human capital is out-weighed by a positive direct effect of a decrease in spending on education so that human capital grows more slowly, decreasing its cumulative rate of growth by 3.4% with respect to the baseline case.

Furthermore, a reduction in spending on education causes the interest rate to grow at a cumulative lower rate with respect to the baseline or an effect of -3.7%. As a result, agents choose to increase their consumption and slow their accumulation of physical capital in this experiment compared to the baseline. The low cumulative rate of growth of the interest rate induces a positive effect on the cumulative rate of growth of consumption of 2.5% but a negative effect on that of physical capital of -9.3%. Moreover, the combination of negative effects of spending on education on the supply of labor , on the accumulation of physical, and on the accumulation of human capital translate into a decrease in the cumulative rate of growth of GDP or an effect of -4.0%.

Moving now on to Experiments 1 and 3, we can see from Table 5 that the effects of a reduction of educational spending to the average of industrialized countries or an elimination of this spending on the supply of labor, human capital, physical capital, and consumption in these two experiments are exactly the opposite of the effects on the corresponding variables in Experiment 2. As mentioned before, a reduction or an elimination of government involvement in the educational sector sends a signal of a decrease in the quality of education. However, agents' reaction will depend on the wage rates. From Table 5, it is obvious that wages of unskilled workers grow faster in these two experiments than they do in the baseline or an effect of reduction in or an elimination of spending on education on the cumulative rate of growth of wage of unskilled workers of 6.3% and 28.80% in Experiments 1 and 3, respectively. At the same time, wages of skilled workers grow slowly in the two experiments compared to the baseline. The effect of a decrease in or an elimination of spending on education on the cumulative rate of growth of the wage of skilled workers is -4.5% in Experiments 1 and -21.0% in Experiment 3, respectively. This implies that working is more attractive than learning in these two experiments respective to the baseline. This is also obvious from the negative effect of a reduction in or an elimination of spending on education on the cumulative rate of growth of the supply of labor, that is, 5.7% in Experiment 1 and 29.2% in Experiment 3. Moreover, a reduction in or an elimination of spending on education causes the interest rate to grow faster in these experiments or a negative effect of 6.3% and 28.8% in Experiments 1 and 3, respectively. Fast growth of the interest rate implies that savings is more attractive than consumption. Consequently, the effect of a reduction in or an elimination of spending on education on the cumulative rate of growth of physical capital is negative in the two experiments (1.3% in Experiment 1 and 8.8% in Experiment 3) but positive on that of consumption (-5.7% in Experiment 1 and -26.2% in Experiment 3). Turning finally to human capital, table 5 indicates that a reduction in or an elimination of spending on education has a negative effect on the cumulative rate of growth of human capital (1.2% in Experiment 1 and of 5.8% in Experiment 3). These effects, however, are not in line with the predictions of the model. Indeed, the cumulative rate of growth of labor supply is higher and the size of spending on education is lower in each of the two experiments compared to the baseline. A fast increase in the supply of labor implies that agents are spending less time on the learning field, and a small size of spending on education means less externalities generated in each of these two experiments compared to the baseline. Consequently, human capital is expected to accumulate faster in the baseline case.

Table 6: Welfare Effects of Public Policies (in %)

	Baseline	**Policy-Exp 1**	**Policy-Exp 2**	**Policy-Exp 3**
Welfare Value	-0.5723116	-0.5770905	-0.5736339	-0.5925917
Welfare Effect	-	-0.835	-0.231	-3.544

Moving on to the welfare effects of spending on education, we can see from Table 6 that each of the three experiments is associated with a negative welfare effect, but this effect is very small. The welfare decreases by 0.8% in Experiment 1, by 0.2% in Experiment 2, and by 3.5% in Experiment 3. It is worth mentioning that the drop in the welfare in Experiment 2 is unexpected since consumption grows at a higher cumulative rate in this experiment respective to the baseline (see Table 5). However, a close examination of the time paths of consumption in the baseline case and in Experiment 2 reveals that it starts and is maintained at low levels in the earlier periods of the process in Experiment 2 relative to the baseline, and its patterns are reversed only in the late period of the process. Thus, large consumption in the late period of the process in Experiment 2 contributes less to the welfare so that in overall, the welfare effect of public spending on education is positive in Experiment 2.

I.8. Conclusion

In this study, I have formulated a model of endogenous growth with human capital accumulation and policies to analyze the growth and welfare effects of public spending on education in South Africa. Since the abolition of its Apartheid regime in 1994, South Africa has launched a massive program of education financed exclusively through fiscal resources representing on average 7% of its GDP or 21% of its national budget. This share is 1.3 times the average of industrialized countries (5.4%) and almost twice that of developing countries (3.9%).

To analyze the growth and welfare effects mentioned above, I have simulated three fiscal policy experiments and then compared their solutions to that of the baseline case. The first and second experiments have consisted of reducing the GDP share of spending on education to the averages of industrialized and developing countries, respectively, while I have eliminated government in the model. in the third experiment.

The numerical solutions to the model have produced results consistent with the predictions the Lucas' model as concerns the baseline case as well as concerns the simulated experiments. In all instances, the solutions are qualitatively similar. In fact, the economy converges after 46 years from 1995 or in 2040. In the long run or BGP, physical capital, human capital, consumption, government expenditures, wage augmented for skill growth, and GDP grow at a positive common constant rate; while the interest rate, wage rate, and labor supply do not grow. The transition dynamics to the BGP are characterized by two different patterns; that is, physical capital, human capital, consumption, wage, wage augmented for skill growth, government expenditures, and GDP grow at increasing rates, whereas human capital, the interest rate, and labor supply grow at decreasing rates.

Additionally, the graphical as well as the statistical analyses have indicated that the model describes the growth process of the South Africa' economy pretty well.

In the next step, three policy experiments were conducted to assess and analyze growth and welfare effects of spending on education. First, a decrease in, or elimination of, spending on education does not have any effect on the long run common rate of growth of labor supply, of the wage and the interest rate but generates a positive effect on the long run common rate of growth of per capita GDP, consumption, human capital, physical capital, and wage of skilled workers. The long run effect of a reduction in, or elimination of, spending on education on the common rate of growth is relatively larger in Experiment 2. It is followed by that of Experiment 1 and then by that of Experiment 3. Further, a reduction in or an elimination of spending on education generates positive effects on the transition cumulative rates of growth of per capita GDP and wage of the skilled workers. For per capita GDP, the transition effect is relatively larger in Experiment 2 and is followed by that of Experiment 1 and then by that of Experiment 3. Regarding the transition effect on the wage of skilled workers, the order goes from the third (the largest) to the first to the second experiment. Also, the transition effects on the cumulative rates of growth of the other variables vary across experiments. In the second experiment, the effects on the cumulative rates of growth of human capital, physical capital, labor supply, wage rate, and interest rate are positive, while that on the cumulative rate of growth of consumption is negative. However, these effects are reversed in experiments 1 and 3.

Furthermore, a decrease in, or elimination of, spending on education generates negative small effects on welfare. The effect is relatively large in the third experiment, followed by the first experiment and then the second experiment. In fact, higher spending on education translates into lower welfare and low spending on education translates into higher welfare.

The results of these experiments can be used to answer critical questions raised in the introduction of this study. First, in all instances, public spending on education does generate positive growth and welfare effects but these effects are small. These results confirm the view of the second strand in the literature of growth with policy, that is, some government expenditures do affect positively the productivity of the economy as a whole as well as the welfare regardless of the distortions that a tax used to finance them may create. Second, the results have shown in all experiments that reducing or eliminating these expenditures does slow growth and lead to welfare losses. This provides a strong support to the intervention of government in the education sector to promote growth and improve welfare. However, these results do not help to determine the optimal size of government in the educational sector. In fact, none of the three experiments ranks better than others in terms of both growth and welfare. However, the welfare loss is smaller when spending is reduced to the average of developing countries. This implies that the second experiment ranks better for a government whose welfare improvement is a priority.

This study can be extended in several directions. First, transfers can be disaggregated in order to identify the other expenditures that qualify as productive so as to include them in the appropriate sectors. Our choice to treat expenditures other than educational ones as transfers was motivated by the concern to ease the derivation of the dynamic system as well as the BGP. This disaggregation may provide new insights and lead possibly to different results. For instance, these transfers include expenditures such as health, social infrastructure (housing, special development initiatives), promotion of industrial development, research and technology development, and competitiveness fund and sectoral partnership facility. Modeled accurately, such productive expenditures can help to capture the interconnections between different areas of government intervention and to allow more effective policies on growth and welfare grounds.

II. Dynamics of Income Inequality, Poverty, and Growth: The Case of Post-Apartheid South Africa

II.1. Introduction

Until the mid of the 20th Century, the interaction between growth, income inequality, and poverty was not a major issue in the agenda of economists. The conventional literature of the efficiency of markets viewed them as temporary outcomes of deregulated growth processes, which growth itself would eliminate through a trickle down mechanism (Bourguignon et al., 2005). This is the view espoused by Kuznets (1955) who argues that inequality in the distribution of income increases in the early stages of the growth process of a country but diminishes after a country reaches a critical level of income. However, continuous increases in the number of the poor and in income inequality in some parts of the world in the post 1960 regardless of the sustained growth achieved by the world economy have brought forth arguments for economists to include distribution considerations into growth analysis.

The dynamics of the interaction between growth, income inequality, and poverty are not fully understood. Although economists are aware of the complexity posed by this interaction, approaching it has proved difficult. In fact, data on growth variables of countries are available from several sources. But the household/individual income or expenditure surveys, which constitute the source data for the construction of income distributions and inequality measures and for the determination of the poverty rates are either unavailable at all or available for only some years in most developing countries.[20] This makes the analysis of these dynamics more complicated since most of those countries are not only the reservoirs of poor people but are also characterized by high

[20] India and China stand apart among countries with large numbers of poor. These two countries have been collecting households surveys for a number of years.

income inequality. Furthermore, the issue of growth and those of income inequality and poverty are of different natures. Growth has a macroeconomic dimension whereas income inequality and poverty have a microeconomic dimension. A more thorough approach consists of integrating them into a framework capable of capturing their interaction.

The relationship between growth, income inequality, and poverty has been approached recently using aggregate data. Ravallion and Chen (1997), de Janvry and Sadoulet (2000), Agenor (2002), and Dollar and Kraay (2002) find in cross-country studies that growth reduces poverty but leaves the distribution of income unchanged. The results accord with those found in earlier studies on the negative effect of growth on poverty, but are not plausible regarding the neutrality of growth on income distribution. For instance, Deninger and Squire (1998) find in a cross-country analysis that income inequality has a negative effect on growth of income of the poor. Ravallion (2001) and Heltberg (2004) revisit some of the above studies and document that growth does indeed change the distribution of income. They also document that higher income inequality makes less effective growth promoting policies on poverty reduction.

The controversial results of the aggregate studies aforementioned indicate how the analysis of the relationship between growth, income inequality, and poverty from a macroeconomic point of view may be missing important determinants of income distribution which are observable only at the household levels. This is the reason why the integration of household heterogeneity into macroeconomic models has permitted some improvement in the analysis of this relationship. The first attempts in this context are the

Computable General Equilibrium (CGE) models.[21] These frameworks disaggregate highly the economy and introduce distribution considerations through a limited number of representative households. They used to be static but have been extended recently to include dynamics in a certain way. However, the CGE models have three shortcomings.[22] First, they require a very small number of representative households in order to be solved numerically, which is unlikely to fully capture the household heterogeneity. Second, the accuracy of their solution is sometimes called into question. Indeed, the values of the behavioral parameters used in most of the cases are arbitrarily set or imported from outside the unit of the analysis. Third, they are not really dynamic models in the sense that their equilibria are sequences of temporary static solutions linked over time by the accumulation of state variables.

More recently, Caselli and Ventura (2000) have overcome the shortcomings of the CGE models by introducing various sources of consumer heterogeneity into one sector representative consumer growth models. Their approach allows for not only the introduction of an unlimited number of households into the model but also a dynamic analysis of the interaction between growth, income inequality, and poverty.

In this paper, I build on the Caselli and Ventura's study to analyze the dynamics of income distribution and inequality, poverty, and growth in the Post-Apartheid South Africa. I introduce household heterogeneity into a growth model through asset holding and skills, and solve numerically the model to the South African economy. Next, I combine the results of the aggregate model and the microeconomic data (1996 South

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[21] See for instance Adelman and Robinson (1978), Lysy and Taylor (1980), Bourguignon et al. (1989), Devarajan and Lewis (1991), Decaluwé et al. (1999), Dorosh and Sahn (2000), Thurlow (2004), and so on.

[22] See Bourguignon et al. (2005).

African Household survey) to construct income distributions over time, and to study the interaction between growth, income distribution and inequality, and poverty.

The results indicate that the dynamics of income distribution are uniquely determined by those of the distribution of physical capital. Poor people lose their relative position in the distribution of income in the early stages of the process but improve thereafter. Furthermore, growth reduces poverty substantially but is not enough to reduce income inequality.

The rest of this study is organized as follows. In Section 2, I formulate a model of growth with heterogeneous households and characterize its equilibrium. In Section 3, I describe the methodology for solving the model using the representative consumer assumption. In Section 4, I solve numerically the model. In Section 5, I combine the solution to the model with the microeconomic data to construct income distributions, and analyze the dynamics of growth, income inequality, and poverty. Section 6 provides concluding remarks.

II.2. Model

II.2.1. Set up

I extend the Ramsey-Cass-Koopmans model to allow for the analysis of the interaction between growth, income inequality, and poverty. The horizon is infinite and time is continuous. The economy is closed and populated by many infinitely lived, rational, and heterogeneous consumers with homothetic preferences, and many competitive firms with identical technology. Consumers are indexed by $i = 1,2,...,I$ and firms are indexed by $g = 1,2,...,G.$ Consumers are heterogeneous in their initial wealth and labor productivity. The population is assumed to be constant over time. A single consumption good is produced in this economy from a technology that combines physical capital and effective labor. Each consumer supplies in each period t one unit of physical capital $k_i(t)$ in exchange for the interest rate $r(t)$ and one unit of effective labor $h_i(t)$ in exchange for the wage rate $w(t)$, $t \geq 0$, where $r(t)$ and $w(t)$ are measured in unit of the consumption good and $h_i(t)$ grows at a constant rate γ $\left(h_i(t) = h_i(0)e^{\gamma t}\right)$. The consumer uses her income to purchase consumption good $c_i(t)$ and to accumulate assets. Since the economy is closed, foreign saving does not exist in this model, so assets are equal to physical capital.

Let consumers be classified into income brackets d based on their incomes, where $d = 1,2,...,D$. This classification allows for heterogeneity in initial wealth and labor productivity within each income bracket as well as across income brackets. A consumer in each income bracket or group derives her utility from consuming $c_{id}(t)$ units of the consumption good in each period t, where $i_d = 1,2,...,I_d$ and $\sum_{d=1}^{D} I_d = I$. We assume that consumer preferences are characterized by a concave utility function $U(c_{id}(t))$ that is twice continuously differentiable, with $U'(c_{id}(t)) > 0$ and $U''(c_{id}(t)) < 0$ for all $c_{id}(t) > 0$, $t \geq 0$, where $U'(c_{id}(t))$ and $U''(c_{id}(t))$ are the first and second derivatives of U with respect to c_{id}, respectively. It is also assumed that $U(c_{id}(t))$ satisfies the Inada conditions, that is:

$\lim_{c_{id}(t)\to 0} U'(c_{id}(t)) = \infty$ and $\lim_{c_{id}(t)\to \infty} U'(c_{id}(t)) = 0.$[23]

The homothetic preference assumption implies that consumers within each group/income bracket have a single consumer representation since the average variables of the consumers within a group behave exactly as those of a single representative consumer (RC) of the group.[24] Using a single consumer representation inside each group, we express the discounted sum of future utilities of an RC of group d as

$$\int_0^\infty U(c_d)e^{-\rho t}dt, \tag{2.1}$$

where ρ is the parameter of time preference and c_d is the consumption of an RC of group d. The budget constraint this RC of group d is given by

$$\dot{k}_d = wh_d(0)e^{\gamma t} + (r-\delta)k_d - c_d + \pi_d, \tag{2.2}$$

where $\dot{k}_d, k_d$ and h_d are respectively the rate of accumulation of physical capital, physical capital, and labor productivity of a RC of group d, π_d is the profit earned from the ownership of the firm shares by a RC of group d; and δ is the constant rate of depreciation of physical capital. A RC of group d starts with some positive endowment of physical capital and effective labor or

$$k_d(0) = k_{d0},\ h_d(0) = h_{d0},\ k_{d0} \text{ and } h_{d0} \text{ are given.} \tag{2.3}$$

Also, assume that all decision variables are non negative or

$$k_d \geq 0, c_d \geq 0. \tag{2.4}$$

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[23] From now on, I omit the time dependency of variables if no ambiguity arises.

[24] See Casseli and Ventura (2000) and Barro and Sala-i-Martin (2004) for a complete treatment of this assumption.

On the supply side there exist many competitive firms indexed by $g = 1,2,...,G$ producing each consumption good y^g from a technology that combines physical capital k^g and effective labor h^g. Assume that all firms are merged into a single competitive firm which is responsible for all the output in the economy, and that this merged firm has a Constant Returns to Scale (CRS) technology. Then its profit function (Π) in each period t is expressed as

$$\Pi = Y - rK - wH, \qquad \forall t \qquad (2.5)$$

where K and H are the firm demand of physical capital and effective labor, respectively; and Y is its output, which is produced according to

$$Y = F(K,H), \qquad \forall t \qquad (2.6)$$

where $F(.,.)$ is a CRS production function. Assume this function is twice continuously differentiable K and H, with $F'(.,.) > 0$ and $F''(.,.) < 0$ (F' and F'' are the first and the second derivatives of F with respect to each of its arguments). Assume that $F(.,.)$ satisfies the Inada conditions, that is,

$$\lim_{K(t)\to\infty} F(.,H) = 0, \ \lim_{K(t)\to 0} F(.,H) = \infty, \ \lim_{H(t)\to\infty} F(K,.) = 0, \ \lim_{H(t)\to 0} F(K,.) = \infty,$$

and that all the firm's decision variables are non-negative, that is

$$Y \geq 0, K \geq 0, H \geq 0. \qquad (2.7)$$

II.2.2 Equilibrium and its Characterization

A competitive equilibrium for this economy is a sequence of allocations of an RC of group d, $\{c_d^*(t), k_d^*(t)\}_{t=0}^{\infty}$, $d = 1,2,...,D$; a sequence of allocations of the firm, $\{Y^*(t), K^*(t), H^*(t)\}_{t=0}^{\infty}$; and a sequence of the rental rates, $\{r(t), w(t)\}_{t=0}^{\infty}$; such that:

i) Given $\{r(t), w(t)\}_{t=0}^{\infty}$, $\{c_d^*(t), k_d^*(t)\}_{t=0}^{\infty}$ maximizes (2.1) subject to $(2.2)-(2.4)$,

ii) The rental rates of physical capital and effective labor in each period t are given by

$$r = \frac{\partial F(K,H)}{\partial K},\; w = \frac{\partial F(K,H)}{\partial H}, \tag{2.8}$$

iii) The motion of the aggregate physical capital is given by

$$\dot{K} = (r-\delta)K + wH - C + \Pi \tag{2.9}$$

iv) The following feasibility conditions hold in each period t:

$$\sum_{d=1}^{D} c_d + \dot{K} + \delta K = Y,\; \sum_{d=1}^{D} c_d = C,\; K = \sum_{d=1}^{D} k_d,\; H = \sum_{d}^{D} h_d(0)e^{\gamma t}. \tag{2.10}$$

Assume that the utility function and the production function take the following forms

$$U(c_d) = \frac{c_d^{1-\sigma}}{1-\sigma},\; F(K,H) = AK^{\alpha}H^{1-\alpha}, \tag{2.11}$$

where σ is the inverse of the inter-temporal elasticity of substitution, A is the technology parameter, and α is the share of physical capital. Then the Current Value Hamiltonian (J_d)[25] of a RC of group d is expressed as

$$J_d = \frac{c_d^{1-\sigma}}{1-\sigma} + \lambda_d \left[wh_d(0)e^{\gamma t} + (r-\delta)k_d - c_d \right], \tag{2.12}$$

Maximizing J_d over c_d (control variable), λ_d (co-state variable), and k_d (state variable) yields:

$$\partial J / \partial c_d = c_d^{-\sigma} - \lambda_d = 0, \qquad \forall t \tag{2.13}$$

$$\dot{k}_d = \partial J / \partial \lambda_d = wh_d(0)e^{\gamma t} + (r-\delta)k_d - c_d, \tag{2.14}$$

$$\dot{\lambda}_d = -\partial J / \partial k_d = -\lambda_d (r - \delta - \rho), \tag{2.15}$$

$$TVCs : \lim_{t\to\infty} \lambda_d(t) k_d(t) e^{-\rho t} = 0. \tag{2.16}$$

Manipulate (2.13) and (2.15) to obtain

$$\dot{c}_d = \sigma^{-1} c_d (r - \delta - \rho). \tag{2.17}$$

From (2.10) and (2.17) I derive the motion of the aggregate consumption, which is given by

$$\dot{C} = \sigma^{-1} C (r - \delta - \rho). \tag{2.18}$$

Given the initial conditions (2.3), the evolution of the economy over time is characterized by the dynamic system formed by $(2.9), (2.14), (2.16), (2.17)$, and (2.18). Furthermore, the following rental rate equations must hold in each period t:

[25] We omit the profit variable π_d in (2.12) because of the CRS specification of the production function, which makes profit to equal zero in each period.

$$r = \alpha A K^{\alpha-1} H^{1-\alpha}, \;\; w = (1-\alpha) A K^{\alpha} H^{-\alpha}. \tag{2.19}$$

II.2.3 Steady State (SS)

Suppose the equilibrium paths converge to the balanced growth path (BGP), that is, a system in which all variables grow at constant (possibly zero) rates. Define these rates of growth by:

$$\gamma_C = \frac{\dot{C}}{C}, \gamma_K = \frac{\dot{K}}{K}, \gamma_{cd} = \frac{\dot{c}_d}{c_d}, \gamma_{kd} = \frac{\dot{k}_d}{k_d}, \gamma_{hd} = \gamma = \frac{\dot{h}_d}{h_d}, \gamma_w = \frac{\dot{w}}{w}, \;\; \gamma_r = \frac{\dot{r}}{r}. \tag{2.20}$$

Then dividing (2.18) by C and substituting (2.19) for r into the resulting expression yields the marginal product of K:

$$\alpha A K^{\alpha-1} H^{1-\alpha} = \sigma \gamma_C + \delta + \rho, \tag{2.21}$$

which is constant along the BGP. Further, dividing (2.9) by K, substituting (2.18) for r and w, and applying (2.20) result in:

$$\frac{C}{K} = \frac{\delta(1-\alpha) + \rho + \gamma_C \sigma - \alpha \gamma_K}{\alpha}. \tag{2.22}$$

Taking the log and time derivative of (2.22) we obtain $\dot{C}/C - \dot{K}/K = 0$, that is, C and K grow at the same constant rate $(\gamma_C = \gamma_K = \gamma')$. Also, it is obvious from (2.17) and (2.18) that c_d and C grow at the same rate $(\gamma_C = \gamma_{cd} = \gamma')$. Moreover, divide (2.14) by k_d, substitute (2.19) for r and w, and apply (2.20) to obtain:

$$\frac{c_d}{k_d} = \left[1 - (1-\alpha)\left(\frac{\delta + \rho + \gamma' \sigma}{\alpha}\right)\left(\frac{h_d(0)}{H(0)}\right)\left(\frac{K}{C}\right)\left(\frac{C}{c_d}\right)\right]^{-1} [\rho + \gamma' \sigma - \gamma_{kd}]. \tag{2.23}$$

Since C, K, and c_d grow at the same rate in BGP, K/C and C/c_d are constant. Thus differentiating (2.23) with respect to time yields $\dot{c}_d / c_d - \dot{k}_d / k_d = 0,$ which implies that c_d and k_d grow at the same rate $(\gamma_{cd} = \gamma_{kd} = \gamma')$. Furthermore, taking the log and time derivative of (2.21) yields $\gamma' = \gamma,$ that is, the rate of growth of C, K, c_d, and k_d is the same as that of the effective labor h_d. γ is also the rate of the aggregate output $(\gamma_Y = \gamma)$ as it can be checked by taking the log and time derivative of the production function in (2.11). It is also easy to verify from (2.19) that the rates of growth of r and w are zero $(\gamma_r = \gamma_w = 0)$.

Next, use γ to normalize variables $\left(\hat{C} = Ce^{-\gamma t},\ \hat{K} = Ke^{-\gamma t},\ \hat{c}_d = c_d e^{-\gamma t},\ \hat{k}_d = k_d e^{-\gamma t}\right)$, and then rewrite our dynamic system in these normalized variables. This new system is expressed as:

$$\dot{\hat{K}} = wH(0) + (r - \delta - \gamma)\hat{K} - \hat{C}, \tag{2.24}$$

$$\dot{\hat{k}}_d = wh_d(0) + (r - \delta - \gamma)\hat{k}_d - \hat{c}_d, \tag{2.25}$$

$$\dot{\hat{C}} = \sigma^{-1}\hat{C}(r - \delta - \rho - \gamma\sigma), \tag{2.26}$$

$$\dot{\hat{c}}_d = \sigma^{-1}\hat{c}_d(r - \delta - \rho - \gamma\sigma), \tag{2.27}$$

$$TVCs: \lim_{t\to\infty} \lambda_d(t)\hat{k}_d(t) = 0, \tag{2.16}$$

where r and w are now given by:

$$r = \alpha A\hat{K}^{\alpha-1}H(0)^{1-\alpha},\ \ w = (1-\alpha)A\hat{K}^{\alpha}H(0)^{-\alpha}. \tag{2.28}$$

The *TVCs* in (2.16) imply that all variables approach their steady state values. Symbolizing e each of these variables by $x(t)$, the steady state of $x(t)$ is defined as:

$$\lim_{t\to\infty} x(t) = x_{ss} < \infty \leftrightarrow \lim_{t\to\infty}\left(\dot{x}(t)/x(t)\right) = 0.$$

Dividing expressions $(2.24)-(2.27)$ by $\hat{K}$, $\hat{k}_d$, $\hat{C}$, and $\hat{c}_d$, respectively, and manipulating yield the following steady state (SS) conditions:

$$\left(\frac{\hat{C}}{\hat{K}}\right)_{ss} = w_{ss}\left(\frac{A\alpha}{\delta+\rho+\gamma\sigma}\right)^{\frac{1}{\alpha-1}} + r_{ss} - \delta - \gamma, \tag{2.29}$$

$$\left(\frac{\hat{c}_d}{\hat{k}_d}\right)_{ss} = \left[1 - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{h_d(0)}{H(0)}\right)\left(\frac{C_0}{c_{d0}}\right)\left(\frac{\hat{C}}{\hat{K}}\right)_{ss} r_{ss}\right]\left(r_{ss} - \delta - \gamma\right), \tag{2.30}$$

where r_{ss} and w_{ss} are:

$$r_{ss} = \delta + \rho + \gamma\sigma,\ w_{ss} = (1-\alpha)A^{\frac{1}{1-\alpha}}\left(\frac{\delta+\rho+\gamma\sigma}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}. \tag{2.31}$$

II.3. Building Consumer Heterogeneity into a One-sector RC Growth Model

The dynamic system $(2.24)-(2.27)$ and (2.16) requires a large number of groups of consumers in order to capture heterogeneity across those consumers. However, a large number of groups imply a large number of differential equations, which make the system complex and difficult to solve. For instance, the number of differential equations that result when there are only two groups is six. This number increases to eight when the number of groups increases to three, to ten when the number of groups rises to four, to twelve when this number of groups increases to five, and so on. As far as I know, solving a dynamic system of four differential equations is already complicated, and methods for solving a dynamic system of six differential equations do not exist yet.

To solve the dynamic system $(2.24)-(2.27)$ and (2.16), I use the methodology developed in Caselli and Ventura (2000). Accordingly, a heterogeneous consumer growth model can be solved by constructing consumer heterogeneity in one-sector RC growth model. Precisely, the methodology consists of using the RC assumption, and then combining it with the results of consumer heterogeneity to generate the cross-section variables for each consumer in the model.

II.3.1 A RC Version of the Model

By the homothetic preference assumption, an economy with infinitely many consumers has a single consumer representation since the average variables of all consumers in this economy behave exactly as those of a single RC of this economy. Admitting a single RC for this economy implies that the dynamic system $(2.24)-(2.27)$ and (2.16) reduces to

$$\dot{\hat{k}} = wh(0) + (r - \delta - \gamma)\hat{k} - \hat{c}, \tag{2.32}$$

$$\dot{\hat{c}} = \sigma^{-1}\hat{c}(r - \delta - \rho - \gamma\sigma), \tag{2.33}$$

$$TVCs : \lim_{t\to\infty} \lambda\hat{k} = 0, \tag{2.34}$$

and the steady state reduces to

$$\hat{k}_{ss} = \left(\frac{\delta + \rho + \gamma\sigma}{\alpha A}\right)^{\frac{1}{\alpha-1}} h(0), \tag{2.35}$$

$$\hat{c}_{ss} = \left(\frac{\delta(1-\alpha)+\rho+\gamma(\sigma-\alpha)}{\alpha} \right) \hat{k}_{ss}. \tag{2.36}$$

All variables and parameters without the index d are those of the single RC of the economy. The static equations are now given by:

$$r = \alpha A \hat{k}^{\alpha-1} h(0)^{1-\alpha}, \quad w = (1-\alpha) A \hat{k}^{\alpha} h(0)^{-\alpha}, \tag{2.37}$$

and their steady values are as in (2.31).

II.3.2 Cross-sections of Consumption, Capital, skills, and Income

Let define the relative variables of the RC of each group with respect to the RC of the economy (cross-section variables) as follows:

$$c_d^R = c_d / c, \; h_d^R = h_d / h, \; k_d^R = k_d / k, \; y_d^R = y_d / y \tag{2.38}$$

where c_d^R, k_d^R, h_d^R, and y_d^R are the cross sections of consumption, physical capital, skills, and income, respectively. Differentiating each cross section variable with respect to time yields:

$$\dot{c}_d^R = c_d^R \left(\frac{\dot{c}_d}{c_d} - \frac{\dot{c}}{c} \right), \tag{2.39}$$

$$\dot{h}_d^R = h_d^R \left(\frac{\dot{h}_d}{h_d} - \frac{\dot{h}}{h} \right), \tag{2.40}$$

$$\dot{k}_d^R = k_d^R \left(\frac{\dot{k}_d}{k_d} - \frac{\dot{k}}{k} \right), \tag{2.41}$$

$$\dot{y}_d^R = y_d^R \left(\frac{\dot{y}_d}{y_d} - \frac{\dot{y}}{y} \right). \tag{2.42}$$

Now if we substitute expressions (2.27) and (2.33) into (2.39), we obtain $\dot{c}_d^R = 0.$ This implies that the cross section of consumption is constant over time and is given by:

$$c_d^R = \frac{c_d(0)}{c(0)}, \tag{2.43}$$

where $c_d(0)$ and $c(0)$ are consumptions of the RC of group d and of the RC of the economy in the initial period, respectively. Likewise, $\dot{h}_d^R = 0$ since labor productivity grows at a constant rate γ for each consumer. As a result, the cross-section of skills is constant over time and is expressed as:

$$h_d^R = \frac{h_d(0)}{h(0)}, \tag{2.44}$$

where $h_d(0)$ is the initial skills of the RC of group d, and $h(0)$ is the initial skills of the RC of the economy. Further, substituting (2.25) and (2.32) into (2.41) and rearranging yield:

$$\dot{k}_d^R = \left(\frac{1-\alpha}{\alpha}\right) r h_d^R - \frac{c}{k} c_d^R + \left[\frac{c}{k} - \left(\frac{1-\alpha}{\alpha}\right) r\right] k_d^R, \tag{2.45}$$

The analytical solution to (3.5) is given by:

$$k_d^R(t) = e^{-\int_0^t \left[\frac{1-\alpha}{\alpha} r(t) - \frac{c(t)}{k(t)}\right] d\tau} \left\{ k_d^R(0) + \int_0^t \left[\left(\frac{1-\alpha}{\alpha}\right) h_d^R r(t) - c_d^R \frac{c(t)}{k(t)}\right] e^{\int_0^t \left[\frac{1-\alpha}{\alpha} r(t) - \frac{c(t)}{k(t)}\right] d\tau} d\tau \right\}, \tag{2.46}$$

$$k_d^R(0) = k_{d0}^R, \ k_{d0}^R \text{ given.} \tag{2.47}$$

To derive the expression of the cross-section of income, we substitute (2.37) for r and w into (2.25) and manipulate to get:

$$y_d^R = (1-\alpha)h_d^R + \alpha k_d^R. \tag{2.48}$$

It is obvious from (2.48) that the time-paths of the cross-section of income depend exclusively on the time-paths of the cross-section of physical capital.

Equations (2.46) and (2.48) can be used to determine conditions under which the cross-section of physical capital and income will converge or diverge. It is apparent from these equations that convergence or divergence[26] in the cross-section of physical capital or income is determined by the sign of $[(\alpha/(1-\alpha))r - (c/k)]$, the sign of $(k_d^R(0)-1)$, and by the patterns of the expression $[k_d^R + \omega(t)]$, where $\omega(t)$ is given by $\omega(t) = \int_0^t \left[\left(\frac{1-\alpha}{\alpha}\right)h_d^R r(t) - c_d^R \frac{c(t)}{k(t)}\right] e^{\int_0^t \left[\frac{1-\alpha}{\alpha} r(t) - \frac{c(t)}{k(t)}\right] d\tau} d\tau.$

Proposition 1: If $r > \alpha(1-\alpha)^{-1}(c/k)$, $k_d^R(t)$ and $y_d^R(t)$ will converge if and only if $k_d^R(0) < 1$ and $\omega(t)$ dominates $k_0^R(t)$. Otherwise, $k_d^R(t)$ and $y_d^R(t)$ will diverge.

Proposition 2: If $r > \alpha(1-\alpha)^{-1}(c/k)$, $k_d^R(t)$ and $y_d^R(t)$ will converge if and only if $k_d^R(0) > 1$ and $k_0^R(t)$ dominates $\omega(t)$. Otherwise, $k_d^R(t)$ and $y_d^R(t)$ will diverge.

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[26] Convergence (divergence) in cross-section of income occurs if and only if $|y_d^R - 1|$ is decreasing (increasing). Convergence is absolute if and only if $|y_d^R - 1|$ approaches zero. Convergence (divergence) in cross-section of physical capital occurs if and only if $|k_d^R - 1|$ is decreasing (increasing). Convergence is absolute if and only if $|k_d^R - 1|$ approaches zero.

Proposition 3: If $r < \alpha(1-\alpha)^{-1}(c/k)$, $k_d^R(t)$ and $y_d^R(t)$ will converge if and only if $k_d^R(0) < 1$ and $k_0^R(t)$ dominates $|\omega(t)|$. Otherwise, $k_d^R(t)$ and $y_d^R(t)$ will diverge.

Proposition 4: If $r < \alpha(1-\alpha)^{-1}(c/k)$, $k_d^R(t)$ and $y_d^R(t)$ will converge if and only if $k_d^R(0) > 1$ and $|\omega(t)|$ dominates $k_0^R(t)$. Otherwise, $k_d^R(t)$ and $y_d^R(t)$ will diverge.

It is obvious from Proposition 1-4 that the dynamics of the cross sections of physical capital and income are complex. It is also evident from these results that the model is consistent with alternating periods of convergence and divergence since $[(\alpha/(1-\alpha))r - (c/k)]$ can possibly change signs a certain number of times.

The intuition of Proposition 1-4 can be described in the following way. Each consumer needs to achieve a consistent growth rate of income to sustain her optimal consumption. But the growth rate of consumption is the same for each consumer, as can be seen from equations (2.17) and (2.33). Also, by the homothetic preference assumption, consumers spend the same fraction of their incomes and thus must exhibit the same rate of growth of income. But this rate of growth is the weighted average of the rates of growth of labor and capital incomes, where the weights are the shares labor and capital incomes. The rate of growth of labor income is the same for each consumer and is exogenously determined. If a consumer's capital income share is less than her labor income share, she must be accumulating physical capital at a lower rate, and thus choosing the growth rate of income in a way that is inconsistent for sustaining her optimal consumption. Consequently, her accumulation of physical capital must increase, and so must her capital income share. Symmetric results obtain when her capital income share is greater than her labor income share.

II.4. Numerical Solution to the Model

The dynamic system $(2.24)-(2.27)$ and (2.16) is too complex to be solved numerically. My strategy for solving it includes two steps. In the first step I solve numerically the dynamic system of the single RC model $((2.32)-(2.34))$. In the next step, I combine the single RC model's solution with the initial consumer characteristics to derive the cross section variables and then the variables of the heterogeneous model.

II.4.1 Numerical Solution to the Single RC Model

I use the relaxation algorithm[27] to solve numerically the dynamic system of the single RC model $((2.32)-(2.34))$, given the parameter estimates as summarized in Table 7 and the boundary conditions. The boundary conditions include the initial conditions on the state variables (k_0, h_0) and a arbitrary guess on the control variable (consumption) in the initial period $(c(0))$ and the steady state conditions $(2.35)-(2.36)$. The numerical solution to the single RC model[28] is given by the time paths of the rates of growth of variables which are depicted in Figure 5, where the initial period is the year 1993.

As it can be noticed, the system globally converges to the steady state after 73 years (1993-2065) from the initial period. The rates of growth of levels of per capita consumption, physical capital, output, and that of wage are increasing, while the rate of growth of the interest rate is decreasing. The rates of growth of per capita consumption,

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[27] See Timborm, Koch, and Steger (2004) for the description and implementation of the algorithm.

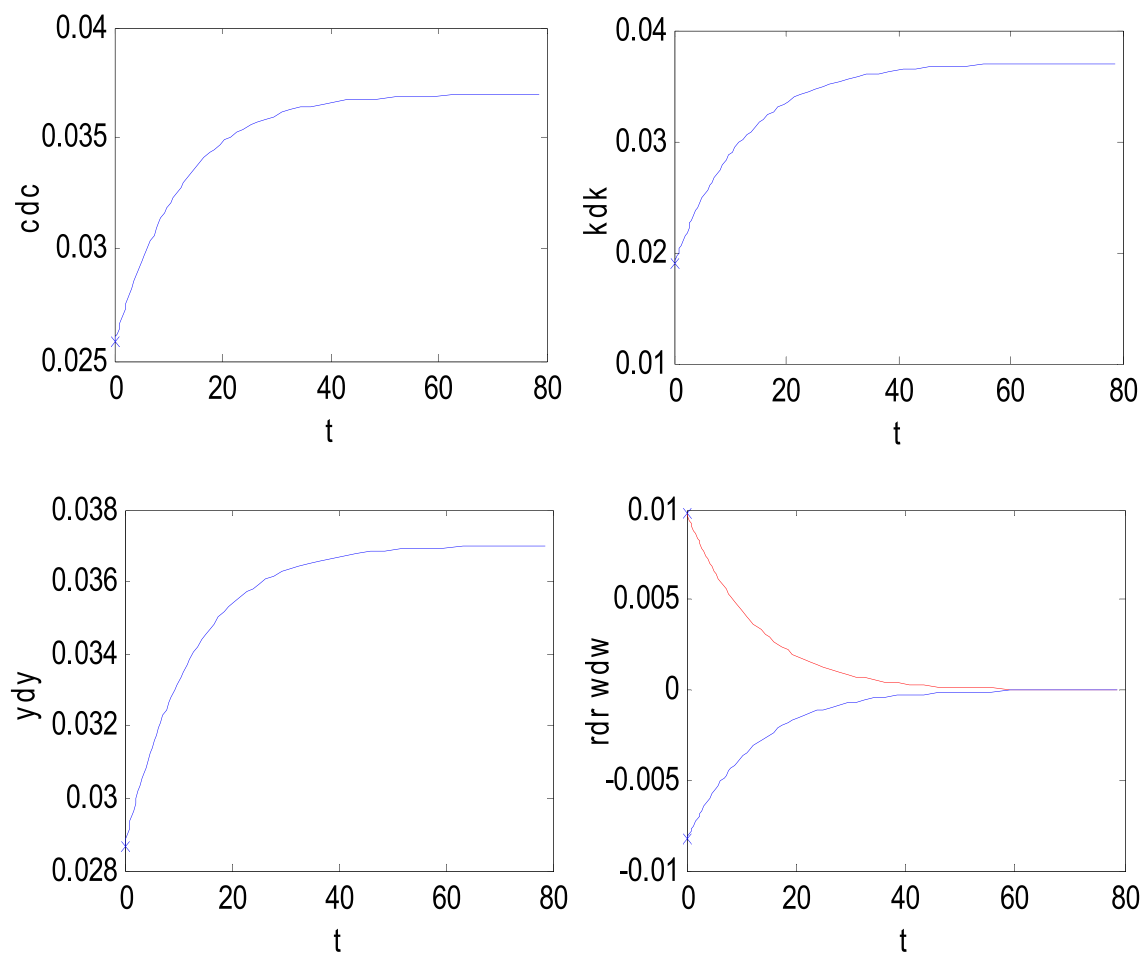
[28] I solve first the system $((2.32)-(2.34))$ and then use the solution to recover the rates of growth of the level of variables, which are given by: $\dot{c}/c = \sigma^{-1}(r-\delta-\rho)$, $\dot{k}/k = r-\delta+(wh(0)-\hat{c})\hat{k}^{-1}$, $\dot{y}/y = \alpha(\dot{k}/k)+(1-\alpha)\gamma$, $\dot{r}/r = (1-\alpha)(\gamma-(\dot{k}/k))$, and $\dot{w}/w = \alpha((\dot{k}/k)-\gamma)$.

Table 7: Parameters' Estimated Values[29]

α	σ	ρ	γ	A	k_0	h_0
0.459	1.75	0.06742	0.037	1	10.527	1

physical capital, and output converge to the common steady state rate of growth, which is the constant rate of growth of skills γ. Furthermore, the rates of growth of wage and interest rate converge to zero.

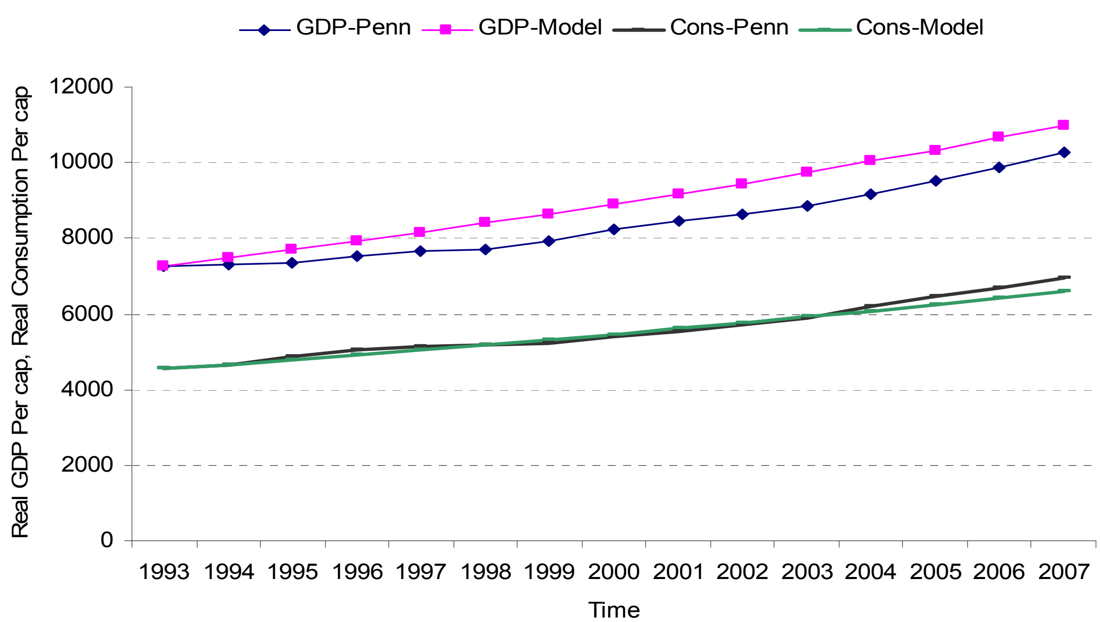
Figure 3: Time-Paths of Rates of Growth of c, k, y, r, and w (1993-2060)



[29] See Appendix C for the parameter estimation's procedure.

This model roughly captures the trend observed in data (see Figure 4). The values of the rate of growth of per capita GDP generated by the model are very close to those from the data in the beginning of the process but differ thereafter. However, the differences are very small and close to zero. Furthermore, the per capita Consumption series from the model and data are almost the same over 1993-2004, period after which they start to show very small differences. Additionally, the test statistic of difference between the two distributions (Theil Inequality coefficient-U)[30] supports the graph analysis. This U coefficient is 0.0014 for per capita GDP and 0.0007 for per capita consumption. The statistic analysis indicates a very good predictive performance of the model as concern the Real GDP per capita as well as concern the Real Consumption per capita over 1993-2007.

Figure 4: Real Per capita GDP and Real Per Capita Consumption (1993-2007)



[30] The expression of Theil Inequality coefficient U as well as its description can be found on p.210 in Pindyck and Rubinfeld (1997).

II.4.2 Dynamics of Cross-Sections of Physical Capital and Income

I now use the results of the single RC model as well as the distribution of consumer characteristics in the initial period to generate the cross-sections of physical capital and income. I begin by classifying consumers into 10 groups or deciles $(d = 1,2,...,10)$. Deciles 1-4 are poor[31], deciles 5-8 are middle-class, and deciles 9-10 are rich. Then, I calculate the characteristics of the RC of each decile in the initial period. These characteristics are summarized in table 8 and the evolution of the cross-sections of physical capital and income are depicted in Figure 5. As this figure shows, the dynamics of the cross-sections of physical capital and income are complex and characterized by episodes of divergence and convergence.

Starting with the RCs of poor deciles (deciles 1-4), it can be seen from Figure 5 that their relative physical capital as well as their relative incomes exhibit convergence overall. Indeed, the RC of each of these 4 deciles improves their relative position in the distributions of physical capital and income by the end of the process in comparison to the beginning of the process. However, their dynamics are slightly different. The relative physical capital and incomes of the RCs of deciles 1, 3, and 4 are characterized by episode of divergence and then convergence. With respect to the initial period, the relative physical capital and incomes of the RCs of deciles 1, 3, and 4 exhibit divergence for a period of 28 years, 13 years, and 20 years, respectively. This episode of divergence is followed by the episode of convergence which prevails up to the steady state. This convergence period lasts longer for each RC. It lasts 44 years for the RC of decile 1, 59

[31] Poverty in this context is defined from the poverty line, that is, a poor is a consumer who lives on a 1$ or 2$ per day. See Barro and Sala-i-Martin (2004) and Sala-i-Martin (2006) for the definition of the poverty line.

years for the RC of decile 3, and 52 years for the RC of decile 4. On the other hand, the relative physical capital and incomes of the RC of decile 2 exhibit convergence for 2 years followed by divergence for 7 years, and then convergence for 63 years.

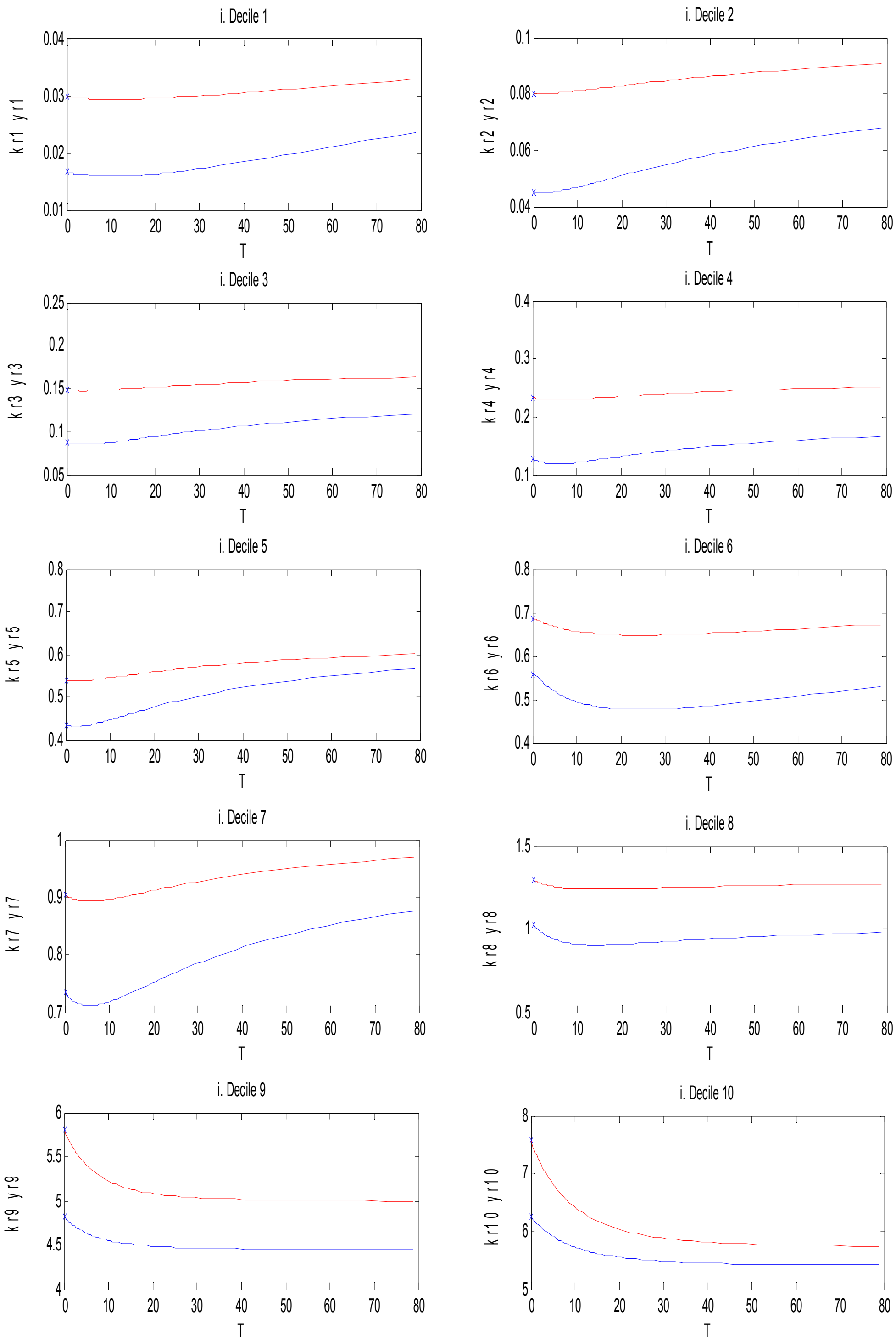
Turning now on to the relative variables of the RCs of the middle class deciles (deciles 5-8), Figure 5 reflects that the relative physical capital and incomes of the RCs of deciles 5 and 6 exhibit episodes of convergence followed by divergence and then convergence, whereas those of the RCs of deciles 7 and 8 exhibit episodes of divergence followed by convergence. For the RC of decile 5, the convergence period lasts 2 years. It is followed by the divergence period of 7 years, and then by the convergence period of 63 years. For the RC of decile 6, the convergence period lasts 2 years and is followed by the divergence period of 42 years and then by the convergence period of 30 years. Further,

Table 8: Distribution of Consumer Characteristics in the Initial Period[32]

	c_d^R	h_d^R	$k_d^R(0)$
RC_1	0.033	0.041	0.017
RC_2	0.088	0.110	0.045
RC_3	0.135	0.200	0.087
RC_4	0.244	0.303	0.127
RC_5	0.553	0.615	0.433
RC_6	0.714	0.794	0.557
RC_7	0.943	1.048	0.735
RC_8	1.321	1.469	1.030
RC_9	4.578	3.953	5.802
RC_{10}	5.970	5.155	7.576

[32] See Appendix C for the estimation procedure of these consumer characteristics.

Figure 5: Time-Paths of Cross-Sections of Physical Capital and of Incomes 1993-60



the divergence for the RCs of deciles 7 and 8 lasts 18 years and 34 years, respectively, and is followed by convergence for 54 years and 38 years, respectively.

Finally, the relative variables of the RCs of deciles 9 and 10 are characterized by only the episode of convergence.

Overall, the relative variables converge. They converge from below for the RCs of deciles 1-7 and from above for the RCs of deciles 8-10. However, this convergence is conditional except in the case of the RC of decile 7 where it is absolute. This is an indication that inequalities in the distribution of physical capital and incomes are decreasing overall.

Using these relative variables I derive the variables of the RC for each decile (not shown). In particular, growth in income is achieved at all levels of incomes. Furthermore, the incomes of low deciles grow faster relative to those of high deciles in the second part of the process. Growth of incomes at all levels implies that poverty is decreasing as it can be seen from table 3. Whether we choose the poverty line of 1$ per day ($570) or of 2$ per day ($1140), poverty is totally eliminated at the RC levels by the end of the process. This result must be taken with caution since the elimination of poverty at the RC level does not necessarily imply the elimination of poverty within a decile. This is the reason why these results needs to be confronted to the ones from a microeconomic analysis of

Table 9: Income of RCs in 1993 and 2065 (in $ US)

	RC_1	RC_2	RC_3	RC_4	RC_5	RC_6	RC_7	RC_8	RC_9	RC_{10}
1993	124	228	422	663	1533	1949	2574	3690	13711	17803
2065	1200	3091	5465	8170	19685	22849	31705	40863	141012	172734

poverty before drawing any conclusion.

II.5. Income Inequality, Poverty, and Growth

The numerical solution to the model of the previous section has indicated that growth occurs at all levels of incomes. Further, growth reduces in overall income inequality as the RCs of the low deciles improve their relative positions in the distribution of income, while the RCs of the high deciles worsen theirs. Plus, growth eliminates poverty totally by the end of the process.

While these results seem appealing, however, they are obtained in the context of a representative consumer framework and thus may underestimate or overestimate the incidence of growth on income inequality and poverty. Indeed, the income distribution evaluated at a few points, representing the incomes of a few RCs, captures heterogeneity across groups but not within each group. If the within group heterogeneity is strong, then, income inequality is guaranteed to be under-estimated in this case, since "within group" inequality is assumed to be equal zero. Likewise, poverty rate is guaranteed to be under-estimated as well, since it will be assessed at the incomes of the RCs of poor deciles, which are always greater than those of the poorest within those deciles.

To get around these problems, I combine the solution of the heterogeneous model with the microeconomic data from the South Africa's 1996 October Household Survey (SA 1996 OHS) to generate the time-paths of income of each household. More precisely, I use the rates of growth of incomes of the RCs of deciles and apply them to the SA 1996 OHS to obtain income for each member of each household over 1996-2065. Then, I use these incomes to construct income distributions and measures of income inequalities and poverty. Armed with these measures, I then assess the interaction between growth, income inequalities, and poverty.

II.5.1 Constructing a Measure of Income from the SA 1996 OHS

I use the SA 1996 OHS to construct a measure of income for the year 1996 in South Africa. This survey is representative of the South African population and includes information on 80,000 individuals (16,000 households) from all population groups across the country. It contains 8 files: HOUSE, PERSON, WORKER, MIGRANT, DEATH, BIRTHS, INCOME, and DOMESTIC. The main file is House which includes economic, social, demographic, and geographic information on each household. The rest of the files provide additional information on each household. The file PERSON contains data on each household member's individual characteristics. The file WORKER provides information on the job status and other job characteristics of each household member aged 15 years or older. The file INCOME covers the sources of income for each household member. Finally, the file DOMESTIC includes information on domestic workers employed in each household.

I use the above files to construct a measure of income. This measure is a combination of the total income per household and the total expenditure per household for the year 1996. Total income per household is the sum of incomes earned by members of a household from employment and self-employment, plus other income and grants. The latter include state pensions, private pensions, social and disability grants, gratuities and other lump sum payments, unemployment benefits, other grants (old age grants, maintenance grants, dependant care grants, etc…), financial support from relatives, and other capital income. Total income per household, however, is reported only in terms of ranges of values. This requires the computation of the middle point which may overestimate or underestimate the income of the household. Plus, there are many missing values for this variable. This is the reason why I start with the total expenditure per household as the measure of income since it is reported for almost all households, and then complement it with a middle point data on total income per household whenever the total expenditure is missing. I call this variable income per household.[33] Furthermore, I exclude any household with missing value on income per household. The total number of households excluded is 725. This leaves a total of 15,295 households with a total of 70,285 consumers.

II.5.2 Estimating Income Distributions

In this sub-section I use the rates of growth of incomes from section 4, as well as the measure of income per household and the number of persons per household in 1996, to generate the time-paths of income for each consumer in the survey. To do this, I start by deriving the mean income per household in 1996, that is, the household income divided by the number of persons per household. Next, I classify consumers from the SA 1996 OHS into 10 deciles, based on their mean incomes, in order to use the income growth rates of the RCs for each decile. Recall that these rates of growth were generated from the cross-section variables and the variables of the RC of the economy starting from the year 1993 to the year 2065 (the steady state).

[33] The use of total expenditure as a proxy of income is justified in this case since the SA 1996 OHS reports a correlation coefficient of 98% between total expenditure and income.

Equipped with mean income per household from 1996 to 2065 and the number of persons per household,[34] I calculate the income per household for each year. Furthermore, I calculate total income in each year by summing the incomes of households, and then divide it by 100 to get the total income per centile for each year.[35] This total income per centile is further divided by the corresponding total number of consumers in the centile to obtain the mean income per centile for each year. This last measure of income, together with its weights (number of consumers in each centile), is the one I use to estimate the income distributions.

I resort to the non-parametric approach (the Gaussian kernel density) to estimate the distribution of income for each year. I estimate first the bandwidth (bw) using the formula $bw = 0.9 * sd * n^{-1/5}$, where sd is the standard deviation of the log of income, and n is the number of observations. The estimates of standard deviations and their corresponding bandwidths for selected years are presented in table 10.

Table 10: Standard Deviations and Bandwidths for Selected Years

Year	**1996**	**2015**	**2035**	**2055**	**2065**
sd	1.242154	1.242106	1.242715	1.245101	1.240334
bw	0.448708	0.448691	0.448912	0.449772	0.448051

Next, I use the estimates of bandwidths to evaluate the Gaussian kernel density at 100 points (mean incomes per centile) for each year and then normalize the distribution so that the area under the curve is 1. Figure 6 – Panels a-e display the distributions of

[34]Recall that population in the heterogeneous model is assumed to be constant over time. This implies that the number of consumers per household is held constant over time. This assumption obtains if we assume that the number of births is exactly equal to the number of deaths for each household in each year.

[35] I use income per centile instead of income per decile to increase the number of points at which the distribution is estimated.

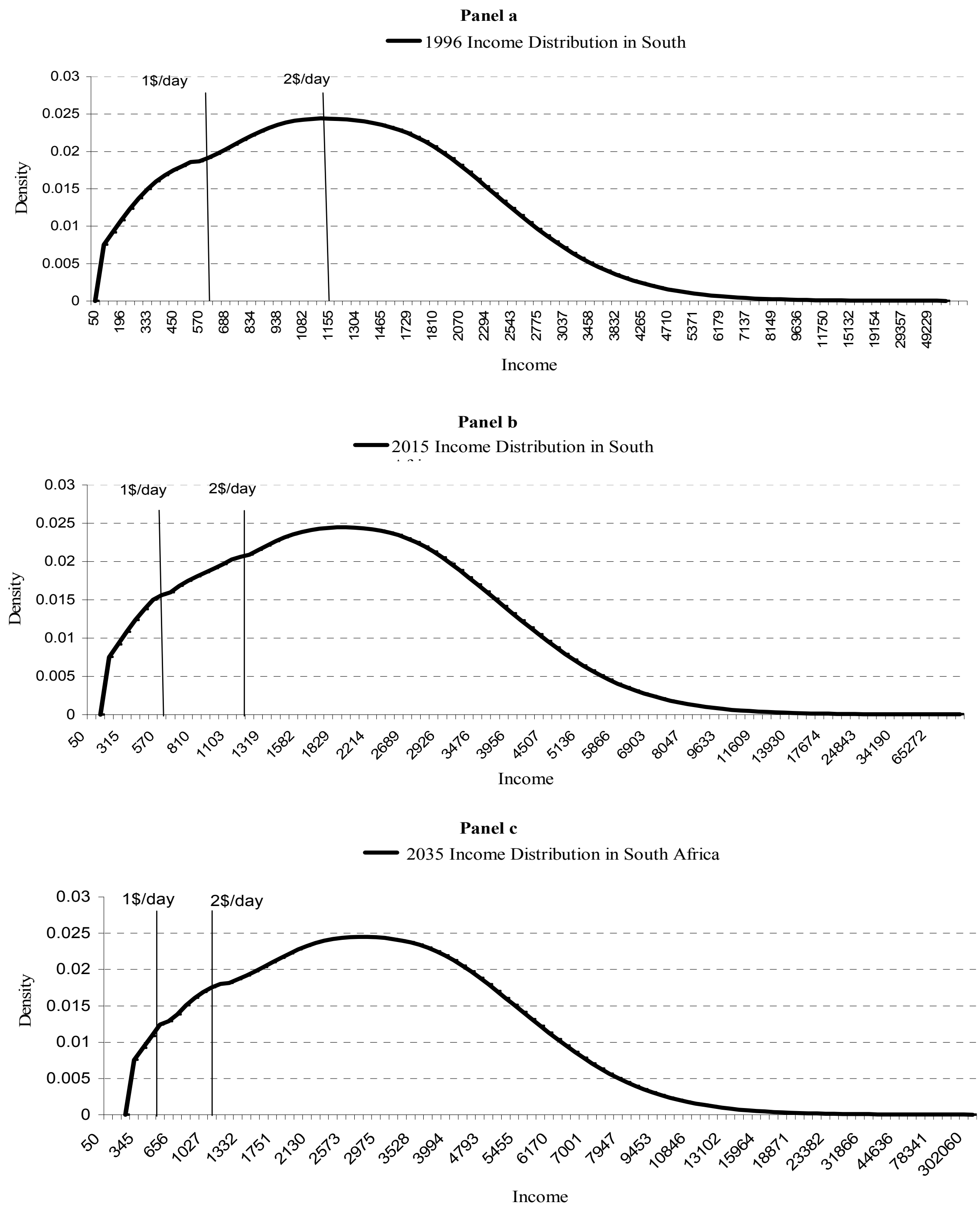
income for the selected years. They also show the poverty line (a vertical line) at a 1$ per day and at a 2$ per day.[36] Starting with the initial income distribution (Figure 6-panel a), we can see that it is positively skewed and has a shape similar to that of the Chi-squared distribution. This shape implies an uneven and large variability of income around its mean and thus large income inequality. Also, this distribution has a mode at $1,155. Furthermore, 46% of the population lies below the poverty line of $2 per day in 1996. The poverty rate is lower (17%) at the $1 per day poverty line.

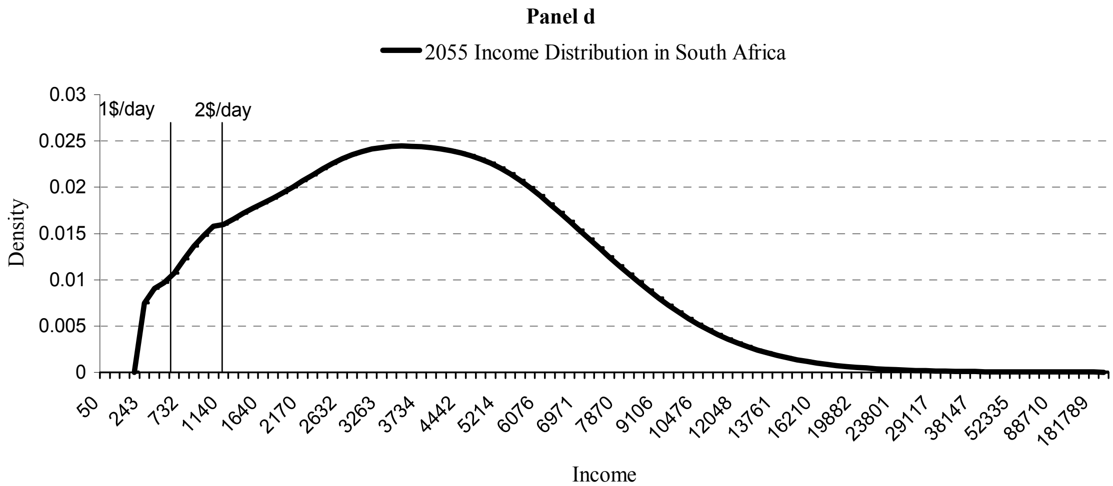
A comparison of the distributions of years 2015 (Figure 6-panel b), 2035 (Figure 6-panel c), 2055 (Figure 6-panel d), and 2065 (Figure 6-panel e) reveals almost no change in the shape of distribution of income over time. However, the distribution as well as its mode shifts to the right. In 1996, the lowest level of income was $82. By 2015, the lowest income level is $131. The lowest income level increases to attain $186 in 2035, $243 in 2055, and $279 in 2065. This shift is the result of increases in the levels of all incomes. Moreover, the mode of the distribution increases from $1155 in 1996 to $1928 in 2015, $2616 in 2035, $3480 in 2055, and $3917 in 2065. Also, poverty is decreasing as panels b to e of Figure 6 show a reduction of the fraction of the distribution at the left of both the $1 per day and $2 per day poverty lines.

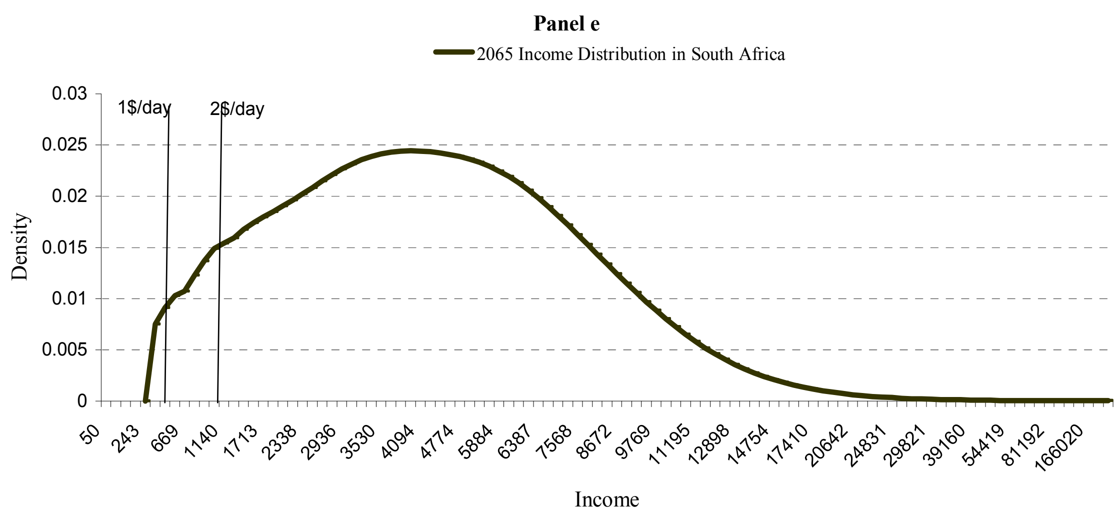
Next, I use Figure 6 to determine the poverty rate, that is, the percentage of population living below the poverty line. This poverty rate is displayed in Figure 7 for both the $1 per day and $2 per day poverty lines. As this figure reflects, the poverty rate at $2 per day poverty line falls from 46% in 1996 to 22%, 12%, 9%, and 7% in 2015,

[36] We use the adjusted poverty line from Sala-i-Martin (2003). Accordingly, one dollar per day represents 570 dollars a year in 1996 dollars. See also Barro and Sala-i-Martin (2004) and Sala-i-Martin (2006) for the definition of the poverty line.

Figure 6: South Africa's Income Distributions for Selected Years





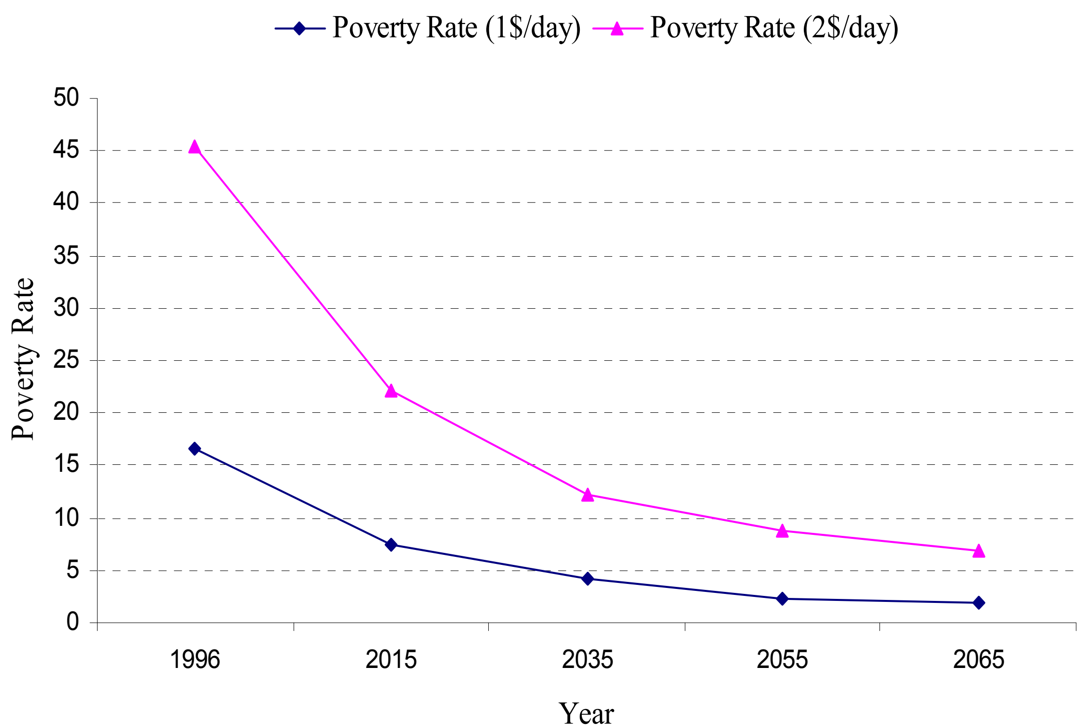


2035, 2055, and 2065, respectively. The poverty rate is lower at $1 per day poverty line. From 16.5%, it falls to 7.5% in 2015, 4.3% in 2035, 2.2% in 2055, and to 1.9% in 2065. Additionally, the poverty rate will continue to fall after 2065 sine all incomes will continue to grow in the steady state. The overall decrease in the poverty rate over 1996-2065 is of 85% at the $2 per day poverty line and of 89% at $1 per day poverty line. Although the overall decrease in the poverty rate is almost the same at both poverty lines, the patterns of this decrease are different. At the $2 per day poverty line, the decrease in the poverty rate is larger in the first part of the process rather than in the last part of the process. It is of 51% from 1996 to 2015 and of 45% from 2015 to 2035, but only of 27% from 2035 to 2055 and of 22% from 2055 to 2065. These patterns, however, are different at the $1 per day poverty line. The decrease in the poverty rate fluctuates over time. It is of 54% from 1996 to 2015, of 43% from 2015 to 2035, of 47% from 2035 to 2055, and of only 16% from 2055 to 2065. This difference in the patterns of reduction in the poverty rate does not have any intuitive explanation and shows how arbitrary can be the choice of the poverty line in the analysis of growth and poverty.

That growth is a key determinant of poverty reduction is strongly supported in this study. To show it, I compute the elasticity of poverty with respect to growth. The average growth elasticity of poverty over 1996-2065 is -3.7% at the $2 per day poverty line and -1.3% at a $1 per day poverty line. So a 1% increase in the rate of growth causes on average the poverty rate to drop by 3.7% and by 1.3% at the $2 per day and at the $1 per day poverty lines, respectively.

Turning now to income inequality, it is not clearly evident from Figure 6 whether income inequality is increasing or decreasing over time. The shape of the income distribution looks similar for all the years depicted. Thus, I need to construct precise measures of income inequality to analyze its evolution. I limit our analysis to 2 measures of income inequality, namely, the Gini coefficient (GC) and the Global Theil index (GTI). Figure 8-panel a displays the GC, and Figure 8-panel b shows the GTI. Both the GC and the GTI exhibit the shape similar to the well known “Kuznets curve” but with the income on the x-axis replaced by the time variable. This figure shows that income inequality increa-

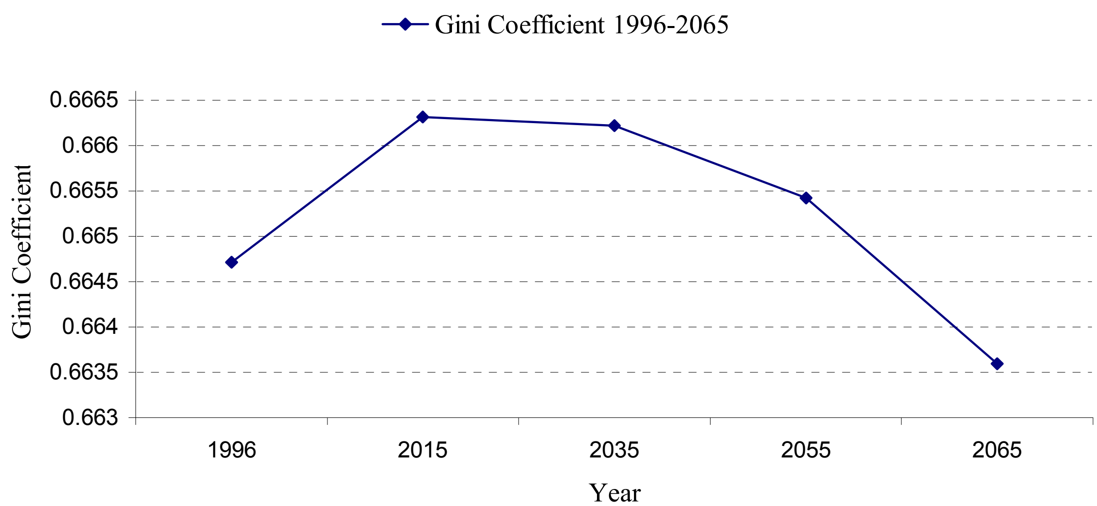
Figure 7: Poverty Rate in South Africa 1996-2065



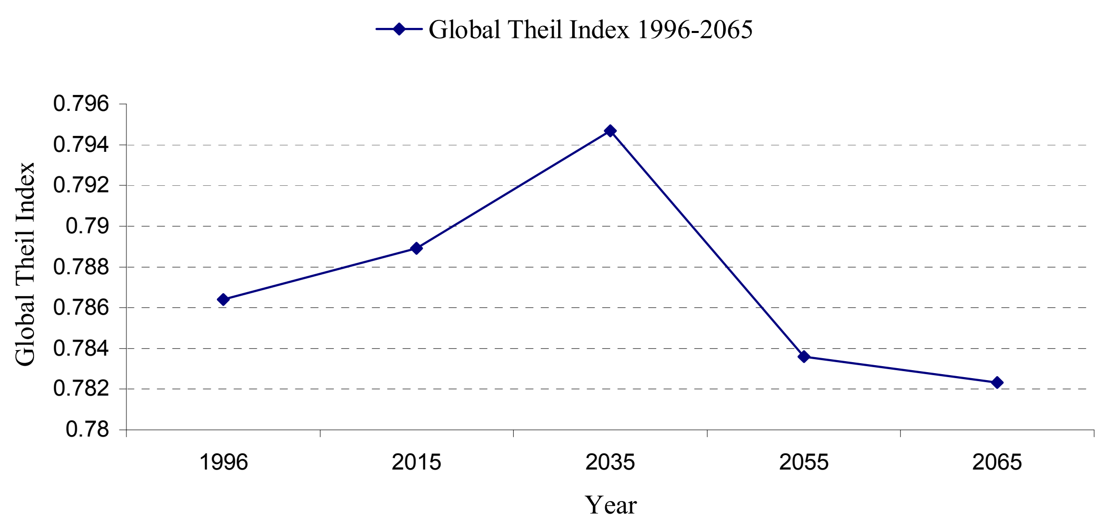
increases first but decreases thereafter. Overall, there is a decrease in income inequalities regardless of the measure used. But the overall decrease is very small. It is of only 0.17% for the GC and of 0.52% for the GTI. Also, the patterns of inequality are slightly different across the 2 measures. The GC curve shows that inequality increases by 0.24% from 1996 to 2015, and then decreases by 0.41% from 2015 to 2065. On the other hand, the GTI curve indicates an increase in inequality of 1.1% from 1996 to 2035 followed by a decline of 1.6% from 2035 to 2065. As for the case of poverty, we calculate the average growth elasticity of inequality. This elasticity is -0.056% and -0.11% for the GC and GTI, respectively. These 2 figures indicate that the effect of growth on inequality is very small. Although growth is achieved at all levels of incomes, incomes of consumers of the high deciles grow faster relative to those of consumers of low deciles in the first part of the process. The patterns are reversed in the second part of the process as

Figure 8: Income Inequality Measures in South Africa 1996-2065

Panel a



Panel b



incomes of consumers of low deciles grow faster. However, the differential in the rates of growth of income between low and high deciles' consumers is not large enough to reduce substantially the income inequality.

II.6. Conclusion

In this study, I have constructed a heterogeneous growth model, and built on the Caselli and Ventura (2000)'s methodology to analyze the dynamics of income inequality, poverty, and growth in the Post-Apartheid South Africa. Using aggregate data I found that growth is achieved at all levels of incomes and poverty is totally eliminated by the end of the process. Furthermore, poor consumers improve their relative positions in the distribution of wealth as well as of income, whereas rich consumers worsen theirs. However, the former category is outperformed in the early stage of the process by the latter category. In the early stage of the process, poor consumers are unable to achieve consistent rates of growth to sustain their optimal consumption in comparison to rich consumers. This situation is explained by the low rates of growth of their capital incomes relative to that of their labor incomes. The patterns are reversed in the middle of the process, allowing poor consumers to outperform rich consumers by the end of the process.

Next, I combined the results of the heterogeneous model with the microeconomic data (the SA 1996 OHS) to estimate the distributions of income and analyze thoroughly the interaction between growth, income inequality, and poverty. I found that growth reduces poverty substantially. A one percent increase in the rate of growth of income causes poverty to drop by 3.7% at the $2 per day poverty line and by 1.3% at the $1 per day poverty line. Overall, poverty drops by 89% at the $2 per day poverty line and by

85% at the \$1 per day poverty line. Moreover, growth causes overall decline in income inequality, but the effect is very small. Indeed, a one percent-increase in the rate of growth of income results on average in a decline in income inequality of only 0.056% by the GC and of 0.11% by the GTI. As in aggregate data, the evolution of income inequality is characterized by 2 phases; a phase of increase and a phase of decrease. During the first phase, income inequality rises by 0.24% (1996-2015) by the GC, and by 1.1% (1996-2035) by the GTI. This phase is followed by a slight reversal in income inequality. Income inequality drops by 0.41% the GC, and by 1.6% by the GTI.

It is apparent from the above analysis that growth is a key determinant of reduction in poverty but has very little effect on income inequality. Indeed, incomes of poor consumers grow faster relative to those of rich consumers in order to sustain their optimal consumption. However, the differential in the rates of growth of incomes of the 2 groups is too small to have a noticeable effect on overall income inequality.

This study can be extended in a number of ways. One extension could consist of building redistributive policies into the heterogeneous model in order to see how effective these policies are in reducing poverty and income inequality. Another extension would be to introduce dynamics in to the cross section of skills through heterogeneity in the rate of skill growth. Such an extension improves the analysis of the relationship among poverty, inequality, and growth since these issues relate primarily to the lack of skills characterizing poor people.

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Appendix A: Derivation of the equation of motion of u

Let rewrite the first order conditions -conditions $(1.14)-(1.19)$ in the text- as:

$$c^{-\sigma} - \lambda_k = 0, \qquad \forall t \qquad (A.1)$$

$$\lambda_k w(1-\tau_u) - \lambda_h \phi(1+\xi\eta_h(g/h)) = 0, \qquad \forall t \qquad (A.2)$$

$$\dot{k} = (1-\tau_u)wuh + (1-\tau_k)rk - \delta_k k - c + (1-\eta_h)g, \qquad (A.3)$$

$$\dot{h} = \phi(1-u)(h+\xi\eta_h g) - \delta_h h, \qquad (A.4)$$

$$\dot{\lambda}_k = -\lambda_k[(1-\tau_k)r - \delta_k - \rho], \qquad (A.5)$$

$$\dot{\lambda}_h = -\lambda_k(1-\tau_u)wu - \lambda_h[\phi(1-u) - \delta_h - \rho]. \qquad (A.6)$$

Plus the boundary conditions, that is, the following *TVC* and initial conditions:

$$\lim_{t\to\infty} \lambda_k(t)e^{-\rho t}k(t) = 0, \ \lim_{t\to\infty} \lambda_h(t)e^{-\rho t}h(t) = 0, \qquad (A.7)$$

$k(0) = k_0$, $h(0) = h_0$, k_0 and h_0 are given. $(A.8)$

Taking the log of $(A.2)$ yields:

$$\ln\lambda_k - \ln\lambda_h + \ln w + \ln(1-\tau_u)\phi^{-1} = \ln(1+\xi\eta_h(g/h)) \approx \xi\eta_h(g/h), \qquad (A.9)$$

In $(A.9)$ we have approximated $\ln(1+\xi\eta_h(g/h))$ by $\xi\eta_h(g/h)$ following a similar approximation used in Enders (2006, p.107). Taking the time-derivative of $(A.9)$ after substituting (1.11) for w, and rearranging yields:

$$\left[\alpha+\xi\eta_h(1-\alpha)\left(\frac{g}{h}\right)\right]\frac{\dot{u}}{u}=\frac{\dot{\lambda}_k}{\lambda_k}-\frac{\dot{\lambda}_h}{\lambda_h}+\alpha\left(1-\xi\eta_h\left(\frac{g}{h}\right)\right)\left(\frac{\dot{k}}{k}-\frac{\dot{h}}{h}\right). \tag{A.10}$$

Divide $(A.5)$ by λ_k and substitute for r to obtain:

$$\dot{\lambda}_k/\lambda_k=\rho+\delta_k-(1-\tau_k)\alpha Ak^{\alpha-1}(uh)^{1-\alpha}. \tag{A.11}$$

Divide $(A.6)$ by λ_h, substitute $(A.2)$ into the resulting expression, and rearrange yields:

$$\dot{\lambda}_h/\lambda_h=\rho+\delta_h-\phi-\phi\xi\eta_h(g/h)u. \tag{A.12}$$

Substituting (1.11) for r and w and (1.9) for g into $(A.3)$ and dividing by k yield:

$$\frac{\dot{k}}{k}=\left[1-\eta_h(\tau_u(1-\alpha)+\tau_k\alpha)\right]Ak^{\alpha-1}(uh)^{1-\alpha}-\delta_k-\frac{c}{k}. \tag{A.13}$$

Dividing $(A.4)$ by h it comes:

$$\dot{h}/h=\phi(1-u)(1+\xi\eta_h(g/h))-\delta_h. \tag{A.14}$$

Substitute $(A.11)-(A.14)$ into $(A.10)$ and rearrange to get:

$$\dot{u}=u\left[\alpha+(1-\alpha)\xi\eta_h\left(\frac{g}{h}\right)\right]^{-1}\left[\phi+\phi\xi\eta_h\left(\frac{g}{h}\right)u-(1-\tau_k)\alpha Ak^{\alpha-1}(uh)^{1-\alpha}+\delta_k-\delta_h+\right.$$

$$\alpha\left(1-\xi\eta_h\left(\frac{g}{h}\right)\right)\left\{(1-\eta_h(\alpha\tau_k+(1-\alpha)(1-\tau_u)))Ak^{\alpha-1}(uh)^{\alpha-1}-\frac{c}{k}-\right.$$

$$\left.\left.\phi(1-u)\left(1+\xi\eta_h\left(\frac{g}{h}\right)\right)\right\}\right]. \tag{A.15}$$

We can see from $(A.15)$ that the expression inside the braces is $\dot{k}/k-\dot{h}/h$. So we rewrite $(A.15)$ as:

$$\dot{u} = u\left[\alpha + (1-\alpha)\xi\eta_h\left(\frac{g}{h}\right)\right]^{-1}\left[\phi + \phi\xi\eta_h\left(\frac{g}{h}\right)u - (1-\tau_k)\alpha A k^{\alpha-1}(uh)^{1-\alpha} + \right.$$

$$\left.\alpha\left(1-\xi\eta_h\left(\frac{g}{h}\right)\right)\left(\frac{\dot{k}}{k}-\frac{\dot{h}}{h}\right)\right], \qquad (A.16)$$

which is the equation (1.22) on p.15 in the text.

Appendix B: Estimation of parameters used to solve the Lucas model

To solve the dynamic system described in (1.37) given the initial conditions (1.4) and the steady state conditions $(1.38)-(1.39)$ for the Post-Apartheid South African economy $(initial - period = 1995)$, we need to obtain the estimated values of the parameters as well as those of initial conditions.

We obtain the parameters of the production function as follows. First, we normalize technology parameter A to 1, and then estimate α using the following formula:

$$\alpha = 1 - (CSUL/GDP), \qquad (B.1)$$

where $CSUL$ is the compensation of skilled plus compensation of unskilled labor -the equivalent of Z in the CRS production function given in (1.13)- and GDP is the aggregate output. South Africa's data on $CSUL/GDP$ is obtained from the Version 5 of GTAP Aggregate Database 2004.

To estimate the initial per capita physical capital stock (k_0), we use the South Africa's data on per capita real investment (i_t) from Heston, Summers, and Aten's Penn Tables Version 6.2 to construct a series of the capital stock according the following rule:

$$k_{t+1} = (1-\delta_k)k_t + i_t, \qquad (B.2)$$

$$k_{T_0} = \bar{k}_{T_0}, \qquad (B.3)$$

where $T_0 = 1995,$ and δ_k is the depreciation rate of capital stock, which is calculated from the South Africa's data from the Version 5 of GTAP Aggregate Database 2004 using the following expression:

$$\delta_k = (VDEP/VKB), \tag{B.4}$$

where VKB and $VDEP$ are the Value of Capital Stock at the Beginning of period and the Value of Depreciation of Capital Stock, respectively. We choose $\bar{k}_{T_0}$ (initial capital stock) such that[37]:

$$k_{T_0+1}/k_{T_0} = \left(k_{T_0+10}/k_{T_0}\right)^{1/10}. \tag{B.5}$$

The initial per capita human capital is the average years of schooling of population aged 15 year old and over for the year 1995 from Barro and Lee (2000), and the human capital technology parameter ϕ is obtained from the following formula:

$$\phi = \left(\frac{1}{5}\frac{h_{t+1}^{25} - h_t^{25}}{h_t^{25}}\right)\left(\frac{1}{5}\right)^{38}, \tag{B.6}$$

where h_t^{25} is the average years of schooling of the population aged 25 and over. Furthermore, ξ is obtained in the following way. First, note that if all effort is allocated to the accumulation of human capital $(u = 0)$, the marginal product of $\dot{h}$ with respect to g is:

$$(\Delta h_t - \Delta h_{t-1})/(g_t - g_{t-1}) = \phi\xi,$$

where the numerator is the change in investment in human capital and the denominator is the change in expenditures on education. Using data on Human capital (measured by average years of schooling from Barro and Lee (2000)) and expenditures on education and training over 1995-2000, we obtain $\phi\xi$, which we divide by ϕ to get ξ.

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[37] This rule is taken from "Econ 8107 Macroeconomics, " Spring 2005, University of Minnesota.

[38] Using the average years of schooling for the population aged 15 and over yields a value of 0.02 for this parameter. This value yields in turn a negative value for the $BGP's$ rate of growth γ. This is the reason why we decide to use the average years of schooling for the population aged 25 and over.

The tax rates as well as the GDP's share of spending on education are obtained from the South Africa's national budgets over 1995-2006. The average national budget share of GDP over 1995-2006 has represented 30.1%. Since we have assumed a balanced budget, we reduce this share to the tax revenue share of GDP, which is 29%. Recall also that we have assumed that the economy is closed. This assumption implies that tax revenue does not include excise duties. The average excise duties' share of GDP is 4%. Subtracting this average excise duties' share of GDP from the tax revenue share of GDP yields the expenditure's share of GDP of 25%. For simplicity, we assume that capital and labor incomes are taxed at the same rate. This implies that $\tau_k = \tau_u = 0.25$. Also, the average GDP share of spending on education η_h from these national budget data is 7%. In terms of the balanced budget with closed economy, this share represents 27% of budget. The budget share of transfers η_T is the complement to unity of η_h, that is $\eta_T = 73\%$.

The preference parameters (ρ,σ) and the savings rate (s) are determined jointly from the SS conditions $(1.38)-(1.39)$. Recall from these conditions that r_{ss} is given by:

$$r_{ss} = (\rho + \delta_k + \gamma\sigma)/(1-\tau_k) \text{ or } r_{ss} = \alpha A \hat{k}_{ss}^{\alpha-1}\left(u_{ss}\hat{h}_{ss}\right)^{1-\alpha} \tag{B.7}$$

Also, the SS expression of $\left(\dot{\hat{k}}/\hat{k}\right)_{ss}$ from (2.38) is given by:

$$\left(\dot{\hat{k}}/\hat{k}\right)_{ss} = 0 = \left[1-\eta_h\left(\tau_u(1-\alpha)+\tau_k\alpha\right)\right]A\left(\hat{k}_{ss}\right)^{\alpha-1}\left(u_{ss}\hat{h}_{ss}\right)^{1-\alpha} - \delta_k - \gamma - \hat{c}/\hat{k}, \tag{B.8}$$

which we can rewrite as:

$$0 = \left[1-\eta_h\left(\tau_u(1-\alpha)+\tau_k\alpha\right)\right]\left(r_{ss}/\alpha\right) - \delta_k - \gamma - \left(\hat{c}/\hat{k}\right)_{ss} \quad \text{or} \tag{B.9}$$

$$r_{ss} = \alpha\left(\gamma + \delta_k + \hat{c}/\hat{k}\right)/\left[1 - \eta_h\left(\tau_u(1-\alpha) + \tau_k\alpha\right)\right]. \qquad (B.10)$$

Setting $(B.7) = (B.10)$ and rearrange yields:

$$\gamma\{[1 - \eta_h(\tau_u(1-\alpha) + \tau_k\alpha)]\sigma - \alpha(1-\tau_k)\} = \alpha(1-\tau_k)\left(\hat{c}/\hat{k}\right)_{ss} - [1 - \eta_h(\tau_u(1-\alpha) + \tau_k\alpha)]\rho. \qquad (B.11)$$

Substitute (2.39) for $\left(\hat{c}/\hat{k}\right)_{ss}$ into $(B.11)$ and rearrange to get:

$$\phi + \phi\xi\eta_h\psi u_{ss}^2 = \rho + \gamma\sigma \qquad (B.12)$$

Substitute (2.39) for ψ and u_{ss} into $(B.12)$ and rearrange to obtain:

$$\Phi(\rho,\sigma,s) = -\frac{\rho\alpha(1-\tau_k)}{\alpha(1-\tau_k) - s\sigma} + \left(\phi + \phi\xi\eta_h[\tau_u(1-\alpha) + \tau_k\alpha]\right)\left[\frac{\rho}{\alpha(1-\tau_k) - s\sigma}\right]^{\frac{\alpha}{\alpha-1}} x$$

$$\left[\frac{\{(2\phi - \rho)(\alpha(1-\tau_k) - s\sigma) - s\rho(1+\sigma)\}(\alpha(1-\tau_k) - s\sigma)^{\frac{1}{\alpha-1}}}{\phi(\alpha(1-\tau_k) - s\sigma)^{\frac{\alpha}{\alpha-1}} - \phi\xi\eta_h[\tau_u(1-\alpha) + \tau_k\alpha]A^{\frac{1}{1-\alpha}}\rho^{\frac{\alpha}{\alpha-1}}}\right]^2 \qquad (B.13)$$

For a given value of ρ, $(B.13)$ is solved for σ and s using Newton iteration method.

The values of parameters from this exercise are summarized in Table 1 on page 20.

Appendix C: Estimation of parameters used to solve the Extended Ramsey-Cass-Koopmans model

We rely on different sources to obtain the estimates of parameters (Table 7) as well as those of the initial conditions on consumer characteristics (Table 8) needed to solve our model to the South Africa economy. The estimation procedure is described in the following paragraphs.

We start with the parameters of the production function which include the technology parameter and the factor shares. We first normalize the technology parameter (A) to one. Then, we pick the physical capital share (α) for the year 1993 from Thurlow (2004, p.19). This parameter is estimated directly from the South Africa 1993 SAM and has an estimated value of 0.459 in 1993. It changes over time (its value is 0.489 in 2000) but we assume for simplicity that its value remains constant during the period under study.

Next, we pick the preference parameters from Tables 1 (the estimation procedure of these parameters is described in Appendix B). The estimated values of the time preference parameter (ρ) and the parameter of substitution (σ) in this table are 0.068 and 1.46, respectively. We use the same estimate for (ρ) but adjust the estimate of σ to 1.75 in order for the model to reproduce the data.

We estimate the average rate of growth of skills (γ) from Barro and Lee (2000). This study reports the average year of schooling for the population aged 15 old and over and for the population aged 25 year old and over. The estimate of γ over 1990-1995 of the population aged 15 old and over is 0.023 and that of the population aged 25 year old and over is 0.114. However, the estimate of 0.023 yields a very low rate of growth of the per capita GDP, while that of 0.114 yields an extremely high rate of growth of per capita GDP. We tried different values in this interval $[0.023 - 0.114]$ until we find an estimate that gets our solution close to data. This value is 0.037.

The estimates of the initial conditions on the physical capital (k_0) and skills (h_0) of the RC of the economy are obtained as follows. We normalize the initial skills of the RC of the economy to 1. Concerning her initial physical capital, we use first the estimate of the year 1995 from Table 1. Then we use the data point on the South Africa population in 1995 from Heston, Summers, and Aten's Penn Tables Version 6.2, and multiply this data point by the 1995 per capita physical capital to obtain the estimate of the aggregate physical capital in 1995. After that, we use data on aggregate investment from Penn Tables Version 6.2 for the year 1994 and 1993 and apply the perpetual inventory method to get the estimate of the aggregate physical capital in 1993. Dividing this estimate by the 1993 value of population from the same source yields the estimate of per capita physical capital in 1993, which is $10,527.

We now turn on to the estimation of the initial consumers characteristics. We use the information on the distributions of population and income, and on the household income and expenditures patterns in South Africa for the year 1993 from Thurlow (2004, pp38-39) to estimate k_d^R, c_d^R, and h_d^R. Recall that k_d^R, c_d^R, and h_d^R are relative quantities which require the knowledge of k, c, h, k_d, c_d, and h_d in 1993. We already have the estimates of k and h for the year 1993. We now have to estimate c, k_d, c_d, and h_d. To obtain the estimate of c in 1993, we first multiply the consumption share of the Gross National

Income (GNI) in 1993 by the GNI in 1993 to get the aggregate consumption in 1993, and then divide the aggregate consumption in 1993 by the total population in 1993. The 1993 data on the consumption share of GNI, the GNI, and population are from the Penn Tables Version 6.2.

Next, we estimate k_d for the year 1993 by using the aggregate physical capital in 1993 calculated above, the distributions of income and population (Thurlow, 2004), the distribution of the expenditure share of savings in 1993 (Thurlow, 2004). We start by assuming that savings in 1993 follows the same distribution as that of physical capital in 1993 (not reported). The distributions of population and income in Thurlow (2004) cover the 10 deciles but that of saving share of expenditures in this study covers only groups of deciles (low, middle class, and high). To get around this problem, we assume further that the distribution of saving share of expenditures inside each group of deciles is the same as that of income inside the group. This allows us to estimate the distribution of saving share of expenditures across the 10 deciles in 1993, which we combine with the distribution of population in 1993 and the 1993 aggregate physical capital to derive the distribution of physical capital in 1993. Dividing physical capital per decile by the corresponding population per decile we get k_d. Also, we use this distribution of saving share of expenditures in 1993 to derive the distribution of consumption share of expenditures across the 10 deciles in 1993. Consumption is a residual and is obtained by subtracting savings from income. Dividing consumption per decile by the corresponding population per decile yields c_d.

Finally, the estimation of h_d is done in 2 steps. In the first step we derive the aggregate skills by multiplying the per capita skills (normalized to 1) by the 1993 population. In the next step we use the distributions of labor income (Thurlow, 2004) and population in 1993 as well as the aggregate skills to estimate h_d. We use the distribution of labor income as a proxy of the distribution of skills. This proxy is relevant since skills (years of schooling) is homogeneous so that labor income is proportional to skills. However, the distribution of labor income covers the 3 groups mentioned above. We use the same assumption as in the case of the savings rate to assess the distribution of skills inside each group. Doing so allows us to estimate the distribution of skills across deciles in 1993, which we combine with the distribution of population to obtain the estimate of h_d.

Dividing k_d by k, c_d by c, and h_d by h we obtain the estimates of k_d^R, c_d^R, and h_d^R, respectively. These estimates are reported in Table 8 on p.63 in the text.