SINGULAR SOLUTIONS FOR A CONVECTION DIFFUSION EQUATION WITH ABSORPTION

By

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Singular Solutions For A Convection Diffusion Equation With Absorption

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Abstract

In this paper we prove the existence of a *very singular solution* of the Cauchy problem

$$u_t = \Delta u + \vec{a} \cdot \nabla u^q - u^p, \qquad u(x,0) = 0 \quad \text{if} \quad x \neq 0, \quad (\vec{a} \text{ constant })$$

which is more singular at (0,0) than the fundamental solution of the heat equation if $1 < p < (N+2)/N$ and $1 \leq q \leq (p+1)/2$. We also prove the nonexistence of singular solutions if $p \geq (N+2)/N$ and $1 \leq q \leq (p+1)/2$.

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§1 Introduction

Consider the Cauchy problem

$$u_t - \Delta u - \vec{a} \cdot \nabla u^q + u^p = 0 \qquad \text{in } Q = R^N \times (0, \infty) \tag{1.1}$$

$$u(x,0) = 0 \qquad \text{for} \quad x \neq 0. \tag{1.2}$$

where $\vec{a}$ is a constant vector and $\vec{a} \neq 0$. By a solution we mean a nonnegative function $u(x,t)$ which is continuous in $\bar{Q} \setminus \{(0,0)\}$, and satisfies (1.1) and (1.2) in the classical sense; in particular, $u \in C^2(Q)$. The behavior of $u(x,t)$ as $(x,t) \to (0,0)$, $(x,t) \in Q$ is not prescribed so that u may exhibit a singularity at the origin. Nontrivial solution with a singularity at (0,0) can be obtained by considering (1.1) with the initial condition

$$u(x,0) = c\delta(x) \qquad \text{in} \quad R^N. \tag{1.3}$$

Indeed, for $\vec{a} = 0$, $1 < p < (N+2)/N$, Brezis and Friedman [2] proved that there exists a unique solution of (1.1), (1.3). On the other hand, for $\vec{a} = 0$, $p \geq (N+2)/N$, Brezis and Friedman showed, in the same paper, that no solution of (1.1) and (1.3) can exist. The case $\vec{a} \neq 0$ was first considered by J. Aguirre and M. Escobedo [1]. They showed that for $1 < p < (N+2)/N$, and $1 < q < (N+1)/N$, there is a unique solution of (1.1), (1.3). However, they did not answer the question what happens if $p \geq (N+2)/N$ or $q > (N+1)/N$. This is the first problem we want to study. A partial answer is the following:

Theorem 1.1. If $\vec{a} \neq 0$, $p \geq (N+2)/N$ and $1 < q \leq (p+1)/2$, then there exists no singular solution of (1.1), (1.2); in particular, no solution of (1.1) and (1.3) can exist.

For the case $\vec{a} = 0$, $1 < p < (N+2)/N$, Brezis, Pletier and Terman [4] found another type of solution of (1.1) and (1.2) which has a stronger singularity at (0,0), namely,

$$\lim_{t \to 0} \int u(x,t) = +\infty. \tag{1.4}$$

such a solution is called a *very singular solution* (VSS). A natural question is whether a VSS exists for the case $\vec{a} \neq 0$. We shall establish the following:

Theorem 1.2. Assume that $\vec{a} \neq 0$, $1 < p < (N+2)/N$.

(i) If $1 < q < (p+1)/2$, then for any $0 < \delta$, $0 < \epsilon < 1/4$, there exist positive constants c and C, depending on ϵ and δ, such that

$$ct^{-\frac{1}{p-1}} e^{-(1/4+\delta)|x|^2/t} \leq w(x,t) \leq Ct^{-\frac{1}{p-1}} e^{-(1/4-\epsilon)|x|^2/t}; \tag{1.5}$$

(ii) If $q = (p+1)/2$, then the conclusion of (i) remains valid under the assumption that $1/4 - \epsilon$ is sufficiently small ; if $q = 1$, then the conclusion of (i) still holds provided $|\vec{a}|$ is sufficiently small.

Next we shall explore the relationship between the solutions of (1.1), (1.3) and VSS and obtain the following result.

Theorem 1.3. Let $1 < p < (N+2)/N$, $1 \leq q \leq (p+1)/2$. If u_c denotes the solution of (1.1), (1.3), then $u_c \to \tilde{w}$ as $c \to \infty$ and $\tilde{w}$ is a VSS of (1.1), (1.2).

Remark 1.4. In the case $\vec{a} = 0$, Theorem 1.3 was proved by Kamin and Pletier [5].

Remark 1.5. We actually presented two different proofs of the existence of a VSS in Theorems 1.2 and 1.3. While Theorem 1.3 connects problem (1.1), (1.3) and VSS, Theorem 1.2 gives quite accurate information of the behavior of $w(x,t)$ near $(0,0)$. However, we are unable to prove that $w = \tilde{w}$ since the uniqueness of VSS remains open. For the case $\vec{a} = 0$, the VSS is unique; see [3], [5] and [6].

Remark 1.6. In all three theorems, we require that $1 \leq q \leq (p+1)/2$. Notice that for $q = (p+1)/2$ the differential equation is invariant under scaling, which explains the upper bound on q.

Our proof of Theorem 1.1 follows quite closely that of Brezis and Friedman [2]. To prove Theorem 1.2, the method in [3] does not carry over. First of all, if $q \neq (p+1)/2$, a VSS is not necessarily self-similar, i.e. we can not write a VSS in the form $t^{-1/(p-1)} f(x/\sqrt{t})$.

Secondly, even in the case $q = (p+1)/2$, the induced equation for f is not spherically symmetric and so the ODE approach does not work. Our method for proving Theorem 1.2 is new and is based on a monotone iteration scheme which requires super and sub solutions of (1.1). As for the proof of Theorem 1.3, we follow the approach used by Kamin and Pletier [5]. However, there are new technical difficulties in the case $\vec{a} \neq 0$.

In §2, we establish some preliminary results. Theorem 1.1 is proved in §3. In §4, super and sub solutions are constructed. We then use monotone iteration scheme in §5 to prove Theorem 1.2. Finally, we prove Theorem 1.3 in §6.

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§2 Preliminaries

Let Ω be a smooth domain in R^N, and $Q_t = \Omega \times (0,t)$. Introduce the Hölder norm

$$|u|_{C^\alpha(Q_T)} = |u|_{L^\infty(Q_T)} + \sup_{P,Q\in Q_T} \frac{|u(P)-u(Q)|}{d(P,Q)^\alpha}$$

where $P=(x,t)$, $Q=(\bar{x},\bar{t})$, and

$$d(P,Q) = \{|x-\bar{x}|^2 + |t-\bar{t}|\}^{1/2}.$$

Set $C^{2+\alpha}(Q_T) = \{u : |u|_{C^{2+\alpha}(Q_T)} < \infty\}$ where the $C^{2+\alpha}(Q_T)$-norm of u is the sum of the C^α-norms of u, u_t, $D_x u$ and $D_x^2 u$. The symbol C will represent a constant, not necessarily the same at each occurrence.

We shall need the following lemma which was proved in [7] (see chap. III, Th. 7.1).

Lemma 2.1. Suppose u satisfies

$$u_t - \Delta u - \sum_{i=1}^N b_i u_{x_i} - au = f.$$

If

$$|\sum_{i=1}^N b_i^2,\ a,\ f^2|_{q,r,Q_T} \le \mu$$

where

$$|g|_{q,r,Q_T} = (\int_0^T (\int |g(x,t)|^q dx)^{r/q} dt)^{1/r}$$

with

$$q \in [\frac{N}{2(1-\kappa)}, \infty],\quad r \in [\frac{1}{1-\kappa}, \infty],\quad 0<\kappa<1 \text{ for } N \ge 2$$

$$q \in [1,\infty],\quad r \in [\frac{1}{1-\kappa}, \frac{2}{1-2\kappa}],\qquad 0<\kappa<1/2, \text{ for } N=1,$$

then

$$|u|_{L^\infty(Q_T)} \leq C$$

where C depends only on N and μ.

Consider the nonlinear parabolic equation:

$$u_t - \Delta u = \vec{a} \cdot \nabla(|u|^{q-1}u) - ku + f \quad \text{in } Q_T; \tag{2.1}$$

$$u(x,t) = \varphi(x,t) \quad \text{on } \partial\Omega \times (0,T) \cup \Omega \times \{0\}. \tag{2.2}$$

where k is a positive constant, $\varphi \in C^{2+\alpha}(Q_T)$, $f \in C^\alpha(Q_T)$. Let $v = u - \varphi$. Then

$$v_t - \Delta v = \vec{a} \cdot \nabla(|v+\varphi|^{q-1}(v+\varphi)) - kv + \tilde{f} \quad \text{in } Q_T; \tag{2.3}$$

$$v = 0 \qquad \text{on } \partial\Omega \times (0,T) \cup \Omega \times \{0\} \tag{2.4}$$

where $\tilde{f} = f - \varphi_t + \Delta\varphi - k\varphi$.

Lemma 2.2. There exists a unique classical solution v of the problem (2.3), (2.4) in Q_{t_0} for some sufficiently small t_0.

Proof. Let $K(x,t;y,s)$ be the Green function of the heat equation on $\Omega \times (0,T)$ with Dirichlet boundary conditions. We first look for the weak solution v of (2.3)-(2.4), i.e.

$$\begin{aligned} v(x,t) &= \int_0^t \int_\Omega K(x,t;y,s)[-kv(y,s) + \tilde{f}(y)] \\ &\quad - \int_0^t \int_\Omega (\nabla_y K(x,t;y,s) \cdot \vec{a})|v+\varphi|^{q-1}(v+\varphi)(y,s)dyds. \end{aligned} \tag{2.5}$$

Let $X_{M,t_0} = \{u \in C^0(Q_T) : \ |u|_{L^\infty(Q_{t_0})} \leq M\}$. Then X_{M,t_0} is a Banach space. For $v \in X_{M,t_0}$, define Lv to be the right side of (2.5). One can easily check that for large M and small t_0, L maps X_{M,t_0} into itself and L is a contraction. Hence L has a unique fixed point v which is the weak solution of (2.3)-(2.4). By J. Aguirre and M. Escobedo [1], the weak solution is classical.

In order to extend the solution globally, we need some a priori estimates.

Lemma 2.3. Let $1 \leq q < 2$ and $k > 0$. Let v be a classical solution of (2.3)-(2.4) in Q_T $(t_0 \leq T < \infty)$ and $u = v + \varphi$. Then

$$|u|_{L^\infty(Q_T)} \leq C \tag{2.6}$$

where C is a constant depending only on data.

Proof. For any positive odd integer p, multiply (2.3) by v^p and integrate over Q_T. Using integration by parts, we get

$$\begin{aligned} \frac{1}{p+1}\int_\Omega v^{p+1}(x,t)dx + p\int_0^T\int_\Omega v^{p-1}|\nabla v|^2 + k\int_0^T\int_\Omega v^{p+1} &= \\ + \int_0^T\int_\Omega [\vec{a} \cdot \nabla(|u|^{q-1}u)](u-\varphi)^p + \int_0^T\int_\Omega \tilde{f}v^p & \\ \stackrel{\text{def}}{=} I_1 + I_2. & \end{aligned} \tag{2.7}$$

Using Young's inequality, we proceed to estimate these integrals as follows:

$$I_2 \le \epsilon \int_0^T \int_\Omega v^{p+1} + C \int_0^T \int_\Omega |\tilde{f}|^{p+1}, \tag{2.8}$$

and

$$\begin{aligned} I_1 &= \int_0^T \int_\Omega [\vec{a} \cdot \nabla(|u|^{q-1}u)]u^p + \int_0^T \int_\Omega [\vec{a} \cdot \nabla(|u|^{q-1}u)][-p\varphi u^{p-1} + \cdots + (-1)^p \varphi^p] \\ &= \int_0^T \int_{\partial\Omega} (\vec{a} \cdot \vec{n}) A_0(\varphi) - p \int_0^T \int_{\partial\Omega} (\vec{a} \cdot \vec{n}) A_1(\varphi)\varphi \\ &\quad + p \int_0^T \int_\Omega A_1(u)(\vec{a} \cdot \nabla\varphi) + \cdots + (-1)^p \int_0^T \int_\Omega A_p(u)\vec{a} \cdot \nabla\varphi^p \end{aligned}$$

where $\vec{n}$ is the unit outer normal to $\partial\Omega$ and $A_i(t) = \int_0^t s^{p-i}|s|^{q-1}ds,\ i = 0, 1, \ldots, p$. Observe that $|A_i(u)| \le C[|u|^{p+q-1} + 1],\ i = 1, \ldots, p$. Since $q < 2$, we have, by Young's inequality again,

$$I_1 \le C + \epsilon \int_0^T \int_\Omega |u|^{p+1}. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we see that

$$\int_0^T \int_\Omega u^{p-1}|\nabla u|^2 + \int_0^T \int_\Omega u^{p+1} \le C \tag{2.10}$$

where C depends only on data. From Lemma 2.1, the conclusion of Lemma 2.3 now follows.

The estimate (2.6) can be used to extend the solution beyond $t = T$. Indeed, we can repeat the proof of Lemma 2.1 with $t = T - \delta$ as the initial time; in view of (2.6), the solution exists for $T - \delta \le t \le T - \delta + t_0$ where t_0 is independent of δ (Actually, t_0 depends only on how large M is, whereas M is independent of δ if $|u(t - \delta)|_{L^\infty} \le C$). It follows that the solution can be extended to $0 \le t \le T + t_0$.

Proceeding step by step, we can extend the weak solution and therefore classical (by [1]) to all $t > 0$. We proved:

Theorem 2.4. Let $1 \le q < 2,\ k > 0$. Then there exists a unique classical solution of (2.1), (2.2) for any $T > 0$.

Next we shall establish an a priori estimate for solutions of (1.1) and (1.2).

Lemma 2.5. Let $1 < p < \infty,\ 1 \le q \le (p+1)/2$. There exists positive constants $C > 0$ and T, such that for any solution $u(x,t)$ of (1.1), (1.2),

$$u(x,t) \le C \frac{(1 + |x|^2)^{(p+1-2q)/2(p-1)(p-q)}}{(t + |x|^2)^{1/(p-1)}} \quad \text{fot} \ \ (x,t) \in R^N \times (0,T).$$

Proof. The following argument is a direct adaptation of the one given in [2]. Fix $0 < R < |x_0|,\ x_0 \in R^N$. Set

$$\Omega = \{(x,t) : |x - x_0|^2 < R^2 + t \ \text{ with } 0 < t < T\}.$$

Let $V(x,t) = C(R)(R^2+t)^{\alpha/2}/(R^2-r^2+t)^\alpha$ where $C(R)$ is to be determined later on, $\alpha = 2/(p-1)$ and $r = |x-x_0|$. It is easy to see that:

$$V_t - \Delta V - \vec{a}\cdot V^q + V^p \geq \frac{\alpha}{2}\frac{C(R)(R^2+t)^{\alpha/2-1}}{(R^2-r^2+t)^\alpha} - \frac{4C(R)\alpha(1+\alpha)r^2(R^2+t)^{\alpha/2}}{(R^2-r^2+t)^{2+\alpha}}$$
$$-\frac{C(R)(2N+1)\alpha(R^2+t)^{\alpha/2}}{(R^2-r^2+t)^{1+\alpha}} - \frac{2q\alpha|\vec{a}|C(R)^q r(R^2+t)^{\alpha q/2}}{(R^2-r^2+t)^{1+\alpha q}} + \frac{C(R)^p(R^2+t)^{\alpha p/2}}{(R^2-r^2+t)^{\alpha p}}.$$

Note that $\alpha = 2/(p-1)$; therefore

$$V_t - \Delta V - \vec{a}\cdot V^q + V^p \geq 0 \quad \text{in } \Omega$$

provided

$$\frac{C(R)^{p-1}(R^2+t)}{2} \geq 4\alpha(1+\alpha)r^2 + (2N+1)\alpha(R^2-r^2+t) \tag{2.11}$$

and

$$\frac{C(R)^{p-q}(R^2+t)^{\alpha(p-q)/2}}{2} \geq 2\alpha q|\vec{a}|r(R^2-r^2+t)^{\alpha p-(1+\alpha q)}. \tag{2.12}$$

By choosing $C(R)$ large (but independent of R), (2.11) is obviously true in Ω and (2.12) follows by:

$$\begin{aligned}\frac{r(R^2-r^2+t)^{\alpha p-(1+\alpha q)}}{(R^2+t)^{\alpha(p-q)/2}} &\leq (R^2+t)^{\alpha p-(1+\alpha q)+1/2-\alpha(p-q)/2} \\ &\leq (R^2+1)^{\alpha(p-q)/2-1/2} \quad \text{(assuming } T \leq 1).\end{aligned}$$

Hence if we choose

$$C(R) \geq C(R^2+1)^{\frac{\alpha(p-q)-1}{2(p-q)}},$$

then (2.12) is valid. Now a comparison argument yields

$$u \leq V \quad \text{in } \Omega.$$

Substituting $x = x_0$ in this inequality and letting $R \uparrow |x_0|$, we obtain the desired estimate.

§3 Nonexistence when $p \geq (N+2)/N,\ 1 \leq q \leq (p+1)/2$

Theorem 1.1 is a consequence of the following theorem.

Theorem 3.1. Let $p \geq (N+2)/N$, $1 \leq q \leq (p+1)/2$. If u is a solution of (1.1) and

$$\lim_{t\to 0}\int u(x,t)\varphi(x)dx = 0 \tag{3.1}$$

for any $\varphi \in C_c(R^N \setminus \{0\})$, then $u \in C^{2,1}(R^N \times [0,\infty))$.

To prove the theorem, we follow the idea of Brezis and Friedman [2]. It was proved in [2] that if u is a solution of (1.1), (3.1), then u satisfies (1.2). We shall first establish an auxiliary result (cf. Step 5 in [2]).

Lemma 3.2 For any $\rho > 0$,

$$\int_{|x|<\rho} \int_0^T u(x,t)^p dx dt < \infty$$

where T is given in Lemma 2.5.

Proof. Let $\eta(t)$ be any smooth non decreasing function on R such that:

$$\eta(t) = \begin{cases} 1 & \text{for} \quad t \geq 2, \\ 0 & \text{for} \quad t \leq 1, \end{cases}$$

and set $\eta_k(t) = \eta(kt)$.

From Lemma 2.5, it follows that

$$\int_0^T \int_{|x|<\rho} u(x,t) dx dt < \infty. \tag{3.2}$$

Take a function $\chi \in C_0^\infty(R^N \times (-2T, 2T))$ with $0 \leq \chi \leq 1$, $\chi = 1$ on $B_\rho \times (0,T)$ and set:

$$\phi_k(x,t) = \eta_k(|x|^2 + t)\chi(x,t).$$

Note that ϕ_k vanishes on a neighborhood of (0,0). Multiplying (1.1) by ϕ_k and integrating over $R^N \times (0, \infty)$, we deduce that

$$\int\int u^p \phi_k = \int\int u(\phi_k)_t - \int\int u^q \vec{a} \cdot \nabla \phi_k + \int\int u \Delta \phi_k. \tag{3.3}$$

Set $D_k = \{(x,t) : 1/k < |x|^2 + t < 2/k\}$. The argument used in [2] yields:

$$\int\int u^p \phi_k \leq C(k \int\int_{D_k} u + \sqrt{k} \int\int_{D_k} u^q) \tag{3.4}$$

By Lemma 2.5,

$$\begin{aligned} \int\int_{D_k} u^q &\leq C \int\int_{D_k} \frac{1}{(|x|^2 + t)^{q/(p-1)}} dx dt \\ &\leq C k^{q/(p-1)} \cdot \text{meas} D_k \leq C k^{-1/2} \end{aligned}$$

provided $q \leq (p+1)/2$, $p \geq (N+2)/N$. Hence $\sqrt{k} \int\int_{D_k} u^q$ remains bounded as $k \longrightarrow \infty$. The other term can be dealt with similarly .

Proof of Theorem 3.1. Define

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t < 0. \end{cases}$$

If we can show that

$$\tilde{u}_t - \Delta\tilde{u} + \vec{a}\cdot\nabla\tilde{u}^q + \tilde{u}^p = 0 \qquad \text{in the distribution sense,}$$

the conclusion of the theorem follows. Let $\chi \in C_0^\infty(R^N \times (-T,T))$ and $\phi_k = \eta_k(|x|^2+t)\chi$, where η_k is defined in the proof of Lemma 3.1. Then (3.3) still holds and it suffices to verify that

$$\int\int u(\eta_k)_t\chi \to 0,$$
$$\int\int u\Delta(\eta_k)\chi \to 0,$$
$$\int\int u\nabla\eta_k\cdot\nabla\chi \to 0,$$
$$\int\int u^q(\vec{a}\cdot\nabla\eta_k)\chi \to 0.$$

The first three terms were treated in [2]. As for the last term, we use Hölder's inequality to deduce that:

$$\begin{aligned}|\int\int u^q(\vec{a}\cdot\nabla\eta_k)\chi| &\le C\sqrt{k}\int\int_{D_k} u^q \\ &\le C\sqrt{k}(\int\int_{D_k} u^p)^{q/p}(\text{meas } D_k)^{1-q/p}.\end{aligned}$$

Recall that meas $D_k = Ck^{-(1+N/2)}$, $p \ge (N+2)/N$ and $1 < q \le (p+1)/2$; therefore $\sqrt{k}(\text{meas } D_k)^{1-q/p} \le C$ and $|\int\int u^q(\vec{a}\cdot\nabla\eta_k)\chi| \le C(\int\int_{D_k} u^p)^{q/p} \to 0$ (by Lemma 3.2). This completes proof of Theorem 3.1.

§4 Constructions of Super and Sub Solutions

Definition 4.1. A positive C^2 function u is called a super (resp. sub) solution of (1.1) if

$$u_t - \Delta u - \vec{a}\cdot\nabla u^q + u^p \ge 0 \quad (\text{resp.}) \le 0 \quad \text{in } R^N \times [0,T]. \tag{4.1}$$

Let $1 \le q \le (p+1)/2$; whence $\theta \triangleq (1-q)/(p-1) + 1/2 \ge 0$. If u has the form $t^{-1/(p-1)}f(|x|/\sqrt{t})$ ("self-similar" form), then (4.1) is reduced to

$$\begin{aligned}&f''(r) + (\frac{N-1}{r}+\frac{r}{2})f'(r) + T^\theta qf^{q-1}(r)|f'(r)\frac{\vec{a}\cdot x}{|x|}| \\ &+\frac{1}{p-1}f(r) - f^p(r) \le 0 \quad (\text{resp.} \ge 0) \text{ in } R^N\end{aligned} \tag{4.2}$$

where $r = |x|/\sqrt{t}$ and the prime denotes the differentiation with respect to r. Consequently, if we can find an f such that

$$f'' + (\frac{N-1}{r}+\frac{r}{2})f' + T^\theta q|\vec{a}|f^{q-1}|f'| + \frac{1}{p-1}f - f^p \le 0 \tag{4.3}$$

$$(\text{resp. } f'' + (\frac{1}{p-1} + \frac{r}{2})f' - T^\theta q|\vec{a}|f^{q-1}|f'| + \frac{1}{p-1}f - f^p \geq 0),$$

then the $u(x,t)$ defined above is a super (resp. sub) solution of (1.1).

Lemma 4.2. For any $0 < \epsilon < 1/4$, let $\bar{u}(r) = Ae^{-\epsilon r^2}$. If $1 < p < \infty$, $1 \leq q < (p+1)/2$, then $\bar{u}$ is a super solution of (1.1) provided A is sufficiently large. If $q = (p+1)/2$, then $\bar{u}$ is still a super solution of (1.1) provided A is large and ϵ is small.

Proof. It is readily verified that (4.3) is equivalent to

$$\bar{K}u \stackrel{\text{def}}{=} (r^{N-1}e^{r^2/4}u')' + r^{N-1}e^{r^2/4}\{T^\theta q|\vec{a}|u^{q-1}|u'| + \frac{1}{p-1}u - u^p\} \leq 0. \tag{4.4}$$

Now for $\bar{u}(r) = Ae^{-\epsilon r^2}$, we compute:

$$\begin{aligned} \bar{K}\bar{u} &\leq [-4\epsilon(\frac{1}{4} - \epsilon)Ar^{N+1} + 2\epsilon T^\theta q|\vec{a}|A^q r^N e^{-\epsilon(q-1)r^2} \\ &\quad + \frac{1}{p-1}Ar^{N-1} - A^p r^{N-1}e^{-\epsilon(p-1)r^2}]e^{(\frac{1}{4}-\epsilon)r^2} \\ &\stackrel{\text{def}}{=} [I_1 + I_2 + I_3 + I_4]e^{(\frac{1}{4}-\epsilon)r^2}. \end{aligned} \tag{4.5}$$

It is easy to see that

$$\frac{1}{2}I_1 + I_3 = -Ar^{N+1}[2\epsilon(\frac{1}{4} - \epsilon) - \frac{1}{(p-1)r^2}] \leq 0 \tag{4.6}$$

for $r \in [r_1(\epsilon), \infty)$, where $r_1(\epsilon)^2 = [2(p-1)\epsilon(1/4 - \epsilon)]^{-1}$. On the other hand, from the fact that $r_1(\epsilon)^2\epsilon \leq M$, we deduce that

$$\begin{aligned} \frac{1}{2}I_4 + I_3 &= -Ar^{N-1}[A^{p-1}e^{-\epsilon(p-1)r^2} - \frac{1}{p-1}] \\ &\leq -Ar^{N-1}[A^{p-1}e^{-(p-1)M} - \frac{1}{p-1}] \qquad (4.7) \\ &\leq 0 \quad \text{for } r \in [0, r_1(\epsilon)] \qquad (4.8) \end{aligned}$$

provided A is sufficiently large. Hence

$$\frac{1}{2}(I_1 + I_4) + I_3 \leq 0 \quad \text{in } [0, \infty) \tag{4.9}$$

for $A \geq A_0$ where A_0 depends only on data..

If $q = 1$, the estimate of I_2 is the same as above for I_3. If $q > 1$, we compute:

$$\frac{1}{2}I_1 + I_2 = -2\epsilon Ar^{N+1}[(\frac{1}{4} - \epsilon) - T^\theta q|\vec{a}|r^{-1}A^{q-1}e^{-\epsilon(q-1)r^2}]. \tag{4.10}$$

Let $\Psi_1(r, A, \epsilon) = r^{-1}A^{q-1}e^{-\epsilon(q-1)r^2}$. Then Ψ_1 is strictly decreasing in r. Consequently, if $r_1(A, \epsilon)$ is the unique solution of $(\frac{1}{4} - \epsilon) - T^\theta q|\vec{a}|\Psi_1(r, A, \epsilon) = 0$, then

$$\frac{1}{2}I_1 + I_2 \leq 0 \quad \text{for } r \in [r_1(A, \epsilon), \infty). \tag{4.11}$$

To estimate I_2 in $[0, r_1(A,\epsilon)]$, we make use of the nonlinear term I_4:

$$
\begin{aligned}
\frac{1}{2}I_4 + I_2 &= -\frac{Ar^{N+1}}{2}[r^{-2}A^{p-1}e^{-\epsilon(p-1)r^2} - 4\epsilon T^\theta q|\vec{a}|\Psi_1(r,A,\epsilon)] \\
&= -\frac{Ar^{N+1}}{2}[r^{\frac{p-1}{q-1}-2}\Psi_1(r,A,\epsilon)^{\frac{p-1}{q-1}} - 4\epsilon T^\theta q|\vec{a}|\Psi_1(r,A,\epsilon)] \\
&= -\frac{Ar^{N+1}}{2}\Psi_1(r,A,\epsilon)[r^\nu \Psi_1(r,A,\epsilon)^\kappa - 4\epsilon T^\theta q|\vec{a}|] \qquad (4.12)
\end{aligned}
$$

where $\kappa = (p-1)/(q-1) - 1 > 0$ and $\nu = (p-1)/(q-1) - 2 \geq 0$. We shall first deal with the case $\nu > 0$. For fixed $\epsilon < 1/4$, let $A > A_0$ be large enough so that

$$
r_1(A,\epsilon)^\nu (1/4 - \epsilon)^\kappa > 4\epsilon (T^\theta q|\vec{a}|)^{1+\kappa}
$$

where we used the fact that $r_1(A,\epsilon) \to \infty$ as $A \to \infty$. Noting that $\Psi_2(r,A,\epsilon) \triangleq r^\nu \Psi_1(r,A,\epsilon)^\kappa = r^{-1}A^{\kappa(q-1)}e^{-\epsilon\kappa(q-1)r^2}$ is monotone decreasing in r and $\Psi_2(r_1(A,\epsilon),A,\epsilon) > 4\epsilon T^\theta q|\vec{a}|$, we conclude from (4.12) that

$$
\frac{1}{2}I_4 + I_2 \leq 0 \quad \text{for } r \in [0, r_1(A,\epsilon)] \qquad (4.13)
$$

if A is sufficiently large. From (4.9), (4.11) and (4.13), the conclusion of the lemma follows in the case $\nu > 0$.

Suppose now that $\nu = 0$. Recall that $T^\theta q|\vec{a}|\Psi_1(r_1(A,\epsilon),A,\epsilon) = \frac{1}{4} - \epsilon$. Hence if ϵ is is chosen so that $(\frac{1}{4} - \epsilon)^\kappa \geq 4\epsilon T^\theta q|\vec{a}|^{(p-1)/(q-1)}$, then $\Psi_1(r_1(A,\epsilon),A,\epsilon)^\kappa - 4\epsilon T^\theta q|\vec{a}| \geq 0$. Since $\Psi_1(r,A,\epsilon)$ is decreasing in r , we conclude from (4.12) again that

$$
\frac{1}{2}I_4 + I_2 \leq 0 \quad \text{for } r \in [0, r_1(A,\epsilon)]. \qquad (4.14)
$$

From (4.9), (4.11) and (4.14), the conclusion of the lemma follows in the case $\nu = 0$.

Next we shall construct sub solutions. If $p < (N+2)/N$, then there exists an ϵ small so that $1/(p-1) > 2(1/4+\epsilon)N$.

Lemma 4.3. Assume that $1 < p < (N+2)/N$. Let $\underline{u} = Ae^{-(\epsilon+1/4)r^2}$.

(i) If $1 < q \leq (p+1)/2$, then $\underline{u}$ is a sub solution of (1.1) provided A is sufficiently small.

(ii) If $q = 1$, then the conclusion of (i) remains valid provided A and $|\vec{a}|$ are both sufficiently small.

Proof. As before it suffices to verify that

$$
\underline{K}\ \underline{u} \stackrel{\text{def}}{=} (r^{N-1}e^{r^2/4}\underline{u}'(r))' + r^{N-1}e^{r^2/4}[-T^\theta q \underline{u}^{q-1}|\underline{u}'(r)| + \frac{1}{p-1}\underline{u} - \underline{u}^p] \geq 0. \qquad (4.15)
$$

It is clear that

$$
\begin{aligned}
\underline{K}\ \underline{u} \ &\geq \ [(\frac{1}{p-1} - 2(\epsilon+1/4)N)A + 4\epsilon(\epsilon+1/4)Ar^2 \\
&\quad -2T^\theta q|\vec{a}|(\epsilon+1/4)A^q r e^{(\epsilon+1/4)(q-1)r^2} - A^p e^{-(\epsilon+1/4)(p-1)r^2}]e^{-\epsilon r^2} r^{N-1} \\
&\stackrel{\text{def}}{=} \ [K_1+K_2+K_3+K_4]e^{-\epsilon r^2} r^{N-1}. \qquad (4.16)
\end{aligned}
$$

Notice that $\delta = \frac{1}{p-1} - 2(\epsilon+1/4)N > 0$ by the assumption on ϵ. Let $\Psi_3(r) = A^{q-1} r^{-1} e^{-(\epsilon+1/4)r^2}$. Then Ψ_3 is decreasing in r. Let $r_2(A)$ be defined by $\Psi_3(r_2(A))|\vec{a}| = \epsilon$. Then

$$
\begin{aligned}
\frac{1}{2}K_2 + K_3 \ &\geq \ 2(\epsilon+1/4)A[\epsilon - |\vec{a}|\Psi_3(r)] \\
&\geq \ 0 \qquad \text{for } \ r \in [r_2(A), \infty). \qquad (4.17)
\end{aligned}
$$

Since we are going to choose A to be small and $r_2(A)$ is increasing in A, we may assume that $A \leq 1$ and $r_2(A) \leq C$. It follows that

$$
\frac{1}{2}K_1 + K_3 \geq A[\frac{1}{2}\delta - 2(\epsilon+1/4)|\vec{a}|A^{q-1}r] \geq 0 \quad \text{for } r \in [0, C] \qquad (4.18)
$$

provided $A \leq A_0$ and A_0 is small. Combining (4.17) and (4.18), we obtain $\frac{1}{2}(K_1+K_2) \geq 0$ in $[0,\infty)$. Since K_4 can be dealt with similarly, the conclusion of (i) follows. The case $q = 1$ is similar. We only need to notice that (4.18) becomes

$$
\frac{1}{2}K_1 + K_3 \geq A[\frac{1}{2}\delta - 2(\epsilon+1/4)|\vec{a}|r] \geq 0 \quad \text{for } r \in [0, C]
$$

provided $|\vec{a}|$ is sufficiently small.

§5 The Proof Of Theorem 1.2

We shall introduce a monotone iteration scheme to prove the existence of a positive solution of the parabolic equation

$$
u_t - \Delta u - \vec{a}\cdot\nabla u^q = g(x,u) \quad \text{in } \ Q_T \qquad (5.1)
$$

$$
u = \varphi \quad \text{on } \ \partial\Omega\times(0,T)\cup\Omega\times\{0\} \stackrel{\text{def}}{=} \partial_p Q_T \qquad (5.2)
$$

provided there exist positive super and sub solutions of (5.1); a super (resp. sub) solution u of (5.1) means that

$$
u_t - \Delta u - \frac{x\cdot\nabla u}{2} - \vec{a}\cdot\nabla(|u|^{q-1}u) \geq g(x,u) \quad (\text{resp. } \ \leq). \qquad (5.3)
$$

Theorem 5.1. Let $g(x,u) \in C^1(\Omega\times R^1)$ and $\varphi \in C^{2+\alpha}(Q_T)$. Let $\underline{u}$, $\bar{u}$ be positive smooth super and sub solutions of (5.1) respectively and $\underline{u} \leq \bar{u}$. If $\underline{u} \leq \varphi \leq \bar{u}$ on $\partial_p Q_T$

and $1 \leq q < 2$, then there exists a unique classical solution u of (5.1,) (5.2). Moreover, $\underline{u} \leq u \leq \bar{u}$.

Proof. The uniqueness of the positive solution follows easily from the maximum principle.

If $q = 1$, this theorem is a standard result, see [8]. Let $1 < q < 2$. Choose K large enough so that

$$\frac{\partial g(x,u)}{\partial u} + K \geq 0$$

for $(x,u) \in \Omega \times [\inf \underline{u},\ \sup \bar{u}]$. For $v \in C^\alpha(Q_T)$, define $u = Lv$ to be the solution of

$$u_t - \Delta u + \vec{a} \cdot \nabla(|u|^{q-1}u) + Ku = g(x,v) + Kv; \tag{5.4}$$

$$u = \varphi \qquad \text{on } \partial_p Q_T. \tag{5.5}$$

The existence and uniqueness of u are ensured by Theorem 2.4. We shall prove the following comparison principle: if $v_1 \leq v_2$, $v_i \in [\inf \underline{u},\ \sup \bar{u}]$ and neither $u_1 \triangleq Tv_1$ nor $u_2 \triangleq Tv_2$ changes sign, then $u_1 \leq u_2$. Let $w = u_2 - u_1$. It is easy to see that

$$\begin{aligned} &w_t - \Delta w + \vec{h_1} \cdot \nabla w + h_2 w + Kw \\ &= (\frac{g(x,v_2) - g(x,v_1)}{v_2 - v_1} + K)(v_2 - v_1) \geq 0 \end{aligned}$$

where $\vec{h_1} = q|u_2|^{q-1}\vec{a}$ and

$$h_2 = \begin{cases} q(\vec{a} \cdot \nabla u_1)(|u_2|^{q-1} - |u_1|^{q-1})/(u_2 - u_1) & \text{if } u_1 \neq u_2 \\ q(\vec{a} \cdot \nabla u_1)|u_1|^{q-2}/(q-1) & \text{otherwise.} \end{cases} \tag{5.6}$$

If neither u_1 nor u_2 changes sign, then h_2 is bounded in Q_T, and we conclude that $w \geq 0$ by the maximum principle.

Defining $u_1 = L\underline{u}$, we shall prove that $\underline{u} \leq u_1$; in particular, u_1 does not change sign. Let $\bar{w} = u_1 - \underline{u}$. Since

$$u_1 - \Delta u_1 - \vec{a} \cdot \nabla(|u|^{q-1}u_1) = g(x,\underline{u}) + K\underline{u}$$

$$\underline{u}_t - \Delta \underline{u} - \vec{a} \cdot \nabla(\underline{u}^q) \leq g(x,\underline{u}) + K\underline{u},$$

we have

$$\bar{w}_t - \Delta \bar{w} - |u_1|^{q-1}\vec{a} \cdot \nabla \bar{w} + h_3 \bar{w} \geq 0$$

where h_3 is defined by similar formulas as in (5.6). We conclude that $\bar{w} \geq 0$ by the maximum principle again.

Similarly, one can prove that $v_1 = L\bar{u} \leq \bar{u}$. Since $\underline{u} \leq \bar{u}$ and u_1 does not change sign, we get that $u_1 \leq v_1$ from the comparison principle. Now defining inductively $u_{n+1} = Lu_n$, $v_{n+1} = Lv_n$, we have

$$\underline{u} \leq u_1 \leq u_2 \leq \cdots u_n \leq v_n \leq \cdots \leq v_1 \leq \bar{u}.$$

Let $u = \lim_{n\to\infty} u_n$ and $v = \lim_{n\to\infty} v_n$. As in the elliptic case (see [8]), u and v are classical solutions of (5.1), (5.2). By the uniqueness of the solution, $u= v$.

Proof of Theorem 1.2. First we note that (1.4) follows from (1.5). Hence it suffices to prove that there exists a w satisfying (1.1), (1.2) and (1.5). Let $\phi_j(x,t) \in C^2(R^N \times [0,T])$ be such that $\underline{u}(x,t+1/j) \le \phi_j(x,t) \le \bar{u}(x,t+1/j)$ where $\underline{u}$ and $\bar{u}$ are constructed in §4. By Theorem 5.1, there is a unique solution u_j^R of (5.1), (5.2) with $g(x,u) = -u^p$, $\Omega = B_R(0)$ and $\varphi = \phi_j$; furthermore, $\underline{u}(x,t+1/j) \le u_j^R(x,t) \le \bar{u}(x,t+1/j)$. By L^p-theory [7],

$$|u_j^R|_{W^{2,p}[B_M(0)\times(0,T)]} \le C$$

for any $M > 0$, $1 < p < \infty$, where C is a constant depending only on M, p. We can therfore extract a sequence $u_j^{R_k}$, with $R_k \to \infty$, which converges to a solution u_j of (1.1) with $u_j(x,0) = \phi_j(x)$. Moreover,

$$\underline{u}(x,t+1/j) \le u_j(x,t) \le \bar{u}(x,t+1/j). \tag{5.7}$$

Now for any compact subset $K \subset R^N \times [0,T] \setminus \{(0,0)\}$, we have $u_j|_K \le C$ where C is independent of j. This can be easily seen from (5.7). By L^p-estimates and Shauder estimates, we can extract a subsequence $\{u_{j'}\}$ which converges in $C^2(K)$ to a function $w(x,t)$. Using a diagonal argument if necessary, we may assume

$$u_{j'} \to w \quad \text{in } C^2(K)$$

for any compact subset $K \subset R^N \times [0,T] \setminus \{(0,0)\}$. In particular, w is a solution of (1.1), (1.2). Letting $j' \to \infty$ in (5.7), we see that

$$\underline{u}(x,t) \le w(x,t) \le \bar{u}(x,t) \quad \text{for } t > 0. \tag{5.8}$$

Since (1.5) follows from (5.8), the proof is complete.

§6 The Proof Of Theorem 1.3

We first recall a result of Aguirre and Escobedo [1]. Assume that $1 < p < (N+2)/N$, $1 \le q \le (N+1)/N$.

Lemma 6.1. Let $S(t)$ denotes the semigroup generated by Δ, i.e. $S(t)\phi_j = E(\cdot,t) * \phi_j$ and

$$E(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}.$$

Let $\phi_j \in L^1(R^N)$ be chosen so that $\phi_j \to c\delta(x)$ in the distribution sense, and

$$|S(t)\phi_j(x)| \le CS(t+\theta_j)\delta(x)$$

where $\theta_j \to 0$ as $j \to \infty$. If u_j is the solution of (1.1) with $u_j(x,0) = \phi_j(x)$, then $u_j(x,t) \to u_c(x,t)$ in $C(K)$ for any compact subset $K \subset R^N \times (0,\infty)$.

Remark 6.2. In the case $\vec{a} = 0$, Brezis and Friedman [2] showed that the conclusion of Lemma 6.1 remains valid under weaker assumptions that $\phi_j \in L^1(R^N)$, $|\phi_j|_{L^1(R^N)} \leq M$ and $\phi_j \to c\delta(x)$ in the distribution sense.

Remark 6.3. If the conclusion of Lemma 6.1 is true under the weaker assumptions of Remark 6.2 in the case $\vec{a} \neq 0$, then the proof of Theorem 1.3 will be much easier. However, we are unable to prove Remark 6.2 in this case.

Let $V(x,t) = At^{-1/(p-1)}e^{-\epsilon|x|^2/4t}$ be a super solution of (1.1) (see §4). Observe that

$$\int_{R^N} V(x,t)dx = \omega'_N At^{-1/(p-1)+N/2}\int_0^\infty e^{-\epsilon\eta^2/4}\eta^{N-1}d\eta \tag{6.1}$$

where ω'_N is the area of the unit ball in R^N. Remembering that $p < (N+2)/N$, we conclude that, for every $c > 0$, there exists a unique τ_c such that

$$\int_{R^N} V(x,\tau_c)dx = c.$$

For $M > 0$, we define the truncated function

$$V_M(x,t) = \begin{cases} V(x,t) & \text{if } V(x,t) \leq M, \\ M & \text{if } V(x,t) > M. \end{cases}$$

By (6.1) for any $\tau \in (0,\tau_c)$ there exists a unique number $M(\tau)$ such that $\int_{R^N} V_{M(\tau)}(x,\tau)dx = c$.

Let us define $\phi_j(x) = V_{M(1/j)}(x,1/j)$, $j = J, J+1, \ldots$, where J has been chosen so that $1/J < \tau_c$, and let u_j be the solution of (1.1) with $u_j(x,0) = \phi_j(x)$. Clearly, there exists a unique $a(j) > 0$ such that

$$\phi_j(x,t) = \begin{cases} V(x,1/j) & \text{if } |x| \geq a(j), \\ M(1/j) & \text{if } |x| < a(j), \end{cases} \tag{6.2}$$

and

$$M(1/j) = V(a(j),1/j) = Aj^{1/(p-1)}e^{-\epsilon a(j)^2/4}. \tag{6.3}$$

Now we can state the key lemma in the proof of Theorem 1.3.

Lemma 6.4. For $1 < p < (N+2)/N$, we have

$$\phi_j(x) \leq \frac{C}{a(j)^N}e^{-|x|^2/4a(j)^2} \tag{6.4}$$

and

$$a(j) \to 0. \tag{6.5}$$

The proof of this lemma will be postponed to the end of this section. We shall also need the following lemma which was established in [4] (see chap. II, Theorem 9).

Lemma 6.5. Let $Lu = u_t - \Delta u + \vec{b}\cdot\nabla u + cu$. Assume that

$$|\vec{b}(x,t)| \leq C(|x|+1),$$

$$|c(x,t) \leq C(|x|^2+1).$$

If $Lu \geq 0$ in $Q \triangleq R^N \times (0,\infty)$, $u(x,0) \geq 0$ and

$$u(x,t) \geq -Ae^{B|x|^2},$$

where A, B are positive constants, then $u \geq 0$ in Q.

Now we are ready to prove Theorem 1.3. First it is easy to see that

$$u_j(x,t) \leq V(x,t+1/j). \tag{6.6}$$

Indeed, if $w = V(x,t+1/j) - u_j(x,t)$, then

$$w_t - \Delta w + u_j^{q-1}\vec{a} \cdot \nabla w + (\vec{a} \cdot \nabla V) f_1 w + f_2 w \geq 0$$

where

$$f_1 = \frac{V^{q-1} - u_j^{q-1}}{V - u_j},$$

$$f_2 = \frac{V^p - u_j^p}{V - u_j}.$$

Since $|\nabla V| \leq C(j)|x|e^{-|x|^2/4(t+1/j)}$ and $|f_1| \leq CV^{q-2} \leq C(j)e^{(2-q)|x|^2/4(t+1/j)}$, we have

$$\begin{aligned} |(\vec{a} \cdot \nabla V) f_1| &\leq C(j)|x|e^{-(q-1)|x|^2/4(t+1/j)} \\ &\leq C(j). \end{aligned}$$

Similarly, f_2 is also bounded in x at ∞. Noting that $w(x,0) = V(x,1/j) - V_{M(1/j)}(x,1/j) \geq 0$, we conclude that $w \geq 0$ by Lemma 6.5 and (6.6) follows.

Let us now verify that ϕ_j satisfy the assumptions of Lemma 6.1. Clearly, $\phi_j \to c\delta(x)$ in the distribution sense. Notice that (6.4) yields $\phi_j(x) \leq CS(a(j)^2)\delta(x)$; whence $S(t)\phi_j(x) \leq CS(t)S(a(j)^2)\delta(x) = CS(t+a(j)^2)\delta(x)$. Since $a(j) \to 0$, Lemma 6.1 is applicable. Letting $j \to \infty$ in (6.6), we obtain

$$u_c(x,t) \leq V(x,t) \tag{6.7}$$

for any $c > 0$.

On the other hand, set $\phi_1^{(j)} = c_1 E(x,1/j)$, $\phi_2^{(j)} = c_2 E(x,1/j)$ and let $u_1^{(j)}$, $u_2^{(j)}$ be the corresponding solutions of (1.1) with $u_k^{(j)}(x,0) = \phi_k^{(j)}(x)$, $k = 1,2$. Then one can easily see that $\phi_1^{(j)}$, $\phi_2^{(j)}$ satisfy the conditions of Lemma 6.1. If $c_1 \leq c_2$, then $\phi_1^{(j)} \leq \phi_2^{(j)}$; whence $u_1^{(j)} \leq u_2^{(j)}$. Letting $j \to \infty$ we see that $u_{c_1} \leq u_{c_2}$, i.e. u_c is monotone increasing in c. From (6.7) it follows that $\tilde{w} = \lim_{c\to\infty} u_c(x,t)$ exists. By standard parabolic regularity theory, one concludes that $\tilde{w}$ is a classical solution of (1.1), (1.2). Finally, observe that $\int \tilde{w}(x,t)dx \geq \int u_c(x,t)dx$ for any $c > 0$ and hence

$$\liminf_{t\downarrow 0} \int \tilde{w}(x,t)dx \geq \lim_{t\downarrow 0} \int u_c(x,t)dx = c.$$

Since c can be chosen arbitrarily large, it follows that $\tilde{w}$ is a VSS.

It remains to prove Lemma 6.4. We first show that

$$a(j)j^{1/2} \to \infty \quad \text{as } j \to \infty. \tag{6.8}$$

By (6.2) and the fact that $\int \phi_j(x)dx = c$, we have

$$Aj^{1/(p-1)}\Big(\int_{|y|\geq a(j)} e^{-\epsilon j|y|^2/4}dy + e^{-\epsilon j a(j)^2/4}\omega_N a(j)^N\Big) = c \tag{6.9}$$

or

$$j^{p-1}\Big(\omega_N' \int_{a(j)}^{\infty} e^{-\epsilon j r^2/4} r^{N-1}dr + \omega_N a(j)^N e^{-\epsilon j a(j)^2/4}\Big) = \frac{c}{A} \tag{6.10}$$

where ω_N is the volume of the unit ball in R^N. It follows that

$$j^{1/(p-1)} e^{-\epsilon j a(j)^2/4} a(j)^N \leq C \tag{6.11}$$

and

$$j^{p-1}\int_{a(j)}^{\infty} e^{-\epsilon j r^2/4} r^{N-1} dr \leq C. \tag{6.12}$$

Rewriting (6.12) as

$$j^{1/(p-1)-N/2}\int_{a(j)\sqrt{j}}^{\infty} e^{-\epsilon\eta^2/4}d\eta \leq C < \infty$$

and noting that $1/(p-1) > N/2$, we get (6.8).

We shall next prove (6.5). If this is not true, there is a subsequence $j' \to \infty$ such that $a(j') \geq \delta > 0$. Therefore,

$$j'^{1/(p-1)} e^{-\epsilon j' a(j')^2/4} a(j')^N \to 0 \quad \text{as } j' \to \infty \tag{6.13}$$

and for some $\zeta < \epsilon/4$ we obtain

$$\begin{aligned} j'^{1/(p-1)}\int_{a(j')}^{\infty} e^{-\epsilon j' r^2/4} r^{N-1} dr &= j'^{1/(p-1)-N/2}\int_{a(j')\sqrt{j'}}^{\infty} e^{\epsilon\eta^2/4}\eta^{N-1}d\eta \\ &\leq j'^{1/(p-1)-N/2} e^{-\zeta j' a(j')^2}\int_{a(j')\sqrt{j'}}^{\infty} e^{-(\epsilon/4-\zeta)\eta^2}\eta^{N-1}d\eta \\ &\longrightarrow 0 \quad \text{as } j' \to \infty. \end{aligned} \tag{6.14}$$

Combining (6.13) and (6.14), we get a contradiction to (6.10).

Lastly, let us prove (6.4). If $|x| \leq a(j)$, then

$$\begin{aligned} \phi_j(x) &\equiv Aj^{1/(p-1)} e^{-\epsilon j a(j)^2/4} \\ &\leq \frac{C}{a(j)^N} \qquad \text{(by (6.11))} \\ &\leq \frac{C}{a(j)^N} e^{-|x|^2/4a(j)^2} \end{aligned}$$

where we used the fact that $e^{-|x|^2/4a(j)^2} \geq e^{-1/4}$ for $|x| \leq a(j)$. For $|x| > a(j)$, we have

$$\begin{aligned} \phi_j(x) &= Aj^{1/(p-1)}e^{-\epsilon j|x|^2/4}e^{-\epsilon j(|x|^2-a(j)^2)} \\ &\leq \frac{C}{a(j)^N}e^{-\epsilon j(|x|^2-a(j)^2)/4} \quad \text{(by (6.11))}. \end{aligned}$$

Hence (6.4) is reduced to

$$e^{-\epsilon j(|x|^2-a(j)^2)/4+|x|^2/4a(j)^2} \leq C. \tag{6.15}$$

For $a(j) \leq |x| \leq Ka(j)$, $K > 1$, (6.15) is true because $|x|^2 - a(j)^2 \geq 0$ and $e^{|x|^2/4a(j)^2} \leq e^{K^2}$. on the other hand, for $|x| > Ka(j)$, we have

$$\begin{aligned} \text{left side of (6.15)} &= e^{-\epsilon j|x|^2[1-a(j)^2/|x|^2-1/\epsilon ja(j)^2]/4} \\ &\leq e^{-\epsilon j|x|^2(1-1/K^2-1/\epsilon ja(j)^2)/4} \\ &\leq C \end{aligned}$$

for j sufficiently large because of (6.8). This completes the proof of Lemma 6.4.

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