A THEORY OF SIGNALLING AUCTIONS

by

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Abstract

The essential structural characteristic of an auction is that a bidder will either win or lose, and there is a different payoff function for each of these two outcomes. In this paper, a symmetric auction model is formulated and analyzed as a game with incomplete information. Special cases of this model include many price auction arrangements, but the specification allows for "bidding" competition on the basis of non-price signals such as design proposals, educational credentials, etc. Common properties and economic applications of such signalling auction models are discussed.

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Many resource allocation mechanisms can be modelled as auctions: agents submit signals or bids, and an auctioneer separates the "winners" from the "losers" on the basis of the signals received. A winning agent's payoff will generally depend on that agent's signal and on the specific attributes that characterize the preferences, opportunities, and abilities of that agent. Losing agents' payoffs may also depend on their signals and attributes. The essential structural characteristic of an auction is that each agent has two payoff functions, one for winning and one for losing. The simplest example is the auction of a prize to the highest bidder. The signal would be a monetary bid, and an individual's payoff function for winning would be the individual's monetary value of the prize minus the sum of the bid price and the cost of preparing the bid. The payoff function for losing would then be the negative of the bid preparation cost. The attributes in this example could be the prize values or bid preparation costs of individual agents.

The relevant signal in many auctions is not a price quotation. For example, "design competition" is probably much more important than price competition for most military procurement contracts in the initial development stages. Similarly, Foster [1] discussed a model of the competition for research grants in which the signal as perceived by the granting agency is a function of both the actual quality of the researcher's ideas and the effort devoted to the preparation of the grant request. The granting agency cannot observe the actual quality attributes and effort input separately, and therefore, grants are awarded on the basis of the signal received.

In equilibrium, it is possible to infer attributes from the signals received if equilibrium signals are uniquely related to agents' attributes. Spence [8], Riley [7], and others have analyzed labor market models in which the observed signal is a level of education, and the unobserved attribute is a prospective employee's ability. The signalling equilibrium is characterized by a one-to-one correspondence between a worker's education and marginal productivity. The competitive outcome is that each worker is paid a real wage that equals the signalled marginal product. In these formulations workers are always employed, and in this sense they always win. In other words, there is only one payoff function - the signalled marginal product minus the cost of education. However, students often compete for a limited number of prized jobs, and the analysis of auction models in which there is a possibility of unemployment or underemployment may turn out to be useful.

The focus of this paper is on the common properties of signalling auctions. In general, there are three relevant payoff functions - one for participating in the auction and winning, one for participating and losing, and one for not participating at all. These payoff functions are discussed in the first section. Differences in individual preferences and opportunities are parameterized by a single attribute variable, and payoffs for each auction outcome may depend on agents' attributes and signals. Agents do not know rival attribute parameters with certainty at the time signalling decisions are made, and the auction rule determines which signals are winners. This signalling competition is formulated as a noncooperative game with incomplete information in sections 2 and 3. The case in which all participants in an auction are automatically winners is considered in the fourth section. It is shown in the fifth section that the payoff function for not participating in the auction is an important factor in determining a "boundary condition" on equilibrium signalling behavior. The concluding section contains a brief discussion of potential applications of signalling auction models.

1. Payoff Functions

Consider an auction with n potential competitors. Each agent is characterized by an attribute which is a real number denoted by a_i, $i = 1, \ldots n$. Each agent has the option of making a signal; signals are denoted by s_i, $i = 1, \ldots n$. This paper considers auctions in which the m agents with the greatest signals are designated as winners; where $m \leqq n$.[1] Finally, technological or institutional considerations may result in a minimal signal $\underline{s}$ with the property that a signal below $\underline{s}$ cannot win under any conditions.

This signalling auction will be formulated as a noncooperative game with incomplete information. The analysis in later sections will focus on the construction of Nash equilibria which are characterized by a strictly increasing, differentiable function $\sigma(\cdot)$ that determines optimal signals in equilibrium: $s_i = \sigma(a_i)$, $i = 1, \ldots n$. In this situation, equilibrium signals will reveal agents' attributes: $a_i = \sigma^{-1}(s_i)$, $i = 1, \ldots n$, where $\sigma^{-1}(\cdot)$ is the inverse of the $\sigma(\cdot)$ function.

The auction payoff functions determine an agent's final wealth position in the event of a win or loss. The analysis in this paper is for a symmetric auction in which all agents have the same monetary payoff functions. For a typical agent i, the payoff for winning may depend on the signal s_i, the attribute a_i, and the signalled atrribute which will be denoted by $\hat{a}_i$. If the equilibrium $\sigma(\cdot)$ function is strictly increasing then $\hat{a}_i = \sigma^{-1}(s_i)$, $i = 1, \ldots n$. However, it is not assumed that signalled attributes must equal true attributes, and therefore both a_i and $\hat{a}_i$ may be arguments in a winner's payoff function. In the event of a loss, the payoff is assumed to depend on the signal and the actual attribute. This attribute dependence can occur if attributes represent initial wealth or opportunities for losers to make money in other ventures. A loser's payoff will depend on the signal when signals involve costs that are not recovered in loss. Finally, agents have the option of not competing in the auction; there would be no signal, but the monetary payoff could depend on the agent's attribute. The payoff functions for winning, losing, and not competing will be denoted by $\pi^W(s, a, \hat{a})$, $\pi^L(s, a)$, and $\pi^N(a)$, where the i subscripts have been suppressed. It is useful to consider several examples of signalling auctions before introducing specific payoff function assumptions.

The first analysis of a signalling auction that I am aware of is a bidding model in Vickrey [9]. In Vickrey's formulation, a prize is awarded to the agent submitting the greatest sealed bid. Thus the monetary bid would be the signal s. An agent's attribute is that agent's monetary value of the prize; bidders may have different values because of differences in tastes. Thus the payoff function for a winning bidder with attribute a_i would be $a_i - s_i$. This model could be modified by adding a fixed bid preparation cost k, and the payoff functions would be:

$$\text{prizes:} \qquad \begin{aligned} \pi^W &= a - s - k \\ \pi^L &= -k \\ \pi^N &= 0 \end{aligned} \tag{1}$$

where $k \geqq 0$. The $\underline{s}$ lower bound on signals would represent a minimal bid; for example, $\underline{s} = 0$ means that negative bids are not allowed.

The formulation in (1) could also be interpreted as the sealed bid auction of a fixed-price procurement contract to the firm submitting the lowest bid: the bid would be $-s$, so the winner (with the lowest bid) is the bidder with the highest signal. A firm's attribute parameter could represent the negative of the firm's production cost, i.e., the bidder with the greatest attribute parameter has the lowest cost. The winner's profit for a fixed price contract is the bid price minus production and bid preparation costs, and this would then be the π^W function in (1).

A second example is the analysis of incentive contract bidding in Holt [5]. A winning firm's payoff determined by an incentive contract is $\alpha p + \beta(p - c)$, where p is the firm's winning bid, and c is the known cost of production.[2] It is assumed that $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, and either α or β is strictly positive. Losing firms are assumed to use their resources in some other operation, and the firm's attribute parameter is determined by the profit in its best alternative operation. Let $s = -p$, and let the attribute a be the negative of the best alternative profit. If the bid preparation cost is k, the payoff functions would be:

$$\text{incentive contracts:} \quad \begin{aligned} \pi^W &= \alpha(-s) + \beta(-s-c) - k \\ \pi^L &= -a-k \\ \pi^N &= -a \end{aligned} \qquad (2)$$

where $k \geqq 0$, $c > 0$, $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, and $\alpha\beta > 0$. Note that high attributes correspond to low opportunity costs, and high signals correspond to low bids. Thus a restriction that $s \geqq \underline{s}$ would mean that $-\underline{s}$ is the maximum bid which will be considered by the procuring agent.

In each of the bidding examples, the payoff in the event of a loss is independent of the signal. However, there are many auctions in which increasing the signal entails real resource costs. Foster's [1] discussion of faculty competition for summer research grants fits into this framework. Let the true quality of the i^{th} professor's research proposal be denoted by a_i. The granting agency is unable to observe a_i directly. By increasing the time devoted to writing a proposal, the professor is able to increase the quality level s_i that is perceived by the granting agency. Foster assumed that the time cost of making a signal s_i is s_i/a_i, where $a_i > \underline{a} > 0$ for $i = 1, \ldots n$. Suppose there is a fixed time cost k associated with the completion of the routine parts of the application forms. If the time value of a grant is denoted by v, the payoffs in time units are:

$$\text{faculty research grants:} \quad \begin{aligned} \pi^W &= v - s/a - k \\ \pi^L &= -s/a - k \\ \pi^N &= 0 \end{aligned} \qquad (3)$$

where $a > \underline{a} > 0$ and $k \geqq 0$. Here it is natural to assume that $\underline{s} \geqq 0$.

The final example is the Spence signalling model discussed in the introduction. In his formulation, s_i would represent a measure of the i^{th} worker's "education", and a_i would be the worker's "ability" as measured by marginal product. The simplest assumption is that education is unproductive in the sense that it does not alter a person's (marginal) productivity. If the equilibrium is characterized by a strictly increasing signalling function: $\sigma(a) = s$, then the i^{th} worker's competitive wage would be the signalled marginal product $\hat{a} = \sigma^{-1}(s_i)$.[3] The minimal educational requirement is $\underline{s}$. For example, s might represent a student's grade point average in a specific degree program, and $\underline{s}$ would be the lowest passing grade.

Signalling is costly in these formulations, and the educational cost function will be denoted by $c(s, a)$. It is assumed that education is expensive in a total and marginal sense: $c(\underline{s}, a) \geqq 0$ and $\partial c/\partial s > 0$. It is also assumed that both the total and marginal cost of education are decreasing functions of the ability attribute: $\partial c/\partial a < 0$ and $\partial^2 c/\partial s\, \partial a < 0$.

A "win" in this example would mean that a worker is employed in the desired occupation, so $\pi^W = \hat{a} - c(s, a)$. Workers who lose and those who do not participate in the auction are assumed to receive a fixed alternative wage $\bar{w}$. For example, $\bar{w}$ may be an unemployment payment or the wage of an unskilled worker in a full employment industry. (If $\bar{w}$ represents the earnings of a self-employed worker, then it would be more natural to assume that $\bar{w}$ is an increasing function of a .)

To summarize:

$$\text{labor market signalling:}\quad \begin{aligned} \pi^W &= \hat{a} - c(s, a) \\ \pi^L &= \bar{w} - c(s, a) \\ \pi^N &= \bar{w} \end{aligned} \qquad (4)$$

where $\bar{w} > 0$, $c(\underline{s}, a) \geqq 0$, $\partial c/\partial s > 0$, $\partial c/\partial a < 0$, and $\partial^2 c/\partial s\, \partial a < 0$. Note that the standard labor market signalling formulation with $m = n$ is not a true auction in the sense that everyone is employed.

The essential common properties of the payoff functions in signalling auction models with $m < n$ can be expressed:

Assume that there is an open interval $(\underline{a}, \bar{a})$ of possible values of agents' attributes. Let

$$X = \{(s, a, \hat{a});\ s \geqq \underline{s},\ a \in [\underline{a}, \bar{a}],\ \hat{a} \in [\underline{a}, \bar{a}]\} .$$

For all $(s, a, \hat{a})$ in some open set Y containing X, the functions π^W, π^L, $\pi^N : Y \to R$ satisfy:

<u>A1</u> π^W, π^L, and π^N are twice continuously differentiable, and $\partial\pi^L/\partial\hat{a} = 0$, $\partial\pi^N/\partial\hat{a} = 0$, and $\partial\pi^N/\partial s = 0$;

<u>A2</u> signals are costly: $\partial\pi^W/\partial s \leqq 0$, $\partial\pi^L/\partial s \leqq 0$; and at least one inequality is strict at each point in Y;

<u>A3</u> high signalled attributes are good: $\partial\pi^W/\partial\hat{a} \geqq 0$;

<u>A4</u> losses are bad: $\pi^L(\underline{s}, a) \leqq \pi^N(a)$;

<u>A5</u> high attributes provide a signalling advantage:
(a) $\partial\pi^W/\partial a \geqq \partial\pi^L/\partial a$,
(b) $\partial^2\pi^W/\partial s\, \partial a \geqq 0$, $\partial^2\pi^L/\partial s\, \partial a \geqq 0$,
(c) $\partial^2\pi^W/\partial\hat{a}\, \partial a \geqq 0$,
and at least one of these four inequalities is strict at each point in Y. (It does not have to be the same inequality that is strict at each point.)

Assumption A5 provides an incentive for agents with high attributes to make high signals. For example, a higher signal will mean a greater chance of a win, and A5(a) makes this more attractive for an agent with a higher attribute. Strict inequalities in A5(b) would mean that higher attributes result in lower marginal costs of signalling. A strict inequality in A5(c) means a higher attribute increases the marginal value of a higher signalled attribute.

These assumptions may be natural for many applications, and they are satisfied for the examples discussed in this section. If $m = n$, the π^L function is irrelevant, and these assumptions must be modified. This case will be considered in more detail in section 4.

2. A Symmetric Signalling Auction Game

Consider a game with n players or agents, each having the same monetary payoff functions π^W, π^L, and π^N which satisfy assumptions A1 - A5. In addition, all agents have the same von Neumann - Morgenstern utility function $U(\cdot)$ satisfying:

> A6 (utility) $U(\cdot): R \to R$ is twice continuously differentiable with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$.

Thus agents differ only with respect to the attributes which parameterize differences in opportunities or preferences.

It is assumed that agents know their own attributes with certainty, but they are uncertain about rival attributes. The assumed informational structure is symmetric in the sense that each agent's beliefs about rival attributes can be represented by the same probability distribution. The specific informational assumptions are:

A7 (attribute information) Each agent knows his own attribute and believes that rival attributes are independent realizations of a continuous random variable with a distribution function (d.f.) denoted by $G(\cdot)$. The corresponding probability density function (p.d.f.) is $g(\cdot)$, and $g(a) > 0$ on $(\underline{a}, \bar{a})$, zero elsewhere.

Harsanyi[3] gives several motivations for this informational structure.

The signalling competition can be analyzed as a "noncooperative game with incomplete information" as presented in Harsanyi [3], Nash equilibria in signalling auctions are typically characterized by a number a^*: $a^* \in [\underline{a}, \bar{a})$, and a strictly monotonic function $\sigma(\cdot)$: $\sigma(a) \geq \underline{s}$ if $a \in (a^*, \bar{a})$. A Nash equilibrium $\sigma(\cdot)$ function with these properties will be called a σ-equilibrium. For this class of equilibria, it is natural to assume that $\hat{a} = \sigma^{-1}(s)$. Acutally, a more precise version of this assumption will be useful:

A8 (signalled attributes) If the equilibrium signalling function $\sigma(\cdot)$ is strictly monotonic, then $\hat{a} = \sigma^{-1}(s)$ if $s \in [\sigma(a^*), \bar{a}]$ and $\hat{a} = a^*$ if $s \in [\underline{s}, \sigma(a^*))$.

In A8, $\sigma(a^*)$ denotes the limit (if it exists) of $\sigma(a)$ as $a \to a^*$ from above.

To summarize, the signalling auction game is characterized by:

a) n agents with attributes $a_i \in (\underline{a}, \bar{a})$, $i = 1, \ldots n$,

b) common monetary payoff and utility functions satisfying A1-A6

c) symmetric information as specified in A7,

d) a constraint that signals must not be less than a specified level $\underline{s}$,

e) an auction rule specifying a number m such that $m < n$ and the m agents with the largest signals will "win" a payoff determined by π^W, and

f) a rule A8 which determines the signalled attribute $\hat{a}$ corresponding to each feasible signal.

A σ-equilibrium is characterized by a number $a^* \in [\underline{a}, \bar{a})$ and a strictly nonotonic function $\sigma(\cdot) : (a^*, \bar{a}) \rightarrow [\underline{s}, \infty)$ which satisfies the appropriate Nash condition: For each agent i, $i = 1, \ldots n$, suppose that the agent's $n - 1$ rivals are known not to signal if $a_j \leqq a^*$, $j \neq i$, and are known to signal $s_j = \sigma(a_j)$ if $a_j > a^*$. Then

(i) the signal $\sigma(a_i)$ is a local maximizer of expected utility if $a_i > a^*$, and

(ii) the expected utility of an equilibrium signal $\sigma(a_i)$ is strictly greater than $U(\pi^N(a_i))$ if $a_i > a^*$, but if $a_i \leqq a^*$ there is no feasible signal which results in an expected utility that exceeds $U(\pi^N(a_i))$.

A σ-equilibrium with a strictly increasing $\sigma(\cdot)$ function will be called a strictly increasing σ-equilibrium.

It is straightforward to show that equilibrium signals will always have a chance of winning, and winning is better than losing if $m < n$. A more precise statement of this result is:

<u>Theorem 1</u>. If a σ-equilibrium exists for a signalling auction game, the equilibrium signal $\sigma(a)$ for $a > a^*$ will have a strictly positive probability of winning, and $\pi^W(\sigma(a), a, a) - \pi^L(\sigma(a), a) > 0$ for $a \in (a^*, \bar{a})$.

<u>Proof</u>. It is convenient to introduce a new random variable y , which is defined to be the attribute that ranks m in an independent sample of $n - 1$ realizations of the attribute variable. For example, if $m = 1$, y would be the maximum attribute obtained by a typical agent's $n - 1$ rivals. In a <u>strictly increasing</u> σ-equilibrium, the i^{th} agent's equilibrium signal can win in two ways: $y \leq a^*$, or $y > a^*$ and $\sigma(a_i) > \sigma(y)$. Obviously, y is an order statistic, and it is straightforward to compute its d.f. $F(y)$ and p.d.f. $f(y)$ from n , m , and the attribute distribution $G(\cdot)$. With $\sigma(\cdot)$ strictly increasing, the event $\sigma(a_i) > \sigma(y)$ occurs when $y < \sigma^{-1}(\sigma(a_i)) = a_i$. Thus the probability of winning with an equilibrium signal $\sigma(a_i)$, $a_i > a^*$, is the probability that $y < \max\{a^*, a_i\} = a_i$, so $F(a_i)$ is the probability of winning with an equilibrium signal. The signalled attribute in equilibrium is $\sigma^{-1}(\sigma(a_i)) = a_i$, so the equilibrium expected utility when $a_i > a^*$ is:

$$U\left(\pi^W(\sigma(a_i), a_i, a_i)\right) F(a_i) + U\left(\pi^L(\sigma(a_i), a_i)\right) [1 - F(a_i)] \quad . \tag{5}$$

It follows from the definition of a σ-equilibrium that this expected utility exceeds $U(\pi^N(a_i))$, or equivalently,

$$F(a_i) \left[U(\pi^W) - U(\pi^L)\right] > U(\pi^N) - U(\pi^L) \tag{6}$$

for $a_i > a^*$, where the payoff functions are evaluated at $(\sigma(a_i), a_i, a_i)$. The next step is to determine the sign of the right side of this inequality. It follows from A2 and A4 that

$$U(\pi^N(a)) \geqq U(\pi^L(\underline{s}, a)) \geqq U(\pi^L(s, a))$$

where $s \geqq \underline{s}$. Therefore, the right side of (6) is non-negative, and the left side is strictly positive. By redefining y and the probability of winning, this proof can be modified to cover the other case of a strictly decreasing $\sigma(\cdot)$ function.

Q.E.D.

3. Equilibrium $\sigma(\cdot)$ Functions

This section considers the determination of an equilibrium $\sigma(\cdot)$ function defined on $(a^*, \bar{a})$. The specification of an appropriate a^* is closely related to the option of not signalling, and this is the topic of section 5.

Not surprisingly, it is useful to begin by analyzing a single agent's optimal signalling behavior when the agent knows the precise $\sigma(\cdot)$ function being used by all $n - 1$ rivals. Specifically, suppose that the i^{th} agent's $n - 1$ rivals are known to use a strictly increasing and differentiable function $\sigma(\cdot)$:

$$s_j = \sigma(a_j) \geqq \underline{s} \text{ if } a_j \in (a^*, \bar{a}), \quad j \neq i .$$

Thus a signal s_i will win if $y \leqq a^*$ or if $y > a^*$ and $s_i > \sigma(y)$, where y is the order statistic defined in the previous section. The probability of winning with $s_i \geqq \underline{s}$ is the probability that $y < \max\{a^*, \sigma^{-1}(s_i)\}$.

Thus there are two cases to be considered: if $\underline{s} \leqq s_i \leqq \sigma(a^*)$, the probability of winning with s_i is $F(a^*)$; if $s_i > \sigma(a^*)$, the probability of winning is $F(\sigma^{-1}(s_i))$.

Consider the optimal choice of s_i subject to an artificial constraint that $s_i > \sigma(a^*)$. In this case, the i^{th} agent's objective function is:

$$U\left(\pi^W(s_i, a_i, \sigma^{-1}(s_i))\right) F(\sigma^{-1}(s_i)) + U\left(\pi^L(s_i, a_i)\right) [1 - F(\sigma^{-1}(s_i))] .$$

This expression will be denoted by $U^*(s_i, a_i, \sigma^{-1}(s_i))$.

The necessary condition for an interior maximum is that $\partial U^*/\partial s_i = 0$, or:

$$U'(\pi^W)[\pi_1^W + \pi_3^W \frac{d}{ds_i} \sigma^{-1}(s_i)] F(\sigma^{-1}(s_i)) + U'(\pi^L) \pi_1^L [1 - F(\sigma^{-1}(s_i))]$$

$$+ [U(\pi^W) - U(\pi^L)] f(\sigma^{-1}(s_i)) \frac{d}{ds_i} \sigma^{-1}(s_i) = 0 \qquad (7)$$

where the subscripts on the payoff functions denote partial derivatives, and the arguments of these functions have been suppressed. The signal s_i that solves (7) will generally depend on the $\sigma(\cdot)$ function used by other agents. But this $\sigma(\cdot)$ function is not arbitrary; it must satisfy the "fixed point" equilibrium condition that the i^{th} agent's best response s_i to the rivals' $\sigma(\cdot)$ function is a signal determined by the <u>same</u> $\sigma(\cdot)$ function:

$$s_i = \sigma(a_i), \quad a_i = \sigma^{-1}(s_i),$$

and

$$\frac{d}{ds_i} \sigma^{-1}(s_i) = \frac{1}{\sigma'(\sigma^{-1}(s_i))} = \frac{1}{\sigma'(a_i)}$$

for $a_i \in (a^*, \bar{a})$, $i = 1, \ldots n$. With these substitutions, equation (7) becomes a condition for determining the <u>equilibrium</u> signalling function $\sigma(a)$:

$$U'\left(\pi^W(\sigma(a), a, a)\right)\left[\pi_1^W(\sigma(a), a, a) + \frac{\pi_3^W(\sigma(a), a, a)}{\sigma'(a)}\right] F(a)$$

$$+ \; U'\left(\pi^L(\sigma(a), a)\right) \pi_1^L(\sigma(a), a)\,[1 - F(a)]$$

$$+ \; \left[U\left(\pi^W(\sigma(a), a, a)\right) - U\left(\pi^L(\sigma(a), a)\right)\right] \frac{f(a)}{\sigma'(a)} = 0 \qquad (8)$$

for $a \in (a^*, \bar{a})$. Thus the equilibrium $\sigma(\cdot)$ function satisfies a first order differential equation:

$$\sigma'(a) = \frac{[U(\pi^W) - U(\pi^L)]\, f(a) + U'(\pi^W)\, \pi_3^W F(a)}{- U'(\pi^W)\, \pi_1^W F(a) - U'(\pi^L)\, \pi_1^L [1 - F(a)]} \qquad (9)$$

for $a \in (a^*, \bar{a})$, where the payoff functions and their partial derivatives are evaluated at the point $(\sigma(a), a, a)$.

Recall that, in the derivation of (9), it was assumed that $s_i > \sigma(a^*)$ and that $\sigma'(\cdot) > 0$ on $(a^*, \bar{a})$. It must now be shown that these assumptions are satisfied in equilibrium. First, $s_i = \sigma(a_i)$ for $a_i > a^*$, so $s_i > \sigma(a^*)$ if $\sigma'(\cdot) > 0$. To check the sign of $\sigma'(\cdot)$, recall the implication of theorem 1 that any equilibrium $\sigma(\cdot)$ function must have the property that winning is better than losing: $\pi^W(\sigma(a), a, a) > \pi^L(\sigma(a), a)$ for $a > a^*$. Then it follows from <u>A1</u>, <u>A2</u>, <u>A3</u>, <u>A6</u>, and <u>A7</u> that any equilibrium $\sigma(\cdot)$ function solving (9) will be strictly increasing. Of course, this is not a proof that <u>any</u> solution to (9) will be strictly increasing, nor is it a proof that a solution exists.

Next, it is necessary to check the requirement that the signals determined by the $\sigma(\cdot)$ function solving (9) will be local maximizers of an agent's expected utility, when all others are known to be using the same $\sigma(\cdot)$ function. This leads to a consideration of the sign of the second derivative of expected utility: $\partial^2 U^*(s_i, a_i, \sigma^{-1}(s_i))/\partial s_i^2$, when $s_i = \sigma(a_i)$ <u>and</u> $\sigma^{-1}(\cdot)$ is the inverse of a solution to (9). At first, this appears to be a messy task, but the equilibrium differential equation in (8) or (9) can be used to simplify matters. Recall that (8) holds for all $a \in (a^*, \bar{a})$, so the derivative of the left side of (8) with respect to the attribute parameter a must equal zero on this interval. It is straightforward but tedious to show that this derivative condition can be expressed:[4]

$$\sigma'(a) \left[\left. \frac{\partial^2 U^*}{\partial s^2} \right|_{s = \sigma(a)} \right] + \Delta(a) = 0 \qquad (10)$$

where

$$\begin{aligned} \Delta(a) \equiv\ & U''(\pi^W)\pi_2^W[\pi_1^W + \pi_3^W/\sigma'(a)]F(a) \\ &+ U''(\pi^L)\pi_2^L \pi_1^L[1 - F(a)] + U'(\pi^W)[\pi_{12}^W + \pi_{23}^W/\sigma'(a)]F(a) \\ &+ U'(\pi^L)\pi_{12}^L[1 - F(a)] + [U'(\pi^W)\pi_2^W - U'(\pi^L)\pi_2^L]\frac{f(a)}{\sigma'(a)} \end{aligned} \qquad (11)$$

and all payoff functions in $\Delta(a)$ are evaluated at the point $(\sigma(a), a, a)$. The formulation in (10) is useful because the second order condition is satisfied for a specific value of the attribute if $\Delta(a) > 0$.

Theorem 2 summarizes this result.

Theorem 2. If $\sigma(\cdot)$ is a strictly increasing function that solves (9), then a necessary and sufficient condition for $\sigma(a)$ to be a local maximizer of expected utility is that $\Delta(a)$ in (11) is strictly positive.

For the case of risk neutral agents $(U''(\cdot) = 0)$, assumption A5 implies that $\Delta(a) > 0$ if $a \in (a^*, \bar{a})$. For models with risk averse agents, a more involved analysis of (8) and (11) may be necessary because the assumption that $\pi_2^W > \pi_2^L$ does not imply that $U'(\pi^W)\,\pi_2^W > U'(\pi^L)\,\pi_2^L$. Additional restrictions on the payoff functions in many applications will ensure that $\Delta(a) > 0$. In the bidding model (1), $\Delta(a) = -\,U''(a - \sigma(a))\,F(a) + U'(a - \sigma(a) - k)\,f(a)/\sigma'(a)$, so $\Delta(a) > 0$ if $U''(\cdot) \leqq 0$. Similarly, it can be shown that $\Delta(a) > 0$ for the contract auction model (2) when $U''(\cdot) \leqq 0$. I have been unable to determine the sign of $\Delta(a)$ for the research grant auction (3) when $U''(\cdot)$ is strictly negative. The sign of $\Delta(a)$ in the labor market signalling model (4) will be considered in section 4.

Given an initial condition, it is often possible to explicitly compute the equilibrium signalling function by solving the differential equation (9). For example (1) with risk neutrality $(U(x) = x)$, equation (9) is: $\sigma' = [a - \sigma]\,f(a)/F(a)$. Given an initial condition $(\sigma(a^*), a^*)$, it is straightforward to verify that the unique solution is:

$$\sigma(a) = \frac{F(a^*)\ \sigma(a^*) + \int_{a^*}^{a} y\ f(y)\ dy}{F(a)}$$

for $a \in (a^*, \bar{a})$. It is not possible to obtain an explicit solution when agents are risk averse in this model, but it is possible to analyze some of the equilibrium effects of changes in risk aversion.[5] In contrast, there is an explicit solution for $\sigma(\cdot)$ in the incentive contract bidding example (2) when bidders are risk averse; see Holt [5].

Finally, consider the research grant example (3). If agents are risk neutral, equation (9) becomes: $\sigma' = v\,a\,f(a)$, so the solution is:

$$\sigma(a) = \sigma(a^*) + v \int_{a^*}^{a} y\, f(y)\, dy$$

for $a \in (a^*, \bar{a})$.

The labor market signalling example (4) is discussed in the next section, and the specification of initial values $(\sigma(a^*), a^*)$ is considered in section 5.

4. Signalling Models with $m = n$

Winning is a certain event when $m = n$, and the previous analysis can be altered to include this case. Recall that $F(\cdot)$ is the d.f. of the order statistic of rank m in a sample of $n - 1$ realizations of the attribute variable. When $m = n$, $F(\cdot)$ is undefined, and it is natural to adopt the convention that $F(a) = 1$ for $a \in (\underline{a}, \bar{a})$ in this case. When the probability of winning is one, the π^L function is irrelevant, and it is convenient to adopt the convention that $\pi^L(s, a) \equiv \pi^W(s, a, \underline{a})$ for $s \geqq \underline{s}$ and $a \in (\underline{a}, \bar{a})$. With these two conventions, a game with $m = n$, which satisfies all of the other requirements in the definition of a signalling auction game, will be called a <u>signalling model with $m = n$</u>.

The convention that $\pi^L(s, a) = \pi^W(s, a, \underline{a})$ alters the implications of some of the payoff function assumptions. Recall that A2 required either $\partial\pi^W/\partial s$ or $\partial\pi^L/\partial s$ to be strictly negative at each point in Y. When $\pi^L(s, a) = \pi^W(s, a, \underline{a})$, A2 implies that $\partial\pi^W/\partial s < 0$. Similarly, A4 implies that $\pi^W(\underline{s}, a, \underline{a}) \leqq \pi^N(a)$ in signalling models with $m = n$. Finally, the implication of A5 is that $\partial^2\pi^W/\partial s\,\partial a \geqq 0$, $\partial^2\pi^W/\partial\hat{a}\,\partial a \geqq 0$, and at least one of these inequalities must be strict. Therefore, there are still two ways of providing a signalling advantage to agents with high attributes when $m = n$: $\pi^W_{12} > 0$ or $\pi^W_{23} > 0$. In the simplest labor market signalling model (4), $\pi^W_{23} = 0$ and it is necessary to assume that $\pi^W_{12} < 0$, i.e., the marginal cost of education is a strictly decreasing function of a worker's ability. On the other hand, it may be possible to have a signalling equilibrium in a signalling auction game $(m < n)$ when both π^W_{12} and π^W_{23} are everywhere zero; this is because an appropriate signalling advantage is present if $\pi^W_2 > \pi^L_2$.

The analysis of σ-equilibria in signalling models with $m = n$ is a special case of the analysis in the previous section. With $F(a) = 1$ and $f(a) = 0$ on $(\underline{a}, \bar{a})$, equations (7), (9), and (11) become:

$$\pi^W_1 + \pi^W_3 \frac{d}{ds_i} \sigma^{-1}(s_i) = 0 \qquad (7')$$

$$\sigma'(a) = \pi^W_3/(-\pi^W_1) \qquad (9')$$

$$\Delta(a) = U'(\pi^W)[\pi^W_{12} + \pi^W_{23}/\sigma'(a)] \qquad (11')$$

If $\pi^W_3 > 0$, it follows from A2 and A5 that $\sigma'(a)$ and $\Delta(a)$ in (9′) and (11′) are strictly positive, and therefore, theorem 2 would be valid. But A3 allows π_3^W to be identically zero, and equation (7′), which is a necessary condition for an interior solution, cannot be satisfied in this case. The optimal signal would always be the lowest possible signal $\underline{s}$, regardless of the signalling behavior of other agents. Indeed, there is nothing to be gained from a signal if $\pi_3^W = 0$, and agents would not signal if given the choice. In contrast, there is a potential benefit associated with higher signals when $m < n$ and $\pi_3^W = 0$, because higher signals have a greater probability of winning in a true signalling auction game with $m < n$. Therefore, it is possible to have a strictly increasing equilibrium $\sigma(\cdot)$ function even when $\pi_3^W = 0$, as is the case in examples (1) - (3).

Next, consider a simple example of a labor market signalling model which is presented in Spence [8]. In this example, $\pi^W(s, a, \hat{a}) = \hat{a} - s/a$, so equation (9′) becomes: $\sigma'(a) = a$. The solutions are a family of functions: $\sigma(a) = \sigma(a^*) + a^2/2$, where $\sigma(a^*)$ is the initial condition that parameterizes this family. Spence argued that the possibility of multiple solutions is important because an increase in the initial boundary value results in higher signals, and therefore, greater educational expenditures at all levels of ability. Education is not productive in this example, so some signalling solutions would presumably be preferred to others. Spence does suggest that competition among employers "on the signalling dimension" might determine a unique initial condition in this example.[6] His suggestion (in my notation) is that $a^* = \underline{a}$ and $\sigma(a^*) = 0$ if $\underline{s} = 0$, so $\sigma(a) = a^2/2$.

The uniqueness issue is also very important in signalling auction models because an upward shift in the $\sigma(\cdot)$ function might correspond to a speculative surge, a price war, or an increase in real resources devoted to signalling activities. The next section presents a more or less systematic method for determining unique initial conditions in a fairly general class of signalling auctions.

5. Boundary Conditions and the Option of Not Signalling

The most difficult step in the analysis of a particular signalling auction is often the specification of an appropriate boundary condition: $(\sigma(a^*), a^*)$. Recall that a^* separates the attributes of agents who compete in the auction from the attributes of those who do not compete. Thus the properties of the payoff function for not participating are likely to be important in any consideration of boundary conditions. Assumptions A2 and A3 imply that not participating is at least as attractive as participating and losing: $\pi^N(a) \geqq \pi^L(\underline{s}, a) \geqq \pi^L(s, a)$ for all $s \geqq \underline{s}$ and $a \in [\underline{a}, \bar{a}]$. If $\pi^N(a)$ is strictly greater than $\pi^L(\underline{s}, a)$, then the entry of a minimal signal results in a cost which is not recovered in the event of a loss. Thus there is <u>an entry fee at $\underline{a}$</u> if the payoff functions satisfy an assumption that

$$\pi^N(\underline{a}) > \pi^L(\underline{s}, \underline{a}) \quad . \tag{12}$$

This inequality holds for the bidding models (1) and (2) if the bid preparation cost k is strictly positive, and it also holds for the research grant competition model (3) if the fixed cost of preparing a grant application is strictly positive.[7]

When $m = n$, the entry fee inequality (12) and the convention that $\pi^L(s, a) = \pi^W(s, a, \underline{a})$ imply:

$$\pi^N(\underline{a}) > \pi^W(\underline{s}, \underline{a}, \underline{a}) \quad . \tag{12'}$$

This inequality is satisfied in the labor market signalling model (4) when $\bar{w} \geqq \underline{a}$ and $c(\underline{s}, \underline{a}) > 0$.

Theorem 3 provides a systematic method for specifying the appropriate boundary conditions for games with entry fees.

<u>Theorem 3</u>. Suppose that a strictly increasing σ-equilibrium exists (for a signalling auction game or a signalling model with $m = n$).

(i) If there is an entry fee at $\underline{a}$, then $a^* > \underline{a}$.

(ii) If $a^* > \underline{a}$, then $\sigma(a^*) = \underline{s}$ and a^* is determined:

$$U(\pi^W(\underline{s}, a^*, a^*))F(a^*) + U(\pi^L(\underline{s}, a^*))[1 - F(a^*)] = U(\pi^N(a^*)) \quad .$$

If $m = n$, the convention that $F(a) = 1$ implies that $\pi^W(\underline{s}, a^*, a^*) = \pi^N(a^*)$ in the conclusion (ii) above.

<u>Proof</u>.

(i) The definition of a strictly increasing σ-equilibrium in a signalling auction game requires that:

$$U\left(\pi^W(\sigma(a), a, a)\right)F(a) + U\left(\pi^L(\sigma(a), a)\right)[1 - F(a)] > U(\pi^N(a)) \tag{13}$$

for $a \in (a^*, \bar{a})$. Recall that $\sigma(\cdot)$ is strictly increasing and bounded from below by $\underline{s}$, so $\sigma(a^*)$ exists and $\sigma(a^*) \geqq \underline{s}$. The $U(\cdot)$, π^W, π^L, π^N, and $F(\cdot)$ functions are continuous by assumption, and it follows from these continuity assumptions and the existence of $\sigma(a^*)$ that (13) holds with a weak inequality when $a = a^*$:

$$U\left(\pi^W(\sigma(a^*), a^*, a^*)\right) F(a^*) \ + \ U\left(\pi^L(\sigma(a^*), a^*)\right) [1 - F(a^*)] \ \geqq \ U(\pi^N(a^*)) \quad (14)$$

Recall that $a^* \geqq \underline{a}$. Suppose that $a^* = \underline{a}$. Then $F(a^*) = 0$, and it follows from (14) that $U(\pi^L(\sigma(\underline{a}), \underline{a})) \geqq U(\pi^N(\underline{a}))$, or equivalently, $\pi^L(\sigma(\underline{a}), \underline{a}) \geqq \pi^N(\underline{a})$. Also, it follows from A2 that $\pi^L(\underline{s}, \underline{a}) \geqq \pi^L(\sigma(\underline{a}), \underline{a})$ because $\underline{s} \leqq \sigma(\underline{a})$. Connecting these inequalities with the entry fee inequality (12), one obtains a contradiction: $\pi^L(\underline{s}, \underline{a}) \geqq \pi^L(\sigma(\underline{a}), \underline{a}) \geqq \pi^N(\underline{a}) > \pi^L(\underline{s}, \underline{a})$. Therefore $a^* > \underline{a}$.

The proof of (i) for the case in which $m = n$ is similar. Suppose $a^* = \underline{a}$. Winning is certain, so $F(a^*) = 1$, and (14) implies that $\pi^W(\sigma(\underline{a}), \underline{a}, \underline{a}) \geqq \pi^N(\underline{a})$. Next, A2 and the inequality $\sigma(\underline{a}) \geqq \underline{s}$ imply that $\pi^W(\underline{s}, \underline{a}, \underline{a}) \geqq \pi^W(\sigma(\underline{a}), \underline{a}, \underline{a})$. The needed contradiction is obtained by connecting these inequalities with the entry fee condition (12′).

(ii) The cutoff attribute a^* in a strictly increasing σ-equilibrium has the property that

$$U(\pi^W(s, a, \hat{a})) F(a) \ + \ U(\pi^L(s, a))[1 - F(a)] \ \leqq \ U(\pi^N(a)) \quad (15)$$

for $a \leqq a^*$ and $s \geqq \underline{s}$. The signalled attribute assumption A8 is that $\hat{a} = a^*$ if $s \leqq \sigma(a^*)$. Then the inequalities (14) and (15) evaluated at $(s = \sigma(a^*),\ a = a^*,\ \hat{a} = a^*)$ imply that a^* satisfies:

$$U(\pi^W(\sigma(a^*), a^*, a^*)) F(a^*) \ + \ U(\pi^L(\sigma(a^*), a^*))[1-F(a^*)] \ = \ U(\pi^N(a^*)) \quad (16)$$

The proof will be complete if it can be shown that $\sigma(a^*) = \underline{s}$. Suppose that $\sigma(a^*) > \underline{s}$. Then there exists a signal $s_0 \in [\underline{s}, \sigma(a^*))$ which signals an attribute a^* because of A8. If $a = a^*$, it follows from (16) and A2 that the expected utility for a signal s_0 is strictly greater than the utility of not participating: $U(\pi^N(a^*))$. This contradicts the inequality in (15), so $\sigma(a^*) = \underline{s}$. It is straightforward to modify this proof for the $m = n$ case by using the convention that $F(a) = 1$.

Q.E.D.

As an example of the application of this theorem, consider the research grant model (3) when agents are risk neutral and $\underline{s} = 0$. There is an entry fee at $\underline{a}$ when $k > 0$. It follows that $\sigma(a^*) = \underline{s} = 0$ and a^* is determined: $F(a^*) = k/v$. This result has the intuitive property that as the ratio of the fixed cost of a grant application to the value of the grant increases to one, the cutoff a^* increases to its upper limit $\bar{a}$.

Next, consider the labor market signalling example when A8 is satisfied. There is an entry fee when $\bar{w} \geqq \underline{a}$ and $c(\underline{s}, \underline{a}) > 0$. In this case, $a^* > \underline{a}$, $\sigma(a^*) = \underline{s}$, and a^* is determined: $a^* - c(\underline{s}, a^*) = \bar{w}$. At the boundary a^*, the wage net of educational expenses equals the alternative wage $\bar{w}$. This is consistent with Riley's analysis of initial conditions in [6].

The conclusion of part (ii) of theorem 3 is not based on an entry fee assumption, so this result can be used in the analysis of auctions without entry fees. Specifically, suppose that losing with a signal $\underline{s}$ is equivalent to not participating:

$$\pi^L(\underline{s}, a) = \pi^N(a) \tag{17}$$

for $a \in (\underline{a}, \bar{a})$. The following theorem is often useful in this situation:

Theorem 4. Suppose that a strictly increasing σ-equilibrium exists in a signalling auction game satisfying (17). If $a^* > \underline{a}$, then a^* is determined: $\pi^W(\underline{s}, a^*, a^*) = \pi^L(\underline{s}, a^*)$.

Proof. When (17) holds, the equation in theorem 3 for determining a^* reduces to:

$$F(a^*)[U(\pi^W(\underline{s}, a^*, a^*)) - U(\pi^L(\underline{s}, a^*))] = 0 .$$

Since $a^* > \underline{a}$, $F(a^*) > 0$, and it follows from the monotonicity of $U(\cdot)$ that $\pi^W(\underline{s}, a^*, a^*) = \pi^L(\underline{s}, a^*)$.

Q.E.D.

This theorem would apply to examples (1) and (2) when $a^* > 0$ and the bid preparation cost k is zero. Then the implication of the theorem is that $a^* = \underline{s}$ in example (1), and $a^* = (\alpha + \beta)(-\underline{s}) - \beta c$ in example (2). Of course, $\underline{s}$ must be great enough so that the a^* determined in this way will exceed $\underline{a}$. Another possible application of this theorem is example (3) when $k = \underline{s} = 0$. The result is that if $a^* > \underline{a}$, a^* must solve the equation: $v = 0$. This is impossible, and therefore, $a^* = \underline{a}$ if a σ-equilibrium exists.

6. Conclusion

In my opinion, the signalling auction game formulation introduced in this paper is a natural and useful way to model many competitive processes with a dichotomous outcome: participants either win or lose. In this formulation, agents compete in a signalling dimension, but signals are not necessarily monetary bids. Therefore, potential applications are not limited to economic markets. For example, elections are either won or lost from a candidate's point of view, and some elements of signalling are probably important.

The emphasis in this paper is on the common structural and equilibrium properties of signalling auctions. The main results, summarized in theorems 1 - 4, should be useful in the construction or characterization of equilibrium signalling functions for specific auction models.

The payoff functions for winners, losers, and non-participants are exogenous structural elements in this analysis. The specification of appropriate payoff functions in any specific application must often be based on the equilibrium behavior of agents who determine the auction payoff structure. For example, in the auction of a procurement contract, the winners' payoff depends on the (equilibrium) contract offered by the procuring agent. Nevertheless, any careful study of determination of structural aspects of auctions should build on an equilibrium analysis of signalling competition under alternative structural assumptions.

FOOTNOTES

1. For an analysis of other auction procedures, see Vickrey [9], Holt [6], and Harris and Raviv [2].

2. Holt's analysis is for the more general case in which the cost of production is not known with certainty at the time bids are made. Auctions with more general contract profit functions are analyzed in Holt [4].

3. An assumption that the wage equals $\hat{a}$ must be based on a careful analysis of the equilibrium behavior of employers. See Riley [7] for a discussion of some difficulties with this assumption.

4. To verify (10), the energetic reader would have to compute $\partial^2 U^*(s_i, a_i, \sigma^{-1}(s_i))/\partial s_i^2$ and evaluate this derivative at $(\sigma(a_i), a_i, a_i)$. Next take the derivative of the left side of (8) and note that $d^2 \sigma^{-1}(s_i)/ds_i^2$, evaluated at $s_i = \sigma(a_i)$, is $-\sigma''(a_i)/[\sigma'(a_i)]^3$. The rest is algebra.

5. See Harris and Raviv [2].

6. Riley [7] discussed initial conditions in a more general labor market signalling model, and some of his results will be mentioned in section 5 of this paper.

7. Most existing analysis of bidding games is for models without entry fees in the form of bid preparation costs. N. J. Simler first suggested the importance of these costs to me.

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