COMPARATIVE STATICS UNDER UNCERTAINTY

by

Clifford Hildreth

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

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1. Introduction

Even in quite simple static models of choice under uncertainty, the response of optimal choice to changes in circumstances presents varied and interesting problems. The examples presented below concern the optimal amount or level of an uncertain venture to be undertaken by an expected utility maximizing risk averse decision maker (dm).

Let $(\Omega, \mathcal{F}, P)$ be a probability space in which the universal event Ω represents all possible developments in dm's environment and P represents his personal probabilities. dm is visualized as having certain initial investments, commitments and plans. If he does nothing to change these, his wealth at some future date is given by the random variable $X(\omega)$ called his initial prospect. ω, an element of Ω, is a particular development or sequence of events in dm's environment.

He is considering a new venture each unit of which will add $Y(\omega)$ to his future wealth if ω is realized in his environment. If he chooses α units of the venture $X(\omega) + \alpha Y(\omega)$ becomes his new prospect. He is presumed to choose α from his admissible set to maximize $E\varphi(X + \alpha Y)$ where φ is utility of wealth and the expectation is with respect to personal probability P.

The venture might be purchase or sale of securities, an insurance policy, a business contract, a position on a futures market or any undertaking whose effect can be approximated by an additive random variable.

Let

$$\eta(\alpha) = E\varphi(X + \alpha Y)$$

be dm's expected utility function. If

(1) $\varphi' > 0$, $\lim_{x\to\infty} \varphi'(x) = 0$

(2) $\varphi'' < 0$, φ'' monotonic

(3) $P(Y > 0) > 0$, $P(Y < 0) > 0$

(4) $\varphi(X + \alpha Y)$, $Y\varphi'(X + \alpha Y)$, $Y^2\varphi''(X + \alpha Y)$ are integrable for all $\alpha \varepsilon R$

then [1] η is strictly concave, assumes a unique maximum on R , and has the continuous derivatives

$$\eta'(\alpha) = EY\varphi'(X + \alpha Y)$$
$$\eta''(\alpha) = EY^2\varphi''(X + \alpha Y) .$$

(1) - (4) are assumed throughout the paper.

The choice $\hat{\alpha}$ that maximizes η for $\alpha \varepsilon R$ lies in an open interval $(0, \alpha^*)$ of favorable choices (better than $\alpha = 0$) if $\hat{\alpha} > 0$, or an interval $(\alpha^*, 0)$ if $\hat{\alpha} < 0$. Let $\langle\alpha^*\rangle$ designate whichever interval is relevant, i.e.,

$$\langle\alpha^*\rangle = \{\alpha : E\varphi(X + \alpha Y) > E\varphi(X)\} .$$

If $\eta'(0) = 0$ then $\hat{\alpha} = 0$, $\langle\alpha^*\rangle = \phi$.

Expected utility functions for $\hat{\alpha} > 0$ and $\hat{\alpha} < 0$ are shown in Figure 1.

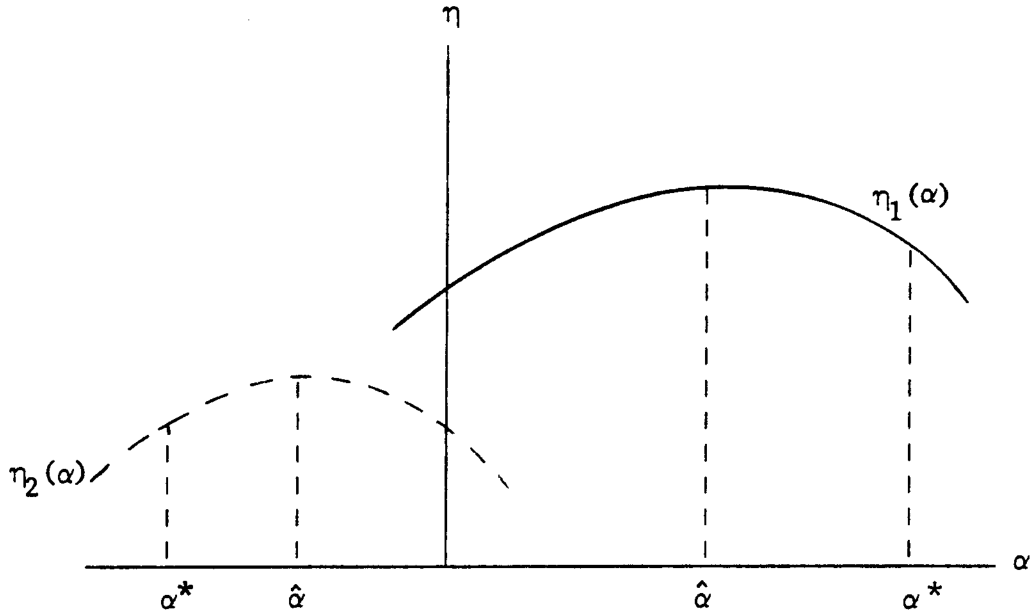

Figure 1

Expected utility functions for $\hat{\alpha} > 0$ $(\eta_1(\alpha))$ and $\hat{\alpha} < 0$ $(\eta_2(\alpha))$.

Inspection of the expected utility function readily verifies

Proposition 1. $\forall \alpha \varepsilon R$, $\eta'(\alpha) \overset{s}{=} (\hat{\alpha} - \alpha)$, where $\overset{s}{=}$ means "is equal in sign to."

In most applications, α must be chosen from some subset of R - e.g., some securities can only be purchased in positive amounts, there will usually be an upper bound on the amount of a venture determined by dm's resources. If $\underline{\mathcal{Q}}$ is the admissible set, let $\hat{\alpha}_{\mathcal{Q}}$ be a restricted optimum if one exists. $\hat{\alpha}_{\mathcal{Q}}$ maximizes $\eta(\alpha)$ for $\alpha \varepsilon \underline{\mathcal{Q}}$.

For problems like the above with just one venture under consideration it is usually convenient to first obtain or characterize $\hat{\alpha}$, α^* and then take account of the restrictions as follows -

(i) if $\hat{\alpha} \varepsilon \underline{\mathcal{A}}$ then $\hat{\alpha}_{\mathcal{A}} = \hat{\alpha}$

(ii) if $\underline{\mathcal{A}} \cap \langle\alpha^*\rangle = \phi$ then $\hat{\alpha}_{\mathcal{A}} = 0$
(we assume $\alpha = 0$ is always admissible)

(iii) if neither (i) nor (ii), let
$\bar{\alpha} = \inf \{\mathcal{A} \cap \langle\alpha^*\rangle \cap (\hat{\alpha}, \infty)\}$ and
$\underline{\alpha} = \sup \{\mathcal{A} \cap \langle\alpha^*\rangle \cap (-\infty, \hat{\alpha}\}$.
Choose α_o so that $\eta(\alpha_o) = \max \{\eta(\bar{\alpha}), \eta(\underline{\alpha})\}$.
If $\alpha_o \varepsilon \mathcal{A}$ then $\hat{\alpha}_{\mathcal{A}} = \alpha_o$. If not, $\hat{\alpha}_{\mathcal{A}}$ doesn't exist but $\forall \varepsilon > 0$ an α_ε can be found $\ni$ $\alpha_\varepsilon \varepsilon \mathcal{A}$ and $\eta(\alpha_\varepsilon) > \eta(\alpha_o) - \varepsilon$.

In the next three sections, we study responses of $\hat{\alpha}$ and α^* to selected kinds of changes in X, Y or P.

2. Uniform Changes in X or Y[1]

Write $X = W + b$, $Y = V + c$ where b, c are real numbers and W, V are random variables defined by the equations. If W remains fixed (as a fn on Ω) while b changes we say there is a uniform change in X. In applications, an increase in b represents a certain or non-contingent increase in dm's wealth. Examples might be a completely unexpected inheritance, cancellation of a debt, or revaluation of a sure asset. The analysis would also be relevant to the consideration of two individuals or groups of individuals who differ mainly in their general levels of wealth. A uniform change in Y is an increase or decrease in c. If Y is a common stock it is natural to think of V as the prospective return (dividends plus eventual sale value) to holding a share and $-c$ as the price of a share. If the price changes with no change in returns associated with various environmental contingencies, there has been a uniform change in Y.

We want to know how $\hat{\alpha}$, α^* respond to changes in b, c so we assume that W, V are fixed and b, c allowed to vary. Expected utility may then be written

$$\eta(\alpha;\ b,c) = E\varphi(W + b + \alpha(V + c))$$

and the unique optimal choice for given b, c denoted by $\hat{\alpha}(b, c)$ which is defined implicitly by the equation

$$(2.1) \qquad D_\alpha\eta(\hat{\alpha};\ b,c) = E(V + c)\ \varphi'(W + b + \hat{\alpha}(V + c)) = 0$$

where $D_\alpha\eta$ is the partial derivative of η with respect to α. If the condition

$$(5)\ \exists\ \varepsilon > 0 \ni |\delta| < \varepsilon \Rightarrow \varphi'(X + \alpha Y + \delta),\quad Y\varphi''(X + \alpha Y + \delta),$$
$$Y^2\varphi''(X + \alpha Y + \delta) \text{ are integrable for all } \alpha \varepsilon R$$

is imposed along with (1) - (4), page 2, then the necessary continuous second partials of η can be shown[2] to exist and, by the implicit function theorem,

$$(2.2) \qquad D_b\hat{\alpha} = -\frac{D^2_{\alpha b}\eta}{D^2_{\alpha\alpha}\eta} = -\Delta^{-1}\ E(V + c)\ \varphi''(W + b + \hat{\alpha}(V + c))$$

$$= -\Delta^{-1}\ E\,Y\varphi''(X + \hat{\alpha}Y)$$

$$(2.3) \qquad D_c\hat{\alpha} = -\frac{D^2_{\alpha c}\eta}{D^2_{\alpha\alpha}\eta} = -\Delta^{-1}[E\varphi'(X + \hat{\alpha}Y) + \hat{\alpha}E\,Y\varphi''(X + \hat{\alpha}Y)]$$

$$= -\Delta^{-1}E\varphi'(X + \hat{\alpha}Y) + \hat{\alpha}D_b\hat{\alpha}$$

where D^2_{gh} stands for second partial derivative with respect to g and h, and

$$\Delta = D^2_{\alpha\alpha}\eta = EY^2\varphi''(X + \hat{\alpha}Y) < 0 \quad \text{since } \varphi'' < 0 .$$

The two terms following the last equality in equation (2.3) are close analogues of the substitution and income effects in Slutsky's equation for consumer equilibrium. Since $\Delta < 0$ and $\varphi' > 0$ the first term is always positive. $D_b \hat{\alpha}$ will be called the wealth response, $D_c \hat{\alpha}$ the venture response, and $\hat{\alpha} D_b \hat{\alpha}$ the wealth effect.

A number of circumstances in which one might determine at least the signs of $D_b \hat{\alpha}$ and/or $D_c \hat{\alpha}$ were examined in [1], [2] and illustrative applications sketched. Among these results are

<u>Proposition 2</u>. If absolute risk aversion $r(x) = -\frac{\varphi''(x)}{\varphi'(x)}$ is constant then $D_b \hat{\alpha} = 0$, $D_c \hat{\alpha} > 0$.

<u>Proposition 3</u>. If $r' < 0$, φ''' is monotonic, X is independent of Y, and $\varphi^{(n)}(X + y)$ is integrable for all $y \in R$ and $n = 0, \cdot\cdot, 3$; then

$$EY \gtreqless \hat{\alpha} \gtreqless D_b \hat{\alpha} \quad .$$

<u>Proposition 4</u>. If $\exists\, \tilde{x} \ni (Y > 0) = (X > \tilde{x})$, if $\hat{\alpha} > 0$, and if $r' < 0$; then $D_b \hat{\alpha} > 0$.

Proofs may be found in [2], pages 10 - 12. The first condition of Proposition 4 might often be realized if the venture is expansion of an existing business. The expansion will only increase net returns under conditions such that the original business would have done pretty well. The proposition says that if such a venture is favorable, the optimal size of the venture is an increasing function of the entrepreneur's wealth.

Now consider the response of α^*, the boundary of the favorable set, to uniform changes in the initial prospect and the venture. α^* is the nonzero solution (if $\nexists$ a nonzero solution set $\alpha^* = 0$) to

$$\eta(\alpha;\, b,c) = E\varphi(W + b + \alpha(V + c)) = \eta(0;\, b,c) = E\varphi(W + b) \quad .$$

Again hold W, V fixed. Let $\alpha^*(b, c)$ be the solution corresponding to a pair (b, c). Let

$$\begin{aligned} \theta(\alpha;\, b,c) &= \eta(\alpha;\, b,c) - \eta(0;\, b,c) \\ &= E\varphi(W + b + \alpha(V + c)) - E\varphi(W + b) \quad . \end{aligned}$$

By the implicit function theorem,

$$(2.4) \qquad D_b\alpha^* = -\frac{D_b\theta}{D_\alpha\theta} = -(D_\alpha\theta)^{-1}\, E[\varphi'(X + \alpha^*Y) - \varphi'(X)]$$

$$(2.5) \qquad D_c\alpha^* = -\frac{D_c\theta}{D_\alpha\theta} = -(D_\alpha\theta)^{-1}\, \alpha^*(E\varphi'(X + \alpha^*Y))$$

where $\quad D_\alpha\theta = EY\varphi'(X + \alpha^*Y) \lesseqgtr -\alpha^* \quad$ (see Figure 1)

and $\quad D_c\theta = \alpha^*(E\varphi'(X + \alpha^*Y)) \lesseqgtr \alpha^* \quad (\varphi' > 0)$.

Thus, by (2.5), $D_c\alpha^* > 0$ except possibly when $\alpha^* = \hat{\alpha} = 0$. When $\alpha^* = 0$, it can be shown[3] that any uniform increment in Y leads to a positive α^* so we may conclude that

<u>Proposition 5</u>. $\alpha^*(b, c)$ is a strictly increasing function of c.

This means that a uniform improvement in the venture enlarges $\langle\alpha^*\rangle$ if positive amounts were originally favorable and may diminish $\langle\alpha^*\rangle$ or shift it from the negative to the positive half line if negative amounts were initially favorable. It appears that sign $D_b\alpha^*$ is indeterminate without further assumptions.

We now consider responses of $\hat{\alpha}$, α^* to other kinds of changes in X, Y, P.

3. <u>Improvement in X or Y on an Event</u>

For any random variables V and W, let $V \gtrsim W$ mean $V \geq W$ a.s. Let $V \gtrsim W$ mean $V \gtrsim W$ and $P(V > W) > 0$. For $V \tilde{>} W$ we say "V exceeds W."

Suppose a dm's initial prospect improves in the following sense - X is replaced by X + Z where $Z \tilde{>} 0$. We are interested in the effect on his optimal choice $\hat{\alpha}$ and his favorable set $\langle\alpha^*\rangle$.

Such a change could come about in many ways. A change in tax laws might mean that dm will have a lower tax liability under some contingencies. These contingencies then comprise the event $(Z > 0)$. If legislation establishes limits on medical malpractice liabilities, the current prospects of many doctors are raised under certain contingencies. Price floors for farm commodities raise the current prospects of many farmers under contingencies that would otherwise be associated with lower prices.

Let $\eta(\alpha)$ be the expected utility function with the original initial prospect and $\theta(\alpha)$ be the expected utility function with the improved initial prospect.

$$\eta(\alpha) = E\varphi(X + \alpha Y)$$

$$\theta(\alpha) = E\varphi(X + Z + \alpha Y)$$

Let $\hat{\alpha}$, α^* be the optimal choice and the boundary of the favorable set for expected utility η and $\tilde{\alpha}$, $\tilde{\alpha}^*$ have corresponding meanings for expected utility θ.

Proposition 6. Consider a dm whose initial prospect changes from X to $X + Z$ where $X + Z$ exceeds X. Let Y be the venture considered in both cases. Let $\hat{\alpha}$, α^* be the optimal choice and boundary of the favorable set before the change and $\tilde{\alpha}$, $\tilde{\alpha}^*$ afterward. Then

(i) $YZ \tilde{>} 0 \Rightarrow \tilde{\alpha} < \hat{\alpha}$ and $\tilde{\alpha}^* < \alpha^*$

(ii) $YZ = 0$ a.s. $\Rightarrow \tilde{\alpha} = \hat{\alpha}$ and $\tilde{\alpha}^* = \alpha^*$

(iii) $YZ \tilde{<} 0 \Rightarrow \tilde{\alpha} > \hat{\alpha}$ and $\tilde{\alpha}^* > \alpha^*$

Proof of (i). By the Mean Value Theorem

$\varphi'(X + Z + \alpha Y) = \varphi'(X + \alpha Y) + Z\varphi''(X + GZ + \alpha Y)$ where $0 \le G(\omega) \le 1$.

Let $\theta(\alpha) = E\varphi(X + Z + \alpha Y)$, $\eta(\alpha) = E\varphi(X + \alpha Y)$. Then

$\eta'(\alpha) = EY\varphi'(X + \alpha Y)$ and

$$\theta'(\alpha) = EY\varphi'(X + Z + \alpha Y) = \eta'(\alpha) + EYZ\varphi''(X + GZ + \alpha Y) \quad (3.1)$$

Recall $\varphi'' < 0$. If $EYZ \tilde{>} 0$, $\theta'(\alpha) < \eta'(\alpha)\ \forall \alpha \in R$. Thus $\theta'(\hat{\alpha}) < 0$ and $\tilde{\alpha} < \hat{\alpha}$ by Proposition 1, page 3. To show $\tilde{\alpha}^* < \alpha^*$ we consider three cases.

(a) $\tilde{\alpha} < 0 < \hat{\alpha}$. Then $\tilde{\alpha}^* < \tilde{\alpha} < 0 < \hat{\alpha} < \alpha^*$ (see Figure 1, page 3).

(b) $0 < \tilde{\alpha} < \hat{\alpha}$. Then $\theta'(\alpha) < \eta'(\alpha)$ implies $\theta(\tilde{\alpha}) - \theta(0) < \eta(\hat{\alpha}) - \eta(0)$ and $\theta(\tilde{\alpha}) - \theta(\alpha^*) > \eta(\hat{\alpha}) - \eta(\alpha^*) = \eta(\hat{\alpha}) - \eta(0)$. Thus $\theta(\tilde{\alpha}) - \theta(0) < \theta(\tilde{\alpha}) - \theta(\alpha^*)$ and $\theta(\alpha^*) < \theta(0)$ which means $\alpha^* \notin \langle\tilde{\alpha}^*\rangle = (0, \tilde{\alpha}^*)$ so $\tilde{\alpha}^* < \alpha^*$.

(c) $\tilde{\alpha} < \hat{\alpha} < 0$. Then $\theta'(\alpha) < \eta'(\alpha) \Rightarrow \theta(\tilde{\alpha}) - \theta(0) > \eta(\hat{\alpha}) - \eta(0)$ and $\theta(\tilde{\alpha}) - \theta(\alpha^*) < \eta(\hat{\alpha}) - \eta(\alpha^*) = \eta(\hat{\alpha}) - \eta(0)$. Thus $\theta(\tilde{\alpha}) - \theta(0) > \theta(\tilde{\alpha}) - \theta(\alpha^*)$ and $\theta(\alpha^*) > \theta(0)$ or $\alpha^* \in \langle\tilde{\alpha}^*\rangle = (\tilde{\alpha}^*, 0)$ so $\tilde{\alpha}^* < \alpha^*$.

Proof of (ii). From (3.1), $YZ = 0$ a.s. $\Rightarrow \theta'(\alpha) = \eta'(\alpha)\ \forall \alpha \in R$ so $\theta(\alpha) - \eta(\alpha)$ is a constant.

Proof of (iii). If $YZ \tilde{<} 0$ then $-YZ \tilde{>} 0$ and the argument for (1) applies to $-Y$. But reversing the sign of a venture reflects η and its points of interest about the η-axis.

It can readily be verified that if X deteriorates, i.e., changes from X to $X - Z$ with $Z \tilde{>} 0$, then the inequalities of (1) and (3) are reversed. Note that Proposition 4 does not apply if $Z > 0$ a.s. since the requirement that Y is not a sure thing $(P(Y > 0) > 0, P(Y < 0) > 0)$ would then preclude any of the conditions.

(i) says that a dm will decrease his (unrestricted) demand for a venture if his initial prospect improves on an event where his venture offers a positive (nonnegative and not a.s. 0) return. This further illustrates the loose, informal characterization in [2] that the attractiveness of a venture may be regarded as a combination of expected return and insurance value, the latter loosely defined as the tendency of the venture to offer rewards on events where the initial prospect is low. Alternatively, one might say that as X improves on an event A , the positive contribution of Y on A is less needed if $\varphi'' < 0$. For example, we would expect demand for disability insurance to diminish if the government provides extensive tax deductions for the disabled.

Now consider dm's responses to an improvement in the venture Y with X remaining unchanged. Again, let $Z \tilde{>} 0$. Compare $\hat{\alpha}$, α^* with $\tilde{\alpha}$, $\tilde{\alpha}^*$ where the latter are optimal amount and boundary of the favorable set for the venture $Y + Z$. For the response of α^*, we have the following generalization of Proposition 5.

<u>Proposition 7</u>. Consider a dm whose prospective venture changes from Y to $Y + Z$ with $Z \tilde{>} 0$. Let α^* be the boundary of the favorable set before the change and $\tilde{\alpha}^*$ afterward. Then $\tilde{\alpha}^* > \alpha^*$.

<u>Proof</u>. Let $\eta(\alpha) = E\varphi(X + \alpha Y)$ and $\theta(\alpha) = E\varphi(X + \alpha(Y + Z))$ be the respective expected utility functions. $\alpha Z \tilde{>} 0$ for $\alpha > 0$ and $\alpha Z \tilde{<} 0$ for $\alpha < 0$. Since φ is increasing, $\theta(\alpha) - \eta(\alpha) \gtreqless \alpha$. Thus $\alpha^* > 0 \Rightarrow \theta(\alpha^*) > \eta(\alpha^*) = \eta(0) = \theta(0)$ so $\alpha^* \varepsilon (0, \tilde{\alpha}^*)$. Also $\alpha^* < 0 \Rightarrow \theta(\alpha^*) < \theta(0)$ so $\alpha^* \notin \langle \tilde{\alpha}^* \rangle$.

We expect the response of the optimal choice to an improvement in Y to typically be positive $(\tilde{\alpha} > \hat{\alpha})$. However, counterexamples can be produced to show this is not universal.[4] The next two propositions give a number of sufficient conditions for positive response. Note that if $Y + Z \tilde{>} 0$, then $\tilde{\alpha} = \infty$ so this trivial case is not included. For any random variable W and any event A, let $W_A = I_A W$ where I_A is the indicator of A.

<u>Proposition 8</u>. Suppose a venture improves from Y to $Y + Z$ where $Z \tilde{>} 0$. Let $\eta(\alpha) = E\varphi(X + \alpha Y)$ and $\theta(\alpha) = E\varphi(X + \alpha Y + \alpha Z)$ be the expected utility functions with $\eta'(\hat{\alpha}) = \theta'(\tilde{\alpha}) = 0$. Let $A = (Z > 0)$. Then

(i) $\theta(\alpha) - \eta(\alpha) \gtreqless \alpha$

(ii) $\hat{\alpha} \geq 0 \Rightarrow \tilde{\alpha} > 0$, $\tilde{\alpha} \leq 0 \Rightarrow \hat{\alpha} < 0$

and each of the following implies $\tilde{\alpha} > \hat{\alpha}$.

(iii) $\hat{\alpha}\tilde{\alpha} \leq 0$

(iv) $\tilde{\alpha} \geq 0$, $Y_A \lesssim 0$

(v) $\hat{\alpha} \leq 0$, $Y_A \gtrsim 0$

(vi) (X, Y) independent of A, $Y_A + Z \tilde{>} 0$

(vii) $Y_A \lesssim 0$, $Y_A + Z \gtrsim 0$

(viii) $\hat{\alpha} \leq 0$, $\varphi''' > 0$

<u>Proof</u>. (i) follows from the definitions of θ, η and the fact that φ is increasing.

(3.2) $$\theta'(\alpha) = E(Y + Z)\,\varphi'(X + \alpha Y + \alpha Z)\,, \quad \eta'(\alpha) = EY\varphi'(X + \alpha Y) \quad .$$

(3.3) $$\theta'(0) - \eta'(0) = EZ\varphi'(X) > 0 \quad .$$

By Proposition 1, page 3, $\hat{\alpha} \geq 0 \Rightarrow \eta'(0) \geq 0$. By (3.3) this implies

$\theta'(0) > 0$ which, by Proposition 1 implies $\tilde{\alpha} > 0$. The second half of (ii) is similar. (iii) follows immediately from (ii).

By the Mean Value Theorem

$$(3.4) \qquad \varphi'(X + \alpha Y + \alpha Z) = \varphi'(X + \alpha Y) + \alpha Z\varphi''(X + \alpha Y + \alpha GZ)$$

where $0 \leq G \leq 1$. Combining (3.2) and (3.4),

$$(3.5) \qquad \theta'(\alpha) - \eta'(\alpha) = EZ\varphi'(X + \alpha Y + \alpha Z) + \alpha EYZ\varphi''(X + \alpha Y + \alpha GZ)$$

φ' is positive and $Z \gtrsim 0$ so $EZ\varphi'(\cdot) > 0$. Since $\varphi'' < 0$; $\alpha YZ \lesssim 0 \Rightarrow \theta'(\alpha) - \eta'(\alpha) > 0$. Thus (iv) implies $\theta'(\tilde{\alpha}) > \eta'(\tilde{\alpha})$ and (v) implies $\theta'(\hat{\alpha}) > \eta'(\hat{\alpha})$. Either justifies $\tilde{\alpha} > \hat{\alpha}$ by Proposition 1.

Note that, using (ii), one could substitute "max $\{\tilde{\alpha}, \hat{\alpha}\}$" for "$\tilde{\alpha}$" in (iv) and "min $\{\tilde{\alpha}, \hat{\alpha}\}$" for "$\hat{\alpha}$" in (v). Let $A = (Z > 0)$. Then $X + \alpha Y + \alpha Z = X + \alpha Y$ on A^c and an alternative expression for the difference in expected utilities is

$$(3.6) \qquad \theta'(\alpha) - \eta'(\alpha) = E(Y_A + Z)\,\varphi'(X + \alpha Y + \alpha Z) - EY_A\,\varphi'(X + \alpha Y) \quad .$$

By definition, $EY\varphi'(X + \hat{\alpha}Y) = 0$ so if (X, Y) are independent of A, the final term of (3.6) is also zero. Then $(Y_A + Z) \tilde{>} 0$ implies that the next to last term is positive so $\theta'(\hat{\alpha}) - \eta(\hat{\alpha}) > 0$ and $\tilde{\alpha} > \hat{\alpha}$. This proves (vi).

The conditions in (vii) make $\theta'(\alpha) > \eta'(\alpha)$ for all α and therefore for $\hat{\alpha}$. Note that $Y_A = 0$, $Y_A + Z = 0$ is ruled out by $Z_A > 0$.

To establish (viii), again use the Mean Value Theorem

$$(3.7) \qquad \varphi''(X + \alpha Y + \alpha Z) = \varphi''(X + \alpha Y) + \alpha Z\varphi'''(X + \alpha Y + \alpha GZ)$$

where, as before, $0 \le G \le 1$. Differentiating (3.6)

$$(3.8) \qquad \theta''(\alpha) - \eta''(\alpha) = E(Y_A + Z)^2 \varphi''(X + \alpha Y + \alpha Z) - EY_A^2 \varphi''(X + \alpha Y)$$

$$= E[(Y_A + Z)^2 - Y_A^2]\varphi''(X + \alpha Y) + \alpha E(Y_A + Z)^2 Z \varphi'''(X + \alpha Y + \alpha GZ)$$

The first term on the last line of (3.8) is always negative and, if $\varphi''' > 0$, the final term has the sign of α. Thus $\theta''(\alpha) - \eta''(\alpha) < 0$ for $\alpha \le 0$. Together with $\theta'(0) > \eta'(0)$ this implies $\theta'(\alpha) > \eta'(\alpha)$ for $\alpha \le 0$. Consequently $\theta'(\hat{\alpha}) > \eta'(\hat{\alpha})$ if $\hat{\alpha} \le 0$ and this means $\tilde{\alpha} > \hat{\alpha}$. Recall that decreasing absolute risk aversion $(r' < 0)$ implies $\varphi''' > 0$.

4. <u>Change in Belief in an Event</u>

Suppose dm's beliefs about events in his environment change in the following way. An event $A \varepsilon \mathcal{F}$ becomes more probable and its complement A^c correspondingly less probable while the conditional probabilities of all events given A (and therefore given A^c) are unchanged.

A 1976 example might have been a piece of news that increased dm's subject probability that Carter would be elected president. If the news were unaccompanied by anything that would change dm's views about what Carter would do if elected or what Ford would do if elected, then unchanged conditional probabilities seem reasonable.

Alternatively suppose a businessman has proposed a contract to another party and is waiting to see if it is accepted. News favoring probable acceptance might not change his ideas about what will happen if acceptance is received or what will happen if rejection is the outcome.

Other examples could concern legislation under consideration that is relevant to dm's affairs, litigation, or a bid to be let by a public agency.

Whatever the context, let $(\Omega, \mathcal{F}, P)$ be the underlying probability space before the change in beliefs and $(\Omega, \mathcal{F}, \tilde{P})$ reflect revised beliefs. For any $B \in \mathcal{F}$ we have

$$\tilde{P}B = \left(\frac{\tilde{P}A}{PA}\right) P(A \cap B) + \left(\frac{\tilde{P}A^c}{PA^c}\right) P(A \cap B^c) \quad .$$

For any random variable W define

$$\tilde{E}W = \int W d\tilde{P} = \left(\frac{\tilde{P}A}{PA}\right) E_A W + \left(\frac{\tilde{P}A^c}{PA^c}\right) E_{A^c} W$$

where E_A is conditional expectation given A. The new expected utility function is

$$\theta(\alpha) = \tilde{E}\varphi(X + \alpha Y) = (\tilde{P}A) E_A \varphi(X + \alpha Y) + (\tilde{P}A^c) E_{A^c} \varphi(X + \alpha Y) \quad .$$

Let $\theta'(\tilde{\alpha}) = 0$, $\theta(\tilde{\alpha}^*) = \theta(0)$. Note that if the vector (X, Y) is independent of A then $\theta(\alpha) = \eta(\alpha)$ so $\tilde{\alpha} = \hat{\alpha}$, $\tilde{\alpha}^* = \alpha^*$. How do $(\tilde{\alpha}, \tilde{\alpha}^*)$ compare with $(\hat{\alpha}, \alpha^*)$ under non-independence?

It will save time to generalize the problem a little before developing some results. Let λ, $0 \le \lambda \le 1$, be the revised probability of A and continue to assume unchanged conditional probabilities. Define

$$P_\lambda(B) = \lambda P_1(B) + \lambda^* P_0(B) \quad \forall B \in \mathcal{F}$$

where $P_1(B)$ is conditional probability of B given A, $P_0(B)$ is conditional probability of B given A^c, and $\lambda^* = 1 - \lambda$. For $\lambda \in [0, 1]$, let

$$(4.1) \qquad \eta(\alpha; \lambda) = E_\lambda \varphi(X + \alpha Y) = \int \varphi(X + \alpha Y)\, dP_\lambda = \lambda \eta(\alpha; 1) + \lambda^* \eta(\alpha; 0) \quad .$$

Define $\hat{\alpha}(\lambda)$ by $D_\alpha \eta(\hat{\alpha}(\lambda), \lambda) = 0$ and let $\alpha^*(\lambda)$ be the nonzero solution (if there is no nonzero solution $\alpha^*(\lambda) = 0$) of $\eta(\alpha^*(\lambda); \lambda) - \eta(0; \lambda) = 0$. Then

<u>Proposition 9</u>. With the above definitions, $\hat{\alpha}(\lambda)$ is monotonic and is strictly monotonic if $\hat{\alpha}(1) \neq \hat{\alpha}(0)$. $\hat{\alpha}$ is continuous on $[0, 1]$ and is continuously differentiable on $(0, 1)$.

<u>Proof</u>.

$$(4.2) \qquad D_\alpha \eta(\alpha; \lambda) = \lambda D_\alpha \eta(\alpha; 1) + \lambda^* D_\alpha \eta(\alpha; 0)$$

Suppose $\hat{\alpha}(1) > \hat{\alpha}(0)$. By definition $D_\alpha \eta(\hat{\alpha}(0); 0) = 0$ and, by Proposition 1, $D_\alpha \eta(\hat{\alpha}(0); 1) > 0$. Thus, sometimes writing $\hat{\alpha}_\lambda$ for $\hat{\alpha}(\lambda)$, $D_\alpha \eta(\hat{\alpha}_0; \lambda) = \lambda D_\alpha \eta(\hat{\alpha}_0; 1) + \lambda^* D_\alpha \eta(\hat{\alpha}_0; 0) > 0$ so $\hat{\alpha}(\lambda) > \hat{\alpha}(0)$ for $0 < \lambda < 1$. Similarly $D_\alpha \eta(\hat{\alpha}_1; \lambda) = \lambda D_\alpha \eta(\hat{\alpha}_1; 1) + \lambda^* D_\alpha \eta(\hat{\alpha}_1; 0) = \lambda^* D_\alpha \eta(\hat{\alpha}_1; 0) < 0$ so $\hat{\alpha}(\lambda) < \hat{\alpha}(1)$.

Now suppose $1 > \mu > \lambda$.

(4.3) $$D_\alpha\eta(\hat{\alpha}_\lambda, \mu) = \mu D_\alpha\eta(\hat{\alpha}_\lambda; 1) + \mu^* D_\alpha\eta(\hat{\alpha}_\lambda; 0) > \lambda D_\alpha\eta(\hat{\alpha}_\lambda; 1) + \lambda^*\eta(\hat{\alpha}_\lambda; 0) = 0$$

so $\hat{\alpha}(\mu) > \hat{\alpha}(\lambda)$ and $\hat{\alpha}$ is strictly increasing. A similar argument reveals $\hat{\alpha}$ strictly decreasing if $\hat{\alpha}(1) < \hat{\alpha}(0)$. If $\hat{\alpha}(1) = \hat{\alpha}(0)$, then putting $\alpha = \hat{\alpha}(0)$ in (4.2) makes both terms on the right vanish and $D_\alpha\eta(\hat{\alpha}_0; \lambda) = 0 \Rightarrow \hat{\alpha}(\lambda) = \hat{\alpha}(0) \quad 0 \le \lambda \le 1$. By assumptions (1) to (4), page 2 and (5), page 5 ; $D_\alpha\eta(\alpha; \lambda)$ has continuous nonzero partial derivative $D^2_{\alpha\alpha}\eta(\alpha; \lambda) < 0$ and from (4.2) $D^2_{\alpha\lambda}\eta(\alpha; \lambda) = D_\alpha\eta(\alpha; 1) - D_\alpha\eta(\alpha; 0)$ so, by the implicit function theorem, $\hat{\alpha}(\lambda)$ is continuously differentiable on $(0, 1)$. Note that for $\hat{\alpha}(1) = \hat{\alpha}(0)$, $\hat{\alpha}(\lambda) =$ constant is continuous at 0 and 1 . To obtain continuity at 0 and 1 when $\hat{\alpha}(1) \ne \hat{\alpha}(0)$, suppose $\lambda_n \uparrow 1$ and take $\bar{\alpha} < \hat{\alpha}(1) < \tilde{\alpha}$. Note $D_\alpha\eta(\bar{\alpha}; 1) > 0$ and $D_\alpha\eta(\tilde{\alpha}; 1) < 0$.
$D_\alpha\eta(\bar{\alpha}; \lambda_n) = \lambda_n D_\alpha\eta(\bar{\alpha}; 1) + \lambda^* D_\alpha\eta(\bar{\alpha}; 0)$ which is positive whenever $\lambda_n > - \lambda_n{}^* \dfrac{D_\alpha(\bar{\alpha}; 0)}{D_\alpha(\bar{\alpha}; 1)}$ so the latter implies $\hat{\alpha}(\lambda_n) > \bar{\alpha}$. Also $D_\alpha\eta(\tilde{\alpha}; \lambda_n) = \lambda_n D_\alpha\eta(\tilde{\alpha}; 1) + \lambda_n{}^* D_\alpha\eta(\tilde{\alpha}; 0)$ which is negative whenever $\lambda_n > - \lambda_n{}^* \dfrac{D_\alpha\eta(\tilde{\alpha}; 0)}{D_\alpha\eta(\tilde{\alpha}; 1)}$ making $\hat{\alpha}(\lambda_n) < \tilde{\alpha}$.

Since $\bar{\alpha}, \tilde{\alpha}$ can be arbitrarily close to $\tilde{\alpha}(1)$ and the necessary inequalities are realized for λ_n sufficiently close to 1 , $\hat{\alpha}(\lambda_n) \to \hat{\alpha}(1)$. Continuity at 0 is similar.

Proposition 9 tells us that an increase in the subjective probability of A, conditional probabilities unchanged, moves $\hat{\alpha}$ in the same direction as if A became certain and that the movement is smooth. These conclusions also hold for the boundary α^* of the favorable set.

<u>Proposition 10</u>. $\alpha^*(\lambda)$ as defined above is monotonic and is strictly monotonic if $\alpha^*(1) \neq \alpha^*(0)$. α^* is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$ except possibly where $\alpha^*(\lambda) = 0$.

<u>Proof</u>. Recall that $\langle\alpha^*\rangle = \{\alpha : \eta(\alpha) > \eta(0)\}$ and, except when $\eta'(0) = 0$, is either $(\alpha^*, 0)$ or $(0, \alpha^*)$. We first show that for $0 < \lambda < 1$, $\alpha^*(\lambda)$ lies between $\alpha^*(0)$ and $\alpha^*(1)$. Let

$$\xi(\alpha, \lambda) = \eta(\alpha, \lambda) - \eta(0, \lambda) \quad . \tag{4.4}$$

Then $\xi(\alpha, \lambda) > 0 \Leftrightarrow \alpha \in \langle\alpha^*(\lambda)\rangle$, i.e., $\langle\alpha^*(\lambda)\rangle = \{\alpha : \xi(\alpha, \lambda) > 0\}$. We have

$$\xi(\alpha, \lambda) = \lambda\, \xi(\alpha, 1) + \lambda^* \xi(\alpha, 0) \quad . \tag{4.5}$$

If $\alpha^*(1) = \alpha^*(0)$, then setting $\alpha = \alpha^*(0)$ in (4.4) shows $\xi(\alpha_0^*, \lambda) = 0 \;\; \forall \lambda \in [0, 1]$ so $\alpha^*(\lambda) = \alpha^*(0)$. If $\alpha^*(0) < \alpha^*(1)$, we consider three cases.

(I) $0 \leq \alpha^*(0) < \alpha^*(1)$. Then, for $0 < \lambda < 1$

$$\xi(\alpha_0^*, \lambda) = \lambda\xi(\alpha_0^*, 1) + \lambda^*\xi(\alpha_0^*, 0) = \lambda\xi(\alpha_0^*, 1) > 0$$

so $\alpha^*(0) \subset \langle\alpha^*(\lambda)\rangle$

$$\xi(\alpha_1^*, \lambda) = \lambda\xi(\alpha_1^*, 1) + \lambda^*\xi(\alpha_1^*, 0) = \lambda^*\xi(\alpha_1^*, 0) < 0$$

so $\alpha^*(1) \not\subset \langle\alpha^*(\lambda)\rangle$

$$\therefore 0 < \alpha^*(0) < \alpha^*(\lambda) < \alpha^*(1)$$

(II) $\alpha^*(0) < \alpha^*(1) \leq 0$. Then

$$\xi(\alpha_0^*, \lambda) = \lambda\xi(\alpha_0^*, 1) + \lambda^*\xi(\alpha_0^*, 0) = \lambda\xi(\alpha_0^*, 1) < 0$$

so $\alpha^*(0) \not\subset \langle\alpha^*(\lambda)\rangle$

$$\xi(\alpha_1^*, \lambda) = \lambda\xi(\alpha_1^*, 1) + \lambda^*\xi(\alpha_1^*, 0) = \lambda^*\xi(\alpha_1^*, 0) > 0$$

so $\alpha^*(1) \subset \langle\alpha^*(\lambda)\rangle$

$$\therefore \alpha^*(0) < \alpha^*(\lambda) < \alpha^*(1) < 0$$

(III) $\alpha^*(0) < 0 < \alpha^*(1)$. Then

$\xi(\alpha_0^*, \lambda) = \lambda\xi(\alpha_0^*, 1) < 0$ so $\alpha^*(0) \not\subset \langle\alpha^*(\lambda)\rangle$

$\xi(\alpha_1^*, \lambda) = \lambda^*\xi(\alpha_1^*, 0) < 0$ so $\alpha^*(1) \not\subset \langle\alpha^*(\lambda)\rangle$

This means $\langle\alpha^*(\lambda)\rangle \subset [\langle\alpha^*(0)\rangle \cup \langle\alpha^*(1)\rangle]$ and

$\alpha^*(0) < \alpha^*(\lambda) < \alpha^*(1)$.

We now show $\mu > \lambda \Rightarrow \alpha^*(\mu) > \alpha^*(\lambda)$ for each of the three cases. This completes the proof of monotonicity since $\alpha^*(1) < \alpha^*(0)$ just involves interchanging 0 and 1 in the proofs for the case of $\alpha^*(0) < \alpha^*(1)$.

(I) $\xi(\alpha^*_\lambda, \mu) = \mu\xi(\alpha^*_\lambda, 1) + \mu^*\xi(\alpha^*_\lambda, 0) > \lambda\xi(\alpha^*_\lambda, 1) + \lambda^*\xi(\alpha^*_\lambda, 0) > 0$

so $\alpha^*(\lambda) \subset \langle\alpha^*(\mu)\rangle$

(II) $\xi(\alpha^*_\lambda, \mu) < 0$ so $\alpha^*(\lambda) \not\subset \langle\alpha^*(\mu)\rangle$

(IIIa) Suppose $\alpha^*(\lambda) < 0$, then

$\xi(\alpha^*_\lambda, \mu) = \mu\xi(\alpha^*_\lambda, 1) + \mu^*\xi(\alpha^*_\lambda, 0) < 0$ and $\alpha^*(\lambda) \not\subset \langle\alpha^*(\mu)\rangle$

(IIIb) $\alpha^*(\lambda) > 0$, then

$\xi(\alpha^*_\lambda, \mu) = \mu\xi(\alpha^*_\lambda, 1) + \mu^*\xi(\alpha^*_\lambda, 0) < 0$ and $\alpha^*(\lambda) \subset \langle\alpha^*(\mu)\rangle$

$\alpha^*(\lambda)$ is defined implicitly by

$$(4.6) \qquad \xi(\alpha^*_\lambda, \lambda) = \eta(\alpha^*_\lambda, \lambda) - \eta(0, \lambda) = 0$$

By the implicit function theorem

$$(4.7) \qquad D_\alpha * = -\frac{D_\lambda \xi}{D_\alpha \xi} = -\frac{\eta(\alpha^*_\lambda, 1) - \eta(\alpha^*_\lambda, 0)}{D_\alpha \eta(\alpha^*_\lambda, \lambda)}$$

which yields a continuous derivative except when $D_\alpha \eta(\alpha^*_\lambda, \lambda) = 0$. From Figure 1, page 3, we recall that $D_\alpha \eta(\alpha^*) \lesseqgtr - \alpha^* \lesseqgtr - \hat{\alpha}$. Thus $D_\alpha \eta(\alpha^*_\lambda, \lambda) = 0 \Rightarrow \alpha^*(\lambda) = \hat{\alpha}(\lambda) = 0$.

To show continuity of $\alpha^*(\lambda)$ at $\mu \ni \alpha^*(\mu) = 0$, let $\bar{\alpha} < 0$ and $\lambda_n \to \mu$.

$$(4.8) \qquad \xi(\bar{\alpha}, \lambda_n) = \lambda_n \, \xi(\bar{\alpha}, 1) + \lambda^*_n \, \xi(\bar{\alpha}, 0)$$

$$= \xi(\bar{\alpha}, \mu) + (\lambda_n - \mu) \, \xi(\bar{\alpha}, 1) + (\lambda^*_n - \mu^*) \, \xi(\bar{\alpha}, 0) \quad .$$

$\xi(\bar{\alpha}, \mu) < 0$ so as n becomes large and the last two terms of (4.8) become negligible, $\xi(\bar{\alpha}, \lambda_n)$ becomes negative implying that $\alpha^*(\lambda_n) > \bar{\alpha}$ for sufficiently large n (recall that for all λ, $\xi(0, \lambda) = 0$ and that $\xi(\alpha, \lambda) \leq 0 \Leftrightarrow \alpha \notin \langle\alpha^*(\lambda)\rangle$.

Also if $\tilde{\alpha} > 0$, $\xi(\tilde{\alpha}, \lambda_n) < 0$ for sufficiently large n. Thus $\langle\alpha^*(\lambda_n)\rangle \subset (\bar{\alpha}, \tilde{\alpha})$ for arbitrary $\bar{\alpha} < 0 < \tilde{\alpha}$ and n sufficiently large; hence the boundary $\alpha^*(\lambda_n) \to 0$. Continuity at $\lambda = 0$ and $\lambda = 1$ can be shown in a similar fashion.

FOOTNOTES

1. The problem of this section has been previously discussed in [1, pages 11-25] and [2, pages 10-12].

2. The ensuing application of the implicit function theorem requires existence and continuity of $D^2_{\alpha\alpha}\eta$, $D^2_{\alpha b}\eta$, $D^2_{\alpha c}\eta$. A proof of continuity of $D^2_{\alpha b}\eta$ is sketched below. The others are similar. For convenience we write X for W + b , Y for V + c .

$D^2_{\alpha b}\eta = EY\varphi''(X + \alpha Y)$. We must show $\lim_{h\to 0} EY\varphi''(X + h + \alpha Y) = EY\varphi''(X + \alpha Y)$.

For $|h| < \delta$ and φ'' increasing the integrand on the left is dominated by $|Y\varphi''(X + \delta + \alpha Y)|$ which is integrable by assumption (5). Equality then follows from the dominated convergence theorem. Note that monotonicity of φ'' was assumed in (2), page 2. For φ'' decreasing the integrand would be dominated by $|Y\varphi''(X + \delta + \alpha Y)|$.

3. By Proposition 1, $\hat{\alpha} \overset{s}{=} \eta'(0) = EY\varphi'(X)$ and from Figure 1, $\alpha^* \overset{s}{=} \hat{\alpha}$. If $\alpha^* = \hat{\alpha} = \eta'(0) = 0$, replacing Y with Y + c changes $\eta'(0)$ to $E(Y + c)\,\varphi'(X) > 0$ and $\hat{\alpha}$, α^* also become positive.

4. Let $PA = \frac{e}{e+1}$ where e is the base of natural logs. Define $X(\omega) = 0$ for $\omega \in A$ and $X(\omega) = 3$ for $\omega \in A^c$. Let $Y(A) = 1$, $Y(A^c) = -1$ and $Z(A) = 1$, $Z(A^c) = 0$. If $\varphi(x) = -e^{-x}$ it can be shown that $\hat{\alpha} = 2$ while $\theta'(2) < 0$ indicating $\tilde{\alpha} < 2$.

REFERENCES

[1] Hildreth, Clifford and Leigh Tesfatsion, "A Model of Choice with Uncertain Initial Prospect," Discussion Paper No. 38, Center for Economic Research, University of Minnesota, 1974.

[2] Hildreth, Clifford, "Expected Utility of Uncertain Ventures," Journal of the American Statistical Association, 69 (1974), pages 9 - 17.