Models of second-order superintegrable systems

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ABSTRACT

The study of superintegrable systems has progressed far beyond analysis of specific examples, especially in the case where the constants of the motion are quadratic in the momenta. In this thesis, I begin with a brief overview of the structure analysis for second order superintegrable systems both in classical and quantum mechanics. In 2d and 3d conformally flat spaces, the algebra generated by the constants of the motion has been proven to be a finitely generated quadratic algebra with closure at finite order. Models are exhibited of the quadratic algebras for each equivalence class of 2d second order quantum superintegrable systems. I also describe some classical models of the algebras and their role in determining the quantum systems. Finally, a model for the 3d singular isotropic oscillator quadratic algebra is given.

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Dedication

To Esther

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Chapter 1

Introduction

Symmetry has played a fundamental role in the study of physical systems and their structures. Sometimes the symmetries are physically obvious. In these cases, the symmetries of a physical system can usually be analyzed using the method of group theory. The symmetries of the system can take the form of discrete symmetries such as reflection or continuous symmetries such as rotation or translation. In the case of continuous symmetries, we have the methods of Lie theory to describe such systems. Lie theory, named after the Norwegian mathematician Sophus Lie (1842-1899), utilizes the algebra of the infinitesimal generators of a symmetry group to understand the group structures. The infinitesimal generators of a continuous symmetry group form a vector space with a bilinear product called the Lie bracket. With this product, the vector space forms an algebra which is easier to analyze than the original group structures. There has been much research on the representation theory of Lie algebras and their associated group structures including the the relation between the representation theory of Lie algebras and physical structures [1, 2, 3].

While many of the best understood physical systems exhibit obvious symmetry in group actions, most also exhibit higher order symmetries which are not obvious in the physical system. These symmetries are understood classically as constants of the motion or as commuting differential operators in the quantum case. If the operators are of first order they will correspond to a group symmetry. However, at higher order there is, in general, no associated group structure and the symmetry is considered to be "hidden." A study of these "hidden symmetries" has been systematized as the study of superintegrable systems, that is systems with more symmetry operators than degrees of freedom [4, 5, 6, 7]. In the case of (maximally) superintegrable systems, the solutions can be determined algebraically. Many of the most common physical systems, such as the Kepler-Coulomb system or the simple harmonic oscillator, are in fact superintegrable and it is exactly this fact which leads to their tractability.

As in the case of Lie theory, the symmetry operators of superintegrable systems form an algebra which, under certain conditions, closes under commutation at a finite order. In this thesis, I will focus on superintegrable systems whose generating symmetries are quadratic in the momenta for classical systems or second-order differential operators in the quantum systems. In these cases, the systems are separable in multiple coordinate systems and are exactly solvable through algebraic methods. Furthermore, the symmetry operators form a quadratic algebra. Representations of this algebra can be used to to solve for the eigenvalues of the symmetry operators. There has been recent work in determining representations of the algebras generated by superintegrable systems [8, 9, 10, 11, 12, 13].

Also, the quadratic algebra structure is well adapted to describe the properties of many special functions and their identities. Though many special functions are associated with group structures and their associated Lie algebras [14], there is a host of other special functions for which the structure of a Lie algebra is too restrictive. However, since most special functions are related to physical systems which are exactly solvable, it is conjectured that the many of the structures of special functions can be recast in terms of superintegrable systems and their algebras. While this thesis does not address this question explicitly, it certainly supports this theory through exhibiting many different special function identities in the models.

In this thesis, I focus solely on second order superintegrable systems and their representation theory. I present some of the structure theory for 2 and 3 dimensional superintegrable systems in chapter 3. For the case of second order superintegrability, most of the classification and structure theory has been worked out and it has been proven that the associated symmetry algebras are in fact quadratic algebras, usually not Lie algebras. The remainder of the thesis contains new results on the representation theory of these quadratic algebras. In chapter 4, I present a model for each equivalence class of superintegrable systems in 2 dimensions. In chapter 5, I then describe in more detail models for four of the quadratic algebras. Finally, in chapter 6 I describes the representation theory for a superintegrable system in 3 dimensions.

Chapter 2

Relevant Physics Theory

In this chapter, I summarize the relevant theory of the dynamics of physical systems in classical and quantum mechanics. In the first section I present classical mechanics with emphasis on the Hamiltonian formalism which was introduced in 1833 by Irish mathematician William Rowan Hamilton. We focus on this formulation of classical dynamics because it is well suited to symmetry analysis and also because it forms the basis for the transition to quantum mechanics. In the next section, I give some basic theory of quantum mechanics and the methods of quantization.

2.1 Classical Mechanics

The Hamiltonian formalism describes the dynamics of a physical system in n dimensions by relating the time derivates of the position coordinates and the momenta to a single function on the phase space. A physical system in n dimensions describing the position of a particle involves n position coordinates, call them $q_i(t)$, and n momentum coordinates, $p_i(t)$ where t is time. In these coordinates, the phase space of a physical system is described by points $(p_i, q_j) \subseteq K^{2n}$, where K is our base field, usually $\mathbb{R}$ or $\mathbb{C}$. Here, we can take the position coordinates to be the standard Cartesian coordinate systems and the momenta to be the mass times the velocity, or we can take a more generalized set of coordinates. In fact, all we require is that the coordinates satisfy the following definitions. As a shorthand, we omit the subscripts in functions which depend on functions of all subscripts. For example, $f(p_i)$ depends on only p_i while $f(p)$ depends on p_j for all j.

Definition 1. *The* **Lagrangian** *of a physical system is given by*

$$\mathcal{L}(q, p, t) \equiv T - V, \tag{2.1}$$

where T is the kinetic energy and V is the potential energy of the system.

Definition 2. *The* **generalized coordinates** *of a Hamiltonian system are given by n generalized position coordinates, q_i, and n generalized momenta which satisfy,*

$$p_i \equiv \frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}_i}, \tag{2.2}$$

where $\dot{q} = \frac{dq}{dt}$.

In the Hamiltonian formalism, a physical system is defined in $2n$-dimensional phase space given by the generalized coordinates. The differential equations which determine the dynamics of this physical system are given in terms of a particular function on the phase space, called the Hamiltonian. The Hamiltonian can be determined from the Lagrangian but is a function of the position and momentum only, not their derivatives.

Definition 3. *The* **Hamiltonian** *$\mathcal{H}(q, p, t)$ is given by the Legendre transform of the Lagrangian. That is,*

$$\mathcal{H}(q, p, t) = \sum_{i=1}^{n} \dot{q}_i p_i - \mathcal{L}(q, \dot{q}(q, p), t) \tag{2.3}$$

The dynamics of the system are now determined by $2n$ first-order differential equations in terms of the Hamiltonian. These are Hamilton's equations and are given by,

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}. \tag{2.4}$$

Solutions of this system of equations give the trajectories of the physical system.

In its simplest form, where the Hamiltonian is independent of time and quadratic in the momenta, the Hamiltonian can be interpreted as the total energy of the system:

$$\mathcal{H} = T + V \tag{2.5}$$

where T and V are kinetic and potential energy respectively. In this thesis, we will only consider Hamiltonians in this form. Explicitly, we require

$$\mathcal{H} = \Sigma_{i,j=1}^{n} g^{ik}(q) p_i p_k + V(q) \tag{2.6}$$

where g^{ij} is the contravariant metric tensor of the manifold for our system and where repeated indices are not summed over. For the remainder of this section, all systems will be given by a Hamiltonian of this form and all definitions will be based on this form. In particular, Hamiltonians of this form have no explicit dependence on time.

We can define a bilinear product on the phase space, called the Poisson bracket.

Definition 4. *Let $\mathcal{R}(p,q,t), \mathcal{S}(p,q,t)$ be two functions on the phase space. We can define a bilinear product $\{,\}$ to be*

$$\{\mathcal{R}, \mathcal{S}\} = \sum_{i=1}^{n} \left(\frac{\partial \mathcal{R}}{\partial q_i} \frac{\partial \mathcal{S}}{\partial p_i} - \frac{\partial \mathcal{R}}{\partial p_i} \frac{\partial \mathcal{S}}{\partial q_i} \right). \tag{2.7}$$

This is called the **Poisson bracket** *of $\mathcal{R}$ and $\mathcal{S}$.*

It is easy to verify that the Poisson bracket obeys the following properties, for $\mathcal{R}, \mathcal{S}, \mathcal{T}$ functions on the phase space and a, b constants.

$$\{\mathcal{R}, \mathcal{S}\} = -\{\mathcal{R}, \mathcal{S}\} \qquad \text{(anti-symmetry)} \tag{2.8}$$

$$\{\mathcal{R}, a\mathcal{S} + b\mathcal{T}\} = a\{\mathcal{R}, \mathcal{S}\} + b\{\mathcal{R}, \mathcal{T}\} \qquad \text{(bilinearity)} \tag{2.9}$$

$$\{\mathcal{R}, \{\mathcal{S}, \mathcal{T}\}\} + \{\mathcal{S}, \{\mathcal{T}, \mathcal{R}\}\} + \{\mathcal{T}, \{\mathcal{R}, \mathcal{S}\}\} = 0 \qquad \text{(Jacobi Identity)} \tag{2.10}$$

$$\{\mathcal{R}, \mathcal{S}\mathcal{T}\} = \{\mathcal{R}, \mathcal{S}\}\mathcal{T} + \mathcal{S}\{\mathcal{R}, \mathcal{T}\} \qquad \text{(Leibniz rule)} \tag{2.11}$$

The coordinates (q, p) satisfy

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \qquad \{p_i, q_j\} = \delta_{i,j}. \tag{2.12}$$

These are called the cannonical commutation relations. We can use them to define a symplectic matrix by writing the coordinates as

$$\zeta_i = \begin{cases} q_i, & i = 0, ..., n \\ p_{i-n}, & i = n+1, ..., 2n \end{cases} \tag{2.13}$$

then we define the matrix

$$c_{ij} = \{\zeta_i, \zeta_j\}, \qquad i, j = 1, \ldots, 2n. \tag{2.14}$$

In this notation, the Poisson bracket becomes

$$\{\mathcal{R}, \mathcal{S}\} = \sum_{i,j=0}^{2n} c_{ij} \frac{\partial \mathcal{R}}{\partial \zeta_i} \frac{\partial \mathcal{S}}{\partial \zeta_j}. \tag{2.15}$$

Definition 5. *We call a set of phase space coordinates* (W_i) **canonical coordinates** *if* $\{W_i, W_j\} = c_{ij}$. *A change of coordinates* $(Q_i(q,p), P_j(q,p))$ *is called a canonical transform if it determines a new set of canonical coordinates.*

It is easy to verify that the definition of the Poisson bracket is independent of the choice of canonical coordinates.

In terms of the Poisson bracket, we can rewrite Hamilton's equations as

$$\dot{q}_i = \{q_i, \mathcal{H}\}, \qquad \dot{p}_i = \{p_i, \mathcal{H}\}. \tag{2.16}$$

For an arbitrary function on the phase space $\mathcal{R}(q,p,t)$ we can determine its dynamics as

$$\frac{d\mathcal{R}}{dt} = \{\mathcal{R}, \mathcal{H}\} + \frac{\partial \mathcal{R}}{\partial t}. \tag{2.17}$$

From this equation, we can readily see that a function on the phase space without explicit time dependence, $\mathcal{R}(q,p)$ will be constant along a trajectory if and only $\{\mathcal{R}, \mathcal{H}\} = 0$. We call such functions constants of the motion.

Definition 6. *A function on the phase space,* $\mathcal{S}(q,p)$ *is called a* **constant of the motion** *if* $\{\mathcal{S}, \mathcal{H}\} = 0$.

Definition 7. *A system with Hamiltonian* $\mathcal{H}$ *is* **integrable** *if it admits* n *constants of the motion*

$$\mathcal{P} = (\mathcal{P}_1 = \mathcal{H}, \mathcal{P}_2, \ldots, \mathcal{P}_n)$$

that are in involution:

$$\{\mathcal{P}_i, \mathcal{P}_j\} = 0, \quad 1 \leq i, j \leq n \tag{2.18}$$

and are functionally independent in the sense that $\det \left(\frac{\partial \mathcal{P}_i}{\partial p_j} \right) \neq 0$.

Note that it is possible for a system to be multi-integrable, i.e., integrable in more than one way, and this will be a common occurence in the thesis.

Now suppose $\mathcal{H}$ is integrable with associated constants of the motion $\mathcal{P}$. Then by the inverse function theorem we can solve the n equations $\mathcal{P}_i(q,p) = c_i$ for the momenta to obtain $p_j = p_j(q,c),\ j = 1, \ldots, n$, where $c = (c_1, \ldots, c_n)$. For an integrable system, if a particle with position q lies on the common intersection of the hypersurfaces $\mathcal{P}_i = c_i$ for constants c_i, then its momentum p is completely determined,

If $F(q,p)$ is a function on phase space and $(q(t), p(t))$ is a solution trajectory of the equations of motion for the Hamiltonian system (2.4) then the time dependence of $F(t) = F(q(t), p(t))$ is determined by

$$\frac{dF}{dt}(t) = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial F}{\partial p_i}\frac{dp_i}{dt} \right) = \{F, \mathcal{H}\}. \tag{2.19}$$

Thus if $\mathcal{S}$ is a constant of the motion it follows that $\mathcal{S}(q(t), p(t))$ is constant along a trajectory. In particular if a particle following a trajectory of an integrable system lies on the common intersection of the hypersurfaces $\mathcal{P}_i = c_i$ at some time t_0 then it must lie on the same common intersection for all t near t_0.

Considering $p_j(q,c)$ and using the conditions (2.18) and the chain rule it is straightforward to verify $\partial_{q_j} p_i = \partial_{q_j} p_j$, where $\partial_{q_j} p_i \equiv \dfrac{\partial p_i}{\partial q_j}$. Therefore, there exists a function $u(q,c)$ such that $p_\ell = \partial_{q_\ell} u$, $\ell = 1, \cdots, n$. Note that

$$\mathcal{P}\left(q, \frac{\partial u}{\partial q}\right) = c,$$

and, in particular, u satisfies the Hamilton-Jacobi equation

$$\mathcal{H}\left(q, \frac{\partial u}{\partial q}\right) = E \tag{2.20}$$

where $E = c_1$. By construction $\det\left(\partial_{q_i}\partial_{c_j} u\right) \neq 0$, and such a solution of the Hamilton-Jacobi equation is called a **complete integral.** Note that this argument is reversible: a complete integral of the Hamilton-Jacobi equation (2.20) determines n constants of the motion in involution via $p_i = \partial_{q_i} u(q,c)$ and solving these equations for the c_i to get $c_i = \mathcal{P}_i(q,p)$. Thus the system is integrable.

Theorem 1. *A system is integrable if and only if the Hamilton-Jacobi equation admits a complete integral.*

One of the most powerful methods for demonstrating that a system is integrable is to explicitly exhibit a complete integral by using the method of additive separation of variables.

Now we describe how to integrate Hamilton's equations for an integrable system, to determine the trajectories. Our goal is to find a set of canonical coordinates $(Q_i(q,p), P_j(q,p))$ for the system. We can do this by defining the new position coordinates,

$$\mathcal{Q}_i(q,p) \equiv \beta_i = \frac{\partial u}{\partial c_i}$$

Recall, that we also have $c_i = \mathcal{P}_i(q,p)$ the constants of the motion.

Next, using (2.18), (2.12) we find

$$\{\beta_i, c_j\} = \{\mathcal{Q}_i, \mathcal{P}_j\} = \delta_{ij}, \quad 1 \leq, i.j \leq n. \tag{2.21}$$

Since $\mathcal{P}_1 = \mathcal{H}$ we see that $\mathcal{Q}_2(q,p), \cdots, \mathcal{Q}_n(q,p)$ are also (local) constants of the motion, so they are constant along a solution to Hamilton's equations, $\mathcal{Q}_i(q,p) = \beta_i$(constant) for $i = 2, ...n$. On the other hand $\mathcal{Q}_1(q,p)$ is not a constant of the motion.

$$\{\mathcal{Q}_1, \mathcal{H}\} = \{\mathcal{Q}_1, \mathcal{P}_1\} = 1.$$

Now let $(q(t), p(t))$ be a solution of Hamilton's equations with position $(q(t_0), p(t_0))$ in phase space at time $t = t_0$. Setting $\mathcal{Q}_i(t) = \mathcal{Q}_i(q(t), p(t))$ and $\mathcal{P}_i(t) = \mathcal{P}_i(q(t), p(t))$ we use (2.19), (2.21) to find

$$\frac{\mathcal{Q}_j(t)}{dt} = \{\mathcal{Q}_j, \mathcal{H}\} = 0, \quad \frac{\mathcal{P}_j(t)}{dt} = \{\mathcal{P}_j, \mathcal{H}\} = 0, \quad j \neq 1,$$

and

$$\frac{d\mathcal{Q}_1(t)}{dt} = \{\mathcal{Q}_1, \mathcal{H}\} = 1.$$

Integrating the last equation and collecting results we have

$$\mathcal{Q}_j(t) \equiv \mathcal{Q}_j(t_0) = \beta_j(0), \quad \mathcal{P}_j(t) \equiv \mathcal{P}_j(t_0) = c_j(0), \quad j \neq 1,$$

and

$$\mathcal{Q}_1(t) = \mathcal{Q}_1(t_0) + t = \beta_1(0) + t.$$

We can now write the phase space in these coordinates to determine the trajectory as $(\mathcal{Q}_j, \mathcal{P}_i) = (Q_1(t), \beta_2, ..., \beta_n, c_1 = E, ..., c_n)$. We can return to the original coordinates by inverting these equations to write the canonical variables q, p as functions of the canonical variables $\mathcal{Q}, \mathcal{P}$. Hence, we have integrated Hamilton's equations to obtain the trajectory $q(\beta_1(0)+t, \beta_j(0), E, c_j)$, $p(\beta_1(0)+t, \beta_j(0), E, c_j)$.

As can been seen from this exposition, one of the main assets of the Hamiltonian formalism is it is well suited to utilizing symmetries of the system through the structures of the Poisson bracket. These basic structures of Hamiltonian mechanics give us useful ways of understanding symmetries as constants of the motion. Additionally, as will be seen in the next section, the Hamiltonian formalism is an important tool in laying the framework for quantum mechanics.

2.2 Quantum Mechanics

Quantum mechanics began in the early 20th century as a tool for explaining physical phenomena below the microscopic level. One of the first attempts to rigorize the mathematics of Quantum mechanics was contained in John von Neumann's 1932 book *Mathematical Foundations of Quantum Mechanics* [15]. To this day, while the theory has progressed far beyond quantum mechanics to field theory and beyond, the basic principles of quantum mechanics and its formulation have remained relatively stable. In this section, I will give a brief introduction to the basic principles necessary to understand quantum superintegrable systems.

In quantum mechanics, we represent physical states as one dimensional subspaces in a complex, projective Hilbert space; that is, two states are identified as equivalent if they differ by a multiplicative factor. In Dirac notation, we denote a state as a "ket" $|\phi\rangle$ and identify its dual as a "bra" $\langle\phi|$. We can then denote the inner product as the quantity $\langle\phi|\psi\rangle$. We require that this product have the characteristics of an inner product: linearity in the first argument, conjugate symmetry and positive definiteness.

Definition 8. *For two normalized state vectors, $|\phi\rangle, |\psi\rangle$ the transition amplitude is*

$$|\langle\phi|\psi\rangle| = \sqrt{\langle\phi|\psi\rangle\overline{\langle\phi|\psi\rangle}} \in \mathbb{R}. \tag{2.22}$$

The transition amplitude is interpreted as the probability of the state ϕ containing a component in the state ψ. Many formulations of quantum mechanics focus their analysis on this probabilistic interpretation.

Observables correspond to quantities which can be measured, such as position or momentum. These are represented by operators. In general, we require that the operators which correspond to observables be Hermitian though there are other formulations that relax this assumption [16]. These requirements are imposed so that the eigenvalues of an observable are real.

The action of taking a measurement of a system corresponds to acting on a (normalized) state vector and transforming it to an eigenvector of the operator corresponding to the observable. The measured value of a state must be an eigenvalue of the operator. We can measure the probability of an arbitrary state, $|\phi\rangle$ being measured with a given eigenvalue λ by computing its transition amplitude, $|\langle\lambda|\phi\rangle|$, where $|\lambda\rangle$ is an eigenvector of an operator A with eigenvalue λ.

We can also define the expected value of an operator which gives the average value of the observable.

Definition 9. *The expected value of an operator A depends on a given (normalized) state vector $|\phi\rangle$ and is given by $\langle A\rangle_\phi = \langle\phi|A|\phi\rangle$.*

As in classical mechanics, we create most of our observables out of the quantities position and momentum. However, instead of being coordinates on a phase space, in quantum mechanics these quantities correspond to operators. That is, X_i gives the value of the coordinate x_i while P_i gives the i^{th} momentum.

We can define a bilinear product on the operators by the Lie bracket $[A, B] = AB - BA$. This bilinear product satisfies the same relations as the Poisson bracket (2.8-2.11).

The quantization of the position and momentum operators satisfy the following commutation relations,

$$[X_j, X_k] = [P_j, P_k] = 0, \qquad [X_j, P_k] = i\hbar\delta_{jk}, \tag{2.23}$$

where $\hbar$ is the reduced Planck constant.

Given a classical operator, such as a Hamiltonian, the quantization procedure is to send $q_i \to x_i$ and $p_i \to -i\hbar\partial_{x_i}$. These operators satisfy the commutation relations (2.23). However, in quantum mechanics the position and momentum operators do not commute so we usually take symmetrized products of any terms with mixed position and momentum, though there are other conventions.

The process of symmetrizing the operators leads to differences between the lower order terms of the classical and quantum systems. In some cases, we make gauge transformations to remove the lower order terms, for example in the general form of Hamiltonian operators. In other cases this is not possible and the differences persist as will be the cases in the symmetry algebras for the classical and quantum systems.

Further complicating the process of quantization is that, in general, the quantization procedure is not unique both because there are alternate symmetrizing conventions and the choice of local coordinates. However, for the case of second-order superintegrable systems, the subject of this thesis, the quantization can be uniquely determined. If the classical operator is quadratic in the momenta as in these cases then the quantization is uniquely defined. Some of the classical models in this thesis have operators which are not polynomial in the momenta; we can use Taylor's theorem to create difference operators, as in sections 5.4 and 5.5 or elliptical operators for more complicated functions [17].

The time evolution of quantum mechanical states is determined by the Hamiltonian. The dynamics of the system is given by Schrödinger's Equation,

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = H|\phi(t)\rangle. \tag{2.24}$$

Using the Hamiltonian, we can define the time evolution operator $U(t) = exp(iHt/\hbar)$ as a one parameter group of unitary operations determined by the operator H. A state at an arbitrary time will be given by $|\phi(t)\rangle = U(t)|\phi(0)\rangle$. If a state is an eigenvector for the Hamiltonian, say $H|\phi_E(0)\rangle = E|\phi_E(0)\rangle$, then the time evolution operator acts on an eigenvector by a (time dependent) scalar function,

$$U(t)|\phi_E(0)\rangle = e^{iEt/\hbar}|\phi_E(0)\rangle. \tag{2.25}$$

Hence, if we can find a basis of eigenvectors for the Hamiltonian, we can expand an arbitrary vector in terms of this basis and know exactly the time evolution of the state. For this reason, finding eigenvalues annd eigenvectors for Hamiltonians is one of the fundamental tasks in quantum mechanics.

We can also consider the time evolution of an operator by looking at its expected value. The expected value obeys the following relation,

$$\frac{d}{dt}\langle A\rangle = \langle[A,H]\rangle + \frac{\partial\langle A\rangle}{\partial t} \tag{2.26}$$

in analogy with the classical relation (2.17).

If an operator commutes with the Hamiltonian, then it it possible to chose a basis which are simultaneous set of eigenfunctions. Hence, for a quantum mechanical system it is important to determine a complete set of commuting observable to completely determine the system and to resolve degeneracies in the spectrum.

Definition 10. *A quantum mechanical system in n dimensions is* **integrable** *if there exist n functionally independent observables, including the Hamiltonian, which commute with each other.*

2.3 Example: The Kepler-Coulomb Potential

As an example of the classical and quantum mechanical structures given above we consider the case of the gravitational or Coulomb potential in 2d Euclidean space. In this section we will set the mass of the objects as 1 and also the value of $\hbar = 1$. With these constants, the total energy due to gravity is given by

$$\mathcal{H} = p_1^2 + p_2^2 + \frac{\alpha}{\sqrt{q_1^2 + q_2^2}}. \tag{2.27}$$

Hamilton's equations become

$$\dot{q}_i = 2p_i, \qquad \dot{p}_i = \frac{-q_i\alpha}{(q_1^2 + q_2^2)^{\frac{3}{2}}}. \tag{2.28}$$

We can immediately observe that this system is radially symmetric and that rotation about the center is a symmetry of the system. This symmetry is generated by the function $\mathcal{X} = \mathcal{M} \equiv q_1p_2 - q_2p_1$. We can immediately verify that this is in involution with the Hamiltonian ($\{\mathcal{H}, \mathcal{X}\} = 0$) and so is a constant of the motion. Since the system is in 2d and we have 2 constants of the motion in involution, we have an integrable system.

When we quantize this system, we obtain the Hamiltonian for the Coulomb potential which gives a good representation of the energy spectrum of the hydrogen atom, since it has one proton and one electron with the Coulomb force between them. The Hamiltonian for this system is,

$$H = \partial_x^2 + \partial_y^2 + \frac{\alpha}{\sqrt{x^2 + y^2}}. \tag{2.29}$$

The classical operator M also has a quantum counterpart given by $X = M \equiv x\partial_y - y\partial_x$ which is the generator of rotation about the origin in the plane. We can see that it commutes with the Hamiltonian, $[H, X] = 0$ and so we can separate off the radial component of the eigenfunctions for H.

Since we will be often be simultaneously talking about classical constants of the motion and quantum commuting observables, it will be convenient to use the term "symmetry operators" when talking about either. To help clarify, I will use script letters for classical functions on the phase space and block letters for quantum operators.

In both the classical and quantum systems, the symmetry $\mathcal{X}, X$ is not the only symmetry of the system. There are two other symmetry operators, which are second order in the p's,

$$\mathcal{L}_1 = \mathcal{M}p_1 - \frac{\alpha q_2}{2\sqrt{q_1^2 + q_2^2}}, \quad \mathcal{L}_2 = \mathcal{M}p_2 - \frac{\alpha q_1}{2\sqrt{q_1^2 + q_2^2}} \tag{2.30}$$

$$L_1 = \frac{1}{2}(M\partial_x + \partial_x M) - \frac{\alpha y}{2\sqrt{x^2 + y^2}}, \quad L_2 = \frac{1}{2}(M\partial_y + \partial_y M) - \frac{\alpha x}{2\sqrt{x^2 + y^2}}. \tag{2.31}$$

We can compute the symmetry algebra generated by these operators. In the classical case, we obtain the Poisson algebra

$$\{\mathcal{L}_1, \mathcal{X}\} = \mathcal{L}_2, \quad \{\mathcal{L}_2, \mathcal{X}\} = -\mathcal{L}_1, \quad \{\mathcal{L}_1, \mathcal{L}_2\} = \mathcal{H}\mathcal{X}. \tag{2.32}$$

Furthermore, there are too many operators to be functionally independent, so we must have a functional relation. It is

$$\mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{X}^2\mathcal{H} = \frac{\alpha^2}{4}. \tag{2.33}$$

The quantum algebra is,

$$[L_1, X] = L_2 \qquad [L_2, X] = -L_1 \qquad [L_1, L_2] = HX \tag{2.34}$$

$$L_1^2 + L_2^2 - HX^2 + \frac{H}{4} = \frac{\alpha^2}{4}. \tag{2.35}$$

These algebra relations can be used to determine the classical trajectories algebraically; in fact the solutions for the orbits of planetary motion under gravitational potential were discovered by Kepler before the invention of calculus. Similarly, the eigenfunctions for the Coulomb potential can be determined using raising and lowering operators on an abstract set of basis functions to solve explicitly the eigenvalue equation for H. A full exposition of the quantum system will be given below in Section 5.1.

The ability to solve these equations explicitly is due to the existence of the extra symmetry operators. To generalize this structure, we consider systems which are superintegrable, sometimes referred to as maximally superintegrable; that is, they have the maximal number of functionally independent symmetry operators.

Chapter 3

Second order superintegrable systems

In this chapter, I define second order superintegrability in the classical and quantum cases and give an overview of the current structure theory for the systems in 2 and 3 dimensions. An important tool in the structure theory and in the classifications of superintegrable systems is the Stäckel transform which gives an equivalence relation for superintegrable systems with isomorphic symmetry algebras. I explain this transform and its role in the structure theory.

3.1 Second order Superintegrable systems: definitions

For the purposes of this thesis, we will be considering classical systems of the form,

$$\mathcal{H} = \sum_{ij} g^{ij}(x)p_i p_j + V(x) = \mathcal{H}_0 + V(x)$$

where $x = (x_1, ..., x_n)$ is a local coordinate system on a Riemannian or pseudo Riemannian manifold with contravariant metric tensor $g^{ij}(x)$. Indeed, to cover both cases we will usually consider the space to be complex Riemannian.

The quantum equivalent of this, as considered in this thesis, is of the form

$$H = \Delta + V(x)$$

where Δ is the Laplace-Beltrami operator on such a manifold.

Definition 11. *A classical Hamiltonian system in n dimensions is (maximally)* **superintegrable** *if it has the maximal number, $2n-1$, of constants of the motion which are functionally independent, globally defined except for isolated singularities and polynomial in the momenta.*

Such a system is of **second-order** *if the constants of the motion are at most quadratic in the momenta.*

We can always find $2n-1$ constants of the motion locally analytic in x and p but to be superintegrable we require that the constants be globally defined except possibly for isolated singularities in the position variables. For the purposes of this thesis, we further require that the constants be polynomial in the momenta so that they can be readily quantized.

The most significant characteristic of superintegrable systems is that superintegrable systems can be solved algebraically, whereas integrable systems can be solved by quadratures. The orbits in the $2n$-dimensional phase space are the common intersections of the $2n-1$ hypersurfaces determined by the constants of the motion.

Definition 12. *A quantum Hamiltonian system in n dimensions is (maximally)* **superintegrable** *if it has the maximal number, 2n-1, commuting observables which are globally defined differential operators.*

Such a system, is of **second-order** *if the differential operators are at most second-order.*

Second-order superintegrable systems are chosen for their tractability and their relation to multiseparability and special function theory. There has also been research into the relations between superintegrable systems and exact-solvability [18, 19, 20]. While higher order superintegrable systems have been found and studied [21, 22, 23], the research has mostly been limited to specific examples rather than structure theories and classification. On the other hand, the structure theory of second-order superintegrable systems has made considerable progress in recent years [24, 25, 26, 27, 27, 28, 29, 30, 31, 32, 33]. In particular, the algebras of the symmetry operators for second-order superintegrable systems in 2d have been proven to close forming a "quadratic algebra;" where further the square of the commutator of any two symmetry operators is expressible as a polynomial of the second order symmetry operators.

Example 1. *In 2d, an example of a superintegrable Hamiltonian is given by*

$$\mathcal{H} = p_x^2 + p_y^2 + \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y}.$$

A basis of generators for the symmetry operators is given by $\mathcal{H}$ and

$$\mathcal{L}_1 = p_x^2 - 4\alpha x^2 - \beta x, \quad \mathcal{L}_2 = (xp_y - yp_x)\, p_y + y^2\left(\frac{\beta}{4} - x\alpha\right) - \gamma\frac{x}{y^2}.$$

In this example, the Poisson bracket of $\mathcal{L}_1$ with $\mathcal{L}_2$ is a new operator that cannot be expressed as a function of the other operators and hence this algebra does not have a Lie algebra structure. In our symmetry algebra we include the new third order symmetry operator $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$.

On the other hand, the Poisson bracket of our new operator $\mathcal{R}$ with the other operators and the square of $\mathcal{R}$ can be expressed as a polynomial in the generators. We call this structure a quadratic algebra. The quadratic algebra for this system is,

$$\{\mathcal{R}, \mathcal{L}_1\} = -2\beta(\mathcal{L}_1 - \mathcal{H}) + 16\alpha\mathcal{L}_2$$

$$\{\mathcal{R}, \mathcal{L}_2\} = 6\mathcal{L}_1^2 + 2\mathcal{H}^2 - 8\mathcal{L}_1\mathcal{H} - 2\beta\mathcal{L}_2 - 8\alpha\gamma$$

$$\mathcal{R}^2 - 4\mathcal{L}_1(\mathcal{L}_1 - \mathcal{H})^2 + 16\alpha\mathcal{L}_2^2 - 4\beta\mathcal{L}_2(\mathcal{L}_1 - \mathcal{H}) + 16\alpha\gamma\mathcal{L}_1 + \gamma\beta^2 = 0$$

Example 2. *Another example of a superintegrable Hamiltonian, this time on a space of non constant curvature is*

$$\mathcal{H} = \frac{1}{x}(p_x^2 + p_y^2) + \alpha\left(4x + \frac{y^2}{x}\right) + \frac{\gamma}{xy^2} - \frac{\delta}{x}.$$

The constants of the motion for this system are $\mathcal{H}$ and

$$\mathcal{L}_1 = -py^2 - \alpha y^2 - \frac{\gamma}{y^2} + \delta,$$

$$\mathcal{L}_2 = \frac{x^2 + y^2}{x}p_y^2 + \frac{y^2}{4x}p_x^2 - yp_xp_y + \alpha\frac{y^4}{4x} + \gamma\frac{4x^2 + y^2}{4xy^2} - \delta\frac{y^2}{4x}.$$

Again, we define $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$. The quadratic algebra for this system is,

$$\{\mathcal{R}, \mathcal{L}_1\} = -2\mathcal{H}(\mathcal{L}_1 + \delta) + 16\alpha\mathcal{L}_2,$$

$$\{\mathcal{R}, \mathcal{L}_2\} = 6\mathcal{L}_1^2 + 8\gamma\mathcal{L}_1 + 2\gamma^2 + 2\mathcal{H}\mathcal{L}_2 - 8\alpha\gamma,$$

$$\mathcal{R}^2 - 4\mathcal{L}_1(\mathcal{L}_1+\delta)^2 + 16\alpha\mathcal{L}_2^2 - 4\mathcal{H}\mathcal{L}_2(\mathcal{L}_1+\delta) + 16\alpha\gamma\mathcal{L}_1 + \gamma\mathcal{H}^2 = 0.$$

By direct comparison, we can see that these two algebras are isomorphic when restricted to constant energy hypersurfaces. In fact, if we make the identification

$$\mathcal{H} \to -\delta, \qquad \beta \to -\mathcal{H}$$

the two algebras are identical. Such a relation between symmetry algebras is called a Stäckel equivalence. This is an equivalence between superintegrable systems given by the Stäckel transform or constant coupling metamorphisis [34, 35].

Definition 13. *Given a classical superintegrable Hamiltonian we can create a new superintegrable Hamiltonian, possibly on a new manifold by the following* **Stäckel transform.** *We separate the Hamiltonian as,*

$$\mathcal{H} = \mathcal{H}_0 + V + \alpha U \qquad \mathcal{L} = \mathcal{L}_0 + W + \alpha W_U \tag{3.1}$$

with

$$\{\mathcal{H}_0 + V, \mathcal{L}_0 + W\} = \{\mathcal{H}, \mathcal{L}\} = 0 \tag{3.2}$$

then we have new constants of the motion, $\{\tilde{\mathcal{H}}, \tilde{\mathcal{L}}\} = 0$

$$\tilde{\mathcal{H}} = \frac{\mathcal{H}_0 + V}{U} \qquad \tilde{\mathcal{L}} = \mathcal{L}_0 - W_U\tilde{\mathcal{H}}. \tag{3.3}$$

Here $\tilde{H}$ *is the new Hamiltonian, possibly on a new manifold.*

Theorem 2. *The Stäckel transform of superintegrable system is superintegrable if and only if the original system is superintegrable.*

The proof of this theorem is immediate since the Stäckel transform preserves the number of constants of the motion and the polynomial condition on the momenta. Furthermore, the functional independence criteria is satisfied by the new constants of the motion [27].

We have an equivalent definition for the Stäckel transform of a quantum system but with the Poisson brackets replaced by Lie brackets. However, in nd for $n > 2$ the transformed potential must be modified by an additive term that depends on the scalar curvature [31]. Notice that the new superintegrable system will have a new metric tensor which differs from the old one by a factor of $1/U$. As will be shown below, any superintegrable system in 2d is Stäckel equivalent to one on a manifold of constant curvature, either Euclidean space or the sphere.

To obtain the system given in example 2, I started with the Hamiltonian in 1 and rewrote it as

$$\mathcal{H} = p_x^2 + p_y^2 + \alpha(4x^2 + y^2) + \frac{\gamma}{y} + \beta x = \mathcal{H}_0 + V + \beta x.$$

I then added a trivial additive constant to the potential. This action corresponds to separating a piece off the energy in the Hamiltonian-Jacobi equations $\mathcal{H} = E \iff \mathcal{H} - \delta = E - \delta$. The additive constant is important so that the new system I obtain from the Stäckel transform still depends on 3 parameters.

I can then obtain my new superintegrable system (2) by following the prescription in the definition of the Stäckel transform (13). Notice that the Stäckel transform does not define an algebra isomorphism because the Hamiltonian is transformed into a parameter but it does become an algebra isomorphism when the algebras are restricted to constant energy hypersurfaces.

3.2 Second Order Superintegrable systems in 2d

Now we focus our attention on non-degenerate systems in 2 dimensions. We give a outline of the proof of the existence of the quadratic algebra. Then, we show that every non-degenerate systems is Stäckel equivalent to a system with constant curvature.

3.2.1 Algebra relations

For any complex 2d Riemannian manifold we can always find local coordinates x, y such that the classical Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{\lambda(x,y)}(p_x^2 + p_y^2) + V(x,y), \qquad (x,y) = (x_1, x_2),$$

where the complex metric is $ds^2 = \lambda(x,y)(dx^2 + dy^2)$ [36].

Necessary and sufficient conditions that

$$\mathcal{L} = \sum a^{ji}(x,y)p_j p_i + W(x,y)$$

be a symmetry of $\mathcal{H}$ are the Killing equations

$$\begin{aligned} a_i^{ii} &= -\frac{\lambda_1}{\lambda}a^{i1} - \frac{\lambda_2}{\lambda}a^{i2}, \quad i = 1,2 \\ 2a_i^{ij} + a_j^{ii} &= -\frac{\lambda_1}{\lambda}a^{j1} - \frac{\lambda_2}{\lambda}a^{j2}, \quad i,j = 1,2,\ i \neq j, \end{aligned} \tag{3.4}$$

and the Bertrand-Darboux (B-D) conditions on the potential $\partial_i W_j = \partial_j W_i$ or

$$(V_{22}-V_{11})a^{12}+V_{12}(a^{11}-a^{22}) = \left[\frac{(\lambda a^{12})_1 - (\lambda a^{11})_2}{\lambda}\right] V_1 + \left[\frac{(\lambda a^{22})_1 - (\lambda a^{12})_2}{\lambda}\right] V_2. \tag{3.5}$$

Here, I have used the notation $f_i = \partial_{x_i} f$. From the 2 second order constants in involution with $\mathcal{H}$ we get 2 B-D equations and can solve them to obtain fundamental PDEs for the potential of the form

$$V_{22} - V_{11} = A^{(22}(\mathbf{x})V_1 + B^{22}(\mathbf{x})V_2, \qquad V_{12} = A^{12}(\mathbf{x})V_1 + B^{12}(\mathbf{x})V_2. \tag{3.6}$$

If the B-D equations and the integrability conditions for these PDEs are satisfied identically, we say that the potential is **nondegenerate**. That means, at each regular point $\mathbf{x}_0$ where the A^{ij}, B^{ij} are defined and analytic, we can prescribe the values of $V,\ V_1,\ V_2$ and V_{11} arbitrarily and there will exist a unique potential $V(\mathbf{x})$ with these values at $\mathbf{x}_0$.

Nondegenerate potentials depend on these 4 parameters, including the trivial additive parameter. Degenerate potentials depend on strictly fewer than 4 parameters. In 2d, degenerate potentials are always restrictions of 4 parameter potentials so we shall ignore these except in the special cases when a Killing vector appears which dramatically changes the algebra.

3.2.2 Structure theory for nondegenerate systems

The structure theory for nondegenerate systems has been determined in recent years by Kalnins et al [36, 27, 24, 25]. The argument classifying such spaces relies on the basis determined by the symmetry operators. Hence, to employ such an argument, we require that the symmetries be functionally linearly independent so that at an arbitrary point we can choose a canonical basis for our symmetry operators. Assume our symmetry operators take the from

$$L_i = \sum_{jk} a^{jk}_{(i)}(x)p_i p_j + W^i(q).$$

Definition 14. *A set of symmetry operators are said to be* **functionally linearly dependent** *if there exists* $c_i(x_1,..x_n)$*, not identically* 0*, such that*

$$\sum_{i=1}^{n} c_i(x)a^{jk}_{(i)}(x)p_jp_k = 0.$$

They are **functionally linearly independent** *if they are not functionally linearly dependent.*

Here we allow the c_i's to be functions of the x_k's only so that the definition works for both classical and quantum systems. This functional linear independence criterion splits superintegrable systems of all orders into two classes with different properties. In $2d$ there is essentially only one functionally linearly dependent superintegrable system, namely $\mathcal{H} = p_zp_{\bar{z}} + V(z)$, where $V(z)$ is an arbitrary function of z alone. This system separates in only one set of coordinates $(z,\bar{z})$. For functionally linearly independent $2d$ systems the theory is much more interesting. These are exactly the systems which separate in more than one set of orthogonal coordinates. Again, from [36] we have the following thereoms.

Theorem 3. *Let* $\mathcal{H}$ *be the Hamiltonian of a 2d superintegrable functionally linearly independent system with nondegenerate potential.*

- *The space of second order constants of the motion is 3-dimensional.*
- *The space of third order constants of the motion is 1-dimensional.*
- *The space of fourth order constants of the motion is 6-dimensional.*
- *The space of sixth order constants is 10-dimensional.*

These results follow from a study of the integrability conditions for constants of the motion that gives these numbers as the upper bounds of the dimensions. To prove that the bounds are achieved we use the following theorem.

Theorem 4. *Let* $\mathcal{K}$ *be a third order constant of the motion for a superintegrable system with nondegenerate potential* V*:*

$$\mathcal{K} = \sum_{k,j,i=1}^{2} a^{kji}(x,y)p_kp_jp_i + \sum_{\ell=1}^{2} b^{\ell}(x,y)p_\ell.$$

Then $b^\ell(x,y) = \sum_{j=1}^2 f^{\ell,j}(x,y)\frac{\partial V}{\partial x_j}(x,y)$ *with* $f^{\ell,j} + f^{j,\ell} = 0, \quad 1 \le \ell, j \le 2$. *The* a^{ijk}, b^ℓ *are uniquely determined by the quantity* $f^{1,2}(x_0,y_0)$ *at some regular point* (x_0,y_0) *of* V.

This follows from a direct computation of the poisson bracket $\{\mathcal{H},\mathcal{K}\}$ using the B-D relations for the a^{ik}'s. Since $\{\mathcal{L}_1,\mathcal{L}_2\}$ is a non-zero third order constant of the motion we can see that the number (1) is achieved and we can solve for the coefficients $a^{ijk}(x,y)$ and b^ℓ. This result enables us to choose standard bases for second and higher order symmetries. Indeed, given any 2×2 symmetric matrix $\mathcal{A}_0$, and any regular point (x_0,y_0) there exists one and only one second order symmetry (or constant of the motion) $\mathcal{L} = a^{kj}(x,y)p_kp_j + W(x,y)$ such that the matrix given by $a^{kj}(x_0,y_0)$ is equal to $\mathcal{A}_0$ and $W(x_0,y_0) = 0$. Further, if

$$\mathcal{L}_1 = \sum a^{kj}_{(1)}p_kp_j + W_{(1)}, \quad \mathcal{L}_2 = \sum a^{kj}_{(2)}p_kp_j + W_{(2)}$$

are second order constants of the the motion and $\mathcal{A}_{(i)}(x,y) = \{a^{kj}_{(i)}(x,y)\}$, $i = 1,2$ are 2×2 symmetric matrix functions, then the Poisson bracket of these symmetries is given by

$$\{\mathcal{L}_1,\mathcal{L}_2\} = \sum_{k,j,i=1}^{2} a^{kji}(x,y)p_kp_jp_i + b^\ell(x,y)p_\ell.$$

Using this relation we can solve for $f^{k,\ell}$, .

$$f^{k,\ell} = 2\lambda\sum_j(a^{kj}_{(2)}a^{j\ell}_{(1)} - a^{kj}_{(1)}a^{j\ell}_{(2)}).$$

Thus $\{\mathcal{L}_1,\mathcal{L}_2\}$ is uniquely determined by the skew-symmetric matrix

$$[\mathcal{A}_{(2)},\mathcal{A}_{(1)}] \equiv \mathcal{A}_{(2)}\mathcal{A}_{(1)} - \mathcal{A}_{(1)}\mathcal{A}_{(2)},$$

hence by the constant matrix $[\mathcal{A}_{(2)}(x_0,y_0),\mathcal{A}_{(1)}(x_0,y_0)]$ evaluated at a regular point, by the Taylor series generated from the Killing equations (3.4) and the Betrand-Darboux equations (3.5). This allows a standard structure by exploiting identification of the space of second order constants of the motion with the space of 2×2 symmetric matrices and identification of the space of third order constants of the motion with the space of 2×2 skew-symmetric matrices.

Indeed given a basis for the 3-dimensional space of symmetric matrices, $\{A^{ij}\}$, we can define a standard set of basis symmetries $\mathcal{S}_{(k\ell)} = \sum a^{ij}(\mathbf{x})p_ip_j + W_{(k\ell)}(\mathbf{x})$ corresponding to a regular point $\mathbf{x}_0$. The third-order symmetry operator is defined by the commutator of the basis elements; these are necessarily skew-symmetric and hence the identification with $\mathcal{K}$ with the skew-symmetric matrix. We change the notation of a second order symmetry operator to $\mathcal{S}_{ij}$ since the choice of basis will determine the form of the operator. We could choose a standard basis

$$\{A^{ij}\} \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{3.7}$$

as implied by the basis $\{\mathcal{S}_{11}, \mathcal{S}_{22}, \mathcal{S}_{12}\}$. Notice, the identity element

$$\mathcal{I} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

corresponds to the Hamiltonian.

The following theorems show the existence of the quadratic algebra [36].

Theorem 5. *The 6 distinct monomials*

$$(\mathcal{S}_{(11)})^2,\ (\mathcal{S}_{(22)})^2,\ (\mathcal{S}_{(12)})^2,\ \mathcal{S}_{(11)}\mathcal{S}_{(22)},\ \mathcal{S}_{(11)}\mathcal{S}_{(12)},\ \mathcal{S}_{(12)}\mathcal{S}_{(22)},$$

form a basis for the space of fourth order symmetries.

We note that since $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$ is the basis for the third order symmetries, then $\{\mathcal{L}_i, R\}$ are of fourth order and so by the above theorem must be polynomials in the 2nd order symmetries.

Theorem 6. *The 10 distinct monomials*

$$(\mathcal{S}_{(ii)})^3,\ (\mathcal{S}_{(ij)})^3,\ (\mathcal{S}_{(ii)})^2\mathcal{S}_{(jj)},\ (\mathcal{S}_{(ii)})^2\mathcal{S}_{(ij)},\ (\mathcal{S}_{(ij)})^2\mathcal{S}_{(ii)},$$

$$\mathcal{S}_{(11)}\mathcal{S}_{(12)}\mathcal{S}_{(22)},$$

for $i, j = 1, 2,\ i \neq j$ form a basis for the space of sixth order symmetries.

Again, since $\mathcal{R}^2$ is a 6th order constant we have immediately the existence of the Casimir relations which forces the algebra to close at order 6. This is a remarkable property of superintegrable systems and gives us finitely generated quadratic algebras.

The analogous results for 5th order symmetries follow directly from the Jacobi identity.

Although there seems to be a large number of possible manifolds for superintegrable systems, in 2d we know that each is Stäckel equivalent to one on a space of constant curvature [37].

Theorem 7. *Every nondegenerate superintegrable 2d system is Stäckel equivalent to a nondegenerate superintegrable system on a constant curvature space.*

This is a very important theorem because it greatly simplifies the task of finding superintegrable systems in 2d; we only need to look on the complex plane and on the 2 sphere. Furthermore, we can use the generator of the symmetries for these spaces to find the symmetries for the superintegrable potential. The exhaustive enumeration of these systems can be found in [26, 24, 25, 36, 27, 30]. There are 19 nondegenerate systems and 11 degenerate systems in 2 dimensional conformally flat space, for a complete list see appendix C .

3.2.3 Structure theory for degenerate systems

Although we know that in 2d degenerate systems are restrictions of degenerate systems sometimes the restriction leads to the appearance of a Killing vector, that is, a first order symmetry operator. When such a Killing vector is present, the structure of the quadratic algebra changes and we have closure at lower order. A systematic study of such systems was published in 2008 by Kalnins et al [38]. We outline some of the important results in this section.

Theorem 8. *Given a degenerate second order superintegrable system, there are three possibilities for constants of the motion*

- *If the potential depends on 2 parameters, it can be extended to a nondegenerate potentional with 3 linearly independent constants of the motion and the symmetry algebra is given by the restricted symmetry algebra of the nondegenerate system.*

- *If the potential depends on 1 parameter and has exactly 3 linearly independent constants of the motion, it can be extended to a nondegenerate potential and the symmetry algebra is given by the restricted symmetry algebra of the nondegenerate system.*

- *If the potential depends on 1 parameter and there are 4 linearly independent constants of the motion then the system admits a Killing vector whose square is one of the constants of the motion and the symmetry algebra closes at a lower degree.*

Hence, if a system exhibits exactly 3 linearly independent constants of the motion we know its algebra is just a restriction of the nondegenerate system and so it is Stäckel equivalent to a system on constant curvature space. However, if it admits a Killing vector, then the structure of the algebra changes. It is important to note that all systems that admit a Killing vector must depend on at most 1 parameter. There is an analogous equation to the equation in theorem 10 for the metrics where the free system admits a Killing vector. These systems are called Darboux spaces and are given in table A.1 in appendix A .

As in the nondegenerate classification theory, we have the following theorem which guarantees that all degenerate systems will be Stäckel equivalent to ones on constant curvature spaces and that the symmetry algebras of the two Stäckel equivalent systems will be isomorphic when restricted to constant energies.

Theorem 9. *All 1 parameter potentials which exhibit 4 linearly independent constants of the motion are Stäckel equivalent to a system on constant curvature space.*

In the case of a 1 parameter potential which exhibits a Killing vector, the symmetry algebra will close at lower order because of the addition of the first order symmetry operator. For example, the Casimir or functional relation between the operators closes at order 4 instead of 6 as with the nondegenerate systems.

These degenerate systems, while being restrictions of nondegenerate systems, are unique and their representation theory is different than simply taking a restriction of the representations for associated nondegenerate system. However, the analysis of such restrictions leads to fruitful insights and special function identities, particularly when a degenerate system is a restriction of more than one nondegenerate system in different Stäckel equivalence classes. In the following sections, I give models for both degenerate and nondegenerate systems but a careful analysis of the relationship between models for nondegenerate systems and their restrictions has yet to be fully explored.

3.2.4 Quantization

In two dimensions, the quantization of a classical superintegrable system is straightforward and bijective. The general method is by sending $p_x \to \partial_x$, then the Poisson bracket changes to a Lie bracket and we often symmetrize non-commuting operators since x and ∂_x no longer commute. A notable exception is that we do not symmetrize the Hamiltonian and instead use the Laplace-Beltrami operator for the non-potential part. It is important to note that the passage from Poisson bracket to Lie bracket conserves the highest order terms, as does the quantization of the operators.

Indeed, in two dimensions we choose a basis such that

$$H = \frac{1}{\lambda(x,y)}(\partial_x^2 + \partial_y^2) + V(x,y) \tag{3.8}$$

then, if we choose a basis for the symmetry operators in self-adjoint form

$$L_k = \sum_{ij} \partial_i a_k^{ij} \partial_j + W_k, \tag{3.9}$$

the Killing equations (3.4) and Bertrand-Darboux equation (3.5) remain unchanged. Similarly, the argument for the existence of the quadratic algebra follows through almost exactly as long as we require that the even order symmetries be formally self-adjoint and the odd order symmetries be formally skew-adjoint [31].

3.2.5 Stäckel equivalence in 2d

Not only is the Stäckel transform important because it allows us to transform all 2d second-order superintegrable systems to ones on spaces of constant curvature but also because it allows us to define equivalence classes of superintegrable systems with similar symmetry algebras. The symmetry algebras are similar in that they are isomorphic when the Hamiltonian is restricted to a constant energy and the irreducible representations of these algebras will be isomorphic.

Let us investigate the effect of a Stäckel transform on the quadratic algebra more closely. The following argument is done in the quantum mechanics operator case, though the analysis is identical for the classical case. Assume we are on a constant energy hypersurface with energy eigenvalue E as a function of α, β, γ and another parameter

m which usually represents the quantization levels of the energy but could also be an arbitrary parameter for continuous spectrum. That is

$$E = E(\alpha, \beta, \gamma, m) \tag{3.10}$$

Further, suppose that a second order symmetry operator L has eigenvectors ϕ_n which obey

$$H\phi_n = E\phi_n, \qquad L\phi_n = \lambda_n \phi_n. \tag{3.11}$$

Suppose further that we have adjusted the constants so that it is convenient to perform the Stäckel transform via a piece of the potential U in the following,

$$H = H_0 + V, \qquad L = L_0 + W. \tag{3.12}$$

$$V(\alpha, \beta, \gamma) = \alpha V^1 + \beta V^2 + \gamma U, \qquad W(\alpha, \beta, \gamma) = \alpha W^1 + \beta W^2 + \gamma W_U$$

We can then rewrite the first of equations 3.11 in the form

$$\left(\frac{1}{U} H_0 + \alpha \frac{V^1}{U} + \beta \frac{V^2}{U} - E \frac{1}{U} \right) \phi_n = -\gamma \phi_n.$$

or

$$\tilde{H}\phi_n = -\gamma\phi_n, \qquad \tilde{H} = \tilde{H}_0 + \tilde{V}$$

where

$$\tilde{H}_0 = \frac{1}{U} H_0, \qquad \tilde{V} = \frac{1}{U} \left(\alpha V^1 + \beta V^2 - E \right).$$

As you can see, here E has become a parameter in the potential and $-\gamma$ has become the new energy value. In fact, we can solve (3.10) for γ to obtain

$$\gamma = \gamma(\alpha, \beta, E).$$

We can also see what happens under the Stäckel transform to the operator L. We can transform the second equation of 3.11 using our new energy value $\tilde{H}\phi_n = -\gamma\phi_n$. We obtain,

$$\tilde{L}\phi_n = \left(L_0 + \alpha W^1 + \beta W^2 - W_U \tilde{H} \right) \phi_n = L\phi_n = \lambda_n \phi_n. \tag{3.13}$$

So the Stäckel transform preserves the eigenvalues of the symmetry operators and so we can think of the symmetry operators L and $\tilde{L}$ as being esssentially the same operators when restricted to energy eigenspaces.

Finally, suppose we have a quadratic algebra structure

$$[L_1, L_2] = R, \qquad R^2 = F(L_1, L_2, H, \alpha, \beta, \gamma) \tag{3.14}$$

$$[R, L_1] = F_1(L_1, L_2, H, \alpha, \beta, \gamma), \qquad [R, L_2] = F_2(L_1, L_2, H, \alpha, \beta, \gamma).$$

Under the Stäckel transform, this goes to a new algebra where the structure is unchanged except for a permutation of parameters.

$$[\tilde{L}_1, \tilde{L}_2] = \tilde{R}, \qquad \tilde{R}^2 = F(\tilde{L}_1, \tilde{L}_2, \tilde{H}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \tag{3.15}$$

$$[\tilde{R}, \tilde{L}_1] = F_1(\tilde{L}_1, \tilde{L}_2, \tilde{H}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}), \qquad [\tilde{R}, \tilde{L}_2] = F_2(\tilde{L}_1, \tilde{L}_2, \tilde{H}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$$

with $\tilde{\alpha} = \alpha, \tilde{\beta} = \beta, \tilde{\gamma} = -E, \tilde{H} = \gamma$.

Notice that this transform is exactly the transform computed in the example of E2 and its Stäckel transform when instead of separating off a piece of the energy δ we separate off the energy value entirely, $\delta = E$ (4.2.1-2).

It is important to note that this computation is only formal and that transformed energy eigenvectors may no longer be square integrable with respect to the new measure λU. This would lead to a change in the energy spectrum, possibly from discrete to continuous or visa versa. Furthermore, because of the interchange of a parameter with the energy it will occur often that a finite dimensional representation will require quantization of a parameter instead of the energy values. While this has no physical sense in the original problem, it indicates that the system is Stäckel equivalent to one where there is quantization of the energy values. This will be the case in system E6 in section 4.3.4.

Let us now return to the Kepler- Coulomb example. This example is Stäckel equivalent to the simple harmonic oscillator which is a well known fact, called the hodograph transform. Though the systems are degenerate and depend on only 1 parameter and exhibit a first order symmetry operator in addition to the 2 second order ones, the Stäckel transform still has the same action on the operators as though it were acting on a nondegenerate system.

Example 3. *The Hamiltonian for the* **quantum simple harmonic oscillator** *in 2d is*

$$H = \Delta + \alpha(x^2 + y^2) \tag{3.16}$$

A basis for the symmetry operators is formed by the Hamiltonian along with

$$X = \frac{1}{2}M = \frac{1}{2}(x\partial_y - y\partial_x)$$

$$L_1 = \frac{1}{4}(\partial_x^2 - \partial_y^2) + \alpha(x^2 - y^2), \qquad L_2 = \frac{1}{2}\left(\partial_x\partial_y + \alpha xy\right)$$

The symmetry algebra is given by,

$$[L_1, X] = L_2, \qquad [L_2, X] = -L_1 \qquad [L_1, L_2] = -\alpha X, \tag{3.17}$$

$$L_1^2 + L_2^2 + \alpha X^2 - \frac{\alpha}{4} - \frac{H^2}{16} = 0. \tag{3.18}$$

This symmetry algebra is isomorphic to the Lie algebra sl_2 when we restrict to constant energy hypersurfaces. We may remember another algebra that had the same structure which was the algebra for the Coulomb potential. Recall, the quantum Coulomb Hamiltonian is of the form

$$H = \Delta_2 + \frac{\alpha}{\sqrt{x^2 + y^2}} \tag{3.19}$$

with symmetry operators

$$X = M = x\partial_y - y\partial_x$$

$$L_1 = \frac{1}{2}(M\partial_x + \partial_x M) - \frac{\alpha y}{2\sqrt{x^2 + y^2}}, \quad L_2 = \frac{1}{2}(M\partial_y + \partial_y M) - \frac{\alpha x}{2\sqrt{x^2 + y^2}}.$$

The algebra is

$$[L_1, X] = L_2 \qquad [L_2, X] = -L_1 \qquad [L_1, L_2] = HX \tag{3.20}$$

$$L_1^2 + L_2^2 - HX^2 + \frac{H - \alpha^2}{4} = 0. \tag{3.21}$$

We can make a Stäckel transform from one Hamiltonian to another by separating off, the only portion of the Hamiltonian that we can, that is associated with the parameter α. We begin with the Hamiltonian for the simple harmonic oscillator, (3.16)

$$H = H_0 + \alpha(x^2 + y^2) \to \tilde{H} = \frac{1}{x^2 + y^2}(H_0 - E). \tag{3.22}$$

We can see this is in fact just the standard Laplace-Beltrami operator written in parabolic cylinder coordinates. If we transform back, using

$$\nu = \frac{1}{4}(x^2 - y^2), \qquad \zeta = xy$$

we obtain

$$\tilde{H} = \partial_\nu^2 + \partial_\zeta^2 - \frac{E}{2\sqrt{\nu^2 + \zeta^2}}.$$

We immediately can see the relation to the Hamiltonian (3.19) and in fact the two symmetry algebras are equivalent if we restrict to constant energy hypersurfaces $H \equiv E, \tilde{H} \equiv \tilde{E}$. We then obtain,

$$[L_1, X] = L_2 \qquad [L_2, X] = -L_1 \qquad [L_1, L_2] = -\alpha X \tag{3.23}$$

$$L_1^2 + L_2^2 + \alpha X^2 - \frac{\alpha}{4} - \frac{E^2}{16} = 0. \tag{3.24}$$

$$[\tilde{L}_1, \tilde{X}] = \tilde{L}_2 \qquad [\tilde{L}_2, \tilde{X}] = -\tilde{L}_1 \qquad [\tilde{L}_1, \tilde{L}_2] = -\tilde{\alpha} \tilde{X} \tag{3.25}$$

$$\tilde{L}_1^2 + \tilde{L}_2^2 - \tilde{E}\tilde{X}^2 + \frac{\tilde{E}}{4} - \frac{\tilde{\alpha}^2}{4} = 0. \tag{3.26}$$

These are isomorphic under the identifications

$$\tilde{L}_1 = L_1, \qquad \tilde{L}_2 = L_2, \qquad \tilde{E} = -\alpha, \qquad \tilde{\alpha} = -\frac{1}{2}E.$$

In fact, if we compare the energy values of the two systems we can also see the duality. The energy levels of hydrogen go as $E_h = -\frac{\alpha^2}{m^2}$ while those for the simple harmonic oscillator are $E_{SHO} = 2\sqrt{-\alpha}m$ where we have normalized out the constants which would typically occur in the physical Hamiltonian, such as charge and mass.

It is important to point out that these are two systems on Euclidean space which are Stäckel equivalent, underscoring the fact that the Stäckel transform does not necessarily map between systems on different spaces. In fact, the Stäckel equivalence cuts down the possibilities of distinct symmetry algebras in 2d from 29 systems in constant curvature spaces to 13 equivalence classes, as shown in tables 4.2 and 4.3.

3.2.6 Stäckel Equivalent Manifolds

We know that in 2d, every superintegrable system is Stäckel equivalent to one on a manifold with constant curvature. Another way to think about this fact is that every superintegrable system is equivalent to a system with a potential term that is proportional to the metric. That is,

$$\mathcal{H} = \frac{1}{\lambda}(p_x^2 + p_y^2) + V(x,y) \underset{\text{Stäckel Equivalent}}{\longleftrightarrow} \tilde{\mathcal{H}} = \frac{1}{\lambda'}(p_x^2 + p_y^2 + k),$$

where $\lambda' = \lambda V(x,y)$.

We can then find immediately the Stäckel equivalence classes by determining these spaces. To do this, we consider the Killing equations (3.4). For nondegenerate systems the Killing equations can then be rewritten as

$$(\lambda_{22} - \lambda_{11})a^{12} - \lambda_{12}(a^{22} - a^{11}) = 3\lambda_1 a^{12}_1 - 3\lambda_2 a^{12}_2 + (a^{12}_{11} - a^{12}_{22})\lambda. \tag{3.27}$$

We can simplify these equations by choosing a standard basis for our symmetry operators. That is, since we have 3 linearly independent symmetry operators, we make a choice of basis as described in (3.7) so that the superintegrable system is uniquely defined by the off diagonal element, call it a^{12}. Furthermore, we can choose coordinates so that our metric obeys the requirement $\lambda_{12} = 0$. With these choices we have simplified the Killing equations to one fundamental equation, given in the following theorem.

Theorem 10. *If $ds^2 = \lambda(dx^2 + dy^2)$ is the metric of a nondegenerate superintegrable system (expressed in coordinates x, y such that $\lambda_{12} = 0$) then $\lambda = \mu$ is a solution of the system*

$$\mu_{12} = 0, \qquad \mu_{22} - \mu_{11} = 3\mu_1(\ln a^{12})_1 - 3\mu_2(\ln a^{12})_2 + \left(\frac{a^{12}_{11} - a^{12}_{22}}{a^{12}}\right)\mu, \tag{3.28}$$

where either

$$I) \quad a^{12} = X(x)Y(y), \quad X'' = \alpha^2 X, \quad Y'' = -\alpha^2 Y,$$

or

$$II) \quad a^{12} = \frac{2X'(x)Y'(y)}{C(X(x)+Y(y))^2},$$

$$(X')^2 = F(X), \quad X'' = \frac{1}{2}F'(X), \quad (Y')^2 = G(Y), \quad Y'' = \frac{1}{2}G'(Y)$$

where

$$F(X) = \frac{\alpha}{24}X^4 + \frac{\gamma_1}{6}X^3 + \frac{\gamma_2}{2}X^2 + \gamma_3 X + \gamma_4,$$

$$G(Y) = -\frac{\alpha}{24}Y^4 + \frac{\gamma_1}{6}Y^3 - \frac{\gamma_2}{2}Y^2 + \gamma_3 Y - \gamma_4.$$

Conversely, every solution λ of one of these systems defines a nondegenerate superintegrable system. If λ is a solution then the remaining solutions μ are exactly the nondegenerate superintegrable systems that are Stäckel equivalent to λ.

We note that these spaces are exactly the spaces classified by Koenigs in [39] where he identified the spaces which admit more than one second order Killing tensor. Following the notation in his paper, we write the requirements on μ and a^{ij} as

$$a^{12}_{11}+a^{12}_{22} = 0, \qquad \mu_{12} = 0, \qquad a^{12}(\mu_{11}-\mu_{22})+3\mu_1 a^{12}_1 - 3\mu_2 a^{12}_2 + (a^{12}_{11}-a^{12}_{22})\mu = 0, \quad (3.29)$$

Using the particular solutions to (3.28), we can recover the metrics discovered by Koenigs which admit two linearly independent, second order Killing tensors. The metrics for these spaces are classified in the tables in appendix A; Tableau VI are solutions to (I) and Tableau VII are solutions to (II) in theorem 10 given by Koenigs.

As noticed by Koenigs, these tables are split up depending on whether they are equivalent to Euclidean space (Tableau VI) or the sphere (Tableau VII.) However, we now know that systems on the sphere can be Stäckel equivalent to each other or to ones on Euclidean space. Furthermore, there are exactly 7 equivalence classes of nondegenerate superintegrable systems in 2 dimensions and so some of the 11 systems given in the two tables must be related to each other by some combination of change of variables or Stäckel transform. The easiest way to identify two such systems is to look at their quadratic algebras.

We can also consider the degenerate potentials. These will correspond to potential free Hamiltonians that allow a Killing vector as well as two linearly independent second order Killing Tensors. The fundamental equations for these are the same as (3.27) but also include the requirement that there exists a Killing vector.

The solutions of these equations correspond to the Darboux spaces which allow exactly one Killing vector [39, 40]. These are also given in appendix A.

The final spaces in 2 dimensions which allow a Killing vector are Euclidean space and the two sphere. Without a potential these spaces allow 3 Killing vectors but this is restricted to one Killing vector when the potential depends on exactly one parameter.

3.3 Superintegrability in 3d

Now we move to the case of second order superintegrable systems in 3 dimensions. We begin with an example.

Example 4. The generic system in 3d.
We define the Hamiltonian operator via the embedding of the unit 3-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ *in four dimensional flat space.*

$$H = \sum_{1 \le i < j \le 4} (x_i \partial_j - x_j \partial_i)^2 + 3 \sum_{k=1}^{4} \frac{a_k}{x_k^2}, \quad \partial_i \equiv \partial_{x_i}. \tag{3.30}$$

A basis for the second order constants of the motion is

$$L_{ij} \equiv L_{ji} = (x_i \partial_j - x_j \partial_i)^2 + \frac{a_i}{x_i^2} + \frac{a_j}{x_j^2}, \tag{3.31}$$

for $1 \le i < j \le 4$*. Here,*

$$H = \sum_{1 \le i < j \le 4} L_{ij}.$$

Notice that there are 6 linearly independent second-order constants of the motion $\{L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}\}$, a characteristic which will always hold for nondegenerate systems in 3d. The symetry algebra of this system will be composed of the second order symmetry operators and the commutators of the second order symmetry operators, $R_i = [L_{ij}, L_{kl}]$, of which 4 are linearly independent. There are 24 fourth order symmetry relations composed of the commutators of the R's with the L's, and 10 sixth order relations rewriting R_i^2 and $R_i R_j$ in terms of the $L_i's$. Additionally, since there can only be 5 second order symmetry operators which are functionally independent, there must be a functional relationship between them. This occurs at the 8th order. A model of this system is currently being investigated.

In terms of the structure theory of nondegenerate 3d systems, much progress has been made recently in the classification of all such spaces. The analysis of such systems is much more complex not only because there are now $2 \times 3 - 1 = 5$ functionally independent symmetry operators but also because of the following characteristics, in 3d

- There may be more than 5 linearly independent second order symmetries.

- Not all degenerate systems are restrictions of nondegenerate systems.
- The 1-1 correlation between classical and quantum systems has not been proven in general, only for nondegenerate systems.

However, despite these difficulties Kalnins et. al. [28, 29, 30, 33] have been able to determine the structure theory for nondegenerate superintegrable systems on comformally flat spaces in 3d. There is an obvious 3d analog for the definition of a nondegenerate potential, that is one that depends on a maximal number of parameters which is 4 plus a trivial additive constant. There are also the obvious 3d versions of the Killing and Bertrand-Darboux Equations which can be derived from computing the Poisson bracket of a constant of the motion with the Hamiltonian and comparing coefficients.

Using these structure equations, we have following results for 3d superintegrable systems.

Theorem 11. ($5 \Longrightarrow 6$) *Let V be a nondegenerate potential corresponding to a conformally flat space in 3 dimensions that is superintegrable, with 5 functionally independent constants of the motion. Then the space of second order symmetries for the Hamiltonian $\mathcal{H} = (p_x^2 + p_y^2 + p_z^2)/\lambda(x, y, z) + V(x, y, z)$ (excluding multiplication by a constant) is of dimension $D = 6$.*

Theorem 12. *Let V be a superintegrable nondegenerate potential on a conformally flat space. Then the space of third order constants of the motion is 4-dimensional and is spanned by Poisson brackets of the second order constants of the motion.*

In the paper [28], the previous theorem is given as a corollary to a theorem identifying the space of third order symmetry operators with 3×3 skew symmetric matrices, similar to the proof in 2d. Again, like the argument in 2d, we use this result to identify the second-order symmetries with 3×3 symmetric matrices, with some complications. We also have the following: as in 2d the $\mathcal{S}_{ij}$ are determined by the choice of basis for the symmetric matrices.

Theorem 13. *The 21 distinct standard monomials $\mathcal{S}_{(ij)}\mathcal{S}_{(jk)}$, defined with respect to a regular point $\mathbf{x}_0$, form a basis for the space of fourth order symmetries.*

Theorem 14. *The 56 distinct standard monomials $\mathcal{S}_{(hi)}\mathcal{S}_{(jk)}\mathcal{S}_{(\ell m)}$, defined with respect to a regular $\mathbf{x}_0$, form a basis for the space of sixth order symmetries.*

From these theorems, one can see that the algebra must close again at 6th order but now there is no longer a single independent Casimir operator but instead 4 of them. That is, given $\mathcal{R}_{ij} = \{\mathcal{L}_i, \mathcal{L}_j\}$ third order constants of the motion then their product $\mathcal{R}_{ij}\mathcal{R}_{kl}$ is a 6th order symmetry operator and so must be expressible in terms of the above monomials. Also in the symmetry algebra is a functional relation between the 6 second-order symmetry operators which has been observed to be of 8th order in all know cases but not yet proven in general.

Using these theorems, all 3d non-degenerate systems with functionally linearly independent symmetry operators can be shown to be multiseparable and Stäckel equivalent to constant curvature space systems [29]. Also, there is a 1-1 correlation between classical and quantum nondegenerate systems [31]. In 3d it has been proven that there are exactly 10 non-degenerate systems in Euclidean space [33], and there are 6 known systems on the sphere but this list has not been proved to be exhaustive.

Despite the similarities in the structure theory for 2d and 3d nondegenerate systems, there are some fundamental differences that come in when we extend to 3d for degenerate systems. First, though there have been no counter-examples found, we also have no proof that there is a bijection between classical and quantum systems for degenerate systems. Furthermore, there are degenerate systems, such as the example below, whose symmetry algebra is not finite dimensional [32].

Example 5. *The quantum* **extended Kepler-Coulomb** *Hamiltonian is*

$$H = \Delta_3 + \frac{\alpha}{\sqrt{x^2+y^2+z^2}} + \frac{\beta}{x^2} + \frac{\gamma}{y^2}.$$

This system admits 5 second order symmetry operator and is separable in 4 orthogonal coordinate systems but the symmetry algebra does not close at a finite order [41].

Chapter 4

Algebras and Representations in the 2d quantum case

4.1 Preliminaries

4.1.1 Conventions

We will make use of the following conventions.

On Euclidean space the generators of the Killing vectors are,

$$p_x = \partial_x, \quad p_y = \partial_y, \quad M = x\partial_y - y\partial_x.$$

We also can define the operator $p_\pm = \partial_x \pm i\partial_y$. From these, we can create second order symmetry operators which are associated with separable coordinates on $E_{2,C}$.

The Laplacian on $E_{2,c}$ in Cartesian coordinates is,

$$\Delta \equiv \partial_x^2 + \partial_y^2.$$

In complex coordinates, $z = x + iy, \overline{z} = x - iy$, we have

$$\Delta \equiv \partial_z \partial_{\overline{z}}.$$

Or, we can take a real form in lightlike coordinates, $\nu = x, \zeta = iy$, to obtain the wave operator,

$$\Delta \equiv \partial_\nu^2 - \partial_\zeta^2.$$

For systems on the two sphere, we use the coordinates of the standard embedding of the sphere into 3 dimensional Euclidean space. We denote these s_1, s_2, s_3 such that $s_1^2 + s_2^2 + s_3^2 = 1$. The basis for the symmetry operators is,

$$J_i = \sum_{i,j,k} \epsilon^{ijk} s_j \partial_{s_k}.$$

The Laplacian on $S_{2,C}$ is,

$$\Delta_{S^2} \equiv \sum_{i=1}^{3} J_i^2$$

For a complete list of second-order symmetry operators and their associated coordinate systems, see the tables in appendix B.

The quantum algebra structures often have symmetrized terms. We define these by $\{a, b\} \equiv ab + ba$ and $\{a, b, c, \} \equiv abc + acb + bac + bca + cab + cba$. Also used is the permutation sign function ϵ_{ijk} and ϵ_{ijkl}, the completely skew-symmetric tensor on three or four variables respectively.

Finally, the potentials depend on 3 parameters for the nondegenerate systems and 1 parameter for the degenerate systems. I will use the Greek letters, α, β, γ for these constants. Sometimes, I make a change of parameters; I use ω as in $\alpha = -\omega^2$ to agree with the conventions of the simple harmonic oscillator. Also, I will use lower case letters for parameters of the form $\beta = \frac{1}{4} - b^2$. Finally, in the cases where a numeric subscript simplifies the structure relations I use the form a_i; this occurs in S9 and later the system in 3d. It is interesting to note that these same parameter changes occur in many different models.

4.1.2 Creating the Models

In the following sections, I exhibit irreducible representations of the quadratic algebras using both differential and difference operator realizations of the operators acting on function spaces. I refer to these realizations as "models." Because the Hamiltonian operator commutes with all of operators, it must be a constant for any irreducible representation. Often, finite dimensional representations will correspond to quantization of the energy levels of this constant. Similarly, as described in Chapter 3, the quadratic algebra admits a functional relation which we refer to a the Casimir relation. For example, the Casimir for E20 is

$$R^2 - 4L_1^2H - 4L_2^2H + 16\alpha^2H + 16(\gamma^2 - \beta^2)L_1 + 32\beta\gamma A_2 = -32\alpha^2(\beta^2 + \gamma^2)$$

The left hand sign of this relations, seen as an operator, will commute with all other operators in the algebra. We can determine a representation of the quadratic algebra whose operators satisfy all the algebra relations except for the Casimir relation. Then, the Casimir operator will commute with all these operators and as such its action in the model will be multiplication by a constant. Thus, we will have a representation of the algebra for the superintegrable systems if and only if the value of the Casimir operator in the model agrees with the value determined by the functional relation.

Practically, we restrict the Hamiltonian to a constant energy $H \equiv h$ (constant) and attempt to find models which will diagonalize operators associated with separation of variables, since these are of immediate interest for explicit solution of the physical system. The relations to be satisfied are then the commutation relations and the Casimir relation.

4.1.3 Classical Models

By 2d classical models, I refer to a set of constants of the motion, given in terms of two conjugate variables, which obey the same quadratic algebra relations as the original system in 2d.

Classical Hamilton-Jacobi theory guarantees the existence of such a model. Recall from the exposition in section 2.1 that the phase space for our system is 4-dimensional, including 2 position coordinates and 2 momenta. Furthermore, since the system is integrable, it is possible to find canonical variables $\mathcal{H}, \mathcal{I}, \mathcal{Q}, \mathcal{P}$ such that $\{\mathcal{I}, \mathcal{H}\} = \{\mathcal{P}, \mathcal{Q}\} = 1$ and all other Poisson brackets vanish; we choose $\mathcal{H}$ as one of our canonical variables so that the constants of the motion, which commute with $\mathcal{H}$ will be independent of it. Let $\mathcal{P}$ be the constant of the motion that we wish to specify. In terms of $\mathcal{H}$ and the other canonical variables the Poisson bracket can be expressed as

$$\{\mathcal{F}, \mathcal{G}\} = -\partial_{\mathcal{H}}\mathcal{F}\partial_{\mathcal{I}}\mathcal{G} + \partial_{\mathcal{I}}\mathcal{F}\partial_{\mathcal{H}}\mathcal{G} - \partial_{\mathcal{Q}}\mathcal{F}\partial_{\mathcal{P}}\mathcal{G} + \partial_{\mathcal{P}}\mathcal{F}\partial_{\mathcal{Q}}\mathcal{G}. \tag{4.1}$$

Then, if we restrict the algebra to the constant energy space $\mathcal{H} = h$, every constant of the motion $\mathcal{F}$ can be expressed in the form $\mathcal{F}(h, \mathcal{Q}, \mathcal{P})$ where the Poisson bracket of two constants of the motion restricts to

$$\{\mathcal{F}, \mathcal{G}\} = -\partial_{\mathcal{Q}}\mathcal{F}\partial_{\mathcal{P}}\mathcal{G} + \partial_{\mathcal{P}}\mathcal{F}\partial_{\mathcal{Q}}\mathcal{G}. \tag{4.2}$$

Thus all functions depend on only two canonically conjugate variables $\mathcal{Q}, \mathcal{P}$ and the parameter h but the algebra relations remain the same. This shows the existence of classical models in two conjugate variables. However the proof is not constructive and, furthermore, it is not unique. To obtain constructive results we will use the strategy of choosing one of the constants of the motion as $\mathcal{P}$ and then use (4.2) for the Poisson bracket and require that the quadratic algebra relations hold. We can then solve the partial differential equations defined by the quadratic algebra relations for these two variables. Classical models have been investigated in detail by Kalnins et al [42, 17].

4.1.4 Quantum Models

Informally, 2d a quantum model is given in terms of a set of differential operators in one variable, usually t, that generate an algebra isomorphic to the quadratic algebra of the superintegrable system when the Hamiltonian is restricted to a constant energy. The basis vectors are given by the spectral resolution of one of the generators of the quadratic algebra, usually one associated with separation of variables.

Though we have no formal proof of the existence of quantum models, we can find guidance from the quantized version of the classical models. However, the quadratic algebra differs by lower-order terms from the classical to quantum case and so knowing a classical model gives an ansatz for the quantum model and even then, as shall be shown below, the classical models often require several changes of variables to put them in a form which can be quantized.

In fact, it is often easier to compute directly the quantum models by assuming a differential operator ansatz. That, is we can use the quantum algebra relations to give restrictions on the possible degrees of the differential operators that will form a basis for the model. For example, we might assume the L_i are of the form

$$L_i = f_2^{(i)}(t)\partial_t^2 + f_1^{(i)}(t)\partial_t + f_0^{(i)}(t).$$

We then use the algebra relations to determine the functional coefficients of the operators. This was the method chosen for all of the following systems, except for the system

S9 (4.2.6) where there is no differential model so we had to use an abstract analysis to determine the model.

A quantum model is especially well suited for determining the spectrum of an operator X if it is of the form $X = t$ or $X = \partial_t$, where t is the single variable used in the model. However, particularly in the former case, this operator may not be the one whose eigenfunctions form the basis for the model. Furthermore, there are many useful models where none of the operators take this form.

Our goals for the quantum models are to create function space representations with an inner product and to use these models to determine the spectra of the symmetry operators. We also can determine expansion coefficients from one basis to another and in the process find interesting special function identities. While the models can give information about the physical system, they also are interesting in their own right.

4.2 Nondegenerate systems quantum systems in $2d$

For nondegenerate systems in $2d$ the symmetry algebra is defined by the second-order operators H, L_1, L_2 and the commutator $R \equiv [L_1, L_2]$. The defining relations of the algebra are then the two commutation relations,

$$[R, L_1] = P_1(L_1, L_2, H), \qquad [R, L_2] = P_2(L_1, L_2, H)$$

where the P_i's are polynomials of the operators of total degree at most 2. The other relation of the algebra is the Casimir relation

$$R^2 - Q(L_1, L_2, H) = c.$$

We often refer to the left side of the equation as the Casimir operator and the right hand side as a constant times the identity operator. Here Q is a polynomial in the arguments of total degree less than 3 and with no constant term.

We can classify the nondegenerate systems in $2d$ by their Stäckel equivalence classes and identify them by the leading order terms of their Casimir, as seen in the accompanying table (4.2). In the following sections, we describe the symmetry operators, quadratic algebras and at least one model for each of the equivalence classes.

Leading terms of Casimir relation		system	Operator models
L_1^3	$+f(\alpha_i, H)L_2^2$	E2, S1	Differential
L_1^3	$+f(\alpha_i, H)L_1L_2$	E9, E10	Differential
L_1^3	$+0$	E15	Differential
$L_1^2L_2$	$+f(\alpha_i, H)L_2^2$	E1, E16, S2, S4	Differential
$L_1^2L_2$	$+0$	E7, E8, E17, E19	Differential
$L_1L_2(L_1+L_2)$	$+f(\alpha_i, H)L_1L_2$	S7, S8, S9	Difference
0	$+f(\alpha_i, H)L_1L_2$	E3, E11, E20	Differential

Table 4.1: Stäckel equivalence classes of non-degenerate systems in 2d

4.2.1 E2

The Hamiltonian is given by,

$$H = \Delta - \omega^2(4x^2+y^2) + bx + \frac{\frac{1}{4}-c^2}{y}$$

A basis for the symmetry operators is given by H and

$$L_1 = p_x^2 + 4\omega^2x^2 - bx,$$
$$L_2 = \frac{1}{2}\{M, p_y\} + y^2(\frac{b}{4} - x\omega^2) - (\frac{1}{4}-c^2)\frac{x}{y^2}.$$

The symmetry algebra relations are,

$$[L_1, R] = 2bH + 16\omega^2L_2 - 2bL_1,$$

$$[L_2, R] = 8L_1H - 6L_1^2 - 2H^2 + 2bL_2 - 8\omega^2(1-c^2),$$

$$R^2 + 4L_1^3 + 4L_1H^2 - 8L_1^2H + 16\omega^2L_2^2 + 4bL_2H - 2b\{L_1, L_2\}$$
$$+16\omega^2(3-c^2)L_1 - 32\omega^2H - b^2(1-c^2) = 0.$$

A suitable differential model yielding spectral information for L_1 is,

$$L_1 = 4t\omega\partial_t + 2\omega + \frac{1}{16}\frac{b^2}{\omega^2},$$

$$L_2 = 32\omega t^2\partial_t^2 + \left((b^2+12\omega^2-4\omega^2h)t^2 + \frac{2b}{\omega}t - \frac{2}{\omega^2}\right)\partial_t$$

$$+\frac{1}{16\omega^3}\left(60\omega^6+16c\omega^6-64h\omega^5+16h^2\omega^4+16b^2\omega^3-8b^2h\omega^2+b^4\right)t+\frac{b}{4\omega^4}(b^2-4\omega^2+4\omega^3).$$

Here, the eigenfunctions of L_1 are monomials and we can define raising and lowering operators as,

$$A = L_2 - \frac{R}{4\omega} - \frac{bL_1}{4\omega^2} + \frac{bh}{4\omega^2} = \frac{1}{\omega^2}\partial_t,$$

$$A^\dagger = L_2 + \frac{R}{4\omega} - \frac{bL_1}{4\omega^2} + \frac{bh}{4\omega^2}$$

$$= 64\omega^3 t^3 \partial_t^2 + 64\omega^3(c-2m+3)t^2\partial_t + 64\omega^3(m-1)(m-1+c)t.$$

These operators obey the following commutation relations

$$[L_1, A] = -4\omega A, \qquad [L_1, A^\dagger] = 4\omega A^\dagger.$$

Here we have used the change of variables $h = 4\omega(m-2c) + \frac{b^2}{16\omega^2}$ and our model determines a finite dimensional irreducible representation under the restriction $m \in \mathbb{N}$.

4.2.2 E10

The Hamiltonian for this system is given by, with $z = x+iy, \overline{z} = x - iy$

$$H = \Delta + \alpha\overline{z} + \beta(z - \frac{3}{2}\overline{z}^2) + \gamma(z\overline{z} - \frac{1}{2}\overline{z}^3).$$

A basis for its symmetry operators is given by H and ,

$$L_1 = p_-^2 + \gamma\overline{z}^2 + 2\beta\overline{z},$$

$$L_2 = 2i\{M, p_-\} - 4\beta z\overline{z} - \gamma z\overline{z}^2 - 2\beta\overline{z}^3 - \frac{3}{4}\gamma\overline{z}^4 + \gamma z^2 + \alpha\overline{z}^2 + 2\alpha z.$$

The algebra relations are given by,

$$[R, L_1] = -32\gamma L_1 - 32\beta^2, \qquad [R, L_2] = 96L_1^2 - 128\alpha L_1 + 32\gamma L_2 + 64\beta H + 32\alpha^2, \quad (4.3)$$

$$R^2 = 64L_1^3 + 32\gamma\{L_1, L_2\} - 128\alpha L_1^2 - 64\gamma H^2 - 128\beta H L_1 + 64\beta^2 L_2 + 64\alpha^2 L_1 - 128\beta\alpha H - 256\gamma^2.$$

We can renormalizes to obtain the Lie algebra sl_2 by using the following invertible transform,

$$K_1 = L_1 - \frac{\beta^2}{\gamma}, \qquad (4.4)$$

$$K_2 = L_2 + \frac{1}{\gamma}L_1^2 - \frac{\beta^2 + 2\alpha\gamma}{\gamma^2}L_1 + \frac{1}{\gamma^3}\left(\beta^4 + 2\beta\gamma h + 2\beta^2\alpha\gamma + \alpha^2\gamma^2\right).$$

This gives us,

$$[R, K_1] = -32\gamma K_1, \qquad [R, K_2] = 32\gamma K_2, \qquad [K_1, K_2] = R,$$

$$R^2 - 32\gamma\{K_1, K_2\} = \frac{64}{\gamma^3}(\beta^6 + 2\alpha\gamma\beta^4 + 2h\gamma^2\beta^3 + \alpha^2\gamma^2\beta^2 + 2\alpha h\gamma^3\beta + h^2\gamma^4 + 4\gamma^5).$$

We can then use the representations of this simple Lie algebra to describe our system. In particular, we can choose finite dimensional representations to obtain quantization conditions. A bounded below model based on the spectral resolution of R is:

$$K_1 = 16\gamma\frac{d}{dt}, \qquad K_2 = 32\gamma t^2\frac{d}{dt} - 64\gamma m t, \qquad R = (32\gamma)^2(t\frac{d}{dt} - m),$$

where m is a constant This yields a finite dimensional representation if $m \in \mathbb{N}$.

A second model, that follows from [43] is

$$R = 16\gamma(2t - 2m - 1 - 2t\frac{d}{dt}), \quad K_1 = 4\sqrt{\gamma}t,$$

$$K_2 = 4\sqrt{\gamma}\left(t - 2m - 1 + 2(m + \frac{1}{2} - t)\frac{d}{dt} + t\frac{d^2}{dt^2}\right).$$

In this model m is any nonnegative real number, the natural Hilbert space is $L_2(0, \infty, \mu)$ where the weight function is $\mu(t) = t^{2m}e^{-2t}$, and the irreducible representations are infinite dimensional but bounded below. This model gives the spectral resolution of K_1 (continuous spectrum) and it is also easy to compute the spectral resolution of R from this (also continuous spectrum). The operator $K_1 - K_2$ has discrete spectrum $8\sqrt{\gamma}(m + n + \frac{1}{2})$ with unnormalized orthogonal eigenbasis $L_n^{(2m)}(2t)$, $n = 0, 1, \cdots$. Here $L_n^{(\alpha)}(x)$ is an associated Laguerre polynomial.

In section 5.3, I will consider this case in more detail.

4.2.3 E15

Here

$$H = \Delta + h(\overline{z})$$

where the potential is an arbitrary function of $\overline{z}$. A basis for the symmetry operators is,

$$L_1 = p_-^2, L_2 = 2i\{M, p_-\} + i\int \overline{z}\frac{dh}{d\overline{z}}\ d\overline{z}.$$

The only nonzero algebra relation is $[L_1, L_2] = iL_1$. This system is unique among all 2d superintegrable systems in that the symmetry operators are not functionally linearly independent and do not correspond to multiseparability. The only separable system is determined by diagonalizing L_1, essentially $z, \overline{z}$, and this coordinate system is not orthogonal. A model is

$$L_1 = \frac{d}{dt} + a, \qquad L_2 = it\frac{d}{dt} + ia,$$

but the irreducible representations of the algebra yield no spectral information about H.

4.2.4 E1

The Hamiltonian for the system is,

$$H = \Delta - \omega^2(x^2 + y^2) + \frac{\frac{1}{4} - a^2}{x^2} + \frac{\frac{1}{4} - b^2}{y^2}.$$

The remaining generators of the symmetry algebra are,

$$L_1 = \partial_x^2 + \frac{1/4 - a^2}{x^2} - \omega^2 x^2, \quad L_2 = M^2 + x^2\frac{1/4 - a^2}{x^2} + y^2\frac{1/4 - b^2}{y^2}.$$

The algebra relations are,

$$[L_1, L_2] = R, \qquad [R, L_1] = 8L_1^2 - 8HL_1 + 16\omega^2 L_2 - 8\omega^2,$$

$$[R, L_2] = 8HL_2 - 8\{L_1, L_2\} + (12 - 16a^2)H + (16a^2 + 16b^2 - 24)L_1,$$

$$R^2 + \frac{8}{3}\{L_1, L_1, L_2\} - 8H\{L_1, L_2\} + (16b^2 + 16a^2 - \frac{200}{3})L_1^2 - 16\omega^2 L_2^2 + (32a^2 - \frac{200}{3})HL_1$$

$$+(16 - 16a^2)H^2 + \frac{176}{3}\omega^2 L_2 = -64\omega^2(b^2 + a^2 - b^2a^2 - \frac{29}{48}).$$

A suitable model for the spectral resolution of L_1 can be chosen as

$$L_1 = -4\omega t\partial_t - 2\omega(1 - 2m - a),$$

$$A = L_2 + \frac{R}{4\omega} - \frac{L_1^2}{2\omega^2} + \frac{h}{2\omega^2}L_1 - \frac{1}{2} = t\partial_t^2 + (1 + b)\partial_t,$$

$$A^\dagger = L_2 - \frac{R}{4\omega} - \frac{L_1^2}{2\omega^2} + \frac{h}{2\omega^2}L_1 - \frac{1}{2}$$

$$= 64t^3\partial_t^2 + (192 - 64a - 128m)t^2\partial_t + (62m^2 + (64a - 128)m + 64 - 64a)t.$$

Notice R, L_2 can be recovered using $L_2 = \frac{1}{2}(A + A^\dagger)$ and the defining equation for R. The commutation relations for our new operators are,

$$[A, A^\dagger] = 764L_1^3 + (510b + 766a - 512)L_1^2 + (160a^2 + 256ba - 256b - 128a + 96)L_1$$
$$-16(a^2 - 4a - 1)(a + b),$$
$$[L_1, A] = 4\omega A, \qquad [L_1, A^\dagger] = -4\omega A^\dagger.$$

Here we have used the change of variables $h = 2\omega(2m + a + b)$ where our model defines a finite dimensional irreducible representation under the restriction $m \in \mathbb{N}$.

In section (5.2), I will consider this case in more detail.

4.2.5 E8

The Hamiltonian is,

$$H = \Delta + \frac{\alpha z}{\overline{z}^3} + \frac{\beta}{\overline{z}^2} + \gamma z\overline{z}.$$

A basis for the symmetry algebra is given by,

$$L_1 = p_-^2 + \frac{\overline{z}^4 - \alpha}{\overline{z}^2}, \qquad L_2 = M^2 + \beta\frac{z}{\overline{z}} + \alpha\frac{z^2}{\overline{z}^2} - \alpha.$$

The algebra relations are,

$$[L_1, R] = 8{L_1}^2 + 32\alpha\gamma, \qquad [L_2, R] = -8\{L_1, L_2\} + 8bH - 16(\alpha + 1)L_1,$$
$$R^2 = -\frac{16}{3}(L_1^2L_2 + L_1L_2L_1 + L_2L_1^2) - (16\alpha + \frac{176}{3})L_1^2$$
$$+16\alpha H^2 - 64\alpha\gamma L_2 + 16\beta L_1 H - 64\gamma\alpha^2 - 16\gamma\beta^2 + \frac{64}{3}\alpha\gamma.$$

A suitable differential model for the spectral resolution of L_1 is given by,

$$L_1 = 2\sqrt{-\alpha\gamma}t,$$
$$L_2 = (t^2 - 1)\partial_t^2 + ((1 + \frac{\beta}{2\sqrt{\alpha}} + 1)t + 2m + \frac{\beta}{2\sqrt{\alpha}} + 2)\partial_t + \frac{(1 + \frac{\beta}{2\sqrt{\alpha}})^2 + \alpha}{4}.$$

Here, the eigenfunctions of L_2 are Jacobi polynomials. This model yields a finite dimensional irreducible representation under the quantization condition $m \in \mathbb{N}$ with energy eigenvalues,

$$h = 2\sqrt{-\gamma}(2m + 2 \pm \frac{\beta}{2\sqrt{\alpha}}).$$

4.2.6 S9: The generic system in 2d

The Hamiltonian is
$$H = \Delta_{S^2} + \frac{\frac{1}{4} - a^2}{s_1^2} + \frac{\frac{1}{4} - b^2}{s_2^2} + \frac{\frac{1}{4} - c^2}{s_3^2}.$$

A basis for the symmetry operators is,

$$L_1 = J_3^2 + (\frac{1}{4} - a^2)\frac{s_1^2}{s_2^2} + (\frac{1}{4} - c^2)\frac{s_2^2}{s_1^2},$$

$$L_2 = J_1^2 + (\frac{1}{4} - a^2)\frac{s_3^2}{s_2^2} + (\frac{1}{4} - b^2)\frac{s_2^2}{s_3^2},$$

$$H = L_1 + L_2 + L_3 + \frac{3}{4} - a^2 - b^2 - c^2.$$

The structure equations can be put in the symmetric form using the following identification,
$$a_1 = \frac{1}{4} - c^2, \ a_2 = \frac{1}{4} - a^2, \ a_3 = \frac{1}{4} - b^2.$$

$$[L_i, R] = \epsilon_{ijk}\left(4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k)\right), \tag{4.5}$$

$$\begin{aligned} R^2 = &\frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 \\ &+\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 \\ &+\frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3) \\ &+48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3. \end{aligned} \tag{4.6}$$

A suitable model diagonalizing L_3 is realized by setting

$$L_3 = -4t^2 + a^2 + c^2$$

We can obtain L_1 in the model by using the following difference operator, based upon the Wilson polynomial algebra. We simplify the model by using the substitutions

$$\alpha = -\frac{a + c + 1}{2} - m, \ \beta = \frac{a + c + 1}{2}, \ \gamma = \frac{a - c + 1}{2}, \ \delta = \frac{a + c - 1}{2} + b + m + 2,$$

$$T^A F(t) = F(t+A),\ \tau = \frac{1}{2t}(T^{1/2} - T^{-1/2}),$$

$$\tau^* = \frac{1}{2t}\left[(\alpha+t)(\beta+t)(\gamma+t)(\delta+t)T^{1/2} - (\alpha-t)(\beta-t)(\gamma-t)(\delta-t)T^{-1/2}\right]$$

to express the action of L_1:

$$L_1 = -4\tau^*\tau - 2(a+1)(b+1) + \frac{1}{2}.$$

The action of the other symmetry operator can be recovered from the following identifications,

$$L_1 + L_2 + L_3 = h = -\frac{1}{4}(4m + 2a + 2b + 2c + 5)(4m + 2a + 2b + 2c + 3).$$

The model yields a finite dimensional irreducible representation with the restriction $m \in \mathbb{N}$.

In this model, we obtain spectral resolution of L_3 with delta functions as eigenfunctions. The eigenfunctions of L_1 are Racah polynomials in the finite dimensional case and Wilson polynomials for the infinite dimensional, bounded below case.

In section 5.5 I will consider this case in more detail.

4.2.7 E20

The Hamiltonian is,

$$H = \Delta + \frac{1}{\sqrt{x^2+y^2}}\left(4\alpha + \beta\frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} + \gamma\frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}}\right).$$

A basis for the symmetry operators is given by H along with,

$$L_1 = \{M, p_x\} + W_1(x,y), \qquad L_2 = \{M, p_y\} + W_2(x,y),$$

where the functions W_i are difficult to determine in rectangular coordinates. Instead, we will use parabolic coordinates (x', y'), with $x' = \sqrt{x^2+y^2} + x, y' = \sqrt{x^2+y^2} - x$. The operators then become,

$$H = \frac{1}{x'^2 + y'^2}(\partial_{x'}^2 + \partial_{y'}^2) + \frac{4a - bx' - cy'}{x'^2 + y'^2},$$

$$L_1 = \frac{1}{x'^2+y'^2}\left(y'^2\partial_{x'}^2 - x'^2\partial_{y'}^2 - 2a(x'^2 - y'^2) - bx'y'^2 + cy'x'^2\right),$$

$$L_2 = \frac{1}{x'^2+y'^2}\left(-x'y'\partial_{x'}^2 + \partial_{x'}\partial_{y'} - x'y'\partial_{y'}^2 - 4ax'y' + \frac{(x'^2-y'^2)(by'-cx')}{2}\right).$$

The algebra relations are given by,

$$[R, L_1] = -4L_2H + 16\beta\gamma, \qquad [R, L_2] = 4L_1H - 8(\beta^2+\gamma^2),$$

$$R^2 = 4L_1^2H + 4L_2^2H - 16\alpha^2H - 16(\gamma^2-\beta^2)L_1 - 32\beta\gamma L_2 - 32\alpha^2(\beta^2+\gamma^2).$$

Notice, these restrict to a Lie algebra on a constant energy surface: $E = constant \equiv h$.

A differential model suited to finding the spectrum of L_1 is,

$$L_1 = -2\sqrt{h}t\partial_t - \sqrt{h} + 2\alpha + 4\frac{\beta^2}{h},$$

$$L_2 = -\frac{1}{2}ht^2\partial_t + 2\partial_t - \frac{1}{2}th + \sqrt{h}at + \left(\frac{\beta^2}{\sqrt{h}} + \frac{\gamma^2}{\sqrt{h}}\right)t - 8\frac{\beta\gamma}{\sqrt{h}}$$

$$R = \frac{1}{2}ht^2\partial_t + 2\partial_t + \frac{1}{2}ht - \sqrt{h}\alpha t - \left(\frac{\beta^2}{\sqrt{h}} + \frac{\gamma^2}{\sqrt{h}}\right)t.$$

Here L_1 is diagonalized by monomials and we have the following raising and lowering operators,

$$A = L_2 + R = 4\partial_t, \qquad A^\dagger = L_2 - R = -ht^2\partial_t - ht + 2\sqrt{h}at + (\frac{\beta^2}{\sqrt{h}} + \frac{\gamma^2}{\sqrt{h}})t.$$

This model is finite dimensional under the quantization condition $m \in \mathbb{N}$ and $-mh - h + 2\sqrt{h}a + \frac{\beta^2}{\sqrt{h}} + \frac{\gamma^2}{\sqrt{h}} = 0$.

4.3 2d Degenerate Systems

Next, we consider degenerate systems whose symmetry algebra includes a first order differential operator. As shown above, this requires that the potential depend on only one constant, not including the trivial additive constant. In these systems, the symmetry algebra is defined by the operators H, L_1, L_2 and X, the first order differential operator. We can rewrite the commutator $R = [L_1, L_2]$ as a polynomial in the other operators of maximal degree 3 in X and 1 in L_1, XL_1, L_2, XL_2, H and XH. In these systems, the commutation relations are in terms of X instead of R. That is, the defining relations are,

$$[L_1, X] = P_1(L_1, L_1, X^2, X, H), \qquad [L_1, X] = P_2(L_1, L_1, X^2, X, H),$$

$$[L_1, L_2] = Q(X^3, XL_1, XL_2, XH, L_1, L_2, H, X).$$

Here, the P_i's and Q are linear in the arguments. Furthermore, the Casimir relation is no longer in terms of R^2 but instead a fourth order identity. However, the remarks about the Casimir relation in the nondegenerate case still apply. That is, we can still use the identity to define a Casimir operator that will commute with all the other operators and hence the model will not correspond to a representation of the quadratic algebra unless the value of the Casimir operator in the model agrees with the value in the original system.

We begin with the table of the equivalence classes of degenerate systems. As proven above, Stäckel equivalent systems will have isomorphic symmetry algebras, up to permutation of parameters. As evident in the accompanying table, there are exactly 6 degenerate systems in 2d. However, since there is no canonical choice of basis, we can only classify them by the leading order terms of the Casimir operator, given in the first column of the table.

Leading order terms	**System**	**Operator models**
$0 \quad + L_1L_2 \qquad + AX^2$	E3, E18	Differential
$X^4 \quad + L_1L_2$	S3, S6	Differential and Difference
$X^4 \quad + X^2L_1 + L_2^2 \quad + 0$	E12, E14	Differential
$0 \quad + X^2L_1 + L_2^2 \quad + AL_1$	E6, S5	Differential
$X^4 \quad + L_1^2$	E5	Differential
$0 \quad + X^2L_1 \qquad + L_2$	E4, E13	Differential

Table 4.2: Stäckel equivalence classes of degenerate systems which admit a Killing vector X, in 2d

4.3.1 E18: The Coulomb Case

This system is defined by the Hamiltonian,

$$H = \Delta + \frac{\alpha}{\sqrt{x^2 + y^2}}.$$

A basis for the symmetry operators is formed by H and

$$L_1 = \{M, p_x\} - \frac{\alpha y}{\sqrt{x^2 + y^2}}, \qquad L_2 = \{M, p_y\} - \frac{\alpha x}{\sqrt{x^2 + y^2}},$$

$$X = M.$$

The symmetry algebra is defined by the following relations

$$[L_1, X] = L_2, \qquad [L_2, X] = -L_1, \qquad [L_1, L_2] = HX$$

$$L_1^2 + L_2^2 - HX^2 + \frac{H - \alpha^2}{4} = 0.$$

We can change basis so that the algebra is in the standard form of the Lie algebra sl_2. We make the substitutions $A = L_1 + iL_2$, $A^\dagger = L_1 - iL_2$. The algebra relations then become,

$$[A, X] = A, \qquad [A^\dagger, X] = -A^\dagger, \qquad [A, A^\dagger] = 2iHX,$$

$$\{A, A^\dagger\} - 2HX^2 + \frac{H}{2} = \frac{\alpha^2}{2}.$$

A suitable model diagonalizing X, is

$$X = i\left(t\partial_t - \frac{m}{2}\right), \quad A = \frac{\alpha^2 m}{4(m+1)^2}\partial_t, \quad A^\dagger = -4u\left(t\partial_t - m\right).$$

Here we have made the substitution $h = \frac{\alpha^2}{(m+1)^2}$ where the model forms a finite dimensional irreducible representation if $m \in \mathbb{N}$.

In section 5.1 I will consider this case in more detail.

4.3.2 S3

This system is defined by the Hamiltonian,

$$H = \Delta_{S^2} + \frac{\alpha}{s_3^2}.$$

A basis for the symmetry operators is formed by H and

$$L_1 = J_1^2 + \frac{\alpha s_2^2}{s_3^2}, \quad L_2 = \frac{1}{2}(J_1 J_2 + J_2 J_1) - \frac{\alpha s_1 s_2}{s_3^2}, \quad X = J_3.$$

The algebra is

$$[L_1, X] = 2L_2, \quad [L_2, X] = -X^2 - 2L_1 + H - \alpha,$$

$$[L_1, L_2] = -\{L_1, X\} - (2\alpha - \frac{1}{2})X,$$

$$\frac{1}{6}\{L_1, X, X\} - HL_1 + L_2^2 + L_1^2 + (\alpha + \frac{11}{12}) + (\alpha - \frac{2}{3})L_1 - \frac{H}{6} = \frac{5\alpha}{6}.$$

A suitable differential model is, where we have used the change of variables $\alpha = \frac{1}{4} - a^2$ and $h = -(m+1-a)^2 + \frac{1}{4}$:

$$L_1 = \left(t^3 + 2t^2 + t\right)\frac{d^2}{dt^2} + \left((2-a-m)t^2 + 2(1-m)t + a - m\right)\frac{d}{dt}$$
$$+m(a-1)t + a(m+1) - m - \frac{1}{2},$$
$$L_2 = i\left(-t^3 + t\right)\frac{d^2}{dt^2} + i\left((a+m-2)t^2 + a - m\right)\frac{d}{dt} - im(a-1)t,$$
$$X = i(2t\frac{d}{dt} - m).$$

We have raising and lowering operators which correspond to

$$A^\dagger = L_1 + iL_2 + \frac{1}{2}(X^2 - H + \alpha) = 2t^3\frac{d^2}{dt^2} + 2(2-a-m)t^2\frac{d}{dt} + 2m(a-1)t,$$
$$A = L_1 - iL_2 + \frac{1}{2}(X^2 - H + \alpha) = 2t\frac{d^2}{dt^2} + 2(a-m)\frac{d}{dt}.$$

The symmtery relations of the orginal symmetry operators in this basis are,

$$[A, X] = 2iA, \quad [A^\dagger, X] = -2iA^\dagger, \quad [A, A^\dagger] = -2iX^3 - \frac{i}{2}(-4H + 4^2 - 3),$$
$$\{A, A^\dagger\} - \frac{1}{2}X^3 + X^2(H - a^2 + \frac{1}{1}2) = -\frac{1}{32}(4H + 4a^2 + 8a + 3)(H + 4a^2 + 8a + 3).$$

X is diagonalized by monomials. The model becomes finite dimensional under the restriction $m \in \mathbb{N}$.

There is also a difference operator model associated with the spectral resolution of L_1. The operators are formed from the operators T^k defined as $T^k f(t) = f(t+k)$. The model is,

$$L_1 = -t^2 + a^2 - \frac{1}{4},$$

$$-iX = \frac{(1/2 - a - t)(m + a - 1/2 - t)}{2t}T - \frac{(1/2 - a + t)(m + a - 1/2 + t))}{2t}T^{-1}, \quad (4.7)$$
$$L_2 = \frac{(1-2t)(1/2 - a - t)(m + a - 1/2 - t)}{4t}T^1$$
$$+\frac{(1+2t)(1/2 - a + t)(\mu + a - 1/2 + t))}{4t}T^{-1}.$$

The basis functions for our representation will be $\nu_n = \delta(t - (n - a + \frac{1}{2}))$ which are eigenfunctions of L_1. In section 5.4 I will consider this system in more detail.

4.3.3 E14

The Hamiltonian is, with $z = x + iy, \overline{z} = x - iy$,

$$H = \Delta + \frac{\alpha}{\overline{z}^2}.$$

A basis for the symmetry operators is formed by H and

$$L_1 = \frac{1}{2}\{M, p_-\} + \frac{\alpha}{i\overline{z}}, \qquad L_2 = M^2 + \frac{\alpha z}{\overline{z}},$$

$$X = p_- = p_x - ip_y.$$

The algebra is

$$[L_1, L_2] = i\{X, L_2\} + \frac{i}{2}X, \quad [L_1, X] = iX^2, \quad [L_2, X] = 2L_2,$$

$$L_1^2 - L_2X^2 + \alpha H - \frac{11}{12}X^2 = 0.$$

A suitable differential model yielding the spectral resolution for X is

$$X = \frac{1}{t}, \qquad L_1 = i\partial_t, \qquad L_2 = -t^2\partial_t^2 - 2t\partial_t + \alpha h - \frac{1}{4}t^2.$$

Eigenfunctions for L_2 are Bessel functions while eigenfunctions of L_1 are exponentials. This model has no obvious finite dimensional restrictions.

4.3.4 E6

The Hamiltonian is,

$$H = \Delta + \frac{\alpha}{x^2}.$$

A basis for the symmetry operators is formed by H and

$$L_1 = \frac{1}{2}\{M, p_x\} - \frac{(\frac{1}{4} - \alpha^2)y}{x^2}, \quad L_2 = M^2 + \frac{(\frac{1}{4} - \alpha^2)y^2}{x^2},$$

$$X = p_y.$$

The quadratic algebra is,

$$[L_1, L_2] = 2X^3 - HX, \quad [L_1, X] = H - X^2, \quad [L_2, X] = 2L_1,$$

$$X^4 - HX^2 + L_1^2 + \alpha L_2 = 0.$$

A suitable model for determining the spectral resolution of L_1 is

$$L_1 = -\sqrt{h}\left(t^3\partial_t^2 + (2t - mt + 2)t\partial_t - m(t+1)\right),$$
$$L_2 = -t^2\partial_t^2 + 2(m(t+1) - 1)\partial_t + \frac{2mt + 3t + 4m + 8}{2t},$$
$$X = \sqrt{h}\left(t^2\partial_t + t + 2\right),$$

which includes the substitution $\alpha = 1/4 - (m+2)^2$.

The eigenfunctions of L_1 are Laguerre polynomials times an exponential, with eigenvalues $\sqrt{h}(2n - m)$. While the eigenfunctions of L_2 are Bessel functions multiplied by a gauge factor and the spectrum is continuous. The model realizes a finite dimensional representation under the restriction $m \in \mathbb{N}$. Notice, a constant is quantized instead of the energy which does not make sense in the physical system, since the constants should be given by the system. However, we can see that under a Stäckel transform the constant and the energy will be interchanged so our model is giving quantization levels of the energy of this Stäckel equivalent system; in this case, S5.

This is also a unique model because the operator determing the spectral resolution of L_1 is not of a simple form t or ∂_t. Such a model could be determined from the expansion coefficients in this model and would probably result in a difference operator realization.

4.3.5 E5

The Hamiltonian is,

$$H = \Delta + \alpha x.$$

A basis for the symmetry operators is formed by H and

$$L_1 = p_x p_y + \frac{1}{2}\alpha y, \quad L_2 = \frac{1}{2}\{M, p_y\} - \frac{1}{4}\alpha y^2,$$
$$X = pl_y.$$

The algebra relations are

$$[L_1, L_2] = 2X^3 - HX, \quad [L_1, X] = -\frac{\alpha}{2}, \quad [L_2, X] = L_1,$$

$$X^4 - HX^2 + L_1^2 + \alpha L_2 = 0.$$

A model adapted to the spectral resolution of either X or L_1 is,

$$X = t \qquad L_1 = -\frac{\alpha}{2}\partial_t,$$

$$L_2 = -\frac{1}{4}\alpha\partial_t^2 + \frac{1}{\alpha}t^2(h - t^2).$$

Here the only nontrivial eigenfunction is that of L_2 which gives us Heun T functions. This model gives no obvious finite dimensional restrictions.

4.3.6 E4

The Hamiltonian is

$$H = \Delta + \alpha(x + iy).$$

A basis for the symmetry operators is formed by H and

$$L_1 = p_x^2 + \alpha x, \quad L_2 = \frac{1}{2}\{M, p_-\} + \frac{i\alpha}{4}(x - iy)^2,$$

$$X = p_-.$$

The algebra is given by,

$$[L_1, X] = -\alpha, \qquad [L_2, x] = iX^2,$$

$$[L_1, L_2] = -X^3 - HX + \{L_1, L_2\},$$

$$X^3 - \frac{4}{3}\{L_1, X, X\} + 2HX^2 + H^2 + 4\alpha i L_2 = 0.$$

A suitable differential operator model for the spectrum of either X or L_1 is,

$$X = i\alpha t, \quad L_1 = i\partial_t + \frac{h}{2}, \quad L_2 = -\alpha t^2\partial_t + \frac{i\alpha^4 t^4 - 4\alpha^2 t + ih^2}{4\alpha}.$$

The eigenfunctions of L_2 are Airy functions and there are no obvious finite dimensional restrictions.

Chapter 5

Discussion of some models

In the following sections I will give a deeper analysis of a few of the superintegrable systems, both to explain how I arrived at the models and to describe some of the information gained from the models. I begin with the Coulomb potential because of its simplicity, since its restriction to a constant energy space is actually a Lie algebra. This example is not only one of the motivating systems for superintegrability in general but also for the representation theory of their algebras, as an analog of the representation theory for Lie algebras.

The next two examples are extensions of the representations of the Coulomb algebra. First, I describe the system E1 whose algebra does not restrict to a Lie algebra but whose representation theory still resembles the Coulomb case. In the next system, E10, we can make a change of basis so that our algebra become that of sl_2. However, the representation theory is not as simple as the Coulomb case because of adjointness conditions and a change in the degree of the operator.

The next example given is S3, which was treated in a paper by Kalnins, Miller and Post [42]. This system was first introduced as an example of a Hamiltonian with position dependent mass by Quesne [13], though in her paper the system was not identified as the system S3 but as a restriction of S9. In our model, we were able to diagonalize all operators and find their eigenvalues. We were also able to describe both differential and difference operator realizations using the methods of quantizing a classical model and what we call the "bootstrap method." That is, using one basis to determine the expansion coefficients of another basis and creating the model from this information.

Next, I describe the generic potential on the two sphere, S9. Here, we were able to prove that no differential operator model exists and were able to determine a difference operator model from an abstract analysis of the symmetry algebra and show that the results fit the algebra associated with Wilson polynomials. A classical model has also been found for this system whose quantization gives the difference model. This model was given in two papers by Kalnins, Miller and Post; the first giving the abstract analysis and the second the classical model [44, 42].

5.1 E18: The Coulomb Potential, a Lie Algebra

We begin our analysis with the Coulomb potential because of its importance in the history of physics. Classically, the Hamiltonian represents the gravitational two-body potential. The existence of the classical constants of the motion allows one to find the orbits algebraically. In the quantum case, the Hamiltonian represents the electrostatic interactions in a hydrogen atom and the eigenvalues give the energy levels of the hydrogen atom. Furthermore, the Coulomb potential is Stäckel equivalent to the simple harmonic oscillator, hence the representation theory of the two will be isomorphic up to permutation of constants. This example is also important since the existence of exact solutions makes it a good candidate for perturbation analysis, as is the case for all superintegrable systems. For these reasons, the Coulomb potential and simple harmonic oscillator are perhaps the most studied and well known examples of physical systems.

In this section, we present an analysis of the algebra for the Coulomb potential whose Hamiltonian is,

$$H = \Delta + \frac{\alpha}{\sqrt{x^2+y^2}}.$$

A basis for the symmetry operators is formed by H along with

$$L_1 = \{M, \partial_x\} - \frac{\alpha y}{\sqrt{x^2+y^2}}, \qquad L_2 = \{M, \partial_y\} - \frac{\alpha x}{\sqrt{x^2+y^2}},$$

$$X = M.$$

The symmetry algebra is defined by the following relations:

$$[L_1, X] = L_2, \qquad [L_2, X] = -L_1, \qquad [L_1, L_2] = HX,$$

$$L_1^2 + L_2^2 - HX^2 + \frac{H}{4} = \frac{\alpha^2}{4}.$$

We can change basis so that the algebra is in the standard form of the Lie algebra sl_2. We make the substitutions $A = L_1 + iL_2$, $A^\dagger = L_1 - iL_2$. The algebra relations then become,

$$[A, X] = A, \qquad [A^\dagger, X] = -A^\dagger, \qquad [A, A^\dagger] = 2iHX,$$
$$\{A, A^\dagger\} - 2HX^2 + \frac{H}{2} = \frac{\alpha^2}{2}.$$

A suitable model diagonalizing X, is

$$X = i\left(t\partial_t - \frac{m-1}{2}\right), \quad A = \frac{\alpha^2(m-1)}{4m^2}\partial_t, \quad A^\dagger = -4u\left(t\partial_t - m\right).$$

Here we have made the substitution $h = \frac{\alpha^2}{m^2}$.

The eigenfunctions of X are monomials; we assume the basis is orthonormal and of the form $\phi_n = c_n t^n$, where c_n is the normalization coefficients. If m is an arbitrary constant, our model forms a bounded below, irreducible representation. That is, there exists a lowest weight vector, in this case $\phi_0 = c_0$ which is annihilated by our operator A. On the other hand, if $m \in \mathbb{N}$, there will be another basis function which is annihilated in the model, $A^\dagger \phi_{m-1} = 0$ and hence the irreducibility condition requires that $\phi_j \equiv 0$, for all $j \geq m$ and the model has dimension m.

Next, we assume an inner product on the space for which L_1 and L_2 are self-adjoint. From the structure equations we can observe that this implies that X must be self-adjoint and that A and $A^\dagger$ are mutual adjoints thus justifying the adjoint notation. In order for this inner product to be positive definite we require $\alpha = -\omega^2$.

With this inner product we thus have,

$$\langle \phi_n, A^\dagger \phi_{n-1}\rangle = \langle c_n t^n, A^\dagger c_{n-1} t^{n-1}\rangle = \langle A c_n t^n, c_{n-1} t^{n-1}\rangle = \langle A\phi_n, \phi_{n-1}\rangle$$
$$\iff \langle c_n t^n, \frac{c_{n-1} 4(m-n)}{m-1} t^n\rangle = \langle \frac{\omega^4 n(m-1)}{4m^2} c_n t^{n-1}, c_n t^n\rangle$$
$$\iff \frac{c_{n-1}}{c_n}\frac{4(m-n)}{m-1}\langle \phi_n, \phi_n\rangle = \frac{c_n}{c_{n-1}}\frac{\omega^4 n(m-1)}{4m^2}\langle \phi_{n-1}, \phi_{n-1}\rangle.$$

Since we assume the basis to be orthonormal, we obtain a recursion relation for the normalization constants. It is,

$$c_n^2 = \frac{16m^2(m-n)}{\omega^4(m-1)^2 n} c_{n-1}^2.$$

This implies,

$$c_n = \left(\frac{4m}{\omega^2(m-1)}\right)^n \sqrt{\frac{(-m-1)_n}{(-n)_n}}.$$

Notice, our normalization is positive and real for $\omega \in \mathbb{R}$, and $m, n \in \mathbb{N}$. Furthermore it cuts off at $n = m$ since for that term, the Pochammer symbol becomes,

$$(-m+1)_n|_{n=m} = (-m+1)(-m+1+1)...(-m+1+n-1)|_{n=m} = 0,$$

in agreement with a finite dimensional representation of dimension m. For infinite dimensional bounded below representations, with an arbitrary complex constant $m \in \mathbb{C}$, the normalization becomes

$$c_n = \frac{4\sqrt{m\overline{m}}}{\omega^2\sqrt{(m-1)\overline{(m-1)}}}\sqrt{\frac{\Gamma(-m-1+n)\Gamma(n)}{\Gamma(-m-1)}}.$$

Finally, we can find an explicit function space representation for the inner product which will make L_1 and L_2 self-adjoint. We assume the inner product is of the form $\langle f(t), g(t)\rangle = \int_\gamma \overline{f(t)}g(t)\rho(t,\overline{t})dtd\overline{t}$ where γ is a path to be determined later. We can determine the weight function using the following relation,

$$\langle L_1 f, g\rangle = \int_\gamma \overline{(L_1 f(t))}g(t)\rho(t,\overline{t})dtd\overline{t} = \int_\gamma \overline{f(t)}(L_1 g(t))\rho(t,\overline{t})dtd\overline{t} = \langle f, L_1 g\rangle.$$

Similarly, we require $\langle Af, g\rangle = \langle f, A^\dagger g\rangle$. These two integral equations can be transformed by integration by parts into differential equation for the weight function who solutions include,

$$\rho(t,\overline{t}) = -\frac{\omega^4(m-1)^2}{32m^2}\left(\frac{1}{16m^2t\overline{t} - \omega^4(m-1)^2}\right)^{-m-1}. \tag{5.1}$$

To obtain the proper normalization, for basis functions $t^n, n = 0, .., m-1$, we integrate over the whole complex plane. The normalization has been chosen so that $\langle 1, 1\rangle = 1$.

In this model, we can find a reproducing kernel for the Hilbert space which in fact is a member of the Hilbert space. It has the characteristic $\langle \delta(t\overline{s}), f(t)\rangle = f(s)$. We form it by computing $\sum \phi_n(t)\phi_n(\overline{s}) = \sum c_n^2 (t\overline{s})^n$ which simplifies to,

$$\delta(t\overline{t}) = \left(1 + \left(\frac{4m}{\omega^4(m-1)}\right)^2 t\overline{s}\right)^m.$$

Finally, we can try to diagonalize the operator L_1 with functions such that $L_1\psi_n = \lambda_k\psi_n$. Its eigenfunctions are rational functions but when the eigenvalues are restricted to $\lambda_k = \omega^2\frac{m-1-2k}{2m}$, for $0 \le k \le m-1$, the eigenfunctions become polynomial of degree $m-1$,

$$\psi_k = \left(\frac{1}{4m}\right)^{m-1}\left(u + \omega^2\frac{m-1}{4m}\right)^{m-1-k}\left(u - \omega^2\frac{m-1}{4m}\right)^k.$$

Similarly, we can diagonalize L_2 with the same eigenvalues. The eigenfunctions for L_2 are,

$$\nu_k = \left(\frac{1}{4m}\right)^{m-1}\left(iu + \omega^2\frac{m-1}{4m}\right)^{m-1-k}\left(iu - \omega^2\frac{m-1}{4m}\right)^k.$$

To summarize, in this model I have diagonalized all the symmetry generators and found quantization restrictions for all of the eigenvalues based on the finite dimensional representation. I also exhibited a kernel function and an integral representation for the inner product on the Hilbert space. Since this quadratic algebra is in fact a (simple) Lie algebra its representation theory was somewhat straightforward to compute. In the following models, I will attempt to find similar structures, though in many cases the representation theory will not be so straightforward.

5.2 E1: The Singular Isotropic Oscillator

In some of the superintegrable systems, the restriction of $H \equiv h$(constant) does not change the quadratic algebra to a Lie algebra but we can still make a change of basis so that the algebra relations become similar to those of sl_2 and so we can attempt to find raising and lowering operators. However, some of the operators in this basis will not be second order and will not correspond to separation of variables. If we are lucky, an operator associated with separation of variables will be one of the basis operators and will be diagonalizable in the model. I treat the example of E1 as a case of this type of model. That is, I will construct a model where the basis functions are monomials and there exist raising and lowering operators as in the Coulomb example just given. Models with this general structure are also possible for E20, S3, and trivially all other systems Stäckel equivalent to E1.

In order to to study the nondegenerate system E1 we create a differential model similar to the model for the Coulomb system, that is we diagonalize one of the operators associated with separation of variables, in this case L_1, as $c_1 t\partial_t + c_2$ with c_1, c_2 constants. Eigenfunctions of this operator, ϕ_n will be monomials and will form a basis for the representation space. In this case, we are able to construct differential raising and lowering operators from the algebra as can be seen in the following theorem.

Theorem 15. *Given two operators L_1 and L_2. Assume L_1 has a basis of eigenvectors whose eigenvalues are linear in a discrete index n; i.e.,*

$$L_1\phi_n = (c_1 n + c_2)\phi_n.$$

Furthermore, assume the action of the other operator is of the form of the 3-term recurrence relation with $C(n,n)$ polynomial in n,

$$L_2\phi_n = C(n+1,n)\phi_{n+1} + C(n,n)\phi_n + C(n-1,n)\phi_{n-1}.$$

Then, there will be raising and lowering operators, $A, A^\dagger$, whose form is independent of the index n.

To prove this, we can immediately compute the action of $R = [L_1, L_2]$ on the eigenvectors for L_1,

$$R\phi_n = [L_1, L_2]\phi_n = c_1 C(n+1,n)\phi_{n+1} - c_1 C(n-1,n)\phi_{n-1}.$$

From this equation, we directly observe the existence of raising and lowering operators as

$$A = -c_1 L_2 - R + C(n,n)\big|_{n=c_1^{-1}(L_1-c_2)}\,, \qquad A^\dagger = -c_1 L_2 + R + C(n,n)\big|_{n=c_1^{-1}(L_1-c_2)}\,.$$

In these equations, we have used the requirement that $C(n,n)$ be polynomial in n so that the corresponding operator will be a polynomial in the operator L_1. This technique will be employed several more times in the models, that is we need to determine an operator corresponding to a certain expansion coefficient so we substitute the operator for its eigenvalue.

It is also important to note that the operators A and $A^\dagger$ will be mutual adjoints as long as the polynomial $C(n,n)$ is real. Raising and lowering operators such as these will play an important role in all of the models presented in the remainder of this thesis.

Returning to the system E1, the Hamiltonian for this system, the singular isotropic oscillator, is

$$H = \Delta - \omega^2(x^2+y^2) + \frac{\frac{1}{4}-a^2}{x^2} + \frac{\frac{1}{4}-b^2}{y^2}.$$

The other operators in the symmetry algebra are,

$$L_1 = \partial_x^2 + \frac{1/4-a^2}{x^2} + \omega^2 x^2, \quad L_2 = M^2 + x^2\frac{1/4-a^2}{x^2} + y^2\frac{1/4-b^2}{y^2}.$$

The algebra relations are

$$[L_1, L_2] = R, \qquad [R, L_1] = 8L_1^2 - 8HL_1 + 16\omega^2 L_2 - 8\omega^2, \tag{5.2}$$

$$[R, L_2] = 8HL_2 - 8\{L_1, L_2\} + (12-16a^2)H + (16a^2+16b^2-24)L_1,$$

$$R^2 + \frac{8}{3}\{L_1, L_1, L_2\} - 8H\{L_1, L_2\} + (16b^2+16a^2-\frac{200}{3})L_1^2 + 16\omega^2 L_2^2 + (32a^2 - \frac{200}{3})HL_1$$

$$+(16-16a^2)H^2 - \frac{176}{3}\omega^2 L_2 - 64\omega^2(b^2+a^2-b^2a^2-\frac{29}{48}) = 0.$$

A suitable model, adapted to the spectral resolution of L_1 can be chosen as

$$L_1 = 4i\omega t\partial_t + 2i\omega(1-2m-a), \tag{5.3}$$

$$A = L_2 - \frac{R}{4i\omega} + \frac{L_1^2}{2\omega^2} - \frac{h}{2\omega^2}L_1 - \frac{1}{2} = t\partial_t^2 + (1+b)\partial_t, \tag{5.4}$$

$$A^\dagger = L_2 + \frac{R}{4i\omega} + \frac{L_1^2}{2\omega^2} - \frac{h}{2\omega^2}L_1 - \frac{1}{2}$$

$$= 64t^3\partial_t^2 + (192-64a-128m)t^2\partial_t + (62m^2 + (64a-128)m + 64 - 64a)t. \tag{5.5}$$

Recall, that we have restricted H to a constant energy eigenspace with eigenvalue h, and the operator R is defined as the commutator of L_1 and L_2. Notice L_2 can be recovered as $L_2 = \frac{1}{2}(A + A^\dagger)$.

Here we note that if L_1 and L_2 are self-adjoint, A and $A^\dagger$ will only be mutual adjoints if ω is completely imaginary, which is reflective of the physical fact that the potential will be attracting in that case. The commutation relations of the raising and lowering operators can either be determined from the quadratic algebra (5.2) or directly from the model. In either case, they are

$$[A, A^\dagger] = 764L_1^3 + (510b + 766a - 512)L_1^2 + (160a^2 + 256ba - 256b - 128a + 96)L_1$$

$$-16(a^2-4a-1)(a+b),$$

$$[L_1, A] = 4\omega A, \qquad [L_1, A^\dagger] = -4\omega A^\dagger.$$

In this model, the eigenfunctions for L_1 are monomials; we obtain a finite dimensional model under the assumption that $A^\dagger t^{m-1} = 0$ for some positive integer m. This leads to the quantization of the energy levels. We obtain $h = -2i\omega(2m+a+b)$, where again we see that ω must be purely imaginary.

With this simplification L_2 becomes

$$L_2 = \frac{1}{2}t(64t^2+16t+1)\partial_t^2 - \frac{1}{2}(8t+1)((16m+8a-24)t-1-b)\partial_t$$
$$+32(m-1)(m+a-1)t + 5/2 - 4m - 2a - 4mb - 2ab + 2b.$$

The eigenvalue equation $(L_2-\lambda)\psi_\lambda = 0$ yields hypergeometric functions, ${}_2F_1$'s [45]. For a finite dimensional irreducible representation, the hypergeometric series must terminate and we obtain a quantization relation on the eigenvalues $\lambda = -3/2 - 2b - 2a - 4k - 2ba - 4bk - 4ak - 4k^2$ and the basis functions become, for $k = 1, ..., m-1$,

$$\psi_k(t) = l_k(8t+1)^{m-1-k}{}_2F_1\left(\begin{array}{cc|c} -k, & -a-k & \\ 1+b & & -8t \end{array}\right).$$

Here the l_k's are normalization constants.

We return to the basis of eigenvectors for L_1, $\phi_n(t) = k_n t^n$. Let us assume an inner product for which L_1 and L_2 are self-adjoint, or practically, we assume A and $A^\dagger$ are mutual adjoints. As mentioned above, this is equivalent to L_1, L_2 being self-adjoint and the constants a, b being purely real and ω being purely imaginary. With this inner product, we can find the normalization for our eigenvectors using the raising and lowering operators. That is, we assume

$$\langle At^n, t^{n-1}\rangle = \langle t^n, A^\dagger t^{n-1}\rangle$$

and $\langle \phi_n, \phi_n\rangle = 1$ to obtain the recursion relation as

$$k_n^2 = \frac{64(m-n)(m-n+a)}{n(b+n)}k_{n-1}^2$$

so that

$$k_n = 16^n\sqrt{\frac{(-m)_n(-m-a)_n}{n!(b)_n}}$$

which is real so long as $a, b > 0$.

From these normalization coefficients, we can find a reproducing kernel for this Hilbert space which in fact is a member of the Hilbert space. It has the characteristic $\langle \delta(t\overline{s}), f(t) \rangle = f(s)$. We form it by computing $\sum \phi_n(t)\phi_n(\overline{s}) = \sum k_n^2 (t\overline{s})^n$ which is exactly the hypergeometric polynomial

$$\delta(t, \overline{s}) = {}_2F_1\left(\begin{array}{cc} -m, & -m-a \\ b & \end{array} \middle| t\overline{s} \right).$$

As in the Coulomb case, we can find an explicit function space representation for the inner product which will make L_1 and L_2 self-adjoint. We assume the inner product is of the form $\langle f(t), g(t) \rangle = \int_\gamma \overline{f(t)} g(t) \rho(t, \overline{t}) dt d\overline{t}$ where γ is a path to be determined later. We can determine the weight function using the following relation,

$$\langle L_1 f, g \rangle = \int_\gamma \overline{(L_1 f(t))} g(t) \rho(t, \overline{t}) dt d\overline{t} = \int_\gamma \overline{f(t)} (L_1 g(t)) \rho(t, \overline{t}) dt d\overline{t} = \langle f, L_1 g \rangle.$$

Similarly, we require $\langle Af, g \rangle = \langle f, A^\dagger g \rangle$. These two integral equations can be transformed by integration by parts into differential equation for the weight function whose solutions can be written as

$$\rho(t\overline{t}) = c_1\ {}_2F_1\left(\begin{array}{cc} 1+m, & m+1+a \\ 1-b & \end{array} \middle| t\overline{t} \right)$$
$$+ c_2\ (t\overline{t})^b {}_2F_1\left(\begin{array}{cc} 1+m+b, & m+1+a+b \\ 1+b & \end{array} \middle| t\overline{t} \right).$$

The exact choice of weight function and contour γ depends on the irreducible representation of the quadratic algebra that we are realizing, see [42].

This model is an interesting example of the simplicity of differential models. We can easily find the eigenvalues for all the operators and the normalizations of the basis vectors. Also, the kernel function and integral representation of the inner product give useful tools to find expansion formulas and possible generating functions for the original quantum mechanical problem.

5.3 E10: sl_2 continued

In this next example, we can make a change of basis so that the algebra is actually isomorphic to sl_2. However, in this case the operator which is diagonalized in the standard finite dimensional representations of sl_2 does not correspond to variable separation in the physical system. Furthermore, the operators corresponding to separation of variables are not diagonalizable in the finite dimensional representation. In order to remedy this, we find a different model which is infinite dimensional and but bounded below; that is, there is a lowest weight vector but no highest weight vector. In this representation, our operators will have continuous spectrum.

Recall, the Hamiltonian for the system E10 is given by,

$$H = \Delta + \alpha\overline{z} + \beta(z - \frac{3}{2}\overline{z}^2) + \gamma(z\overline{z} - \frac{1}{2}\overline{z}^3).$$

A basis for its symmetry operators is given by H along with the operators

$$L_1 = p_-^2 + \gamma\overline{z}^2 + 2\beta\overline{z},$$
$$L_2 = 2i\{M, p_-\} + p_+^2 - 4\beta z\overline{z} - \gamma z\overline{z}^2 - 2\beta\overline{z}^3 - \frac{3}{4}\gamma\overline{z}^4 + \gamma z^2 + \alpha\overline{z}^2 + 2\alpha z.$$

The algebra relations are given by,

$$[R, L_1] = -32\gamma L_1 - 32\beta^2, \qquad [R, L_2] = 96L_1^2 - 128\alpha L_1 + 32\gamma L_2 + 64\beta H + 32\alpha^2, \quad (5.6)$$

$$R^2 = 64L_1^3 + 32\gamma\{L_1, L_2\} - 128\alpha L_1^2 - 64\gamma H^2 - 128\beta H L_1 + 64\beta^2 L_2 + 64\alpha^2 L_1 - 128\beta\alpha H - 256\gamma^2.$$

We can renormalizes to obtain the Lie algebra sl_2 by using the following invertible transform,

$$K_1 = L_1 - \frac{\beta^2}{\gamma}, K_2 \quad = L_2 + \frac{1}{\gamma}L_1^2 - \frac{\beta^2 + 2\alpha\gamma}{\gamma^2}L_1 + \frac{1}{\gamma^3}\left(\beta^4 + 2\beta\gamma h + 2\beta^2\alpha\gamma + \alpha^2\gamma^2\right). \quad (5.7)$$

This gives us,

$$[R, K_1] = -32\gamma K_1, \qquad [R, K_2] = 32\gamma K_2, \qquad [K_1, K_2] = R,$$
$$R^2 - 32\gamma\{K_1, K_2\} = \frac{64}{\gamma^3}(\beta^6 + 2\alpha\gamma\beta^4 + 2h\gamma^2\beta^3 + \alpha^2\gamma^2\beta^2 + 2\alpha h\gamma^3\beta + h^2\gamma^4 + 4\gamma^5).$$

We can then use the representations of this simple Lie algebra to describe our system. In particular, we can chose finite dimensional representations to obtain quantization conditions. A bounded below model based on the spectral resolution of R is:

$$K_1 = 4\sqrt{-\gamma}\partial_t, \qquad K_2 = 4\sqrt{-\gamma}t(t\partial_t - m), \qquad R = -16\gamma(2t\partial_t - m),$$

where m is an arbitrary constant satisfying the equation

$$h = \pm 2(m+1)\sqrt{(-\gamma)} - \frac{1}{\gamma^2}(\alpha\beta\gamma - \beta^3).$$

This yields a finite dimensional representation if $m \in \mathbb{N}$. We can also diagonalize $K_1 - K_2$ in this model. The eigenvalues for this operator are $\lambda = 4\sqrt{c}(m-2n)$, with (unnormalized) eigenfunctions $\phi_n = (u-1)^n(u+1)^{m-n}$. Though this is a standard model for the Lie algebra, we are interested in diagonalizing L_1 and L_2 since they correspond to separation of variables in the physical systems. We can see immediately that L_1 cannot be diagonalized in this finite dimensional representation, since its action on the basis functions t^n will yield nontrivial Jordan blocks. Furthermore, the action of L_2 on basis functions is a four term recursion relation, indicating that it is not related to a family of orthogonal polynomials. In fact, if we try to diagonalize the original operators in this model, we obtain exponential functions for L_1 and Heun T functions for L_2.

A second model, that follows from [43] is

$$R = 16\gamma(2t - 2m - 1 - 2t\frac{d}{dt}), \quad K_1 = 4\sqrt{\gamma}t,$$
$$K_2 = 4\sqrt{\gamma}\left(t - 2m - 1 + 2(m + \frac{1}{2} - t)\frac{d}{dt} + t\frac{d^2}{dt^2}\right).$$

In this model m is any nonnegative real number, the natural Hilbert space is $L_2(0,\infty,\mu)$ where the weight function is $\mu(t) = t^{2m}e^{-2t}$, and the irreducible representations are infinite dimensional but bounded below. This model gives the spectral resolution of K_1 (continuous spectrum) and it is also easy to compute the spectral resolution of R from this (also continuous spectrum). The operator $K_1 - K_2$ has discrete spectrum $8\sqrt{\gamma}(m+n+\frac{1}{2})$ with unnormalized orthogonal eigenbasis $L_n^{(2m)}(2t)$, $n = 0, 1, \cdots$. Here $L_n^{(\alpha)}(x)$ is an associated Laguerre polynomial. In this model, the Casimir operator for the Lie algebra has the form

$$C \equiv \frac{1}{32\gamma}\{K_1, K_2\} - R^2 = \frac{1}{4} - m^2. \tag{5.8}$$

L_1 and L_2 in the original model have continuous spectrum.

As is evident from this example, even if we can make a change of basis to recast the algebra in terms of a Lie algebra the physical considerations from the physical problem give restrictions on the types of representations and the operators which we chose to diagonalize.

5.4 S3: Quesne's Variable Mass Hamiltonian

This next example first appeared in the literature as an example of a variable-mass Hamiltonian; mathematically, a position dependent mass corresponds to a position dependent metric. Such Hamiltonians have been utilized in the fields of semiconductors, quantum dots, nuclei, and quantum liquids to name a few [13]. We consider the Hamiltonian first described by Quesne [13] and analyzed by Kalnins, Miller and Post [42]. In her 2007 paper, Quesne treated the system as a restriction of a nondegenerate superintegrable system using a model based on parafermionic algebras. As we shall see, the more appropriate system is in fact a degenerate superintegrable system on the 2 sphere that admits a first order symmetry operator which was not utilized in Quesne's treatment.

I start with a simple differential model similar to those given above where the operators are modeled by differential operators and the operator to be diagonalized has monomials as eigenfunctions. I then use a bootstrapping method to create a difference model which gives a representation in which another operator is diagonalized. By bootstrapping method, I mean that we begin with an initial model with a given set of basis functions and use the basis functions to compute the action of the other operators on the basis. We then use this information to create a new model where a different operator represents the operator to be diagonalized. This method is particularly useful in determining difference operators because of the difficulty in computing a difference operator model directly. The other way to determine a difference operator model would be to quantize a difference model and at the end of this section I describe how to create such a difference model from a classical model.

A general variable mass Hamiltonian can be written the form

$$H = \partial_x \frac{1}{M(x,y)} \partial_x + \partial_y \frac{1}{M(x,y)} \partial_y + V(x,y)$$

where in the particular case of Quesne's potential we have $M(x,y) = -\cosh^{-2} qx$ so that the Hamiltonian becomes

$$H_Q = -\cosh^2 qx(\partial_x^2 + \partial_y^2) - 2q \cosh qx \sinh qx \partial_x - q^2 \cosh^2 qx + \frac{q^2 k(k-1)}{\sinh^2 qx}.$$

Here we use the subscript Q to indicate that this was the Hamiltonian given in Quesne's 2007 paper. In order to match the non-potential part of the Hamiltonian with a Laplace

Beltrami operator in two dimensions, we see that the metric must be

$$d\mathbf{s}^2 = q^2 \frac{dx^2 + dy^2}{\cosh^2 qx}.$$

We can immediately see that this corresponds to a system on the sphere through the change of variables,

$$s_1 = \frac{\sin qy}{\cosh qx}, \quad s_2 = \frac{\cos qy}{\cosh qx}, \quad s_3 = \tanh qx.$$

Notice, this system is on the two sphere as evident from the equality $s_1^2 + s_2^2 + s_3^2 = 1$.

We can rewrite the Hamiltonian for the superintegrable system S3 in these coordinates. The Hamiltonian then becomes,

$$H_{S3} = \frac{\cosh^2 qx}{q^2}(\partial_x^2 + \partial_y^2) + \frac{\frac{1}{4} - \alpha^2}{\tanh^2 qx},$$

where the subscript S3 indicated this is the Hamiltonian for the superintegrable system S3. In order to compare H_{S3} and H_Q we must introduce a gauge transformation to cancel out the first order derivative term. To do this, we set $G = \cosh(qx)$ and then we can determine the exact relation between the two operators. Explicitly this is,

$$H_Q = q^2 \left(G^{-1} H_{S3} G - \frac{1}{4} + \alpha^2 \right), \qquad k(k-1) = \frac{1}{4} - \alpha^2.$$

For the remainder of this section, we will deal with just the system S3 understanding that we can use this relation to return to the Hamiltonian given by Quesne.

As shown in [42] the basis for the symmetry operators for S3 is comprised of H and

$$L_1 = J_1^2 + \frac{(\frac{1}{4} - a^2)s_2^2}{s_3^2}, \quad L_2 = \frac{1}{2}(J_1 J_2 + J_2 J_1) - \frac{(\frac{1}{4} - a^2)s_1 s_2}{s_3^2}, \quad X = J_3.$$

For this system, X^2 corresponds to separation of variables in spherical coordinates and L_1 corresponds to separation in parabolic coordinates.

The symmetry algebra is determined by

$$[L_1, X] = 2L_2, \quad [L_2, X] = -X^2 - 2L_1 + H - (\frac{1}{4} - a^2),$$

$$[L_1, L_2] = -\{L_1, L_2\} - (-\frac{1}{4} - a^2)X,$$

$$\frac{1}{6}\{L_1,X,X\}-HL_1+L_2^2+L_1^2+((\frac{7}{6}-a^2))-(\frac{5}{12}+a^2)L_1-\frac{H}{6}=\frac{5(\frac{1}{4}-a^2)}{6}.$$

To find a model we use the monomial eigenfunctions of X. For simplicity, and with hindsight, we write $h=-(\mu+a-1)^2+\frac{1}{4}$, where μ is an arbitrary complex number. If $\mu=-m \quad m\in\mathbf{N}$, then our representation becomes finite dimensional and reads

$$L_1=\left(t^3+2t^2+t\right)\frac{d^2}{dt^2}+\left((2-a-m)t^2+2(1-m)t+a-m\right)\frac{d}{dt} \tag{5.9}$$

$$+m(a-1)t+a(m+1)-m-\frac{1}{2},\quad X=i(2t\frac{d}{dt}-m),$$

$$L_2=i\left(-t^3+t\right)\frac{d^2}{dt^2}+i\left((a+m-2)t^2+a-m\right)\frac{d}{dt}-im(a-1)t.$$

This model also holds for $m=-\mu$ for arbitrary complex number μ but then the model will only be bounded below and not finite dimensional. We note that the finite dimensional representations give us the quantization for the energy levels.

We have raising and lowering operators which correspond to

$$A^\dagger=L_1+iL_2+\frac{1}{2}(X^2-H+\frac{1}{4}-a^2)=2t^3\frac{d^2}{dt^2}+2(2-a-m)t^2\frac{d}{dt}+2m(a-1)t,$$

$$A=L_1-iL_2+\frac{1}{2}(X^2-H+\frac{1}{4}-a^2)=2t\frac{d^2}{dt^2}+2(a-m)\frac{d}{dt}.$$

The algebra relations in this basis are,

$$[A,X]=2iA,\quad [A^\dagger,X]=-2iA^\dagger,\quad [A,A^\dagger]=-2iX^3-\frac{i}{2}(-4H+4^2-3),$$

$$AA^\dagger+A^\dagger A-\frac{1}{2}X^3+X^2(H-a^2+\frac{1}{1}2)=-\frac{1}{32}(4H+4a^2+8a+3)(H+4a^2+8a+3).$$

We can use the raising and lowering operators to find normalization coefficients for our eigenfunctions, $\phi_n=k_nt^n \quad n=0,...,m$

$$k_n=\sqrt{\frac{(-m)_n(1-a)_n}{n!(-m+a)_n}}=\sqrt{\frac{m!(1-a)_n(1-a)_{m-n}}{(1-a)_m n!(m-n)!}}.$$

as well as a weight function and reproducing kernel, see [42]. Note that the normalization for the finite dimensional model is positive only for $a<1$ or $a-m>0$.

We can also find the spectral resolution of L_1 in terms of hypergeometric functions. If we are in the finite dimensional model, we have the requirement that our hypergeometric functions be polynomials of order m; this gives us quantization of energy levels, $\lambda_n = -(n-a+\frac{1}{2})^2 + a^2 - \frac{1}{4}$, where $L_1\psi_n = \lambda_n\psi_n$. Then the eigenfunctions become, for $n = 0, ..., m$

$$\psi_n = l_n(t+1)^n {}_2F_1\left(\begin{array}{cc} -m+n & 1-a+n \\ -m+a & \end{array} \middle| -t\right)$$

where l_n is a normalization coefficient.

Now, suppose we would like to compute a new model adapted to the spectral resolution of L_1. We already have a basis of eigenfunctions for L_1 in the previous model so we know the eigenvalues for that operator in a finite dimensional representation. Next, we need to find the action of the other operators in terms of a basis of eigenfunctions for L_1. Luckily, we can easily compute the action of these operators using the previous model; that is, we compute the action of the operators on the ψ_n's and the use the expansion coefficients to determine our new model. For example, we can use recurrence relations of the hypergeometric equations to expand the action of operator $X = i(2t\frac{d}{dt} - m)$ on the (unnormalized) eigenfunctions

$$\tilde{\psi_n} = (t+1)^n {}_2F_1(\left(\begin{array}{cc} -m+n & 1-a+n \\ -m+a & \end{array} \middle| -t\right).$$

We then can obtain the two expansion,

$$X\tilde{\psi_n} = C(n+1,n)\tilde{\psi_{n+1}} + C(n-1,n)\tilde{\psi_{n-1}},$$

where

$$C(n+1,n) = -i\frac{(m-n)(n+2a)}{2n-1-2a}, \quad C(n,n-1) = -i\frac{(m+n)n}{2n-2a-1},$$
$$C(n,n) = 0.$$

Armed with the eigenvalues for L_1 and the expansion coefficients for the symmetry operators on a basis of eigenfunctions for L_1, we can then attempt to create a model adapted to the spectral resolution of L_1. We can see from above that the eigenvalues of L_1 are $\lambda = -(n-a+\frac{1}{2})^2 + a^2 - \frac{1}{4}$. With an eye towards a model using difference operators, we assume that L_1 is of the form $L_1 = -t^2 + a^2 - \frac{1}{4}$ and the (unnormalized) basis functions are of the form

$$\nu_n = \delta(t - (n - a + \frac{1}{2})). \tag{5.10}$$

Notice that $\nu_{n+1} = \delta(t - (n + 1 - a + \frac{1}{2})) = \delta(t - 1 - (n - a + \frac{1}{2})) = T^{-1}\nu_n$, where T^k is the difference operator $T^k f(t) = f(t + k)$. Hence, we can try to construct our difference operator model using the substitution

$$\nu_{n+\sigma} = T^{-\sigma}\nu_n.$$

In the model, the expansion coefficients become rational functions of t by replacing t with a function of n. That is, in the example X, we can see it has the form $X = c_{+1}(t)T + c_{-1}T^{-1}$, and $c_\sigma(t)|_{t=n-\sigma-a+\frac{1}{2}} = C(n+\sigma, n)$, for $\sigma = \pm 1$. Finally, we also need to find the expansion coefficients for L_2 and use them to compute the form of L_2 in the difference model.

Using these computations, we can determine a difference operator model of the form

$$\begin{aligned} L_1 &= -t^2 + a^2 - \frac{1}{4} \\ -iX &= \frac{(1/2 - a - t)(m + a - 1/2 - t)}{2t}T - \frac{(1/2 - a + t)(m + a - 1/2 + t))}{2t}T^{-1}, \\ L_2 &= \frac{(1 - 2t)(1/2 - a - t)(m + a - 1/2 - t)}{4t}T^1 \\ &\quad + \frac{(1 + 2t)(1/2 - a + t)(\mu + a - 1/2 + t))}{4t}T^{-1}. \end{aligned} \tag{5.11}$$

The basis functions for our representation will be $\nu_n = \delta(t - (n - a + \frac{1}{2}))$ which are eigenfunctions of L_1.

In this model, we can then also diagonalize X with $Xf_n = i(2n - m)f_n$. This corresponds to bootstrapping in the other direction; that is, these are basis functions which give the expansion coefficients for the previous model. In this difference model, the functions diagonalizing X are $f_n(t) = (-1)^n p_n(-t^2)$ where

$$p_n(-t^2) = {}_3F_2\left(\begin{matrix} -n & \frac{1}{2} - a + t & \frac{1}{2} - a - t \\ m & 1 - a & \end{matrix} ; 1\right).$$

Here f_n is a polynomial of order n in the variable $-t^2$, a special case of the family of dual Hahn polynomials, [46, 47] whose recurrence relations can be found in the appendix

E.1. These polynomials are orthogonal with respect to a measure with support at the values $t+a-\frac{1}{2}=0,1,\cdots,m$, in agreement with equation (5.10) for the support of the basis functions. Indeed, we have (for $a<1$)

$$\sum_{j=0}^{m}\frac{(1-2a)_j(3/2-a)_j(-m-1)_j(-1)^j}{(1/2-a)_j(2+m-2a)_j j!}p_n(-t^2)p_{n'}(-t^2)$$

$$=\frac{(2-2a)_m(a-m)_n n!}{(1-a)_m(1-a)_n(-m)_n}\delta_{nn'}.$$

For the infinite dimensional, bounded below representation we can use the same model with $m=\mu$ and arbitrary complex number. The only differences between the two representations will be that the basis functions are *continuous* dual Hahn and the inner product will be an integral instead of a finite sum. A model that realizes this infinite dimensional representation then becomes,

$$\begin{aligned}
L_1 &= -t^2+a^2-\frac{1}{4}, \\
-iX &= \frac{(1/2-a-t)(\mu+a-1/2-t)}{2t}T-\frac{(1/2-a+t)(\mu+a-1/2+t))}{2t}T^{-1}, \\
L_2 &= \frac{(1-2t)(1/2-a-t)(\mu+a-1/2-t)}{4t}T \\
&\quad+\frac{(1+2t)(1/2-a+t)(\mu+a-1/2+t))}{4t}T^{-1}.
\end{aligned}\tag{5.12}$$

The eigenfunctions for X in this representation are now $f_n(t)=(-1)^n s_n(t^2)$ where

$$s_n(t^2)={}_3F_2\left(\begin{matrix}-n & \frac{1}{2}-a+t & \frac{1}{2}-a-t \\ \mu & 1-a & \end{matrix};1\right).\tag{5.13}$$

Here f_n is a polynomial of order n in the variable t^2, a special case of the family of continuous dual Hahn polynomials, [46]. The orthogonality and normalization are given by

$$\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(1/2-a+t)\Gamma(\mu+a-1/2+t)\Gamma(1/2+t)}{\Gamma(2t)}\right|^2 s_n(t^2)s_{n'}(t^2)\,dt\tag{5.14}$$

$$=\frac{\Gamma(n+\mu)\Gamma(n+1-a)\Gamma(n+\mu+a)n!}{(\mu)_n^2|(1-a)_n|^2}\delta_{nn'},$$

where either 1) $\mu>1/2-a>0$ or 2) $\mu>0$ and $a=((1-\mu)/2+i\gamma$ is complex.

This bootstrapping method of using the expansion coefficients in one basis to determine the model in another basis will be seen again in the analysis below of the model in 3d. This method is particularly helpful if we are trying to find a difference model, because of the computations of difference models is fairly difficult without some hint as to the structure of the model.

Another way to get a good understanding of the basic form of a possible difference model is to compute the classical model and then quantize it. In this system, we can construct classical models which corresponds to both of our quantum models.

The classical operators for the system are the Hamilonian and the constants of the motion:

$$\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2 + \frac{\alpha(s_1^2 + s_2^2 + s_3^2)}{s_3^2}, \tag{5.15}$$

$$\mathcal{L}_1 = \mathcal{J}_1^2 + \alpha\frac{s_2^2}{s_3^2}, \qquad \mathcal{L}_2 = \mathcal{J}_1\mathcal{J}_2 - \alpha\frac{s_1 s_2}{s_3^2}, \qquad \mathcal{X} = \mathcal{J}_3, \tag{5.16}$$

where $\mathcal{J}_1 = s_2 p_3 - s_3 p_2$ and $\mathcal{J}_2, \mathcal{J}_3$ are cyclic permutations of this expression. Here we have used the standard embedding of the sphere in 3 dimensional Euclidean space, $s_1^2 + s_2^2 + s_3^2 = 1$. The structure relations are, where here the brackets indicate Poisson brackets not the symmetrizer,

$$\{\mathcal{X}, \mathcal{L}_1\} = -2\mathcal{L}_2, \qquad \{\mathcal{X}, \mathcal{L}_2\} = 2\mathcal{L}_1 - \mathcal{H} + \mathcal{X}^2 + \alpha, \tag{5.17}$$

$$\{\mathcal{L}_1, \mathcal{L}_2\} = -2(\mathcal{L}_1 + \alpha)\mathcal{X},$$

and the Casimir relation is

$$\mathcal{L}_1^2 + \mathcal{L}_2^2 - \mathcal{L}_1\mathcal{H} + \mathcal{L}_1\mathcal{X}^2 + \alpha\mathcal{X}^2 + \alpha\mathcal{L}_1 = 0. \tag{5.18}$$

We can model this algebra by restricting the Hamiltonian to a constant energy hypersurface and setting a constant of the motion as a variable. We then solve for the remaining constants of the motion in terms of this variable and its conjugate, as described in section 4.1.3. In the models below, there have been several changes of variables so that the quantization of the operators matches the models given above for the quantum models.

A suitable classical model is,

$$L_1 = \frac{1}{2}(E - c^2 - \alpha) + \frac{1}{2}\left((c^2 - (E+\alpha)^2)\sin 2\beta + 2i\sqrt{\alpha}\cos 2\beta\right),$$

$$L_2 = \frac{1}{2}\left((c^2-(E+\alpha)^2)\cos 2\beta - 2i\sqrt{\alpha}\sin 2\beta\right), \quad X = c.$$

Now the prescription $\beta \to t, c \to -\partial_t$ leads to a quantum realization of L_1, L_2 by second order differential operators. We can see that this model is the same as the differential model above by a change of variable $\tilde{t} = e^{2it}$ and a gauge transform of $\tilde{t}^m$.

On the other hand, if we try to create a classical model diagonalizing L_1, that is we assume $L_1 = c$, then we obtain a model of the form,

$$L_1 = c, \qquad L_2 = \sqrt{c(E-c-\alpha)}\sin(2\sqrt{c+\alpha}\beta),$$
$$X = \sqrt{\frac{c(E-c-\alpha)}{c+\alpha}}\cos(2\sqrt{c+\alpha}\beta).$$

We can see that this classical model will provide us with a difference model using Taylor's theorem. That is, we use the identity, $e^{a\partial_t}f(t) = f(t+a)$. We can then use the change of variables, $2\sqrt{c+\alpha}\partial_c = \partial_{\mathcal{C}}$ to obtain a difference operator realization which corresponds to the one given above.

The process of finding classical models and quantizing is useful beacuse classical Hamilton-Jacobi theory guarantees us the existence of a model, so it offers a good place to start if one is having difficulties determining the quantum model. This is particularly true in the cases where there are only difference operator models for the quantum system, such as the generic system on the two sphere, as given below.

5.5 S9: The generic system on the two sphere

System S9 is the generic system in 2 dimensions. All other systems are limiting cases of this one, although the limit proceedures may be very complicated. The model of the system was discovered using an abstract analysis of the symmetry algebra and comparing the structure to that of the algebra structure of the Wilson polynomials. A full description of the algebraic method of determining a model can be found in the paper of Kalnins, Miller and Post [44]. Here, I also give an example of a classical model whose quantization leads to the given difference model. A full exposition of the classical models can be found in [42].

Recall, the Hamiltonian for the system S9 is

$$H = \Delta_{S^2} + \frac{\frac{1}{4}-a^2}{s_1^2} + \frac{\frac{1}{4}-b^2}{s_2^2} + \frac{\frac{1}{4}-c^2}{s_3^2}.$$

A basis for the symmetry operators is,

$$L_1 = J_3^2 + (\frac{1}{4} - a^2)\frac{s_1^2}{s_2^2} + (\frac{1}{4} - c^2)\frac{s_2^2}{s_1^2},$$

$$L_2 = J_1^2 + (\frac{1}{4} - a^2)\frac{s_3^2}{s_2^2} + (\frac{1}{4} - b^2)\frac{s_2^2}{s_3^2},$$

$$H = L_1 + L_2 + L_3 + \frac{3}{4} - a^2 - b^2 - c^2.$$

The structure equations can be put in the symmetric form using the following identification,

$$a_1 = \frac{1}{4} - c^2,\ a_2 = \frac{1}{4} - a^2,\ a_3 = \frac{1}{4} - b^2.$$

The structure equations are,

$$[L_i, R] = \epsilon^{ijk}\Big(4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k)\Big), \quad (5.19)$$

$$\begin{aligned} R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 \\ + \frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 \\ + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3) \\ + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3. \end{aligned} \quad (5.20)$$

To determine a model, we restrict to a constant energy surface, $H \equiv h$ and assume an basis of eigenvectors for one of the symmetry operators L_1. We then obtain,

$$L_1 f_n = \left(\mathcal{K} - (2n + B)^2\right) f_n, \quad L_2 f_n = \sum_{\ell} C(\ell, n) f_\ell, \quad (5.21)$$

where $\mathcal{K}$ and B are constants to be determined.

The quadratic dependence on n is chosen both because we have prior knowledge of the eigenvalues of the operator that corresponds to separation of variables in the spherical basis and also because we find a linear dependence on n to be incompatible with the structure equations.

Here, B is not yet fixed. We do not require that the basis be orthonormal. From these assumptions we can compute the action of R and $[L_1, R]$ on the basis. Indeed,

$$Rf_n = [L_1, L_2]f_n = \sum_{\ell} 4(n-\ell)(n+\ell+B)C(\ell,n)f_\ell, \tag{5.22}$$

$$[L_1, R]f_n = \sum_{\ell} 16(n-\ell)^2(n+\ell+B)^2C(\ell,n)f_\ell. \tag{5.23}$$

On the other hand, from (5.19) with $i=1, j=2, k=3$ we have

$$[L_1, R]f_n = 8\sum_{\ell} \left((2\ell+B)^2 + (2n+B)^2 - 2\mathcal{K} + 2a^2 + 2b^2 - 3\right) C(\ell,n)f_\ell \tag{5.24}$$

$$+8\left[-((2n+B)^2-\mathcal{K})^2 + ((2n+B)^2-\mathcal{K})(\frac{9}{4} - a^2 - 3b^2 - c^2 - H)\right.$$

$$\left. +(\frac{3}{2} - 2b^2)(h - \frac{3}{4} + a^2 + b^2 + c^2) + b^2 - a^2\right] f_n.$$

Now we equate (5.23) and (5.24). For $n \neq \ell$, equating coefficients of f_ℓ in the resulting identity yields the condition

$$C(\ell,n)\left(((2\ell+B)^2 - (2n+B)^2)^2 - 8(2\ell+B)^2 - 8(2n+B)^2 + 16\mathcal{K} + 24 - 16a^2 - 16b^2\right] = 0.$$

We see from this that in order for $C(\ell,n) \neq 0$ we must have $\ell = n, n\pm 1$ and $\mathcal{K} = -\frac{1}{2} + a^2 + b^2$. Equating coefficients of f_n in the identity, we can solve for $C(n,n)$ and obtain

$$C(n,n) = \frac{w}{2} + \left(\frac{h}{2} + \frac{3}{8} - \frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}\right) + \frac{Q_1}{w} \tag{5.25}$$

where $w = (2n+B+1)(2n+B-1)$ and

$$Q_1 = \frac{1}{2}(h - \frac{3}{4} + a^2 + b^2 + c^2)(-a^2 + b^2) + \frac{a^4}{2} - \frac{a^2}{4} - \frac{b^4}{2} + \frac{b^2}{4}.$$

It is straightforward to show that the action of $[L_2, R]$ on the basis is

$$[L_2, R]f_n = \sum_{\ell,j} K(n,j,\ell)C(j,\ell)C(\ell,n)f_j \tag{5.26}$$

where

$$K(n,j,\ell) = (2n+B)^2 + (2j+B)^2 - 2(2\ell+B])^2.$$

For fixed n there are 8 nonzero terms in the double sum:

j	ℓ	$K(n,j,\ell)$
$n+2$	$n+1$	8
$n-2$	$n-1$	8
$n+1$	n	$8n+4+4B$
$n+1$	$n+1$	$-8n-4-4B$
$n-1$	n	$-8n+4-4B$
$n-1$	$n-1$	$8n-4+4B$
n	$n+1$	$-16n-8-8B$
n	$n-1$	$16n-8+8B.$

On the other hand, the structure equation for $[L_2, R]$ is

$$[L_2,R] = 8(L_1L_2+L_2L_1)+8L_2^2-8(h-\frac{3}{4}+a^2+b^2+c^2)(L_2-\frac{3}{2}+2b^2) \tag{5.27}$$
$$+16(\frac{3}{2}-b^2-c^2)L_1+16(\frac{3}{2}-b^2)L_2+8(-b^2+c^2).$$

Comparing (5.26) and (5.27) and equating coefficients of $f_{n\pm2}$, $f_{n\pm1}$, respectively, on both sides of the resulting identities, we do not obtain new conditions. However, equating coefficients of f_n results in the condition

$$-(2n+B+2)C(n,n+1)C(n+1,n)+(2n+B-2)C(n-1,n)C(n,n-1) = \tag{5.28}$$
$$C(n,n)^2+(-2w-h-\frac{3}{4}+a^2-b^2-c^2)C(n,n)+2(-\frac{3}{2}+b^2+c^2)w+Q_2$$

where

$$Q_2=(h-\frac{3}{4}+a^2+b^2+c^2)(-\frac{3}{2}+2b^2)-\frac{9}{2}+3a^2+5b^2+4c^2-2b^4-2a^2b^2-2a^2c^2-2b^2c^2.$$

We can regard this as an inhomogeneous recurrence relating F_n and F_{n-1} for the sequence

$$F_n = C(n,n+1)C(n+1,n),\ n=0,1,\cdots m.$$

Using (5.25) we find that the general solution is

$$F_n = C(n,n+1)C(n+1,n) = \tag{5.29}$$

$$\frac{A_8(2n+B+1)^8+A_6(2n+B+1)^6+A_4(2n+B+1)^4+A_2(2n+B+1)^2+A_0}{(2n+B+2)(2n+B+1)^2(2n+B)}$$

where

$$A_8=\frac{1}{16},\qquad A_6=\frac{1}{8}h-\frac{1}{32}-\frac{1}{8}(a^2+b^2+c^2),$$
$$4A_0=\Big(\frac{1}{2}(h-\frac{3}{4}+a^2+b^2+c^2)(b^2-a^2)+\frac{a^4}{2}-\frac{a^2}{4}-\frac{b^4}{2}+\frac{b^2}{4}\Big),$$
$$A_4=\frac{h^2}{16}-\frac{h}{32}(1+8a^2+8b^2-8c^2)+\frac{1}{16}(a^2+b^2-\frac{1}{2}c^2+a^4+b^4+c^4)$$
$$+\frac{1}{4}(-\frac{1}{2}a^2b^2+a^2c^2+b^2c^2)+\frac{1}{256},$$

and A_2 is arbitrary.

We determine A_2 by the requirement $F_{-1}\equiv 0$, so that

$$A_8(B-1)^8+A_6(B-1)^6+A_4(B-1)^4+A_2(B-1)^2+A_0=0.$$

At this point we can already see that there are parameter-dependent raising and lowering relations. Indeed

$$\left(R\ \mp\ 4(2n+B\ \mp\ 1)L_2\ \pm\ 4C(n,n)\frac{2n+B\mp\ 1}{-1/2+a^2+b^2-(2n+B)^2}L_1\right)f_n \qquad (5.30)$$
$$=\mp 8(2n+B)\ C(n\ \pm 1,n)\ f_{n\ \pm\ 1}.$$

So far, we have made use of the structure equations (5.19) and our model in fact forms a representation of the algebra determined by these relations. However, our superintegrable system admits an algebra with a specific value for the Casimir operator, determined by the Casimir relation (5.20). Thus, we will obtain a model of our superintegrable system from equations (5.25) and (5.29) if and only if the eigenvalue μ of the Casimir operator C vanishes; that is, if and only if the Casimir relation holds. To determine μ it is enough to compute

$$Cf_0=\mu f_0,$$

i.e., to evaluate C on the lowest weight vector f_0. A straightforward computation, using the fact that $C(-1,0)C(0,-1)=0$ leads to the result

$$\mu=-\frac{(B-1+a+b)(B-1+a-b)(B-1-a+b)(B-1-a-b)}{(B-1)^2}\times$$

$$(2h+3+4c-2a^2-2b^2-4B-4cB+2B^2)(2h+3-4c-2a^2-2b^2-4B+4cB+2B^2).$$

Thus, in order to achieve a model of the superintegrable system, we must have B equal to one of the four roots:

$$B = 1 \pm a \pm b.$$

To be definite, we make the standard choice

$$B = 1 + a + b.$$

The final requirement that uniquely determines the sequence F_n is the highest weight vector condition $F_m \equiv 0$, i.e.,

$$A_8(2m+B+1)^8 + A_6(2m+B+1)^6 + A_4(2m+B+1)^4 + A_2(2m+B+1)^2 + A_0 = 0$$

where m is a fixed nonnegative integer. We can then solve for H to determine the quantization condition on the energy values,

$$h = -\frac{1}{4}(4m+2a+2b+2c+5)(4m+2a+2b+2c+3). \tag{5.31}$$

There is a second solution with c replaced by $-c$. Taking the first solution as standard, we obtain the following values for the expansion coefficients:

$$\begin{aligned} C(n,n) = \frac{1}{2}\Big(&(2n+a+b+2)(2n+a+b) - (2m+a+b+c+2)^2 \\ &+ \frac{(a^2-b^2)(a+b+2m+2)(a+b+2c+2m+2)}{(2n+a+b+2)(2n+a+b)} + a^2 - b^2 - c^2 - 1\Big), \end{aligned} \tag{5.32}$$

$$\begin{aligned} &C(n,n+1)C(n+1,n) = \\ &\quad 16(n+1)(n-m)(n-c-m)(n+b+1)(n+a+1)(n+a+b+1) \\ &\quad \times \frac{(n+m+a+b+2)(n+m+a+b+c+2)}{(2n+a+b+3)(2n+a+b+2)^2(2n+a+b+1)}. \end{aligned} \tag{5.33}$$

Note that only the product $C(n,n+1)C(n+1,n)$ is determined uniquely. The values of the individual factors depend on the normalization of the basis vectors f_n. This result is in basic agreement with the expansion formula for products of Lamé or Heun polynomials in terms of products of Jacobi polynomials, obtained by Kalnins and Miller in 1991 [48] . In the earlier paper the coefficients were derived using recurrence formulas for Jacobi polynomials (and there were some typographical errors in the formulas). Here the derivation is directly from the structure formulas for the quadratic algebra. The condition that there is an inner product with respect to which the f_n form an orthogonal basis is $C(n,n+1)C(n+1,n) > 0$ for $n = 0, 1, \ldots, m-1$, and this is satisfied, for example, if $a, b, c > 0$. Although these representations are finite dimensional for m a positive integer in the expression (5.31) for H (the bound state energy levels) we can view (5.31) as a parameterization for H corresponding to arbitrary values of m. In these cases our representation is infinite dimensional but bounded below.

Due to the symmetry of the structure equations, it follows that the corresponding eigenvalues of the operator L_2 must be

$$\xi_n = -(2n+b+c+1)^2 - \frac{1}{2} + b^2 + c^2, \;\; n = 0, 1, \ldots, m, \tag{5.34}$$

whereas the eigenvalues of the operator L_3 must be

$$\eta_n = -(2n+a+c+1)^2 - \frac{1}{2} + a^2 + c^2, \;\; n = 0, 1, \ldots, m, \tag{5.35}$$

The bounded below representations of the superintegrable system S9 are intimately connected with the Wilson polynomials. The connection between these polynomials and the representation theory of S9 is the three term recurrence formula for the action of L_3 on and L_1 basis:

$$L_3 f_n = C(n+1,n) f_{n+1} + C(n,n) f_n + C(n-1,n) f_{n-1}$$

where the coefficients are given by (5.32) and (5.33). To understand the relationship we recall some facts about the Wilson polynomials. They are given by the expressions

$$\begin{aligned} p_n(t^2) \equiv p_n(t^2, \alpha, \beta, \gamma, \delta) = {} & (\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n \\ & \times {}_4F_3\left(\begin{matrix} -n, & \alpha+\beta+\gamma+\delta-n-1, & \alpha-t, & \alpha+t \\ \alpha+\beta, & \alpha+\gamma, & \alpha+\delta & \end{matrix} ; 1 \right) \end{aligned} \tag{5.36}$$

where $(a)_n$ is the Pochammer symbol and ${}_4F_3(1)$ is a generalized hypergeometric function of unit argument. The polynomial $p_n(t^2)$ is symmetric in $\alpha, \beta, \gamma, \delta$. For a more complete exposition of the Wilson polynomials and their recurrences, see the appendix D.

We consider the action of t^2 on Wilson polynomials as,

$$t^2 f_n = K(n+1,n)f_{n+1} + K(n,n)f_n + K(n-1,n)f_{n-1}, \tag{5.37}$$

where the $K(m,m')$ are given by (D.4-D.6).

We can see that with the choices,

$$\alpha = -\frac{a+c+1}{2} - m,\ \beta = \frac{a+c+1}{2},\ \gamma = \frac{a-c+1}{2},\ \delta = \frac{a+c-1}{2} + b + m + 2,$$

we have an exact match with

$$C(n+1,n) = 4K(n+1,n), \quad C(n-1,n) = 4K(n-1,n).$$

The diagonal elements are related by

$$C(n,n) = h + 4K(n,n) - \lambda_n - \frac{1}{4} + b^2.$$

Thus, we can form a model for the algebra with

$$L_3 = -4t^2 - \frac{1}{2} + a^2 + c^2.$$

Now we can construct a one variable model for the realization of these representations. The L_1 basis functions are the Wilson polynomials $f_n = p_n(t)$ and t^2 is multiplication by the transform variable. We can use the divided difference operator eigenvalue equation for the Wilson polynomials [49],

$$\tau^*\tau p_n = n(n+\alpha+\beta+\gamma+\delta-1)p_n$$

where

$$E^A F(t) = F(t+A), \qquad \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}),$$

$$\tau^* = \frac{1}{2t}\left[(\alpha+t)(\beta+t)(\gamma+t)(\delta+t)E^{1/2} - (\alpha-t)(\beta-t)(\gamma-t)(\delta-t)E^{-1/2}\right]$$

to express the action of L_1:

$$L_1 = -4\tau^*\tau - 2(a+1)(b+1) + \frac{1}{2}.$$

The inner product is given in the appendix (D.2).

When m is a nonnegative integer then $\alpha+\beta=-m<0$ so that the above continuous Wilson orthogonality does not apply. The representation becomes finite dimensional and the orthogonality is a finite sum,

$$\frac{(\alpha-\gamma+1)_m(\alpha-\delta+1)_n}{(2\alpha+1)_m(1-\gamma-\delta)_m}\sum_{k=0}^{m}\frac{(2\alpha)_k(\alpha+1)_k(\alpha+\beta)_k(\alpha+\gamma)_k(\alpha+\delta)_k}{(1)_k(\alpha)_k(\alpha-\beta+1)_k(\alpha-\gamma+1)_k(\alpha-\delta+1)_k}$$

$$\times p_n((\alpha+k)^2)p_{n'}((\alpha+k)^2)$$

$$=\delta_{nn'}\frac{n!(n+\alpha+\beta+\gamma+\delta-1)_n(\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n(\beta+\gamma)_n(\beta+\delta)_n(\gamma+\delta)_n}{(\alpha+\beta+\gamma+\delta)_{2n}}.$$

Thus, the spectrum of t^2 is the set $\{(\alpha+k)^2 : \ k=0,\ldots,m\}$, in agreement with equation (5.35) for the eigenvalues of L_3. In the original quantum mechanics eigenvalue problem the eigenfunctions of L_1 and L_3 each separate in suitable versions of spherical coordinates to give Karlin-McGregor polynomials. It follows from this derivation that the expansion coefficients relating one eigenbasis to the other are just the general Racah polynomials.

These relations are derived in equations (3.4) and (4.2) of Wilson's 1980 paper [50]. These finite discrete polynomials, suitably renormalized, are called the Racah polynomials. Thus the Racah polynomials are those associated with the bound state energy levels of the S9 Schrödinger eigenvalue equation, whereas the continuous Wilson polynomials are those associated with the continuous (but infinitely degenerate) spectrum of the Schrödinger operator.

As we shall also see, classical models can also provide guidance on determining a model. The classical S9 system has a basis of symmetries

$$\mathcal{L}_1=\mathcal{J}_1^2+a_2\frac{s_3^2}{s_2^2}+a_3\frac{s_2^2}{s_3^2},\ \mathcal{L}_2=\mathcal{J}_2^2+a_3\frac{s_1^2}{s_3^2}+a_1\frac{s_3^2}{s_1^2},\ \mathcal{L}_3=\mathcal{J}_3^2+a_1\frac{s_2^2}{s_1^2}+a_2\frac{s_1^2}{s_2^2}, \quad (5.38)$$

where $\mathcal{H}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+a_1+a_2+a_3$ and the $\mathcal{J}_i$ are defined by $\mathcal{J}_3=s_1p_{s_2}-s_2p_{s_1}$ and cyclic permutation of indices.

The structure relations for the classical algebra are,

$$\{\mathcal{L}_1,\mathcal{R}\}=8\mathcal{L}_1(\mathcal{H}+a_1+a_2+a_3)-8\mathcal{L}_1^2-16\mathcal{L}_1\mathcal{L}_2-16a_2\mathcal{L}_2+16a_3(\mathcal{H}+a_1+a_2+a_3-\mathcal{L}_1-\mathcal{L}_2), \quad (5.39)$$

$$\{\mathcal{L}_2,\mathcal{R}\}=-8\mathcal{L}_2(\mathcal{H}+a_1+a_2+a_3)+8\mathcal{L}_2^2+16\mathcal{L}_1\mathcal{L}_2+16a_1\mathcal{L}_1-16a_3(\mathcal{H}+a_1+a_2+a_3-\mathcal{L}_1-\mathcal{L}_2),$$

with $\{\mathcal{L}_1, \mathcal{L}_2\} = \mathcal{R}$ and

$$\mathcal{R}^2 - 16\mathcal{L}_1\mathcal{L}_2(\mathcal{H}+a_1+a_2+a_3) + 16\mathcal{L}_1^2\mathcal{L}_2 + 16\mathcal{L}_1\mathcal{L}_2^2 + 16a_1\mathcal{L}_1^2 + 16a_2\mathcal{L}_2^2 + 16a_3(\mathcal{H}+a_1+a_2+a_3)^2$$

$$-32a_3(\mathcal{H}+a_1+a_2+a_3)(\mathcal{L}_1+\mathcal{L}_2) + 16a_3\mathcal{L}_1^2 + 32a_3\mathcal{L}_1\mathcal{L}_2 + 16a_3\mathcal{L}_2^2 - 64a_1a_2a_3.$$

Taking $\mathcal{L}_1 = \gamma$, $\mathcal{H} = h$ with γ, β as conjugate variables, we find the model

$$\mathcal{L}_2 = \frac{1}{2}(a_1 + 2a_2 + h - \gamma) - \frac{(a_2 - a_3)(a_1 + 2a_2 + 2a_3 + h)}{2(\gamma + a_2 + a_3)} + \tag{5.40}$$

$$\frac{\sqrt{(4a_1a_2 + 4a_1a_3 + 2\gamma(h + a_1 + a_2 + a_3) + 4\gamma a_1 - (h + a_1 + a_2 + a_3)^2 - \gamma^2)(4a_2a_3 - \gamma^2)}}{2(a_2 + a_3 + \gamma)}$$

$$\times \cos(4\beta\sqrt{a_2 + a_3 + \gamma}).$$

This suggests a difference operator realization of the quantum model by quantizing $\gamma \to x \quad \beta \to \partial_x$, and using an appropriate change of variables.

In this example, we created a difference operator model where the eigenfunctions for L_1 are the Wilson polynomials. The model was created by analyzing the abstract structure relations and fitting a difference model to it. However, as explained, one could also have begun with the classical model and quantized it to obtain the general form of the difference model.

This system is also of particular note because of the current research into the most generic system in 3d which is an obvious generalization of this system and whose symmetry algebra includes some of the operators from the 2d system. The representation theory of the generic system is expected to admit a two variable generalization of the Wilson polynomials. In the next section, I give a discussion of a 3d model where there is a differential operator model suitable to determining the spectrum of two of the operators.

Chapter 6

A 3d Model: the singular isotropic oscillator

In this section, I work out several models for the quadratic symmetry algebra of the 3d singular isotropic oscillator. The original problem separates in three coordinate systems and I give three different models suited to the symmetry operators that define each system. The results from this section are summarized in [51].

6.1 The system in 3d

The Hamiltonian operator is

$$H = \partial_1^2 + \partial_2^2 + \partial_3^2 - \omega^2(x_1^2 + x_2^2 + x_3^2) + \frac{b_1}{x_1^2} + \frac{b_2}{x_2^2} + \frac{b_3}{x_3^2} \quad \partial_i \equiv \partial_{x_i}. \tag{6.1}$$

A basis for the second order constants of the motion is (with $H = M_1 + M_2 + M_3$)

$$M_i = \partial_i^2 - \omega^2 x_i^2 + \frac{b_i}{x_i^2}, \qquad L_i = (x_j\partial_k - x_k\partial_j)^2 + \frac{b_j x_k^2}{x_j^2} + \frac{b_k x_j^2}{x_k^2}, \tag{6.2}$$

where i, j, k are pairwise distinct and run from 1 to 3. There are 4 linearly independent commutators of the second order symmetries:

$$S_1 = [L_1, M_2] = [M_3, L_1], \qquad S_2 = -[M_3, L_2] = [M_1, L_2], \tag{6.3}$$

$$S_3 = -[M_1, L_3] = [M_2, L_3], \qquad R = [L_1, L_2] = [L_2, L_3] = [L_3, L_1],$$

$$[M_i, M_j] = [M_i, L_i] = 0, \quad 1 \le i, j \le 3.$$

Here we define the commutator of linear operators F, G by $[F, G] = FG - GF$. (Thus a second order constant of the motion is a second order partial differential operator K in the variables x_j such that $[K, H] = 0$, where 0 is the zero operator.)

The fourth order structure equations are $[M_i, S_i] = 0,\ 1 = 1, 2, 3$, and

$$\begin{aligned}
\epsilon_{ijk}[M_i, S_j] &= 8M_iM_k + 16\omega^2 L_j - 8\omega^2, \\
\epsilon_{ijk}[M_i, R] &= 8(M_jL_j - M_kL_k) + 4(M_k - M_j), \\
\epsilon_{ijk}[S_i, L_j] &= 8M_iL_i - 8M_kL_k + 4(M_k - M_i), \\
\epsilon_{ijk}[L_i, S_i] &= 4\{L_i, M_k - M_j\} + 16b_jM_k - 16b_kM_j + 8(M_k - M_j), \\
\epsilon_{ijk}[L_i, R] &= 4\{L_i, L_k - L_j\} - 16b_jL_j + 16b_kL_k + 8(L_k - L_j + b_j - b_k).
\end{aligned} \tag{6.4}$$

Here, $\{F, G\} = FG + GF$ and ϵ_{ijk} is the completely antisymmetric tensor and we are not summing over repeated indices.

The fifth order structure equations are obtainable directly from the fourth order equations and the Jacobi identity. The sixth order structure equations are

$$S_i^2 - \frac{8}{3}\{L_j, M_j, M_k\} - 16\omega^2 L_i^2 + (16b_k + 12)M_j^2 + (16b_j + 12)M_k^2 - \frac{104}{3}M_jM_k$$

$$+\frac{176}{3}\omega^2 L_i + \frac{16}{3}\omega^2(2 + 9b_j + 9b_k + 12b_jb_k) = 0, \tag{6.5}$$

$$\frac{1}{2}\{S_i, S_j\} + \frac{4}{3}\{L_i, M_k, M_i\} + \frac{4}{3}\{L_j, M_k, M_j\} - 8L_kM_k^2 + 8\omega^2\{L_i, L_j\} - (16b_k + 12)M_iM_j$$

$$+4M_k^2 - 4M_k(M_i + M_j) - \omega^2(32b_k + 24)L_k - 8\omega^2(L_i + L_j) + 16\omega^2(b_k + 1) = 0,$$

$$\frac{1}{2}\{S_i, R\} - 8L_i^2M_i + \frac{4}{3}\{L_k, L_i, M_k\} + \frac{4}{3}\{L_i, L_j, M_j\} - (8b_k + 6)\{L_k, M_j\}$$

$$-(8b_j + 6)\{L_j, M_k\} - 2\{L_i, M_k + M_j\} + \frac{88}{3}L_iM_i + \frac{52}{3}(L_kM_k + L_jM_j)$$

$$+(32b_kb_j + 24b_k + 24b_j + \frac{16}{3})M_i + (8b_j - \frac{8}{3})M_k + (8b_k - \frac{8}{3})M_j = 0,$$

$$R^2 - \frac{8}{3}\{L_1, L_2, L_3\} + (16b_1 + 12)L_1^2 + (16b_2 + 12)L_2^2 + (16b_3 + 12)L_3^2 - \frac{52}{3}\{L_1, L_2\}$$

$$-\frac{52}{3}\{L_1, L_3\} - \frac{52}{3}\{L_2, L_3\} - \frac{16}{3}(11b_1 + 1)L_1 - \frac{16}{3}(11b_2 + 1)L_2$$

$$-\frac{16}{3}(11b_3+1)L_3-\frac{32}{3}\left(6b_1b_2b_3+\frac{9}{2}(b_1b_2+b_1b_3+b_2b_3)+b_1+b_2+b_3\right).$$

Here, $\{A,B,C\}=ABC+ACB+BAC+BCA+CAB+CBA$ and i,j,k are pairwise distinct.

The eighth order functional relation is

$$L_1^2M_1^2+L_2^2M_2^2+L_3^2M_3^2-\frac{1}{12}\{L_1,L_2,M_1,M_2\}-\frac{1}{12}\{L_1,L_3,M_1,M_3\} \tag{6.6}$$

$$-\frac{1}{12}\{L_2,L_3,M_2,M_3\}-\frac{7}{3}L_1M_1^2-\frac{7}{3}L_2M_2^2-\frac{7}{3}L_3M_3^2-\frac{2}{3}\omega^2\{L_1,L_2,L_3\}$$

$$-\frac{1}{18}\{L_1,M_1,M_2\}-\frac{1}{18}\{L_1,M_1,M_3\}-\frac{1}{18}\{L_2,M_1,M_2\}-\frac{1}{18}\{L_2,M_2,M_3\}$$

$$-\frac{1}{18}\{L_3,M_1,M_3\}-\frac{1}{18}\{L_3,M_2,M_3\}+\frac{1}{6}(4b_1+3)\{L_1,M_2,M_3\}$$

$$+\frac{1}{6}(4b_2+3)\{L_2,M_1,M_3\}+\frac{1}{6}(4b_3+3)\{L_3,M_1,M_2\}+\omega^2(4b_1+3)L_1^2+\omega^2(4b_2+3)L_2^2+\omega^2(4b_3+3)L_3^2$$

$$-\frac{\omega^2}{3}(\{L_1,L_2\}+\{L_1,L_3\}+\{L_2,L_3\})-(4b_2b_3+3b_2+3b_3+\frac{4}{3})M_1^2$$

$$-(4b_1b_3+3b_1+3b_3+\frac{4}{3})M_2^2-(4b_1b_2+3b_1+3b_2+\frac{4}{3})M_3^2+\frac{2}{3}(b_3+2)M_1M_2$$

$$+\frac{2}{3}(b_2+2)M_1M_3+\frac{2}{3}(b_1+2)M_2M_3-\frac{4}{3}\omega^2(7b_1+4)L_1-\frac{4}{3}\omega^2(7b_2+4)L_2$$

$$-\frac{4}{3}\omega^2(7b_3+4)L_3-\frac{4}{3}\omega^2(12b_1b_2b_3+9b_1b_2+9b_1b_3+9b_2b_3+4b_1+4b_2+4b_3)=0.$$

Here, $\{A,B,C,D\}$ is the 24 term symmetrizer of 4 operators.

6.2 Cartesian basis: A quantum model with M_1, M_2 diagonal

For the model we choose variables u, v in which the eigenfunctions are polynomials, and write the parameters as $b_j = 1/4 - k_j^2$. Then we have

Model 1. Cartesian Model

$$M_1 = 2\omega(2u\partial_u + k_1 + 1), \qquad M_2 = 2\omega(2v\partial_v + k_2 + 1), \tag{6.7}$$

$$M_1 + M_2 + M_3 = E,$$

$$
\begin{aligned}
L_1 &= 4v\left(u^2\partial_u^2 + 2uv\partial_u\partial_v + (v^2+1)\partial_v^2 - (\frac{E}{2\omega} - k_1 - k_2 - 4)(u\partial_u - v\partial_v)\right)\\
&\quad + 4(1+k_2)\partial_v + v\left((-\frac{E}{2\omega} + k_1 + k_2 + 3)^2 - k_3^2\right) + \frac{1}{2a^2}M_2M_3 + \frac{1}{2},\\
L_2 &= 4u\left(v^2\partial_v^2 + 2uv\partial_u\partial_v + (u^2+1)\partial_u^2 - (\frac{E}{2\omega} - k_1 - k_2 - 4)(v\partial_v + u\partial_u)\right)\\
&\quad + 4(1+k_1)\partial_u + u\left((-\frac{E}{2\omega} + k_1 + k_2 + 3)^2 - k_3^2\right) - \frac{1}{2\omega^2}M_1M_3 + \frac{1}{2},\\
L_3 &= 4\left(uv\partial_u^2 + uv\partial_v^2 + (k_1+1)v\partial_u + (k_2+1)u\partial_v\right) - \frac{1}{2\omega^2}M_1M_2 + \frac{1}{2}.
\end{aligned}
$$

In the model, the monomials $f_{m,n} = u^n v^{m-n}$ such that the m and n are nonnegative integers satisfying $0 \le n \le m$ are simultaneous eigenfunctions of the operators M_i:

$$
M_1 f_{m,n} = 2\omega(2n + k_1 + 1)f_{m,n},\ M_2 f_{m,n} = 2\omega(2m - 2n + k_2 + 1)f_{m,n}. \tag{6.8}
$$

Further, we have the expansion formulas,

$$
\begin{aligned}
L_1 f_{m,n} &= \left((2m + 3 - \frac{E}{2\omega} + k_1 + k_2)^2 - k_3^2\right) f_{m+1,n} + 4(m-n)(m-n+k_2)f_{m-1,n}\\
&\quad + \left(2(\frac{E}{2\omega} - 2m - 2k_1 - 2k_2 - 2)(2m - 2j + k_2 + 1) + \frac{1}{2}\right) f_{m,n},\\
L_2 f_{m,n} &= \left((2m + 3 - \frac{E}{2\omega} + k_1 + k_2)^2 - k_3^2\right) f_{m+1,n+1} + 4j(n + k_1)f_{m-1,n-1}\\
&\quad + \left(2(\frac{E}{2\omega} - 2m - 2k_1 - 2k_2 - 2)(2j + k_1 + 1) + \frac{1}{2}\right) f_{m,n},\\
L_3 f_{m,n} &= 4(m-n)(m-n+k_2)f_{m,n+1} + 4j(n+k_2)f_{m,n-1}\\
&\quad + \left(2(2j + k_1 + 1)(2m - 2j + k_2 + 1) + \frac{1}{2}\right) f_{m,n}.
\end{aligned}
$$

We can also define raising and lowering operators based on these operators and their commutators given by (6.3). They are,

$$
\begin{aligned}
A_v &= \left(L_1 + \frac{1}{4\omega}S_1 + \frac{1}{2\omega^2}M_2M_3 - \frac{1}{2}\right)\\
&= \left(8v\partial_v^2 + 8(1+k_2)\partial_v\right),\\
A_v f_{m,n} &= 8(m-n)(m-n+k_2)f_{m-1,n},\\
A_v^\dagger &= \left(L_1 - \frac{1}{4\omega}S_1 + \frac{1}{2\omega^2}M_2M_3 - \frac{1}{2}\right)\\
&= 8v\left((u\partial_u + v\partial_v)^2 + (k_3+1-2M)(u\partial_u + v\partial_v) + M(M-k_3)\right),\\
A_v^\dagger f_{m,n} &= 2\omega((\frac{E}{\omega} + 4m + 2k_1 + 2k_2 + 6\omega)^2 - k_3^2)f_{m+1,n},
\end{aligned}
$$

$$
\begin{aligned}
A_u &= \left(L_2 + \frac{1}{4\omega}S_2 + \frac{1}{2\omega^2}M_1M_3 - \frac{1}{2}\right)\\
&= \left(8u\partial_u^2 + 8(1+k_1)\partial_u\right),\\
A_u f_{m,n} &= 8n(n+k_1)f_{m-1,n-1},\\
A_u^\dagger &= \left(L_2 + \frac{1}{4\omega}S_2 + \frac{1}{2\omega^2}M_1M_3 - \frac{1}{2}\right)\\
&= 8u\left((u\partial_u + v\partial_v)^2 + (k_3+1-2M)(v\partial_v + u\partial_u) + M(M-k_3)\right),\\
A_u^\dagger f_{m,n} &= 2\omega((\frac{E}{\omega} + 4m + 2k_1 + 2k_2 + 6\omega)^2 - k_3^2)f_{m+1,n+1},
\end{aligned}
$$

$$
\begin{aligned}
A_{uv} &= \left(S_3 + 4\omega L_3 + \frac{1}{\omega}\{L_1, L_2\} - 2\omega\right)\\
&= 32\alpha v\left(u\partial_u^2 + (1+k_1)\partial_u\right),\\
A_{uv} f_{m,n} &= -32\omega n(n+k_1)f_{m,n-1},\\
A_{uv}^\dagger &= \left(S_3 - 4\omega L_3 - \frac{1}{\omega}\{L_1, L_2\} + 2\omega\right)\\
&= 32\omega u\left(v\partial_v^2 + (1+k_2)\partial_v\right),\\
A_{uv}^\dagger f_{m,n} &= -32\omega(m-n)(m-n+k_2)f_{m,n+1}.
\end{aligned}
$$

These operators satisfy the following symmetry equations,

$$[A_v, M_1] = 0, \qquad [A_v, M_2] = 4\omega A_v,$$

$$[A_v^\dagger, M_1] = 0, \qquad [A_v^\dagger, M_2] = -4\omega A_v^\dagger,$$

$$[A_u, M_1] = 4\omega A_u, \qquad [A_u, M_2] = 0,$$

$$[A_u^\dagger, M_1] = -4\omega A_u^\dagger, \qquad [A_u^\dagger, M_2] = 0,$$

$$[A_{uv}, M_1] = 4\omega A_{uv}, \qquad [A_{uv}, M_2] = -4\omega A_{uv},$$

$$[A_{uv}^\dagger, M_1] = -4\omega A_{uv}^\dagger, \qquad [A_{uv}^\dagger, M_1] = 4\omega A_{uv}^\dagger,$$

$$[A_u, A_v] = [A_u^\dagger, A_v^\dagger] = 0,$$

$$[A_{uv}, A_v^\dagger] = [A_{uv}, A_u] = [A_{uv}^\dagger, A_v] = [A_{uv}^\dagger, A_u^\dagger] = 0,$$

$$[A_u^\dagger, A_v] = -\frac{4}{\omega} A_{vu} M_3 + 16, \quad [A_v^\dagger, A_u] = -\frac{4}{\omega} A_{uv} M_3 - 16,$$

$$[A_{uv}, A_v] = \frac{4}{\omega} A_u M_2, \qquad [A_{uv}^\dagger, A_u] = \frac{4}{\omega} A_v M_1,$$

$$[A_{uv}, A_u^\dagger] = -\frac{4}{\omega} M_1 A_v^\dagger, \qquad [A_{uv}^\dagger, A_v^\dagger] = -\frac{4}{\omega} M_2 A_u^\dagger,$$

$$[A_{uv}^\dagger, A_{uv}] = \frac{2M_1M_2}{\omega^3}(M_1 - M_2) - \frac{8}{\omega}\left((k_2^2-1)M_1 - (k_1^2-1)M_2\right),$$

$$[A_v, A_v^\dagger] = \frac{2M_2M_3}{\omega^3}(2M_2 + M_1) + \frac{8M_2}{\omega}((\frac{E}{2\omega})^2 - k_2^2 - k_3^2 + 2)$$
$$- \frac{8}{\omega}\left((k_2^2-1)M_1 + 4E(k_2^2-1)\right),$$

$$[A_u, A_u^\dagger] = \frac{2M_1M_3}{\omega^3}(2M_1 + M_2) + \frac{8M_1}{\omega}((\frac{E}{2\omega})^2 - k_1^2 - k_3^2 + 2)$$
$$- \frac{8}{\omega}\left((k_1^2-1)M_2 + 4E(k_1^2-1)\right).$$

These commutation relations can either be determined from the structure equations (6.4) or directly from the model. It is immediately obvious from the relations that, though the operators seem to follow an sl_2 structure, the algebra is truely not a Lie algebra since the commutator of some operators are more than first order in the operators.

Notice that if $n = 0$ the basis functions are annihilated by A_u, and A_{uv} and if $n = m$ the basis functions are annihilated by A_v and $A_{uv}^\dagger$. Thus, in order for the model to form an irreducible representation, we have the requirements $0 \le n \le m$. Furthermore, if we demand that the representation be finite dimensional, we obtain quantization levels for the energy. That is, the energy satisfies

$$E = 2\omega(2M + k_1 + k_2 + k_3 + 3). \tag{6.9}$$

The dimension of the representation is $(M+2)(M+1)/2$. With this requirement, we can see that $f_{M,n}$ is annihilated by $A_u^\dagger$ and $A_v^\dagger$, hence in our model we realize irreducible finite dimensional representation under the restrictions $0 \leq n \leq m \leq M$.

Next, we introduce an inner product such that the operators M_i, L_i are formally self-adjoint for $i = 1, 2, 3$. This forces ω to be real. Under these requirements, we can see that A_* and $A_*^\dagger$ are indeed mutual adjoints and we can then use these raising and lowering operators to determine the coefficients.

That is, let $\hat{f}_{m,n} = K_{m,n} u^n v^{m-n} = K_{m,n} f_{m,n}$ such that $||\hat{f}_{m,n}|| = 1$. We can use the defining property of adjoint operators to obtain recurrence coefficients,

$$\begin{gathered}
\langle f_{m,n}, A_v f_{m+1,n} \rangle = \langle A_v^\dagger f_{m,n}, f_{m+1,n} \rangle \\
\Rightarrow K_{m+1,n}^2 = \frac{(M-m)(M-m-k_3)}{(m-n+1)(m-n+1+k_2)} k_{m,n}^2 \\
\langle f_{m,n}, A_{uv} f_{m,n+1} \rangle = \langle A_{uv}^\dagger f_{m,n}, f_{m,n+1} \rangle \\
\Rightarrow K_{m,n+1}^2 = \frac{(m-n)(m-n+k_2)}{(n+1)(n+1+k_1)} k_{m,n}^2.
\end{gathered}$$

The other operators A_u and $A_u^\dagger$ give no new information that cannot be determined by the two recursions above. If we assume $K_{0,0} = 1$ then the coefficients become,

$$K_{m,n} = \frac{(-M)_m(-M-k_3)_m}{(m-n)!n!(k_2+1)_{m-n}(k_1+1)_n}$$

We next attempt to find eigenfunctions for L_3 in this model. Recall from the structure relations (6.3) that $[L_3, M_3] = 0$ so we search for simultaneous eigenvectors for these operators, i.e. of M_3, L_3. M_3 is already diagonalized in the Cartesian basis so we expect to find eigenfunctions for L_3 in this model. Furthermore, we already know the eigenvalues of M_3 from this model since $M_3 = H - M_1 - M_2$. Hence, the eigenvalues for M_3 must be $-4\omega(M - N - \frac{1}{2})$, where $m = N$ is the total degree of the monomial.

The eigenfunctions for L_3 are , for $0 \leq j \leq N \leq M$,

$$\phi_{N,j} = c_{N,j} v^j (u+v)^{N-j} {}_2F_1 \left(\begin{matrix} -j, & -j-k_2 \\ 1+k_1 & \end{matrix} \middle| -\frac{u}{v} \right)$$

with eigenvalues

$$\lambda_{N,j} = -4j^2 - 4(k_1 + k_2 + 1)j - 2(k_1 + 1)(k_2 + 1) + \frac{1}{2}.$$

Notice, these are polynomials of highest total degree N.

To determine the normalization for these basis function, we make use of the kernel functions. That is, we know that in the monomial basis

$$\sum k_{m,n}^2 (u\bar{s})^m (v\bar{t})^n = F_4 \left(\begin{array}{cc} -M, & -M + k_3 \\ 1 + k_2, & 1 + k_1 \end{array} \middle| u\bar{s}, v\bar{t} \right).$$

While in the hypergeometric basis we require,

$$F_4 \left(\begin{array}{cc} -M, & -M + k_3 \\ 1 + k_2, & 1 + k_1 \end{array} \middle| u\bar{s}, v\bar{t} \right) = \sum c_{N,j}^2 (v\bar{t})^j (u + v)^{N-j} (\overline{s + t})^{N-j}$$

$$\times {}_2F_1 \left(\begin{array}{cc} -j, & -j - k_2 \\ 1 + k_1 & \end{array} \middle| -\frac{u}{v} \right) {}_2F_1 \left(\begin{array}{cc} -j, & -j - k_2 \\ 1 + k_1 & \end{array} \middle| -\frac{\bar{s}}{\bar{t}} \right).$$

We can compare coefficients on both sides of the equation to obtain the requirement that

$$c_{N,j}^2 = \frac{(-M)_N (-M + k_3)_N (1 + k_1)_j (k_1 + k_2 + 1)_j (k_1 + k_2 + 2j + 1)}{j!(N - j)!(1 + k_2)_j (k_1 + k_2 + 1)_{N+j} (k_1 + k_2 + N + j + 1)}.$$

The normalizations are important, because the next step is to determine the action of the other operators on this basis and knowing the normalization helps simplify the computations. To further simplify this computation, we use a change of variables $x = v$ and $t = \frac{v-u}{v+u}$. In these variables, the basis functions take the form of Jacobi polynomials in t multiplied by a gauge function,

$$\psi_{N,j} = c_{N,j} x^N \frac{j!}{(1 + k_1)_j} \left(\frac{t + 1}{2} \right)^{-N} P_j^{k_1, k_2}(t).$$

We can then use the recurrence relations of the Jacobi polynomials [45] to aid in the computation. Also notice that the basis function are separated into two functions dependent only on the individual variables; that is, $f_N(x) g_{N,j}(t)$. We then make a change of variables in the operators M_1, M_2 and $A_u, A_u^\dagger, A_v, A_v^\dagger$ to determine their action on the basis functions. Notice, we do not compute the action of A_{uv} or its adjoint because they are related to the operator L_3 which acts as a constant in this basis.

Using the recurrence relations between Jacobi polynomials we obtain,

$$
\begin{aligned}
\frac{M_i}{2\omega}\psi_{N,j} &= m^{(i)}_{0,+1}(N,j)\psi_{N,j+1} + m^{(i)}_{0,0}(N,j)\psi_{N,j} + m^{(i)}_{0,-1}(N,j)\psi_{N,j-1}, \quad i = 1,2,\\
2A_v\psi_{N,j} &= l^{(1)}_{-1,0}(N,j)\psi_{N-1,j} + \ l^{(1)}_{-1,+1}(N,j)\psi_{N-1,j+1} + l^{(1)}_{-1,-1}(N,j)\psi_{N-1,j-1},\\
2A^{\dagger}_v\psi_{N,j} &= l^{(1)}_{+1,0}(N,j)\psi_{N+1,j} + l^{(1)}_{+1,+1}(N,j)\psi_{N+1,j+1} + l^{(1)}_{+1,-1}(N,j)\psi_{N+1,j-1},\\
2A_u\psi_{N,j} &= l^{(2)}_{-1,0}(N,j)\psi_{N-1,j} + \ l^{(2)}_{-1,+1}(N,j)\psi_{N-1,j+1} + l^{(2)}_{-1,-1}(N,j)\psi_{N-1,j-1},\\
2A^{\dagger}_v\psi_{N,j} &= l^{(2)}_{+1,0}(N,j)\psi_{N+1,j} + l^{(2)}_{+1,+1}(N,j)\psi_{N+1,j+1} + l^{(2)}_{+1,-1}(N,j)\psi_{N+1,j-1}.
\end{aligned}
$$

We name the coefficients this way because

$$
\begin{aligned}
L_1 &= \frac{1}{2}\left(A_v + A^{\dagger}_v\right) - \frac{M_2M_3}{2\omega^2} + \frac{1}{2},\\
L_2 &= \frac{1}{2}\left(A_u + A^{\dagger}_u\right) - \frac{M_1M_3}{2\omega^2} + \frac{1}{2}.
\end{aligned}
$$

Thus, the expansion coefficients of L_1 and L_2 will be exactly

$$
L_i\psi_{N,j} = \sum_{\sigma,\sigma'\in\{-1,0,1\}} l^{(i)}_{N+\sigma,N;j+\sigma'}\psi_{N+\sigma,j+\sigma}, \qquad i = 1,2,
$$

with $l_{\pm1,\sigma'}(N,j)^{(i)}$ given by the coefficients of the raising and lowering operators. The remaining coefficients are given by,

$$
\begin{aligned}
l^{(1)}_{0,\pm1}(N,j) &= (M-N-\frac{1}{2})m^{(2)}_{0,\pm1}(N,j), \quad l^{(1)}_{0,0}(N,j) = (M-N-\frac{1}{2})m^{(2)}_{0,0}(N,j) + \frac{1}{2},\\
l^{(2)}_{0,\pm1}(N,j) &= (M-N-\frac{1}{2})m^{(1)}_{0,\pm1}(N,j), \quad l^{(2)}_{0,0}(N,j) = (M-N-\frac{1}{2})m^{(1)}_{0,0}(N,j) + \frac{1}{2}.
\end{aligned}
$$

The expansion coefficients are, for $i = 1,2$,

$$m_{0,0}^{(i)}(N,j) = \frac{(k_i^2 + k_1 + k_2 + k_1k_2 + 2j(k_1 + k_2 + 1 + j))(k_2 + k_1 + 2 + 2N)}{(k_1 + k_2 + 2j + 2)(k_1 + k_2 + 2j)},$$

$$\begin{aligned} m_{0,+1}^{(i)}(N,j) = &(-1)^i \sqrt{(k_1 + k_2 + j + 1)(k_1 + j + 1)(k_2 + j + 1)} \\ &\times \sqrt{\frac{(j+1)(N-j)(k_1 + k_2 + N + j + 2)}{(k_1 + k_2 + 2j + 3)(k_1 + k_2 + 2j + 2)^2(k_1 + k_2 + 2j + 1)}}, \end{aligned}$$

$$\begin{aligned} l_{+1,0}^{(i)}(N,j) = &\sqrt{(M-N)\,(M-N-k_3)(N-j+1)(k_1 + k_2 + N + j + 2)} \\ &\times \frac{(2j(k_1 + k_2 + j + 1) + k_i^2 + k_1k_2 + k_1 + k_2)}{(k_1 + k_2 + 2j)(k_1 + k_2 + 2j + 1)}, \end{aligned}$$

$$\begin{aligned} l_{+1+1}^{(i)}(N,j) = &\sqrt{(k_1 + k_2 + N + j + 2)(k_1 + k_2 + N + j + 3)(k_1 + k_2 + j + 1)} \\ &\times \sqrt{\frac{(M-N)(M-N-k_3)(j+1)(k_1 + j + 1)(k_2 + j + 1)}{(k_1 + k_2 + 2j + 1)(k_1 + k_2 + 2j + 2)^2(k_1 + k_2 + 2j + 3)}}, \end{aligned}$$

$$\begin{aligned} l_{+1,-1}^{(i)}(N,j) = &\sqrt{(k_1 + j)\,(k_2 + j)\,(k_1 + k_2 + j)} \\ &\times \sqrt{\frac{j\,(M-N)\,(M-N-k_3)\,(N-j)\,(N-j+1)}{(k_1 + k_2 + 2j - 1)\,(k_1 + k_2 + 2j)^2\,(k_1 + k_2 + 2j + 1)}}. \end{aligned}$$

We can also recover the coefficients of the lowering operator by the adjoint property. That is, for example,

$$\left\langle \psi_{N+\sigma,j+\sigma'}, L_i \psi_{N,j} \right\rangle = \left\langle L_i \psi_{N+\sigma,j+\sigma'}, \psi_{N-\sigma,j-\sigma'} \right\rangle,$$

which tells us that

$$l_{\sigma,\sigma'}^{(i)}(N,j) = l_{-\sigma,-\sigma'}^{(i)}(N+\sigma, j+\sigma').$$

This relation is due to the fact that L_i are self adjoint operators and the basis functions are normalized so a matrix representation of the operator will be symmetric. Furthermore, if we consider the product $l_{\sigma,\sigma'}^{(i)}(N,j)l_{-\sigma,-\sigma'}^{(i)}(N+\sigma, j+\sigma')$ we can remove the square roots. This product is constant despite the choice of normalization.

We can quickly verify this fact by taking a different normalization $\tilde{\psi}_{N,j} = l_{N,j}\psi_{N,j}$. Then, suppose a self-adjoint operator takes the form

$$\Omega\tilde{\psi}_{N,j} = \sum_{\sigma,\sigma'} \tilde{O}_{\sigma,\sigma'}(N,j)\tilde{\psi}_{N+\sigma,j+\sigma'},$$

while,

$$\Omega\psi_{N,j} = \sum_{\sigma,\sigma'} O_{\sigma,\sigma'}(N,j)\psi_{N+\sigma,j+\sigma'}.$$

Comparing coefficients, we obtain

$$\tilde{O}_{\sigma,\sigma'}(N,j)\frac{l_{N+\sigma,j+\sigma'}}{l_{N,j}} = O_{\sigma,\sigma'}(N,j),$$

and,

$$\tilde{O}_{-\sigma,-\sigma'}(N+\sigma,j+\sigma')\frac{l_{N+\sigma,j+\sigma'}}{l_{N,j}} = O_{-\sigma,-\sigma'}(N+\sigma,j+\sigma').$$

Hence we have the equality

$$\tilde{O}_{\sigma,\sigma'}(N,j)\tilde{O}_{-\sigma,-\sigma'}(N+\sigma,j+\sigma') = O_{\sigma,\sigma'}(N,j)O_{-\sigma,-\sigma'}(N+\sigma,j+\sigma').$$

This is important because when we use the recurrence relations to determine a new model we can see the choice of factorizations of these products determine the normalization constants. So, we must make choices of factorizations that yield consistent recurrences in the normalization constants.

We are now ready to construct a model based on the spectrum of M_3 and L_3.

6.3 Cylindrical Basis; A quantum model with M_3, L_3 diagonal

Because the eigenvalues of M_3 are linear in N and the eigenvalues of L_3 are quadratic in j, we attempt to define a model where M_3 is a first order differential operator and L_3 is multiplication by a variable squared. That is, we assume an orthonormal basis of eigenvectors for M_3 and L_3 of

$$\psi_{N,j} = d_{N,j}t^N\delta(y-\lambda_j)$$

with $0 \leq j \leq N \leq M$, $\lambda_j = j + (k_1+k_2+1)/2$ and d_n a normalization constant.

Then, the action of M_3 on $\psi_{N,j}$ is

$$M_3\psi_{N,j} = -4\omega(M - N - \frac{1}{2})\psi_{N,j}$$

so we take

$$M_3 = 4\omega(t\partial_t - M + \frac{1}{2}).$$

Also, since the eigenvalues for L_3 are $-4\lambda_j^2 + k_1^2 + k_2^2$, we take the action of L_3 to be

$$L_3 = -4y^2 + k_1^2 + k_2^2.$$

To compute the action of the other operators on this basis, we try to reproduce the expansions from the differential/differential model previously analysed. That is, we know that neither M_1 nor M_2 change the index N associated with the power of t so we assume the M_i's, $i = 1, 2$ have the form,

$$\frac{1}{2\omega}M_i = T^{-1}\mu^{(i)}_{0,+1}(t\partial_t, y) + T^1\mu^{(i)}_{0,-1}(t\partial_t, y) + \mu^{(i)}_{0,0}(t\partial_t, y), \tag{6.10}$$

$$\frac{1}{2\omega}M_i t^N\delta(y-\lambda_j) = \mu^{(i)}_{0,+1}(N,j)t^N\delta(y-\lambda_j-1) + \mu^{(i)}_{0,-1}(N,j)t^N\delta(y-\lambda_j+1)$$
$$+\mu^{(i)}_{0,0}(N,j)t^N\delta(y-\lambda_j),$$

where T^k is a difference operator in t. It acts on functions as $T^k f(t) = f(t+k)$. In these equations $\mu^{(i)}_{0,\sigma}(t\partial_t, y)$ are operators while

$$\mu^{(i)}_{0,\sigma}(N,j) = \mu^{(i)}_{0,\sigma}(t\partial_t, y+\sigma)\Big|_{t\partial_t = N, y = \lambda_j - \sigma}$$

is a constant. Here $\sigma \in \{-1, 0, 1\}$.

These constants become more clear if we assume an inner product that makes the operators L_i, M_i self-adjoint. Then we can see that these constants are more simply defined as,

$$\mu^{(i)}_{0,\sigma}(N,j) = \frac{d_{N,j+\sigma}}{d_{N,j}}\langle\phi_{N,j+\sigma}, M_1\phi_{N,j}\rangle.$$

Next, we can use this inner product to determine recurrence relations for the normalization constant, i.e.

$$\langle\psi_{N,j+\sigma}, M_i\psi_{N,j+\sigma}\rangle = \langle M_i\psi_{N,j+\sigma}, \psi_{N,j}\rangle,$$

$$\Rightarrow \langle \psi_{N+\sigma}, \mu^{(i)}_{0,\sigma}(N,j)\frac{d_{N,j}}{d_{N,j+\sigma}}\psi_{N,j+\sigma}\rangle = \langle \frac{d_{N,j+\sigma}}{d_{N,j}}\mu^{(i)}_{0,-\sigma}(N,j+\sigma)\psi_{N,j}, \psi_{N,j}\rangle,$$

where in the equation above I have left out orthogonal terms. This then implies,

$$\frac{\mu^{(i)}_{0,\sigma}(N,j)}{\mu^{(i)}_{0,-\sigma}(N,j+\sigma)} = \frac{d^2_{N,j+\sigma}}{d^2_{N,j}}.$$

But also, we know the recurrence relations from the other model are

$$\frac{1}{2\omega}M_i\psi_{N,j} = m^{(i)}_{0,+1}(N,j)\psi_{N,j+1} + m^{(i)}_{0,-1}(N,j)\psi_{N,j-1} + m^{(i)}_{0,0}\psi_{N,j}$$

which implies

$$\frac{m^{(i)}_{0,j+\sigma,j}(N,j)}{\mu^{(i)}_{0,\sigma}(N,j)} = \frac{d_{N,j+\sigma}}{d_{N,j}}.$$

From these two equations, we obtain

$$\mu^{(i)}_{0,\sigma}(N,j)\mu^{(i)}_{0,-\sigma}(N,j+\sigma) = \left(m^{(i)}_{0,\sigma}(N,j)\right)^2 = m^{(i)}_{0,\sigma}(N,j)m^{(i)}_{0,-\sigma}(N,j+\sigma).$$

Additionally, we assume the operators L_1 and L_2 are of the form,

$$L_i = \sum_{\sigma,\sigma'\in\{-1,0,1\}} t^{\sigma'}T^{-\sigma}\ell^{(i)}_{\sigma',\sigma}(t\partial_t, y), \qquad i = 1,2.$$

Then their actions on the unnormalized basis vectors are,

$$L_i t^N\delta(y-\lambda_j) = \sum_{\sigma,\sigma'\in\{-1,0,1\}} \ell^{(i)}_{\sigma',\sigma}(N,j)t^{N+\sigma'}\delta(y-\lambda_j-\sigma), \qquad i = 1,2$$

and we have similar factorization equations,

$$\ell^{(i)}_{\sigma',\sigma}(N,j)\ell^{(i)}_{-\sigma',-\sigma}(N+\sigma',j+\sigma) = \left(l^{(i)}_{N+\sigma',N;j+\sigma,j}\right)^2,$$

and recursion relations on the normalization constants,

$$\frac{\ell^{(i)}_{\sigma',\sigma}(N,j)}{\ell^{(i)}_{-\sigma',-\sigma}(N+\sigma',j+\sigma)} = \frac{d^2_{N+\sigma',j+\sigma}}{d^2_{N,j}}.$$

From these factorization equations, we determine the form of the operators by choosing appropriate factorizations which can then be made into operators. Most importantly, the factors must be polynomial in N so that we can make the substitution $N \to t\partial_t$.

We also use the recursion relations on the normalization constants to ensure that the model corresponds to our choice of representation. That is, for the bounded below representation we want the normalization constants to cut off at $j = 0$ and $N - j = 0$. For the finite dimensional representations, we want the normalization constants to cut of at $M - N = 0$. Finally, we also need the recurrence relations on the normalization constants to be consistent. That is, we need

$$\begin{aligned}\frac{\mu_{\sigma}^{(i)}(N,j)}{\mu_{-\sigma}^{(i)}(N,j+\sigma)} &= \frac{d^2_{N,j+\sigma}}{d^2_{N,j}} \\ &= \frac{d^2_{N,j+\sigma}}{d^2_{N+\sigma',j+\sigma}} \times \frac{d^2_{N+\sigma',j+\sigma}}{d^2_{N,j}}, \\ &= \frac{\ell^{(i)}_{-\sigma',0}(N+\sigma',j+\sigma)}{\ell^{(i)}_{\sigma',0}(N,j+\sigma)} \times \frac{\ell^{(i)}_{\sigma',\sigma}(N,j)}{\ell^{(i)}_{-\sigma',-\sigma}(N+\sigma',j+\sigma)}.\end{aligned}$$

Luckily, the normalization is similar to the one for the Cartesian case and can be intuited from the finite dimensionality restrictions. We choose recursions for the normalization constants as,

$$d^2_{N,j} = \frac{(N-j+1)(k_1+k_2+2j-1)(k_1+k+2+2j)^2(k_1+k_2+2j+1)}{(k_1+k_2+N+j+1)j(j+k_1)(j+k_2)(j+k_1+k+2)} d^2_{N,j-1}$$

$$d^2_{N,j} = \frac{(M-N+1)(M-N+1+k_3)}{(N-j)(k_1+k_2+N+j+1)} d^2_{N-1,j}.$$

We can use these to determine the factorizations. Finally, since we want to be able to find the eigenfunctions of these operators in terms of orthogonal polynomials, we choose a factorization which is balanced; that is, one that has the same degree in y on the top and the bottom of the rational function.

We make the following choices of factorization,

$$
\begin{aligned}
m^{(i)}_{0,+1}(N,j) =& (-1)^i \frac{(k_1+k_2+j)(k_1+j)(N-j+1)}{(k_1+k_2+2j+2)(k_1+k_2+2j+3)},\\
m^{(i)}_{0,-1}(N,j) =& (-1)^i \frac{j(k_2+j)(k_1+k_2+N+j+1)}{(k_1+k_2+2j)(k_1+k_2+2j-1)},\\
m^{(i)}_{0,0}(N,j) =& \frac{(k_i^2+k_1+k_2+k_1k_2+2j(k_1+k_2+1+j))(k_2+k_1+2+2N)}{(k_1+k_2+2j+2)(k_1+k_2+2j)},
\end{aligned}
$$

$$
\begin{aligned}
l^{(i)}_{1,1}(N,j) =& \frac{(-1)^i(M-N)(M-N-k_3)(k_1+j)(k_1+k_2+j)}{(k_1+k_2+2j)(k_1+k_2+2j+1)},\\
l^{(i)}_{1,-1}(N,j) =& \frac{(-1)^i j(k_2+j)(M-N)(M-N-k_3)}{(k_1+k_2+2j)(k_1+k_2+2j-1)},\\
l^{(i)}_{1,0}(N,j) =& \frac{(M-N)(M-N-k_3)}{4(k_1+k_2+2j+1)(k_1+k_2+2j)},\\
& \times (k_i^2+k_1+k_2+k_1k_2+2j(k_1+k_2+1+j)),
\end{aligned}
$$

$$
\begin{aligned}
l^{(i)}_{-1,1}(N,j) &= \frac{(-1)^i(N-j+1)(N-j)(k_1+j)(k_1+k_2+j)}{(k_1+k_2+2j+2)(k_1+k_2+2j+3)},\\
l^{(i)}_{-1,-1}(N,j) &= \frac{(-1)^i j(k_2+j)(k_1+k_2+N+j+1)(k_1+k_2+N+j)}{(k_1+k_2+2j)(k_1+k_2+2j-1)},\\
l^{(i)}_{-1,0}(N,j) &= \frac{(N-j)(k_1+k_2+N+j+1)}{4(k_1+k_2+2j+1)(k_1+k_2+2j)},\\
& \times (k_i^2+k_1+k_2+k_1k_2+2j(k_1+k_2+1+j)),
\end{aligned}
$$

$$
\begin{aligned}
l^{(i)}_{0,1}(N,j) &= \frac{(-1)^i(2(M-N)-k_3+1)(k_1+k_2+j)(k_1+j)(N-j+1)}{(k_1+k_2+2j+2)(k_1+k_2+2j+3)},\\
l^{(i)}_{0,-1}(N,j) &= \frac{(-1)^i(2(M-N)-k_3+1)j(k_2+j)(k_1+k_2+N+j+1)}{(k_1+k_2+2j)(k_1+k_2+2j-1)},\\
l^{(i)}_{0,0}(N,j) &= \frac{(2(M-N)-k_3+1)(k_2+k_1+2+2N)}{4(k_1+k_2+2j+1)(k_1+k_2+2j)},\\
& \times (k_i^2+k_1+k_2+k_1k_2+2j(k_1+k_2+1+j)).
\end{aligned}
$$

We can then make operators out of these factors under the identifications

$$
N \to t\partial_t, \qquad j \to \frac{2y-1-k_1-k_2}{2}.
$$

The functional version of the coefficients is then,

$$m^{(i)}_{0,+1}(t\partial_t, y) = (-1)^i \frac{(2y+1+k_1+k_2)(2y+1+k_1-k_2)(2t\partial_t-2y+1+k_1+k_2)}{4(2y+1)(y+1)},$$

$$m^{(i)}_{0,-1}(t\partial_t, y) = (-1)^i \frac{(2y-1-k_1-k_2)(2y-1+k_2-k_1)(2t\partial_t+2y+1+k_1+k_2)}{4(2y-1)(y-1)},$$

$$m^{(i)}_{0,0}(t\partial_t, y) = \frac{(-1)^i(k_1+k_2)(k_1-k_2)+4y^2-1)(k_1+k_2+2t\partial_t+2)}{4(2y+1)(2y-1)},$$

$$l^{(i)}_{1,1}(t\partial_t, y) = \frac{(-1)^i(2y-1+k_1+k_2)(2y-1-k_1+k_2)(t\partial_t-M)(t\partial_t-M+k_3)}{2^3(2y+1)(y+1)},$$

$$l^{(i)}_{1,-1}(t\partial_t, y) = \frac{(-1)^i(2y+1-k_1-k_2)(2y+1+k_2-k_1)(t\partial_t-M)(t\partial_t-M+k_3)}{2^3(2y-1)(y-1)},$$

$$l^{(i)}_{1,0}(t\partial_t, y) = \frac{(-1)^i(k_1+k_2)(k_2-k_1)+4y^2-1)(M-t\partial_t)(M-t\partial_t-k_3)}{2^3(2y+1)(2y-1)},$$

$$\begin{aligned} l^{(i)}_{-1,1}(t\partial_t, y) &= (-1)^i(2t\partial_t-2y+1+k_1+k_2)(2t\partial_t-2y-1+k_1+k_2) \\ &\times \frac{(2y+1+k_1+k_2)(2y+1-k_1+k_2)}{2^3(2y+1)(y+1)}, \\ l^{(i)}_{-1,-1}(t\partial_t, y) &= (-1)^i(2t\partial_t+2y+1+k_1+k_2)(2t\partial_t+2y-1+k_1+k_2) \\ &\times \frac{(2y-1-k_1-k_2)(2y-1+k_2-k_1)}{2^3(2y-1)(y-1)}, \\ l^{(i)}_{-1,0}(t\partial_t, y) &= (2t\partial_t-2y+k_1+k_2+1)(2t\partial_t+2y+k_1+k_2+1) \\ &\times \frac{((-1)^i(k_1+k_2)(k_2-k_1)+4y^2-1)}{2^3(2y+1)(2y-1)}, \end{aligned}$$

$$\begin{aligned} l^{(i)}_{0,1}(t\partial_t, y) &= (-1)^i(2(M-t\partial_t)-k_3+1) \\ &\times \frac{(2y+1+k_1+k_2)(2y+1+k_1-k_2)(2t\partial_t-2y+1+k_1+k_2)}{4(2y+1)(y+1)}, \\ l^{(i)}_{0,-1}(t\partial_t, y) &= (-1)^i(2(M-t\partial_t)-k_3+1) \\ &\times \frac{(2y-1-k_1-k_2)(2y-1+k_2-k_1)(2t\partial_t+2y+1+k_1+k_2)}{4(2y-1)(y-1)}, \\ l^{(i)}_{0,0}(t\partial_t, y) &= \frac{1}{2} + ((-1)^i(k_1+k_2)(k_2-k_1)+4y^2-1) \\ &\times \frac{(2(M-t\partial_t)-k_3+1)(2t\partial_t+k_1+k_2+2)}{4(2y+1)(2y-1)}. \end{aligned}$$

We first consider M_1. In this model it takes the form,

$$M_1 = \frac{\omega(2y+1-k_1-k_2)(2y+1+k_2-k_1)(2t\partial_t+2y+3+k_1+k_2)}{2^3(2y+1)y}T^1$$
$$+\frac{\omega(2y-1+k_1+k_2)(2y-1-k_2+k_1)(2t\partial_t-2y+3+k_1+k_2)}{2^3(2y-1)y}T^{-1}$$
$$+\omega\frac{((k_1+k_2)(k_1-k_2)+4y^2-1)(k_1+k_2+2t\partial_t+2)}{2(2y+1)(2y-1)}.$$

This yields the eigenfunction equation for the dual Hahn polynomials. For a full exposition of the operators associated with the dual Hahn polynomials see appendix E. The polynomials are defined by,

$$h_n(y^2) = (\alpha+\beta)_n(\alpha+\gamma)_{n3}F_2\left(\begin{matrix} -n, & \alpha-y, & \alpha+y \\ \alpha+\beta, & \alpha+\gamma & \end{matrix};1\right). \tag{6.11}$$

We can write the operators of this model in terms of the operators of the dual Hahn polynomials. To summarize, if we set

$$\alpha = -N - \frac{k_1+k_2+1}{2} \qquad \beta = \frac{k_1+k_2+1}{2} \qquad \gamma = \frac{k_1-k_2+1}{2},$$

we have the model.

Model 2. Cylindrical Model

$$M_1 = 2\omega\left(2\mathbf{L}R - 2t\partial_t + \beta - \gamma\right), \quad M_3 = 2\omega(2t\partial_t - 2M - k_3 + 1), \tag{6.12}$$

$$M_1 + M_2 + M_3 = E, \quad L_3 = -4y^2 + k_1^2 + k_2^2, \tag{6.13}$$
$$L_1 = \frac{1}{2}\left(\frac{1}{t}R^\alpha R + 8t(M-t\partial_t)(M-t\partial_t-k_3)\mathbf{L}_\alpha\mathbf{L} + \frac{1}{2\omega^2}M_2M_3 + \frac{1}{2}\right),$$

$$\begin{aligned} L_1+L_2+L_3 &= t(M-t\partial_t)(M-t\partial_t-k_3) \\ &\quad + \frac{1}{4t}(2t\partial_t-2y-k_1-k_2-1)(2t\partial_t+2y+k_1+k_2+1) \\ &\quad + 2(2t\partial_t+k_1+k_2+2)(2M-2t\partial_t-k_3+1)+1. \end{aligned}$$

In these equations, the operators in bold typeface correspond to difference operators associated with the dual Hahn polynomial algebra where a parameter has been replaced with a differential operator; that is $N \to t\partial_t$.

Notice, that the eigenvalues of M_1 are linear in n and also that the action of L_3 gives a three term recurrence in n where the center term is quadratic in n. Hence, by theorem 15 there must be raising and lowering operators which are independent of the index n.

The raising and lowering operators are, respectively,

$$A^\dagger = -4\omega L_3 + [M_1, L_3] + 2(M_1 + 2t\partial_t - \beta + \gamma)^2 \tag{6.14}$$
$$- 2(2\alpha + 2\beta + 2\gamma - 1)(M_1 + 2t\partial_t - \beta + \gamma) + 2\beta^2 + 2\gamma^2 + 4(\alpha\gamma + \alpha\beta + \beta\gamma) - 2\beta - 2\gamma + 1.$$

$$A = -4\omega L_3 - [M_1, L_3] + 2(M_1 + 2t\partial_t - \beta + \gamma)^2 \tag{6.15}$$
$$- 2(2\alpha + 2\beta + 2\gamma - 1)(M_1 + 2t\partial_t - \beta + \gamma) + 2\beta^2 + 2\gamma^2 + 4(\alpha\gamma + \alpha\beta + \beta\gamma) - 2\beta - 2\gamma + 1,$$

These operators are nontrivial and of 2nd order as difference operators in y and differential operators in t.

So far, we have found a suitable model based on separation of variables in cartesian coordinates (M_1, M_2) and parabolic coordinates (M_3, L_3). The final basis to be expanded is associated with horospherical coordinates whose symmetry operators are L_3 and $L_1 + L_2 + L_3$. Since the eigenfunctions of L_3 already form a basis for the parabolic model, we can attempt to find eigenfunctions for $L_1 + L_2 + L_3$ in this model.

We find that the eigenfunctions of $L_1 + L_2 + L_3$ in this model are

$$\phi_{N,j} = k_{N,j}(t-1)^{M-N} t^j {}_2F_1 \left(\begin{matrix} -N+j, & -N+j+k_3 \\ 2j+2+k_1+k_2 & \end{matrix} \middle| t \right) \delta(y - \lambda_j)$$

with eigenvalues $(2N+k_1+k_2-k_3+2)^2+k_1^2-k_2^2-k_3+\frac{3}{2}$. Here the $k_{N,j}$ is a normalization constant.

We can make a change of variable $t = \frac{x+1}{x-1}$ to obtain Jacobi polynomials multiplied by a gauge factor. They are,

$$\phi_{N,j} = \tilde{k}_{N,j}(-1)^{M-N} \left(\frac{x+1}{2}\right)^{-M} \left(\frac{x-1}{2}\right)^{j} P_{N-j}^{2j+A+1,-k_3}(x)\delta(y - \lambda_j).$$

We can then compute the action of the operators on these basis functions. Since we already know the action of $L_1+L_2+L_3$ and L_3 on the basis, it suffices to compute the action of M_2, M_3 and L_1 on the eigenfunctions of $L_1+L_2+L_3$. To do this, we must use the recurrence relations of the Jacobi polynomials [45]. That is, to change the value of j we must change not only the gauge factor out front but also the order of the polynomial and the parameter as well as the delta function. On the other hand, to change the value of N we need only change the order of the polynomial.

The action of our operators on the basis can be computed as,

$$\frac{1}{2\omega}M_i\phi_{N,j} = \sum_{\sigma,\sigma'\in\{-1,0,1\}} \pi^{(i)}_{\sigma,\sigma'}(N,j)\phi_{N+\sigma,j+\sigma'},$$

where the $\pi^{(i)}$'s are as follows.

$$\begin{aligned}
\pi^{(3)}_{0,0}(N,j) &= -(2M+k_1+k_2+3-k_3)\\
&\times \frac{(-3-2j(j+k_1+k_2+1)-2k_3+A+2(N+1)^2-(k_1+k_2-k_3)(2N+k_3))}{(2N+k_1+k_2+3-k_3)(2N+k_1+k_2+1-k_3)},\\
\pi^{(3)}_{1,0}(N,j) &= \pi^{(3)}_{-1,0}(N+1,j)\\
&= 4\frac{\sqrt{(M+N+3+k_1+k_2-k_3)(M-N)(N-j+k_3+1)}}{(2N+k_1+k_2+3-k_3)}\\
&\times \sqrt{\frac{(N-j+1)(N+j+k_1+k_2+2)(N+j+k_1+k_2+2-k_3)}{(2N+k_1+k_2+4-k_3)(2N+k_1+k_2+2-k_3)}},
\end{aligned}$$

$$\begin{aligned}
\pi^{(1)}_{1,1}(N,j) &= \pi^{(1)}_{-1,-1}(N+1,j+1)\\
&= \sqrt{\frac{(N+j+k_1+k_2-k_3+2)(N+j+k_1+k_2-k_3+3)}{(2N+k_1+k_2-k_3+4)(2N+k_1+k_2-k_3+2)}}\\
&\times \sqrt{\frac{(j+1)(j+k_1+1)(j+k_2+1)(j+k_1+k_2+1)}{(2j+k_1+k_2+3)(2j+k_1+k_2+1)}}\\
&\times \frac{\sqrt{(N+j+2)(N+j+3)(M-N)(M+N+k_1+k_2-k_3+3)}}{(2N+k_1+k_2-k_3+3)(2j+k_1+k_2+2)},
\end{aligned}$$

$$\begin{aligned}
\pi^{(1)}_{1,-1}(N,j+1) &= \pi^{(1)}_{-1,1}(N+1,j)\\
&= \frac{\sqrt{(M-N)(M+N+k_1+k_2-k_3+3)}}{(2N+k_1+k_2-k_3+3)(2j+k_1+k_2+2)}\\
&\times \sqrt{\frac{(N-j+1)(N-j)(N-j-k_3+1)(N-j-k_3)}{(2N+k_1+k_2-k_3+2)(2N+k_1+k_2-k_3+4)}}\\
&\times \sqrt{\frac{(j+k_1+1)(j+1)(j+k_2+1)(j+k_1+k_2+1)}{(2j+k_1+k_2+3)(2j+k_1+k_2+1)}},
\end{aligned}$$

$$\begin{aligned}
\pi^{(1)}_{0,1}(N,j) &= \pi^{(1)}_{0,-1}(N,j+1)\\
&= (2M+k_1+k_2-k_3+3)\sqrt{(N-j)(N-j-k_3)}\\
&\times \frac{\sqrt{(N+j+k_1+k_2+2)(N+j+k_1+k_2-k_3+2)}}{(2N+k_1+k_2-k_3+3)(2N+k_1+k_2-k_3+1)}\\
&\times \sqrt{\frac{(j+1)(j+k_1+1)(j+k_2+1)(j+k_1+k_2+1)}{(2j+k_1+k_2+1)(2j+k_1+k_2+2)^2(2j+k_1+k_2+3)}},
\end{aligned}$$

$$\begin{aligned}
\pi^{(1)}_{1,0}(N,j) &= \pi^{(1)}_{-1,0}(N+1,j)\\
&= \frac{(4j^2+4j+4jA+2A+A^2-k_1^2+k_2)}{(2N+k_1+k_2-k_3+3)(2j+k_1+k_2)(2j+k_1+k_2+2)}\\
&\times \sqrt{\frac{(N-j+1)(N-j-k_3+1)(M+N+k_1+k_2-k_3+3)}{(2N+k_1+k_2-k_3+2)(2N+k_1+k_2-k_3+4)}}\\
&\times \sqrt{(N+j+k_1+k_2+2-k_3)(N+j+k_1+k_2+2)(M-N)},
\end{aligned}$$

$$\begin{aligned}
\pi^{(1)}_{0,0}(N,j) = &\frac{(2N^2+(4-2k_3+2A)N+2j^2+(2+2A)j-2k_3+1+3A-k_3A)}{(2N+k_1+k_2-k_3+3)(2N+k_1+k_2-k_3+1)}\\
&\times \frac{(2M+k_1+k_2-k_3+3)(4j^2+4j+4jA+2A+A^2-k_1^2+k_2)}{(2j+k_1+k_2+2)(2j+k_1+k_2)}.
\end{aligned}$$

The last operator to compute is L_1, which we expand as,

$$L_1\phi_{N,j} = \sum_{\sigma,\sigma'\in\{-1,0,1\}} \zeta^{(1)}_{\sigma,\sigma'}(N,j)\phi_{N+\sigma,j+\sigma'}$$

with the coefficients as follows,

$$\begin{aligned}
\zeta^{(1)}_{0,1}(N,j) &= \pi^{(1)}_{0,-1}(N,j+1)\\
&= \sqrt{(N-j)(N-j-k_3)(N+j+k_1+k_2+2)(N+j+k_1+k_2-k_3+2)}\\
&\times\sqrt{\frac{(j+1)(j+k_1+1)(j+k_2+1)(j+k_1+k_2+1)}{(2j+k_1+k_2+1)(2j+k_1+k_2+2)^2(2j+k_1+k_2+3)}},
\end{aligned}$$

$$\begin{aligned}
\zeta^{(1)}_{0,0}(N,j) &= (2N^2+(4-2k_3+2A)N+2j^2+(2+2A)j-2k_3+1+3A-k_3A)\\
&\times\frac{(4j^2+4j+4jA+2A+A^2-k_1^2+k_2)}{(2j+k_1+k_2+2)(2j+k_1+k_2)}.
\end{aligned}$$

We are now ready to create a new model suited to the spectrum of L_3 and $L_1+L_2+L_3$.

6.4 Spherical Basis: A quantum model with $L_1, L_1+L_2+L_3$ diagonal

To calculate this model we will try to fit difference operators in two variables to the expansion coefficients we have obtained above. Furthermore, we know the general form of some of our operators from the difference model for S9 which was constructed using the recurrence relations for the Wilson polynomials, see appendix D. With these clues, we expect that the weights of the variables will be indexed by the integers N and j and we choose these in a way so that the denominators of our coefficients will match those of the Wilson polynomial operators. Most importantly, we look at the coefficients which are completely determined, that is those with $\sigma=\sigma'=0$. We want the variables in the denominators of these to be of the form $(2y+1)(2y-1)$, and so we make the following identifications,

$$\lambda_j = j+\frac{k_1+k_2+1}{2}\qquad \lambda_N = N+1+\frac{k_1+k_2-k_3}{2},$$

and the basis functions become

$$\theta_{N,j} = C_{N,j}\delta(y-\lambda_j)\delta(z-\lambda_N).$$

Since we know the eigenvalues of the operators L_3 and $L_1+L_2+L_3$, we can immediately determine the support of our measure, the λ_j, λ_N.

We also want to make a choice of factorization that matches the Wilson polynomial operators. That is, we want the rational function coefficients of the pure difference operators to match each other under the transformation $y \to -y$. We choose expansion coefficients for the operators as follows, for $i = 1, 3$,

$$M_i = \sum_{\sigma,\sigma' \in \{-1,0,1\}} Z^{-\sigma} T^{-\sigma'} \pi^{(i)}_{\sigma,\sigma'}$$

and

$$L_1 = \sum_{\sigma,\sigma' \in \{-1,0,1\}} Z^{-\sigma} T^{-\sigma'} \zeta^{(1)}_{\sigma,\sigma'}.$$

We make a change of variables as,

$$\alpha = z + \frac{k_3+1}{2} \qquad \delta = -z + \frac{k_3+1}{2} \qquad \beta = \frac{k_1 - k_2 + 1}{2} \qquad \gamma = \frac{1 - k_1 - k_2}{2},$$

so that the relation with the Wilson polynomial algebra is more explicit. We rewrite the coefficients of M_3 and the other operators as,

$$\begin{aligned}
\pi^{(3)}_{-1,0}(z,y) &= 2\frac{(\alpha - y)(\alpha + y)(1 - \delta - \gamma + M)}{(\alpha - \delta - 2)(\alpha - \delta - 3)},\\
\pi^{(3)}_{1,0}(z,y) &= 2\frac{(\delta + y)(\delta - y)(1 - \alpha - \gamma + M)}{(\alpha - \delta + 2)(\alpha - \delta + 3)},\\
\pi^{(3)}_{0,0}(z,y) &= \frac{1}{(\alpha - \delta + 1)(\alpha - \delta - 1)}\Big(\alpha^3 + \delta^2 - 2(\alpha^2 + \delta^2)(M - \gamma + 2)\\
&\qquad + 3(\alpha^2\delta + \alpha\delta^2 - 2\alpha\delta) + (\alpha + \delta)(2M + 2\gamma + 7) - 2\Big),
\end{aligned}$$

$$\begin{aligned}
\pi^{(1)}_{-1,-1}(z,y) &= -\frac{(2 - \delta + M - \gamma)(\alpha + y)(\alpha - 1 + y)(y + \beta)(y - \gamma)}{(\alpha - \delta)(\alpha - \delta - 1)(y - 1)(2y - 1)},\\
\pi^{(1)}_{-1,1}(z,y) &= -\frac{(2 - \delta + M - \gamma)(\alpha - y)(\alpha - 1 - y)(-y + \beta)(-y - \gamma)}{(\alpha - \delta)(\alpha - \delta - 1)(y + 1)(2y + 1)},\\
\pi^{(1)}_{-1,0}(z,y) &= -2\frac{(2 - \delta + M - \gamma)(\alpha - 1 - y)(\alpha - 1 + y)(2\gamma\beta - \gamma - \beta - 2y^2)}{(\alpha - \delta)(\alpha - \delta - 1)(4y^2 - 1)},
\end{aligned}$$

$$\begin{aligned}
\pi^{(1)}_{1,1}(z,y) &= -\frac{(M + 2 - \gamma - \alpha)(-y + \delta)(-y + \delta - 1)(-y + \beta)(-y + \gamma)}{(\alpha - \delta + 1)(\alpha - \delta)(y + 1)(2y + 1)},\\
\pi^{(1)}_{1,-1}(z,y) &= -\frac{(M + 2 - \gamma - \alpha)(y + \delta)(y + \delta - 1)(y + \beta)(y + \gamma)}{(\alpha - \delta + 1)(\alpha - \delta)(y - 1)(2y - 1)},\\
\pi^{(1)}_{1,0}(z,y) &= -\frac{2(M + 2 - \gamma - \alpha)(\delta - 1 + y)(\delta - 1 - y)(2\gamma\beta - \gamma - \beta + 1 - 2y^2)}{(\alpha - \delta)(\alpha - \delta + 1)(4y^2 - 1)},
\end{aligned}$$

$$\pi^{(1)}_{0,1}(z,y)=\frac{(\alpha-y)(\beta-y)(\gamma-y)(\delta-y)(2M-2\gamma-\alpha-\delta+5)}{(y+1)(2y+1)(\alpha-\delta+1)(\alpha-\delta-1)},$$
$$\pi^{(1)}_{0,-1}(z,y)=\frac{(\alpha+y)(\beta+y)(\gamma+y)(\delta+y)(2M-2\gamma-\alpha-\delta+5)}{(y-1)(2y-1)(\alpha-\delta+1)(\alpha-\delta-1)},$$
$$\pi^{(1)}_{0,0,}(z,y)=\frac{(2M-2\gamma-\alpha-\delta+5)}{(\alpha-\delta+1)(\alpha-\delta-1)} \times\frac{(2\alpha\delta-\alpha-\delta+1-2y^2)(2\gamma\beta-\gamma-\beta+1-2y^2)}{4y^2-1}.$$

$$\zeta^{(1)}_{0,1}(z,y)=-\frac{(\alpha-y)(\beta-y)(\gamma-y)(\delta-y)}{(y+1)(2y+1)},$$
$$\zeta^{(1)}_{0,-1}(z,y)=-\frac{(\alpha+y)(\beta+y)(\gamma+y)(\delta+y)}{(y-1)(2y-1)},$$
$$\zeta^{(1)}_{0,0,}(z,y)=\frac{(2\alpha\delta-\alpha-\delta+1-2y^2)(2\gamma\beta-\gamma-\beta+1-2y^2)}{4y^2-1}.$$

In terms of the operators for the Wilson polynomials, we have the following definition of the operators.

Model 3. Spherical Model

$$L_3=-4y^2+2(\gamma^2-\gamma+\beta^2-\beta)+1, \tag{6.16}$$
$$L_1+L_2+L_3=-8y^2-4(\gamma\beta+\alpha\gamma)+2(\alpha+\beta+\gamma+\delta)-\frac{1}{2},$$
$$L_1=\frac{2(\alpha+\delta-5-2M+2\gamma)}{(\alpha-\delta-1)(\alpha-\delta+1)}\left(2LR+(\alpha+\delta)(\beta+\gamma)\right),$$

$$\frac{M_1}{2\omega}=\frac{2(\alpha+\delta-5-2M+2\gamma)}{(\alpha-\delta-1)(\alpha-\delta+1)}\left(2LR+(\alpha+\delta)(\beta+\gamma)\right) +2Z\frac{2-\delta+M-\gamma}{(\alpha-\delta)(\alpha-\delta-1)}L_{\alpha\beta}L_{\alpha\beta}+2Z^{-1}\frac{2-\delta-\gamma+M}{(\alpha-\delta)(\alpha-\delta+1)}R_{\alpha\beta}R_{\alpha\gamma},$$

$$\frac{M_3}{2\omega}=-\frac{(\alpha^2+\delta^2-\alpha-\delta-2y^2)(2M+5-2\gamma-\alpha-\delta)}{(\alpha-\delta-1)(\alpha-\delta+1)}-2(\delta+\alpha-1) +2\frac{(\alpha-1-y)(\alpha-1+y)(2-\delta-\gamma+M)}{(\alpha-\delta)(\alpha-\delta-1)}Z +2\frac{(\delta-1+y)(\delta-1-y)(2-\alpha-\gamma+M)}{(\alpha-\delta)(\alpha-\delta+1)}Z^{-1},$$

$$M_1 + M_2 + M_3 = E.$$

where Z is a difference operator in the variable z that acts on functions as, $Z^k f(z) = f(z+k)$.

Notice, in this model the eigenfunctions of L_1 and $L_1 + L_2 + L_3$ are Wilson polynomials in y multiplied by a delta function in z. However, this will not give us any new information since the model which diagonalizes L_1 and $L_1 + L_2 + L_3$ will be isomorphic, up to permutation of constants, with the spherical model we already have because of the symmetry of the physical system.

On the other hand, the eigenfunctions of M_3 are dual Hahn polynomials in the variable z as can be seen if we return to the original variables where M_3 is,

$$\begin{aligned}\frac{M_3}{2\omega} = &-\frac{(2z+1+k_3+2y)(2z+1+k_3-2y)(2z-1-k_3-k_1-k_2-2M)}{8z(2z+1)}Z\\ &-\frac{(-2z+1+k_3+2y)(-2z+1+k_3-2y)(-2z-1-k_3-A-2M)}{8z(2z-1)}Z^{-1}\\ &+\frac{(2M+2+k_1+k_2+k_3)(4z^2-4y^2+k_3^2-1)}{2(2z-1)(2z+1)}+2k_3.\end{aligned}$$

These eigenfunction correspond to moving back to the parabolic model.

In this section, I have determined 3 different models, each corresponding to a pair of commuting symmetry operators that defines separation of variables for the original system. I used a bootstrapping technique of moving from one basis to another by using the eigenvalues of a shared operator for the two models to determine the expansion coefficients of the action of the other operators on this basis. By doing this, I was able to find a pure differential, pure difference, and mixed difference-differential operator realization of the quadratic algebra. This method is particularly useful when there exists a differential operator model which can begin the bootstrapping procedure, as it is difficult to a priori determine a difference operator model.

Chapter 7

Conclusions and Future Research

Superintegrable systems are unique systems because of their tractability and because of their relatively simple algebraic structure. In this thesis, I have described the structure theory of second order superintegrable systems and the classification theory in 2d and also in 3d for nondegenerate systems. In both cases, the symmetry algebras close to form a quadratic algebra. This symmetry algebra can be used to find eigenvalues of the physical operators and to find expansion coefficients between bases. Also important in the theory of superintegrable systems is the Stäckel transform which relates two different superintegrable systems with isomorphic symmetry algebras.

Chapters 4, 5 and 6 include new results on models of quadratic algebras for superintegrable systems. I have demonstrated at least one model for all 2d Stäckel equivalence classes and discussed a few of the models in more detail. In fact, all of the models found were different and interesting in their own right. I hope that the exposition gives a glimpse at the different types of models and methods of determining them. In the example in 3d, I gave an extended description of a "bootstrapping" method to determine new models moving from a double differential operator model to a mixed model and finally to a pure double difference operator model. The majority of the models given are associated with special functions, either as basis functions or arrived at while trying to determine expansion coefficients. The special function relations enabled me to simplify my computations in many of the models but also, in the example in 3d, I was able to find new nontrivial raising and lowering operators for the dual Hahn polynomials.

Research has already begun on the generic system in 3d, in analogy with the generic system in 2d given in Section 4.2.6. Like the system in 2d, it is expected that no differential operator model will be possible but instead a difference operator model in two variables. We expect the cylindrical model will have two variable Wilson polynomials as eigenfunctions for some of the operators.

Further in the future, I would like to explore the relation between special function theory and superintegrable systems and using superintegrable systems as a classifying tool for special functions. Related to this is the concept of a superintegrable Laplace system, where the Hamiltonian in replaced by the Laplace equation with a potential. The symmetries of such a system would be conformal symmetries with an analogous definition of superintegrability. It is expected that there would be new symmetry algebras, in fact one has already been found but not modeled. Furthermore, it would enlarge the category of superintegrable systems and their associated special functions.

Additionally, there has been recent work in superintegrable systems with symmetry operators of degree higher than two. While the structure theory for these cases has not been worked out, there are many examples of such systems whose symmetry operators form polynomial algebras. For example, there are some examples of third-order superintegrabilty whose symmetry algebras are cubic and higher order algebras [52, 41, 21, 22]. There has been some research in finding representations for such models using the methods of deformed parafermionic algebras [53, 13, 8, 23] but not using the more general methods described in this thesis. Our methods have the advantage of being able to describe the spectrum of any operator in the symmetry algebra and being able to use the bootstrapping method to move between different models.

Finally, an interesting project of future research would be to relate the representation theory for these quadratic algebras to other types of algebras currently being researched in other areas. For the 3d model given, there were some relations to Hecke algebras [54, 55, 56] but also it would be both interesting and hopefully fruitful to examine the relation between the quadratic algebras and other noncommutative algebras.

In conclusion, the symmetry of superintegrable systems gives them a unique structure. The existence of a quadratic algebra helps simplify the analysis of these systems and leads to interesting models. These models are not only useful but objects of interest in their own right, particularly in their relation to special functions.

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Appendix A

Metrics of manifolds that admit second order superintegrable systems in 2d

We have made a choice of coordinates, $a = x, b = iy$ so that the tables match those given by Koenigs [39].

All of the metric must satisfy the fundamental equations,

$$\mu_{12} = 0, \qquad \mu_{22} - \mu_{11} = 3\mu_1(\ln a^{12})_1 - 3\mu_2(\ln a^{12})_2 + \left(\frac{a^{12}_{11} - a^{12}_{22}}{a^{12}}\right)\mu, \tag{A.1}$$

where a^{12} is the off-diagonal element in the basis of symmetry operators.

Table A.1: Darboux spaces and associated metrics

$$\begin{aligned}
[D1]\ ds^2 &= 4a(da^2 - db^2)\\
[D2]\ ds^2 &= \frac{(a^2+1)(da^2 - db^2)}{a^2}\\
[D3]\ ds^2 &= (4 + a^2 - b^2)(da^2 - db^2)\\
[D4]\ ds^2 &= \frac{a^2+b^2}{a^2b^2}(da^2 - db^2)
\end{aligned}$$

Table A.2: TABLEAU VI: Metrics of Koenigs spaces given by (I)

$$\begin{aligned}
[1]\ ds^2 &= \left[\frac{c_1\cos a + c_2}{\sin^2 a} + \frac{c_3\cos b + c_4}{\sin^2 b}\right](da^2 - db^2)\\
[2]\ ds^2 &= \left[\frac{c_1\cosh a + c_2}{\sinh^2 a} + \frac{c_3 e^b + c_4}{e^{2b}}\right](da^2 - db^2)\\
[3]\ ds^2 &= \left[\frac{c_1 e^a + c_2}{e^{2a}} + \frac{c_3 e^b + c_4}{e^{2b}}\right](da^2 - db^2)\\
[4]\ ds^2 &= \left[c_1(a^2 - b^2) + \frac{c_2}{a^2} + \frac{c_3}{b^2} + c_4\right](da^2 - db^2)\\
[5]\ ds^2 &= \left[c_1(a^2 - b^2) + \frac{c_2}{a^2} + c_3 b + c_4\right](da^2 - db^2)\\
[6]\ ds^2 &= \left[c_1(a^2 - b^2) + c_2 a + c_3 b + c_4\right](da^2 - db^2)
\end{aligned}$$

Table A.3: TABLEAU VII: Metrics of Koenigs spaces given by (II)

$$\begin{aligned}
[1]\ ds^2 &= \left[c_1\left(\frac{1}{\mathrm{sn}^2(a,k)} - \frac{1}{\mathrm{sn}^2(b,k)}\right) + c_2\left(\frac{1}{\mathrm{cn}^2(a,k)} - \frac{1}{\mathrm{cn}^2(b,k)}\right)\right.\\
&+ \left. c_3\left(\frac{1}{\mathrm{dn}^2(a,k)} - \frac{1}{\mathrm{dn}^2(b,k)}\right) + c_4\left(\mathrm{sn}^2(a,k) - \mathrm{sn}^2(b,k)\right)\right](da^2 - db^2)\\
[2]\ ds^2 &= \left[c_1\left(\frac{1}{\sin^2 a} - \frac{1}{\sin^2 b}\right) + c_2\left(\frac{1}{\cos^2 a} - \frac{1}{\cos^2 b}\right) + c_3\left(\cos 2a - \cos 2b\right)\right.\\
&+ \left. c_4\left(\cos 4a - \cos 4b\right)\right](da^2 - db^2)\\
[3]\ ds^2 &= [c_1(\sin 4a - \sin 4b) + c_2(\cos 4a - \cos 4b) + c_3(\sin 2a - \sin 2b)\\
&+ c_4(\cos 2a - \cos 2b)]\,(da^2 - db^2)\\
[4]\ ds^2 &= \left[c_1\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + c_2(a^2 - b^2) + c_3(a^4 - b^4) + c_4(a^6 - b^6)\right](da^2 - db^2)\\
[5]\ ds^2 &= \left[c_1(a - b) + c_2(a^2 - b^2) + c_3(a^3 - b^3) + c_4(a^4 - b^4)\right](da^2 - db^2)
\end{aligned}$$

Appendix B

Generators for orthogonal coordinate systems

In the tables below, I give the separable coordinates for the free Hamiltonian-Jacobi equation and the free Laplace-Beltrami eigenvalue equation on 2d flat space and the 2d sphere.

Table B.1: Separable Coordinates on $E_2(\mathbb{C})$

Name	Coordinate System	Associated Symmetry Operator
Cartesian	x,y	p_x^2
Spherical	$x = r\cos\theta, y = r\sin\theta$	M^2
Parabolic	$x_p = \frac{1}{2}(\zeta^2 - \xi^2)$ $y_p = \zeta\xi$	$\{M, p_x\}$
Lightlike	$x = x, t = iy$	$p_\pm^2$
Elliptic	$x_E = c(u-1)(v-1)$ $y_E = c^2uv$	$M^2 + cp_x^2$
Hyperbolic	$x_H = (r^2 + r^2s^2 + s^2)/2rs$ $y_H = (r^2 - r^2s^2 + s^2)/2rs$	$M^2 + cp_\pm^2$
Semi-Hyperbolic	$x_{SH} = -c^2(w-u)^2 + c(w+u)$ $iy_{SH} = -c^2(w-u)^2 - c(w+u)$	$\{M, p_+\} + cp_-^2$

Table B.2: Separable Coordinates on $S_2(\mathbb{C})$

Name	Coordinate System	Associated Symmetry Operator
Spherical	$x = \sin\theta\cos\phi$ $y = \sin\theta\sin\phi$ $z = \cos\phi$	J_3^2
Parabolic	$x^2 = (ru-1)(rv-1)/(1-r)$ $y^2 = r(u-1)(v-1)/(1-r)$ $z^2 = ruv$	$J_1^2 + rJ_2^2$
Horospherical	$x = \frac{i}{2}(v^2+u^2-1)/v$ $y = \frac{1}{2}(v^2+u^2+1)/v$ $z = iu/v$	$(J_1 + iJ_2)^2$
Degenerate Elliptic I	$x + iy = 4cuv/(u^2+1)(v^2+1)$ $x - iy = (u^2v^2+1)(u^2+v^2)/cuv(u^2+1)(v^2+1)$ $z = (u^2-1)(v^2-1)/(u^2+1)(v^2+1)$	$(J_1 + iJ_2)^2 - c^2J_3^2$
Degenerate Elliptic II	$x + iy = -iuv$ $x - iy = (u^2+v^2)^2/u^3v^3$ $z = \frac{i}{2}(u^2-v^2)/uv$	$\{J_3, (J_1 - iJ_2)\}$

Appendix C

Superintegrable Systems on 2d spaces of constant curvature

In this section, I reproduce the list of superintegrable potentials given in [57].

C.1 Euclidean Space Potentials

- E1
$$V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}$$
- E2
$$V = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2}$$
- E3
$$V = \alpha(x^2 + y^2)$$
- E4
$$V = \alpha(x + iy)$$
- E5
$$V = \alpha x$$
- E6
$$V = \frac{\alpha}{x^2}$$

- E7 For arbitrary c, with $z = x + iy, \quad \overline{z} = x - iy$,

$$V = \frac{\alpha \overline{z}}{\sqrt{\overline{z}^2 - c^2}} + \frac{\beta z}{\sqrt{\overline{z}^2 - c^2}(\overline{z} + \sqrt{\overline{z}^2 - c^2})^2} + \gamma \overline{z} z$$

- E8

$$V = \frac{\alpha z}{\overline{z}^3} + \frac{\beta}{\overline{z}^2} + \gamma \overline{z} z$$

- E9

$$V = \frac{\alpha}{x - iy} + \beta x + \gamma \frac{2x - iy}{\sqrt{x - iy}}$$

- E10

$$V = \alpha \overline{z} + \beta (z + \frac{3}{2} \overline{z}^2) + \gamma (\overline{z} z - \frac{1}{2} \overline{z}^3$$

- E11

$$V = \alpha z + \beta \frac{z}{\sqrt{\overline{z}}} + \gamma \frac{1}{\sqrt{\overline{z}}}$$

- E12

$$V = \alpha \frac{\overline{z}}{\sqrt{\overline{z}^2 - c^2}}$$

- E13

$$V = \alpha \frac{1}{\sqrt{\overline{z}}}$$

- E14

$$V = \frac{\alpha}{\overline{z}^2}$$

- E15

$$V = h(\overline{z}) \quad \text{for h an arbitrary function}$$

- E16

$$V = \frac{1}{\sqrt{x^2 + y^2}} \left(\alpha + \frac{\beta}{x + \sqrt{x^2 + y^2}} + \frac{\gamma}{x - \sqrt{x^2 + y^2}} \right)$$

- E17

$$V = \frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{\beta}{(x + iy)^2} + \frac{\gamma}{(x + iy)\sqrt{x^2 + y^2}}$$

- E18

$$V = \frac{\alpha}{\sqrt{x^2 + y^2}}$$

- E19

$$V = \frac{\alpha\bar{z}}{\sqrt{\bar{z}^2 - 4}} + \frac{\beta}{\sqrt{z(\bar{z}+2)}} + \frac{\gamma}{\sqrt{z(\bar{z}-2)}}$$

- E20

$$V = \frac{1}{\sqrt{x^2+y^2}}\left(\alpha + \beta\sqrt{x + \sqrt{x^2+y^2}} + \gamma\sqrt{x - \sqrt{x^2+y^2}}\right)$$

C.2 Sphere potentials

We use the coordinates, (x, y, z), such that $x^2 + y^2 + z^2 = 1$.

- S1

$$V = \frac{\alpha}{(x-iy)^2} + \frac{\beta z}{(x-iy)^3} + \gamma\frac{1-4z^2}{(x-iy)^4}$$

- S2

$$V = \frac{\alpha}{z^2} + \frac{\beta}{(x-iy)^2} + \gamma\frac{x+iy}{(x-iy)^2}$$

- S3

$$V = \frac{\alpha}{z^2}$$

- S4

$$V = \frac{\alpha}{(x-iy)^2} + \beta\frac{x+iy}{\sqrt{x^2+y^2}} + \frac{\gamma}{(x-iy)\sqrt{x^2+y^2}}$$

- S5

$$V = \frac{\alpha}{(x-iy)^2}$$

- S6

$$V = \frac{\alpha z}{\sqrt{x^2+y^2}}$$

- S7

$$V = \alpha\frac{x}{\sqrt{y^2+z^2}} + \beta\frac{y}{z^2\sqrt{y^2+z^2}} + \gamma\frac{1}{z^2}$$

- S8

$$V = \alpha\frac{x}{\sqrt{y^2+z^2}} + \beta\frac{x+iy-z}{\sqrt{(x+iy)(z-iy)}} + \gamma\frac{x+iy+z}{\sqrt{(x+iy)(z+iy)}}$$

- S9

$$V = \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2}$$

Appendix D

Recurrence relations for Wilson polynomials

The following treatment of the recurrences for the Wilson polynomials is based on [51] where we modified some of the results of [49]. The unnormalized Wilson polynomials are

$$w_n(y^2) \equiv w_n(y^2, \alpha, \beta, \gamma, \delta) = (\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n \times \tag{D.1}$$

$$_4F_3\left(\begin{array}{llll} -n, & \alpha+\beta+\gamma+\delta-n-1, & \alpha-y, & \alpha+y \\ \alpha+\beta, & \alpha+\gamma, & \alpha+\delta & \end{array};1\right)$$

$$= (\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n Phi_n^{(\alpha,\beta,\gamma,\delta)}(y^2),$$

where $(a)_n$ is the Pochhammer symbol and $_4F_3(1)$ is a generalized hypergeometric function of unit argument.

For fixed $\alpha, \beta, \gamma, \delta > 0$ the Wilson polynomials are orthogonal with respect to the inner product

$$< w_n, w_{n'} >= \frac{1}{2\pi}\int_0^\infty w_n(-y^2)w_{n'}(-y^2)\left|\frac{\Gamma(\alpha+iy)\Gamma(\beta+iy)\Gamma(\gamma+iy)\Gamma(\delta+iy)}{\Gamma(2iy)}\right|^2 dy \tag{D.2}$$

$$= \delta_{nn'} n!(\alpha+\beta+\gamma+\delta+n-1)_n \times$$

$$\frac{\Gamma(\alpha+\beta+n)\Gamma(\alpha+\gamma+n)\Gamma(\alpha+\delta+n)\Gamma(\beta+\gamma+n)\Gamma(\beta+\delta+n)\Gamma(\gamma+\delta+n)}{\Gamma(\alpha+\beta+\gamma+\delta+2n)}.$$

The Wilson polynomials $\Phi_n(y^2) \equiv \Phi_n^{(\alpha,\beta,\gamma,\delta)}(y^2)$, satisfy the three term recurrence formula

$$y^2\Phi_n(y^2) = K(n+1,n)\Phi_{n+1}(y^2) + K(n,n)\Phi_n(y^2) + K(n-1,n)\Phi_{n-1}(y^2) \qquad \text{(D.3)}$$

where

$$K(n+1,n) = \frac{\alpha+\beta+\gamma+\delta+n-1}{(\alpha+\beta+\gamma+\delta+2n-1)(\alpha+\beta+\gamma+\delta+2n)}, \qquad \text{(D.4)}$$

$$K(n-1,n) = \frac{n(\alpha+\beta+n-1)(\alpha+\gamma+n-1)(\alpha+\delta+n-1)}{(\alpha+\beta+\gamma+\delta+2n-2)(\alpha+\beta+\gamma+\delta+2n-1)}\times \qquad \text{(D.5)}$$
$$(\beta+\gamma+n-1)(\beta+\delta+n-1)(\gamma+\delta+n-1),$$

$$K(n,n) = \alpha^2 - K(n+1,n)(\alpha+\beta+n)(\alpha+\gamma+n)(\alpha+\delta+n) \qquad \text{(D.6)}$$
$$-\frac{K(n-1,n)}{(\alpha+\beta+n-1)(\alpha+\gamma+n-1)(\alpha+\delta+n-1)}.$$

Moreover, they satisfy the following parameter-changing recurrence relations when acting on the basis polynomials $\Phi_n \equiv \Phi_n^{(\alpha,\beta,\gamma,\delta)}$. Here $T^\tau f(y) = f(y+\tau)$.

1.

$$R = \frac{1}{2y}[T^{1/2} - T^{-1/2}], \; R\Phi_n = \frac{n(n+\alpha+\beta+\gamma+\delta-1)}{(\alpha+\beta)(\alpha+\gamma)(\alpha+\delta)}\Phi_{n-1}^{(\alpha+1/2,\beta+1/2,\gamma+1/2,\delta+1/2)}.$$

2.

$$L = \frac{1}{2y}\Big[(\alpha-\frac{1}{2}+y)(\beta-\frac{1}{2}+y)(\gamma-\frac{1}{2}+y)(\delta-\frac{1}{2}+y)T^{1/2}$$
$$-(\alpha-\frac{1}{2}-y)(\beta-\frac{1}{2}-y)(\gamma-\frac{1}{2}-y)(\delta-\frac{1}{2}-y)T^{-1/2}\Big].$$

$$L\Phi_n = (\alpha+\beta-1)(\alpha+\gamma-1)(\alpha+\delta-1)\Phi_{n+1}^{(\alpha-1/2,\beta-1/2,\gamma-1/2,\delta-1/2)}.$$

3.

$$L_{\alpha\beta} = \frac{1}{2y}\left[-(\alpha-\frac{1}{2}+y)(\beta-\frac{1}{2}+y)T^{1/2} + (\alpha-\frac{1}{2}-y)(\beta-\frac{1}{2}-y)T^{-1/2}\right].$$

$$L_{\alpha\beta}\Phi_n = -(\alpha+\beta-1)\Phi_n^{(\alpha-1/2,\beta-1/2,\gamma+1/2,\delta+1/2)}.$$

4.

$$R^{\alpha\beta} = \frac{1}{2y}\left[-(\gamma - \frac{1}{2} + y)(\delta - \frac{1}{2} + y)T^{1/2} + (\gamma - \frac{1}{2} - y)(\delta - \frac{1}{2} - y)T^{-1/2}\right].$$

$$R^{\alpha\beta}\Phi_n = -\frac{(n+\gamma+\delta-1)(n+\alpha+\beta)}{\alpha+\beta}\Phi_n^{(\alpha+1/2,\beta+1/2,\gamma-1/2,\delta-1/2)}.$$

The operators $L_{\alpha\gamma}, L_{\alpha\delta}, R_{\alpha\gamma}, R_{\alpha\delta}$ are obtained by obvious substitutions.

Appendix E

Recurrence relations for dual Hahn polynomials

The dual Hahn polynomials are given by the formula

$$h_n(y^2) \equiv h_n(y^2, \alpha, \beta, \gamma) = (\alpha+\beta)_n(\alpha+\gamma)_n \; {}_3F_2\left(\begin{matrix} -n, & \alpha - y, & \alpha + y \\ \alpha+\beta, & \alpha+\gamma & \end{matrix} ; 1\right) \tag{E.1}$$
$$= (\alpha+\beta)_n(\alpha+\gamma)_n \; \Omega_n^{(\alpha,\beta,\gamma)}(y^2).$$

Note that the discrete dual Hahn polynomials [46] are $p_j(y^2) = \Omega^{(-Q/2,(A+1)/2,(B+1)/2)}(y^2)$ in this notation. The dual Hahn polynomials are obtained from the Wilson polynomials by letting $\delta \to \infty$. Indeed

$$h_n(y^2, \alpha, \beta, \gamma) = \lim_{\delta\to\infty} \frac{w_n(y^2, \alpha, \beta, \gamma, \delta)}{(\alpha+\delta)_n}, \quad \Omega_n^{(\alpha,\beta,\gamma)}(y^2) = \lim_{\delta\to\infty} \Phi_n^{(\alpha,\beta,\gamma,\delta)}(y^2). \tag{E.2}$$

It follows immediately that $h_n(y^2, \alpha, \beta, \gamma)$ is symmetric in α, β, γ.

The recurrence relations for Wilson polynomials presented in the previous section go in the limit to parameter changing recurrences for dual Hahn polynomials. The three term recurrence relation for the dual Hahn polynomials is

$$(-\alpha^2+y^2)\Omega_n(y^2) = K(n+1,n)\Omega_{n+1}(y^2) + K(n,n)\Omega_n(y^2) + K(n-1,n)\Omega_{n-1}(y^2) \tag{E.3}$$

where $K(n+1,n) = (n+\alpha+\beta)(n+\alpha+\gamma)$, $K(n-1,n) = n(n+\beta+\gamma-1)$, $K(n,n) = -K(n+1,n) - K(n-1,n)$. There are 8 basic raising and lowering operators for the dual Hahn polynomials. We list them here and describe their actions on the basis polynomials $\Omega_n \equiv \Omega_n^{(\alpha,\beta,\gamma)}$.

1.

$$R = \frac{1}{2y}[T^{1/2} - T^{-1/2}], \; R\Omega_n = \frac{n}{(\alpha+\beta)(\alpha+\gamma)}\Omega_{n-1}^{(\alpha+1/2,\beta+1/2,\gamma+1/2)}.$$

2.

$$L = \frac{1}{2y}\left[P(y-\frac{1}{2})T^{1/2} - P(-y-\frac{1}{2})T^{-1/2}\right],$$

$$P(y) = (\alpha+y)(\beta+y)(\gamma+y), \; L\Omega_n = (\alpha+\beta-1)(\alpha+\gamma-1)\Omega_{n+1}^{(\alpha-1/2,\beta-1/2,\gamma-1/2)}.$$

3.

$$R^\alpha = \frac{1}{2y}\left[-(\beta+y-\frac{1}{2})(\gamma+y-\frac{1}{2})T^{1/2} + (\beta-y-\frac{1}{2})(\gamma-y-\frac{1}{2})T^{-1/2}\right],$$

$$R^\alpha\Omega_n = -(\beta+\gamma+n-1)\Omega_n^{(\alpha+1/2,\beta-1/2,\gamma-1/2)}.$$

4.

$$R^\beta = \frac{1}{2y}\left[-(\alpha+y-\frac{1}{2})(\gamma+y-\frac{1}{2})T^{1/2} + (\alpha-y-\frac{1}{2})(\gamma-y-\frac{1}{2})T^{-1/2}\right],$$

$$R^\beta\Omega_n = -(\alpha+\gamma-1)\Omega_n^{(\alpha-1/2,\beta+1/2,\gamma-1/2)}.$$

5.

$$R^\gamma = \frac{1}{2y}\left[-(\alpha+y-\frac{1}{2})(\beta+y-\frac{1}{2})T^{1/2} + (\alpha-y-\frac{1}{2})(\beta-y-\frac{1}{2})T^{-1/2}\right],$$

$$R^\gamma\Omega_n = -(\alpha+\beta-1)\Omega_n^{(\alpha-1/2,\beta-1/2,\gamma+1/2)}.$$

6.

$$L_\alpha = \frac{1}{2y}\left[-(\alpha-\frac{1}{2}+y)T^{1/2} + (\alpha-\frac{1}{2}-y)T^{-1/2}\right], \; L_\alpha\Omega_n = -\Omega_n^{(\alpha-1/2,\beta+1/2,\gamma+1/2)}.$$

7.
$$L_\beta = \frac{1}{2y}\left[-(\beta - \frac{1}{2} + y)T^{1/2} + (\beta - \frac{1}{2} - y)T^{-1/2}\right],$$
$$L_\beta \Omega_n = -\left(\frac{n+\alpha+\gamma}{\alpha+\gamma}\right)\Omega_n^{(\alpha+1/2,\beta-1/2,\gamma+1/2)}.$$

8.
$$L_\gamma = \frac{1}{2y}\left[-(\gamma - \frac{1}{2} + y)T^{1/2} + (\gamma - \frac{1}{2} - y)T^{-1/2}\right],$$
$$L_\gamma \Omega_n = -\left(\frac{n+\alpha+\beta}{\alpha+\beta}\right)\Omega_n^{(\alpha+1/2,\beta+1/2,\gamma-1/2)}.$$