ON THE LEBESQUE-AUMANN DOMINATED CONVERGENCE THEOREM IN INFINITE DIMENSIONAL SPACES

by

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ABSTRACT

The Lebesgue-Aumann dominated convergence Theorem (see Aumann [1]) is generalized to correspondences taking values in a Banach space.

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1. INTRODUCTION

In a pioneering paper Aumann [1] generalized the classical dominated convergence theorem of Lebesgue to correspondences taking values in a Euclidean n-space. This result of Aumann was motivated by the problem of the existence of a competitive equilibrium in economies with a measure space of agents and finitely many commodities, (see Aumann [4]). Recent work in economies with a measure space of agents and infinitely many commodities (see for instance [12] and [19]) has necessitated a similar result with that of Aumann [1] for correspondences taking value in a Banach space.

The purpose of this paper is to present such a result, i.e., generalize the Lebesgue-Aumann dominated convergence theorem to correspondences taking values in a Banach space. It should be noted that since some of the results in Aumann [1] (needed to prove his generalization of the Lebesgue dominated convergence theorem) fail in infinite dimensional spaces, our arguments of necessity are quite different than his.

We believe that in addition to applications in game theory and economics, our extension of the Lebesgue-Aumann dominated convergence theorem in Banach spaces may be useful in other fields as well.

The paper is organized as follows: Section 2 contains notation and definitions. The main result of the paper, i.e., the extension of the Lebesgue-Aumann dominated convergence theorem to infinite dimensional spaces is stated in Section 3. The proof of this result is given in Section 4. In particular, several parts of the argument are isolated as Lemmata or Propositions, which in turn extend some results of Aumann [1, 2]. Finally some concluding remarks are given in Section 5.

2. NOTATION AND DEFINITIONS

2.1 Notation

$\mathbb{R}^n$ denotes the n-fold Cartesian product of the set of real numbers $\mathbb{R}$.

$\overline{\text{con}}A$ denotes the closed convex hull of the set A.

2^A denotes the set of all nonempty subsets of the set A.

ϕ denotes the empty set.

/ denotes the set theoretic subtraction.

dist denotes distance.

If $A \subset X$, where X is a Banach space, $c\ell A$ denotes the norm closure of A.

If F_n, (n=1,2,...) is a sequence of nonempty subsets of a Banach space X, we will denote by LsF_n and LiF_n the set of its (strong) limit superior and (strong) limit inferior points respectively, i.e.,

$$LsF_n = \{x \in X : x = \lim_{k\to\infty} x_{n_k},\ x_{n_k} \in F_{n_k},\ k=1,2,\ldots\}, \text{ and}$$

$$LiF_n = \{x \in X : x = \lim_{n\to\infty} x_n,\ x_n \in F_n,\ n=1,2,\ldots\}.$$

2.2 Definitions

Let X and Y be sets. The graph of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. Let (T,τ,μ) be a complete finite measure space, (i.e., it is a real-valued, non-negative countably additive measure defined on a complete σ-field τ of subsets of T such that $\mu(T) < \infty$) and X be a separable Banach space. The correspondence $\phi : T \to 2^X$ is said to have a measurable graph if $G_\phi \in \tau \otimes \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes product σ-field. The correspondence $\phi : T \to 2^X$ is said to be lower measurable if for every open subset V of X, the set

$\{t \in T : \phi(t) \cap V \neq \phi\}$ is an element of τ. Recall (see for instance Himmelberg [11, p. 47] or Debreu [7, p. 359]) that since T is a complete measure space if $\phi : T \to 2^X$ has a measurable graph, then ϕ is lower measurable. Furthermore, if $\phi(\cdot)$ is closed valued and lower measurable then $\phi : T \to 2^Y$ has a measurable graph. A well-known result of Aumann [3], which will be of fundamental importance in this paper, (see also Himmelberg [11, Theorem 5.2, p. 60]) says that if (T,τ,μ) is a complete measure space and X is a separable metric space and $\phi : T \to 2^X$ is a nonempty valued correspondence having a measurable graph, then $\phi(\cdot)$ admits a <u>measurable selection</u>, i.e., there exists a measurable function $f : T \to X$ such that $f(t) \in \phi(t)$ μ - a.e.

We now define the notion of a Bochner integrable function. We will follow closely Diestel-Uhl [9]. Let (T,τ,μ) be a finite measure space and X be a Banach space.

A function $f : T \to X$ is called <u>simple</u> if there exist $x_1, x_2, \ldots, x_n$ in X and $\alpha_1, \alpha_2, \ldots, \alpha_n$ in τ such that $f = \sum_{i=1}^{n} x_i \chi_{\alpha_i}$, where $\chi_{\alpha_i}(t) = 1$ if $t \in \alpha_i$ and $\chi_{\alpha_i}(t) = 0$ if $t \notin \alpha_i$. A function $f : T \to X$ is said to be μ-<u>measurable</u> if there exists a sequence of simple functions $f_n : T \to X$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in T$. A μ-measurable function $f : T \to X$ is said to be <u>Bochner integrable</u> if there exists a sequence of simple functions $\{f_n\}$, $(n=1,2,..)$ such that

$$\lim_{n\to\infty} \int_T \|f_n(t) - f(t)\| d\mu(t) = 0.$$

In this case we define for each $E \in \tau$ the integral to be $\int_E f(t)d\mu(t) = \lim_{n\to\infty} \int_E f_n(t)d\mu(t)$. It can be shown (see Diestel-Uhl [9, Theorem 2, p. 45]) that, if $f : T \to X$ is a μ-measurable function then f is Bochner integrable if and only if $\int_T \|f(t)\| d\mu(t) < \infty$. We denote by $L_1(\mu,X)$ the space of equivalence classes of X-valued Bochner integrable functions $x : T \to X$ normed by

$$\|x\| = \int_T \|x(t)\| d\mu(t).$$

It is a standard result that normed by the functional $\|\cdot\|$ above, $L_1(\mu,X)$ becomes a Banach space, (see [9, p. 50]).

We denote by $\mathcal{L}_\phi$ the <u>set of all X-valued Bochner integrable selections</u> from $\phi : T \to 2^X$, i.e.,

$$\mathcal{L}_\phi = \{x \in L_1(\mu,X) : x(t) \in \phi(t) \quad \mu - \text{a.e.}\}.$$

Moreover, as in Aumann [1] the <u>integral of the correspondence</u> $\phi : T \to 2^X$ is defined as follows:

$$\int_T \phi(t) d\mu(t) = \{\int_T x(t) d\mu(t) : x \in \mathcal{L}_\phi\}.$$

Recall that the correspondence $\phi : T \to 2^X$ is said to be <u>integrably bounded</u> if there exists a map $h \in L_1(\mu)$ such that $\sup\{\|x\| : x \in \phi(t)\} \leq h(t)$ μ - a.e. Moreover, note that if T is a complete measure space, X is a separable Banach space and $\phi : T \to 2^X$ is integrably bounded, nonempty valued having a measurable graph, by the Aumann measurable selection theorem we can conclude that $\mathcal{L}_\phi$ is nonempty and therefore $\int_T \phi(t) d\mu(t)$ is nonempty as well.

Let A_n, (n=1,2,..) be a sequence of nonempty subsets of a Banach space X. Following Kuratowski [14, p. 339] we say that A_n <u>converges</u> in A (written as $A_n \to A$) if $LiA_n = LsA_n = A$. It may be useful to remind the reader that LiA_n and LsA_n are both closed sets and $LiA_n \subset LsA_n$ (see [14, p. 336-338]). Let X be a metric space and Y be a Banach space. The correspondence $\phi : X \to 2^Y$ is said to be <u>upper-semicontinuous</u> (u.s.c.) at $x_0 \in X$, if for any neighborhood $N(\phi(x_0))$ of $\phi(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that for all $x \in N(x_0)$, $\phi(x) \subset N(\phi(x_0))$. We say that ϕ is u.s.c. if ϕ is u.s.c. at every point $x \in X$. (Recall that this definition is equivalent to the fact that the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y, see for instance [14, Theorem 3, p. 176]).

Let ε be a small positive number and let B be the open unit ball in Y. The correspondence $\phi : X \to 2^Y$ is said to be quasi upper-semicontinuous (q.u.s.c.) at $x_0 \in X$, if whenever the sequence x_n, (n=1,2,..) in X converges to x_0, then for some n_0, $\phi(x_n) \subset \phi(x_0) + \varepsilon B$ for all $n \geq n_0$. We say that ϕ is q.u.s.c. if ϕ is q.u.s.c. at every point $x \in X$. It can be easily checked that if ϕ is compact valued, quasi upper-semicontinuity implies upper-semicontinuity and vice versa.

With all these preliminaries out of the way we can now turn to the statement of the main theorem.

3. THE MAIN THEOREM

We now state the main theorem of the paper, i.e., an infinite dimensional generalization of the well-known Lebesgue dominated convergence theorem. This theorem may also be considered as an extension of Aumann's [1, Theorem 5] main result, which in turn is the finite dimensional generalization of Lebesgue's dominated convergence theorem.[1] In that sense our main theorem may be seen as an extension of the Lebesgue-Aumann dominated convergence results to infinite dimensional spaces.

Main Theorem: Let (T,τ,μ) be a complete finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$, (n=1,2,..) be a sequence of integrably bounded[2] nonempty valued correspondences taking values in a compact nonempty subset of X, such that:

(i) $\phi_n(\cdot)$, (n=1,2,..) have a measurable graph, i.e., $G_{\phi_n} \in \tau \otimes \mathcal{B}(X)$, and

(ii) $\phi_n(t) \to \phi(t)$ μ - a.e.

Then

$$\int_T \phi_n(t)d\mu(t) \to c\ell \int_T \phi(t)d\mu(t).$$

Moreover, if $\phi(\cdot)$ is convex valued then

$$\int_T \phi_n(t)d\mu(t) \to \int_T \phi(t)d\mu(t).$$

Notice that the first conclusion of the main theorem is an approximate version of the Aumann dominated convergence theorem [1, Theorem 5]. The approximate nature of this result arises from the fact that, if $\phi_n : T \to 2^X$ is a sequence of correspondences satisfying the assumptions of the main theorem, then $\int_T \phi_n(t)d\mu(t)$ does not necessarily converge to $\int_T \phi(t)d\mu(t)$, (unless of course $\phi(\cdot)$ is convex valued as our Main Theorem indicates), a result which is known to be true if $X = \mathbb{R}^n$, (see Aumann [1, Theorem 5]), but false in infinite dimensional spaces. In particular, Uhl's counterexample (see Diestel-Uhl [9, p. 262]) on the failure of the Lyapunov theorem in infinite dimensional spaces can be easily modified to indicate this, (see for instance Rustichini [15]). Hence, without additional assumptions other than those of Aumann [1] an exact version of the first conclusion of the main theorem cannot be obtained. However, with the additional assumption, i.e., by assuming that $\phi(\cdot)$ is convex valued, we are indeed able to obtain an exact version of the Aumann dominated convergence theorem in infinite dimensional spaces, and this is the second conclusion of the main theorem.

As in Aumann [1] the proof of the main theorem (which is given in the next section) is based on two infinite dimensional versions of the Fatou Lemma, which in turn have found interesting applications in general economic equilibrium theory, and may be useful to other areas as well. Also it should be noted that in the process of proving our main theorem we offer several infinite dimensional extensions of some results of Aumann [1, 2].

4. PROOF OF THE MAIN THEOREM

We begin by isolating parts of the argument as Lemmata or Propositions.

Lemma 4.1: Let (T,τ,μ) be a complete finite measure space and X be a separable Banach space. Let $\phi : T \to 2^X$ be a correspondnece satisfying the following condition:

(i) $\phi(t) \subset K$ for all $t \in T$, where K is a compact, nonempty subset of X.

Then

$$c\ell \int_T \overline{\text{con}}\phi(t)d\mu(t) = \int_T \overline{\text{con}}\phi(t)d\mu(t).$$

Proof: Let $\tilde{K} = \overline{\text{con}}K$. Note that $\tilde{K}$ is compact [10, Theorem 6, p. 416] nonempty and convex. Hence, from Diestel's theorem [8, Theorem 2] we have that $\mathcal{L}_{\tilde{K}}$ is weakly compact in $L_1(\mu,X)$. Since $\overline{\text{con}}\phi(\cdot)$ is norm closed and convex valued so is $\mathcal{L}_{\overline{\text{con}}\phi}$. It is a consequence of the Separation Theorem that the weak and norm topologies coincide on closed convex sets. Hence, $\mathcal{L}_{\overline{\text{con}}\phi}$ is weakly closed. Since $\mathcal{L}_{\overline{\text{con}}\phi} \subset \mathcal{L}_{\tilde{K}}$ and the latter set is weakly compact we can conclude that $\mathcal{L}_{\overline{\text{con}}\phi}$ is weakly compact. Define the mapping $\gamma : L_1(\mu,X) \to X$ by $\gamma(x) = \int_T x(t)d\mu(t)$. Certainly γ is linear and norm continuous. It follows from Theorem 15 in Dunford-Schwartz [10, p. 422] that γ is also weakly continuous. Therefore, $\gamma(\mathcal{L}_{\overline{\text{con}}\phi}) = \{\gamma(x) : x \in \mathcal{L}_{\overline{\text{con}}\phi}\} =$

$\int_T \overline{\text{con}}\phi(t)d\mu(t)$ is weakly compact, and we can conclude that $c\ell \int_T \overline{\text{con}}\phi(t)d\mu(t) = \int_T \overline{\text{con}}\phi(t)d\mu(t)$.

Notice that the above proof of the Lemma showed that $\int_T \overline{\text{con}}\phi(t)d\mu(t)$ is weakly compact. Hence, the above Lemma may be seen as the infinite dimensional extension of Theorem 4 of Aumann [1].

The proposition below generalizes to separable Banach spaces a very useful result of Aumann [2] which has found applications in general economic equilibrium theory.

Proposition 4.1: Let (T,τ,μ) be a complete finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T \times P \to 2^X$ be a nonempty valued and integrably bounded correspondence, such that for each fixed $t \in T$, $\psi(t,\cdot)$ is q.u.s.c. and for each fixed $p \in P$, $\psi(\cdot,p)$ has a measurable graph. Then

$$\int_T \psi(t,\cdot) \text{ is q.u.s.c.}$$

Proof: Without loss of generality we may assume throughout the argument that $\int_T d\mu(t) = 1$. Let B be the open unit ball in X, and let ε be a small positive number. We must show that if $\{p_n : n = 1,2,\ldots\}$ is a sequence in P converging to $p \in P$, then for a suitable n_0,

$$\int_T \psi(t,p_n)d\mu(t) \subset \int_T \psi(t,p)d\mu(t) + \varepsilon B \text{ for all } n \geqslant n_0.$$

Define the mapping $\mathcal{L}_\psi : P \to L_1(\mu,X)$ by $\mathcal{L}_\psi(p) = \{x \in L_1(\mu,X) : x(t) \in \psi(t,p)$ μ - a.e.$\}$. Let B and $\tilde{B}$ be the open unit balls in X and $L_1(\mu,X)$ respectively. We first show that for a suitable n_0, $\mathcal{L}_\psi(p_n) \subset \mathcal{L}_\psi(p) + \varepsilon\tilde{B}$ for all $n \geqslant n_0$.

We begin by finding the suitable n_0. Since for each fixed $t \in T$, $\psi(t,\cdot)$ is q.u.s.c. we can find a minimal M_t such that

(4.1) $\psi(t,p_n) \subset \psi(t,p) + \delta B$ for all $n \geq M_t$,

where $\delta = \dfrac{\varepsilon}{3\mu(T)}$.

We now show that M_t is a measurable function of t. However, first we make a few observations. By assumption for each fixed p and n, $G_{\psi(\cdot,p)}$ and $G_{\psi(\cdot,p_n)}$ belong to $\tau \otimes \mathcal{B}(X)$. Hence, $G_{\psi(\cdot,p_n)} + \delta B \in \tau \otimes \mathcal{B}(X)$ and so does $(G_{\psi(\cdot,p_n)} + \delta B)^c$ (where S^c denotes the complement of the set S).
It is easy to see that $G_{\psi(\cdot,p)} \cap (G_{\psi(\cdot,p_n)} + \delta B)^c \in \tau \otimes \mathcal{B}(X)$. Therefore, the set

$$U = \{(t,x) \in T \times X : (t,x) \in G_{\psi(\cdot,p)} \cap (G_{\psi(\cdot,p_n)} + \delta B)^c\}$$

belongs to $\tau \otimes \mathcal{B}(X)$.
It follows from the projection theorem (see [6, Theorem III.23, p. 75]) that

$$\text{proj}_T(U) \in \tau.$$

Notice that,

$$\text{proj}_T(U) = \{t \in T : \psi(t,p) \not\subset \psi(t,p_n) + \delta B\}$$
$$= \{t \in T : \psi(t,p) / (\psi(t,p_n) + \delta B) \neq \phi\}.$$

By virtue of the measurability of the above set we can now conclude that M_t is a measurable function of t. In particular, simply notice that,

$$\{t \in T : M_t = m\} = \bigcap_{n \geq m} \{t \in T : \psi(t,p_n) \subset \psi(t,p) + \delta B\} \cap \{t \in T : \psi(t,p_{m-1}) \not\subset \psi(t,p) + \delta B\}.$$

We are now in a position to choose the desired n_0. Since $\psi(\cdot,\cdot)$ is integrably bounded there exists $h \in L_1(\mu)$ such that for almost all $t \in T$, $\sup\{\|x\| : x \in \psi(t,p)\} \leq h(t)$ for each $p \in P$.

Choose δ_1 such that if $\mu(S) < \delta_1$, $(S \subset T)$, then $\int_S h(t)d\mu(t) < \frac{\varepsilon}{3}$. Since M_t is a measurable function of t, we can choose n_0 such that $\mu(\{t \in T : M_t \geq n_0\}) < \delta_1$. This is the desired n_0.

Let $n \geq n_0$ and $y \in \mathcal{L}_\psi(p_n)$. We must show that

$$y \in \mathcal{L}_\psi(p) + \delta\tilde{B}.$$

By assumption, for each fixed $p \in P$, $\psi(\cdot,p)$ has a measurable graph and $\psi(\cdot,\cdot)$ is nonempty valued. Hence, by the Aumann measurable selection theorem there exists a measurable function $f_1 : T \to X$ such that $f_1(t) \in \psi(t,p)$ for almost all $t \in T$. Define the correspondence $\theta : T \to 2^X$ by $\theta(t) = (\{y(t)\} + \delta B) \cap \psi(t,p)$. It follows from (4.1) that for all $t \in T_0 = \{t : M_t \leq n_0\}$, $\theta(t) \neq \phi$. Moreover, $\theta(\cdot)$ has a measurable graph. Another application of the Aumann measurable selection theorem allows us to guarantee the existence of a measurable function $f_2 : T \to X$ such that $f_2(t) \in \theta(t)$ for almost all t in T_0. Define $f : T \to X$ by

$$f(t) = \begin{cases} f_1(t) & \text{for } t \notin T_0 \\ f_2(t) & \text{for } t \in T_0. \end{cases}$$

Then $f(t) \in \psi(t,p)$ for almost all t in T and since $\psi(\cdot,\cdot)$ is integrably bounded we can conclude that $f \in \mathcal{L}_\psi(p)$. If we show that $\|f-y\| < \varepsilon$ then $y \in \mathcal{L}_\psi(p) + \varepsilon\tilde{B}$ and we will be done. But this is easy to see. We have

$$\begin{aligned} \|f-y\| &= \int_{T \setminus T_0} \|f_1(t) - y(t)\| d\mu(t) + \int_{T_0} \|f_2(t)\| d\mu(t) \\ &< 2\int_{T \setminus T_0} h(t)d\mu(t) + \int_{T_0} \delta d\mu(t) \\ &< \frac{2}{3}\varepsilon + \delta\mu(T) = \frac{2}{3}\varepsilon + \frac{\varepsilon}{3\mu(T)} \cdot \mu(T) = \varepsilon. \end{aligned}$$

This completes the proof of the fact that, if the sequence $\{p_n : n=1,2..\}$ in P converges to $p \in P$, then for a suitable n_0

(4.2) $\mathcal{L}_\psi(p_n) \subset \mathcal{L}_\psi(p) + \varepsilon \tilde{B}$ for all $n \geq n_0$.

Define now the mapping $\gamma : L_1(\mu,X) \to X$ by $\gamma(x) = \int_T x(t)d\mu(t)$. It follows from (4.2) that for all $n \geq n_0$,

$$\gamma(\mathcal{L}_\psi(p_n)) = \{\gamma(x) : x \in \mathcal{L}_\psi(p_n)\}$$

$$= \int_T \psi(t,p_n)d\mu(t) \subset \gamma(\mathcal{L}_\psi(p) + \varepsilon \tilde{B}) = \gamma(\mathcal{L}_\psi(p)) + \gamma(\varepsilon \tilde{B})$$

$$= \int_T \psi(t,p)d\mu(t) + \varepsilon B.$$

Hence,

$$\int_T \psi(t,p_n)d\mu(t) \subset \int_T \psi(t,p)d\mu(t) + \varepsilon B \quad \text{for all } n \geq n_0.$$

i.e., $\int_T \psi(t,\cdot)d\mu(t)$ is q.u.s.c. as was to be shown.

Remark 4.1: If in addition to the assumptions of Proposition 4.1 it is assumed that $\int_T \psi(t,\cdot)d\mu(t)$ is compact valued, then we can conclude that $\int_T \psi(t,\cdot)d\mu(t)$ is u.s.c.[3]

Proposition 4.2: Let (T,τ,μ) be a complete finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$, (n=1,2,..) be a sequence of integrably bounded nonempty valued correspondences having a measurable graph, i.e., $G_{\phi_n} \in \tau \otimes \mathcal{B}(X)$. Then

$$\mathcal{L}_{Li\phi_n} \subset Li\mathcal{L}_{\phi_n}.$$

Proof: A closely related result was proved by Aumann [1, Proposition 5.1, p. 9]. Let $x \in \mathcal{L}_{Li\phi_n}$, i.e., $x(t) \in Li\phi_n(t)$ μ - a.e. We wish to show that $x \in Li\mathcal{L}_{\phi_n}$ Notice that $x(t) \in Li\phi_n(t)$ μ - a.e., implies that there exists a sequence $\{x_n : n=1,2..\}$ such that $\lim_{n\to\infty} x_n(t) = x(t)$ μ - a.e. and $x_n(t) \in \phi_n(t)$ μ - a.e., which in turn is equivalent (by Kuratowski's Theorem [13, p. 335]) to the fact that $\lim_{n\to\infty} \text{dist}(x(t), \phi_n(t)) = 0$ μ - a.e. Define $F_n : T \to 2^X$, $(n=1,2,..)$ by

$$F_n(t) = \{y \in \phi_n(t) : \|y - x(t)\| \leq \text{dist}\ (x(t), \phi_n(t)) + \frac{1}{n}\}.$$

Since $x(t) \in Li\phi_n(t)$ μ - a.e. it follows that $F_n(t) \neq \phi$ μ - a.e. Observe that for $(t,y) \in T\times X$ the function $g(t,y) = \|y - x(t)\|$ is measurable in t and continuous in y and thus, by a standard result (see for instance [14, Theorem 2, p. 378]) $g(\cdot,\cdot)$ is jointly measurable, i.e., measurable with respect to the product σ-algebra $\tau \otimes \mathcal{B}(X)$. Note that for each $t \in T$, the function $u(t) = \text{dist}(x(t), \phi_n(t))$ (where $\text{dist}(x(t),\phi) = +\infty$) is measurable. Therefore, the function $h(t,y) = g(t,y) - u(t)$ is jointly measurable, i.e., measurable with respect to the product σ-algebra $\tau \otimes \mathcal{B}(X)$. It is easy to see that

$$h^{-1}((-\infty, \frac{1}{n}]) \cap G_{\phi_n} = \{(t,y) \in T \times X : h(t,y) \leq \frac{1}{n}\} \cap G_{\phi_n} = G_{F_n}.$$

Since $h(\cdot,\cdot)$ is jointly measurable, $h^{-1}((-\infty,\frac{1}{n}])$ belongs to $\tau \otimes \mathcal{B}(X)$ and so does G_{ϕ_n} since by assumption $\phi_n(\cdot)$ has a measurable graph. Therefore, G_{F_n} belongs to $\tau \otimes \mathcal{B}(X)$, i.e., $F_n(\cdot)$ has a measurable graph. Since (T,τ,μ) is a complete finite measure space, we can appeal to the Aumann measurable selection theorem to ensure the existence of a measurable function $f_n : T \to X$ such that $f_n(t) \in F_n(t)$ μ - a.e. Since $x(t) \in Li\phi_n(t)$ μ - a.e., $\lim_{n\to\infty} \text{dist}(x(t), \phi_n(t)) = 0$ μ - a.e. which implies that $\lim_{n\to\infty} \|f_n(t) - x(t)\| = 0$ μ - a.e. Since $f_n(t) \in \phi_n(t)$ μ - a.e. and $\phi_n(\cdot)$ is integrably bounded, by the Lebesgue dominated convergence theorem [9, p. 45] we have that $f_n(\cdot)$ is Bochner integrable, i.e., $f_n \in L_1(\mu,X)$. Therefore, $x \in \mathrm{Li}\,\mathcal{L}_{\phi_n}$ and this completes the proof of the Proposition.

The following Corollary of Proposition 4.2 is an infinite dimensional version of the Fatou Lemma. It extends Proposition 5.1 of Aumann [1] to separable Banach spaces.

Corollary 4.1: Let $\phi_n : T \to 2^X$, (n=1,2,..) be a sequence of correspondences satisfying all the assumptions of Proposition 4.2. Then

$$\int_T \mathrm{Li}\phi_n(t)d\mu(t) \subset \mathrm{Li} \int_T \phi_n(t)d\mu(t).$$

Proof: It follows directly from Proposition 4.2. In particular, define the mapping $\gamma : L_1(\mu,X) \to X$ by $\gamma(x) = \int_T x(t)d\mu(t)$. By virtue of Proposition 4.2 we have that $\gamma(\mathcal{L}_{\mathrm{Li}\phi_n}) = \{\gamma(x) : x \in \mathcal{L}_{\mathrm{Li}\phi_n}\} = \int_T \mathrm{Li}\phi_n(t)d\mu(t) \subset \gamma(\mathrm{Li}\,\mathcal{L}_{\phi_n}) = \mathrm{Li} \int_T \phi_n(t)d\mu(t)$.

We now prove another infinite dimensional analogue of the Fatou Lemma which extends Proposition 4.1 of Aumann [1] to separable Banach spaces.

Corollary 4.2: Let (T,τ,μ) be a complete finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$ (n=1,2..) be a sequence of nonempty valued and integrably bounded correspondences, taking values in a compact nonempty subset of X, having a measurable graph. Then

$$\mathrm{Ls} \int_T \phi_n(t)d\mu(t) \subset c\ell \int_T \mathrm{Ls}\phi_n(t)d\mu(t).$$

Moreover, if $\mathrm{Ls}\phi_n(\cdot)$ is convex valued, then

$$\mathrm{Ls} \int_T \phi_n(t)d\mu(t) \subset \int_T \mathrm{Ls}\phi_n(t)d\mu(t).$$

Proof:[4] Denote by P the interval [0,1). Define the correspondence $\psi : T \times P \to 2^X$ by

$$\psi(t,p) = \begin{cases} \phi_n(t) & \text{if } \frac{1}{1+n} < p < \frac{1}{n} \\ \phi_n(t) \cup \phi_{n+1}(t) & \text{if } p = \frac{1}{n+1} \\ Ls\phi_n(t) & \text{if } p = 0. \end{cases}$$

It can be easily checked that for each fixed $t \in T$, $\psi(t,\cdot)$ is u.s.c. and that for each fixed $p \in P$, $\psi(\cdot,p)$ has a measurable graph. Moreover, ψ is integrably bounded. Hence, ψ satisfies all the assumptions of Proposition 4.1 and thus, $\int_T \psi(t,\cdot)d\mu(t)$ is q.u.s.c. Let now $x \in Ls \int_T \phi_n(t)d\mu(t)$, i.e., there exists x_{n_k} such that $\lim_{k\to\infty} x_{n_k} = x$, $x_{n_k} \in \int_T \phi_{n_k}(t)d\mu(t)$, k=1,2,... We wish to show that $x \in c\ell \int_T Ls\phi_n(t)d\mu(t)$.

Since $\int_T \psi(t,\cdot)d\mu(t)$ is q.u.s.c. it follows that if p_{n_k} converges to 0 then $\int_T \psi(t,p_{n_k})d\mu(t) \subset \int_T \psi(t,0)d\mu(t) + \varepsilon B$ for all sufficiently large k. Consequently, $x_{n_k} \in \int_T \psi(t,0)d\mu(t) + \varepsilon B$ for all sufficiently large k and therefore, $x \in c\ell \int_T \psi(t,0)d\mu(t) \equiv c\ell \int_T Ls\phi_n(t)d\mu(t)$ as was to be shown. If now $Ls\phi_n(\cdot)$, is convex valued

(recall that $Ls\phi_n(\cdot)$ is closed valued as well) it follows from Lemma 4.1 and the first conclusion of the Corollary that

$$Ls \int_T \phi_n(t)d\mu(t) \subset c\ell \int_T Ls\phi_n(t)d\mu(t) = \int_T Ls\phi_n(t)d\mu(t).$$

The proof of the Corollary is now complete.

We are now ready to complete the proof of the Main Theorem.

Since by assumption $\phi_n(t) \to \phi(t)$ μ - a.e., i.e.,

$\phi(t) = Li\phi_n(t) = Ls\phi_n(t)$ μ - a.e., by Corollary 4.1 and Corollary 4.2 we have that:

(4.3) $\int \phi = \int \mathrm{Li}\phi_n \subset \mathrm{Li} \int \phi_n \subset \mathrm{Ls} \int \phi_n \subset c\ell \int \mathrm{Ls}\phi_n = c\ell \int \phi.$

Therefore,

$$c\ell \int_T \phi(t)d\mu(t) = \mathrm{Li} \int_T \phi_n(t)d\mu(t) = \mathrm{Ls} \int_T \phi_n(t)d\mu(t),$$

i.e.,

$$\int_T \phi_n(t)d\mu(t) \rightarrow c\ell\int_T \phi(t)d\mu(t).$$

If now $\phi(\cdot)$ is convex valued, taking values in a weakly compact convex nonempty set, (4.3) can be written (recall the second conclusion of Corollary 4.2) as:

$$\int \phi = \int \mathrm{Li}\phi_n \subset \mathrm{Li} \int \phi_n \subset \mathrm{Ls} \int \phi_n \subset \int \mathrm{Ls}\phi_n = \int \phi.$$

Thus,

$$\int_T \phi(t)d\mu(t) = \mathrm{Li} \int_T \phi_n(t)d\mu(t) = \mathrm{Ls} \int_T \phi_n(t)d\mu(t),$$

i.e., $\int_T \phi_n(t)d\mu(t) \rightarrow \int_T \phi(t)d\mu(t),$

and this completes the proof of the Main Theorem.

5. CONCLUDING REMARKS

We conclude by obtaining some useful Corollaries which follow easily from the results proved in the previous section.

5.1 Lower Semicontinuity of the Set of Integrable Selections and of the Aumann Integral

Let P and X be metric spaces. The correspondence $F : P \rightarrow 2^X$ is said to be lower-semicontinuous (l.s.c.) if the sequence p_n, (n=1,2,..) in P converges to $p \in P$, then $F(p) \subset \mathrm{Li}F(p_n)$.

The following corollary follows directly from Proposition 4.2.

Corollary 5.1: Let (T,τ,μ) be a complete finite measure space, X be a separable Banach space and P be a metric space. Let $\phi : T \times P \to 2^X$ be an integrably bounded nonempty valued correspondence such that for each fixed $t \in T$, $\phi(t,\cdot)$ is l.s.c. and for each fixed $p \in P$, $\phi(\cdot,p)$ has a measurable graph. Then the correspondence $\mathcal{L}_\phi : P \to 2^{L_1(\mu,X)}$ defined by

$$\mathcal{L}_\phi(p) = \{y \in L_1(\mu,X) : y(t) \in \phi(t,p) \ \mu - \text{a.e.}\}$$

is l.s.c.

Proof: Let $\{p_n : n=1,2,..\}$ be a sequence in P converging to $p \in P$. We must show that $\mathcal{L}_\phi(p) \subset \mathrm{Li}\,\mathcal{L}_\phi(p_n)$. Since by assumption for each fixed $t \in T$, $\phi(t,\cdot)$ is l.s.c. we have that $\phi(t,p) \subset \mathrm{Li}\,\phi(t,p_n)$ for all $t \in T$, and therefore,

$$(5.1) \quad \mathcal{L}_\phi(p) \subset \mathcal{L}_{\mathrm{Li}\phi}(p_n).$$

It follows now from Proposition 4.2 that (5.1) can be written as:

$$\mathcal{L}_\phi(p) \subset \mathcal{L}_{\mathrm{Li}\phi}(p_n) \subset \mathrm{Li}\,\mathcal{L}_\phi(p_n).$$

The proof of the Corollary is now complete.

Corollary 5.2: Let $\phi : T \times P \to 2^X$ be a correspondence satisfying all the assumptions of Corollary 5.1. Then

$\int_T \phi(t,\cdot)d\mu(t)$ is l.s.c.

Proof: The result follows from Corollary 4.1 by adopting a similar argument with that used in the proof of Corollary 5.1 above. Alternatively it can be proved by means of Corollary 5.1 as follows. Define the mapping $\gamma : L_1(\mu,X) \to X$ by $\gamma(x) = \int_T x(t)d\mu(t)$. Then γ is linear and norm continuous.

Notice that $\gamma(\mathcal{L}_\phi(p)) = \{\gamma(x) : x \in \mathcal{L}_\phi(p)\} = \int_T \phi(t,p)d\mu(t)$. Since by Corollary 5.1 $\mathcal{L}_\phi(\cdot)$ is l.s.c. so is $\gamma(\mathcal{L}_\phi)$ (see for instance Kuratowski [14, Theorem 7, p. 178]), i.e., $\int_T \phi(t,\cdot)d\mu(t)$ is l.s.c. as was to be shown.

Recall now that a correspondence $\phi : X \to 2^Y$, (where X, Y are metric spaces) is said to be continuous if and only if it is u.s.c. and l.s.c.

The following result follows directly from Remark 4.1 and Corollary 5.2.

Corollary 5.3: Let (T,τ,μ) be a complete finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T \times P \to 2^X$ be an integrably bounded, nonempty valued correspondence such that for each fixed $p \in P$, $\psi(\cdot,p)$ has a measurable graph and for each fixed $t \in T$, $\psi(t,\cdot)$ is continuous. Moreover, suppose that $\int_T \psi(t,\cdot)d\mu(t)$ is compact valued. Then

$$\int_T \psi(t,\cdot)d\mu(t) \text{ is continuous.}$$

Corollary 5.3 is the infinite dimensional extension of the corresponding Corollary 5.2 of Aumann [1].

5.2 A Dominated Convergence Theorem

If X is a Banach space and A_n, (n=1,2,..) is a sequence of nonempty subsets of X, w-LsA_n will now denote the set of its weak limit superior points, i.e., w-LsA_n = $\{x \in X : x = \text{w-}\ell im\, x_k,\ x_k \in A_{n_k},\ k=1,2,..\}$.

The following version of Fatou's Lemma which generalizes a result of Khan-Majumdar [13] was proved in [18].

Theorem 5.1: Let (T,τ,μ) be a complete finite atomless measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$ (n=1,2,..) be a sequence of nonempty closed valued correspondences such that:

(i) For all n, $\phi_n(t) \subset F(t)$ for all $t \in T$, where $F : T \to 2^X$ is an integrably bounded, weakly compact, convex, nonempty valued correspondence, and

(ii) $w\text{-}Ls\phi_n(\cdot)$ is lower measurable.

Then

$$w\text{-}Ls \int_T \phi_n(t)d\mu(t) \subset c\ell \int_T w\text{-}Ls\phi_n(t)d\mu(t).$$

Moreover, if $w\text{-}Ls\phi_n(\cdot)$ is closed and convex valued, then

$$w\text{-}Ls \int_T \phi_n(t)d\mu(t) \subset \int_T w\text{-}Ls\phi_n(t)d\mu(t).$$

Observe that Theorem 5.1 is the w-Ls version of Corollary 4.2. A version of Theorem 5.1 has also been proved in Balder [5].

By means of Corollary 4.1 and the above theorem, we can prove a similar result with our Main Theorem. We will first need the following definition.

Let F_n, (n=1,2..) be a sequence of nonempty subsets of the Banach space X. The sequence F_n is said to converge in a <u>Kuratowski-type sense</u> to F (witten as $F_n \overset{K}{\to} F$), if $LiF_n = w\text{-}LsF_n = F$.

Below we state a dominated convergence result which is similar in spirit with our Main Theorem.

<u>Theorem 5.2</u>: Let (T,τ,μ) be a finite atomless complete measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$, (n=1,2,..) be a sequence of nonempty closed valued correspondences having a measurable graph such that:

(i) For all n, (n=1,2,..) $\phi_n(t) \subset F(t)$ for all $t \in T$, where $F : T \to 2^X$ is an integrably bounded, weakly compact, convex, nonempty valued correspondence, and

(ii) $\phi_n(t) \overset{K}{\to} \phi(t)$ μ - a.e.

Then

$$\int_T \phi(t)d\mu(t) \overset{K}{\to} c\ell \int_T \phi(t)d\mu(t).$$

Moreover, if $\phi(\cdot)$ is convex valued, then

$$\int_T \phi_n(t)d\mu(t) \overset{K}{\to} \int_T \phi(t)d\mu(t).$$

For the proof of Theorem 5.2 we will need the following simple lemma.

<u>Lemma 5.1</u>: Let (T,τ,μ) be a complete finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$ (n=1,2,..) be a sequence of nonempty valued correspondences having a measurable graph, i.e., $G_{\phi_n} \in \tau \otimes \mathcal{B}(X)$. Then $Li\phi_n(\cdot)$ has a measurable graph, i.e., $G_{Li\phi_n} \in \tau \otimes \mathcal{B}(X)$.

<u>Proof</u>: First notice that $Li\phi_n(\cdot)$ is closed valued (recall from Kuratowski [14, p. 336-337] that if A_n is a sequence of sets, LiA_n and LsA_n are both closed sets). By definition (see [14, p. 335]), $Li\phi_n(t) = \{f \in X : \lim_{n\to\infty} dist(f, \phi_n(t)) = 0\}$. Since by assumption $\phi_n(\cdot)$ has a measurable graph and (T,τ,μ) is a complete measure space, $\phi_n(\cdot)$ is lower measurable. It follows from Himmelberg [11, Theorem 3.3, p. 50]that dist $(f, \phi_n(t))$ is continuous in f and measurable in t, i.e., dist$(\cdot,\cdot)$ is jointly measurable, with respect to the product σ-algebra $\tau \otimes \mathcal{B}(X)$. Define the function $g : T \times X \to \mathbb{R}$ by $g(t,f) = \lim_{n\to\infty}$ dist $(f,\phi_n(t))$. Then, $g(\cdot,\cdot)$ is jointly

measurable. Moreover,

$$g^{-1}(0) = \{(t,f) \in T \times X : \lim_{n\to\infty} \text{dist } (f,\phi_n(t)) = 0\}$$

$$= G_{Li\phi_n},$$

and the latter set belongs to $\tau \otimes \mathcal{B}(X)$ since $g(\cdot,\cdot)$ is jointly measurable. Therefore, $Li\phi_n(\cdot)$ has a measurable graph and this completes the proof of the Lemma.

Remark 5.1: Under the assumptions of Lemma 5.1, $Ls\phi_n(\cdot)$ has a measurable graph as well. Simply notice that (see [14, p. 337]) $Ls\phi_n(t) = \{f \in X : Li \text{ dist } (f,\phi_n(t)) = 0\}$.

Proof of Theorem 5.2: By assumption, $\phi(t) = Li\phi_n(t) = w - Ls\phi_n(t)$ μ - a.e. and by the above Lemma $\phi(\cdot)$ has a measurable graph. Hence, all the assumptions of Corollary 4.1 and Theorem 5.2 are satisfied and it follows that

$$(5.2) \quad \int \phi = \int Li\phi_n \subset Li \int \phi_n \subset w\text{-}Ls \int \phi_n \subset c\ell \int w\text{-}Ls\phi_n = c\ell \int \phi .$$

Therefore,

$$c\ell \int_T \phi(t)d\mu(t) = Li \int \phi_n(t)d\mu(t) = w\text{-}Ls \int_T \phi(t)d\mu(t),$$

i.e.,

$$\int_T \phi_n(t)d\mu(t) \xrightarrow{K} c\ell \int_T \phi(t)d\mu(t).$$

Furthermore, if $\phi(\cdot)$ is convex valued it follows from the second conclusion of Theorem 5.2 that (5.2) can be written as:

$$\int \phi = \int \mathrm{Li}\phi_n \subset \mathrm{Li} \int \phi_n \subset w - \mathrm{Ls} \int \phi_n \subset c\ell \int w - \mathrm{Ls}\phi_n = \int\phi$$

and thus we can conclude that

$$\int_T \phi(t)d\mu(t) = \mathrm{Li} \int_T \phi_n(t)d\ (t) = w - \mathrm{Ls} \int_T \phi_n(t)d\mu(t),$$

i.e.,

$$\int_T \phi_n(t)d\mu(t) \overset{K}{\rightarrow} \int_T \phi(t)d\mu(t).$$

This completes the proof of the Theorem.

FOOTNOTES

1. Schmeidler [17] has obtained a finite dimensional version of Aumann's dominated convergence theorem, as well.

2. For the sequence $\phi_n : T \to 2^X$ to be integrably bounded we mean that there exists $h \in L_1(\mu)$ such that $\sup\{\|x\| : x \in \phi_n(t)\} \leqslant h(t)$ μ - a.e., for all n.

3. If in Proposition 4.1 we add the assumption that $\psi(\cdot,\cdot)$ is convex valued and that for all $(t,p) \in T \times P$, $\psi(t,p) \subset K$, where K is a weakly compact, convex nonempty subset of X, then it follows from Lemma 4.1 that $\int_T \psi(t,\cdot)d\mu(t)$ is weakly compact valued and we can conclude that $\int_T \psi(t,\cdot)d\mu(t)$ is weakly u.s.c., i.e., the set $\{p \in P : \int_T \psi(t,p)d\mu(t) \subset V\}$ is open in P for every weakly open subset V of X.

4. The first part of the proof is the infinite dimensional extension of an argument used in [16].

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