REVELATIONS OF A GAMBLER

by

Marcel K. Richter and Leonard Shapiro

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

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I. Introduction

If you are offered a desired price depending on the outcome of a coin toss, and you express a definite preference for tails over heads, you reveal -- in a very natural sense -- that you consider tails "more likely than" heads. Can you reveal more complicated statements by your other choices -- for example, that you consider tails more than twice as likely as heads? Or more than $\sqrt{2}$ times as likely as heads? Or more than π times as likely as heads?

In this paper we delineate precisely what revelations are possible. This turns out to be a surprisingly rich class, even in our simple case of a single two-sided coin and a finite number of prizes. Our methods are based on a hypothesis of expected utility maximization, but we impose no topological, metric, or linear structure on the set of prizes or the set of events. In addition to its own intrinsic interest, we feel

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that such a characterization of the expressive power of gambling choices may play a role in axiomatizing expected utility-maximizing behavior, under our hypotheses of a finite set of gambles and a partial order on the set of choices between gambles [cf. 9, section 8].

Let us outline the paper. Section II contains definitions. In Section III we present several examples. They show that choice behavior can reveal these progressively more complicated relations between the likelihoods P_H and P_T of the two events heads and tails: $P_H > \gamma P_T$, first for $\gamma = \frac{3}{2}$ (subsection A) then for $\gamma = \sqrt{2}$ (subsection B); $P_H \neq \frac{2}{3} P_T$ (subsection C); and $P_H/P_T \in [0,1) \cup (\frac{3}{2},\infty)$. Section IV contains the two theorems which characterize what can be revealed, namely that the ratio P_H/P_T is in a "polynomial set." For additional insight into the structure of polynomial sets we go on in Section V to characterize them as unions of finitely many intervals with algebraic endpoints. In particular this allows us to answer the question raised about π in the first paragraph of the Introduction. We close, in Section VI, with some open problems.

II. Definitions

Let X be a finite set; we shall call its elements prizes. We imagine a two-event world, such as our coin-tossing example earlier. We call the events H and T.* Then a gamble (also called in the literature a lottery ticket, or act) is an ordered pair of elements of X, e.g. (x,y) which is interpreted as yielding prize x in the first event, and prize y in the second. Let $\mathcal{A} = X \times X$ be the set of gambles.

We imagine a decision maker, or gambler, who makes various choices between gambles. We call an expression of the form

$$(1) \qquad (x,y) \succ (z,w) \; ,$$

where x, y, z, w are in X, a gamble-choice on X.**

There are several methods one might use to choose among gambles. For example, given a real function $U: X \to \mathbb{R}$ (which we can think of as a "utility," i.e. $U(z) > U(w)$ means z is "preferred" to w), one might choose the gamble (x,y) which maximizes (over some domain)

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* In Section V we comment on the extension of our results to the n-event case.

** We could motivate this terminology by letting $\mathcal{B}$ be a family of subsets of the set $\mathcal{A}$ of gambles and letting $h: \mathcal{B} \to \mathcal{A}$ be a function such that,

$$h(B) \in B \quad \text{for each} \quad B \in \mathcal{B} \; .$$

(The intuition is that $h(B)$ is the decision maker's choice out of the set B.) Then we could interpret the gamble choice $(x,y) \succ (z,w)$ to mean that for some $B \in \mathcal{B}$, $(x,y) = h(B)$ and $(x,y) \neq (z,w) \in B$. Cf. [4].

(2) $\min(U(x),U(y))$.

This is the "maximin" method.

Alternatively, suppose one has a utility $U: X \to \mathbb{R}$ and "probabilities" P_H and P_T (i.e. P_H , P_T are in $[0,1]$ with $P_H + P_T = 1$ -- we can think of P_H , P_T as the probability of H, T, respectively). Let P denote the pair (P_H,P_T) and define the <u>expected utility</u> function $E_{P,U}: X \times X \to \mathbb{R}$ from gambles to real numbers by

(3) $$E_{P,U}(x,y) = P_H U(x) + P_T U(y), \text{ for all} (x,y) \in X \times X .$$

Then a subject might choose (x,y) to maximize $E_{P,U}(x,y)$ over some domain.

Of course other methods than maximin or expected utility maximization are possible, and they need not involve maximizing a numerical function. Here we will be concerned only with expected utility maximization.

We say that (P,U) <u>rationalizes</u> a set $\mathcal{S}$ of gamble choices* if, for all $(x,y) \succ (z,w)$ in $\mathcal{S}$,

(4) $$E_{P,U}(x,y) > E_{P,U}(z,w) .$$

*We could justify this terminology by noting that if the gamble choices in $\mathcal{S}$ are those derived from some $h: \mathcal{B} \to \mathcal{G}$ as in the previous footnote then (4) holds for all choices $(x,y) \succ (z,w)$ in $\mathcal{S}$ if and only if for each $B \in \mathcal{B}$, $h(B)$ is the unique gamble in B with highest expected utility; i.e., for each $B \in \mathcal{B}$ and each $(z,w) \in B$, if $h(B) \neq (z,w)$, then $E_{P,U}(h(B)) > E_{P,U}(z,w)$. Cf. [4].

Given a set $\mathcal{S}$ of gamble-choices, let us now assume these were made by a gambler who maximized expected utility, i.e. we assume there is a (P,U) which rationalizes $\mathcal{S}$. (Obviously this implies some restrictions on $\mathcal{S}$.) What does $\mathcal{S}$ "reveal" about the pair (P_H, P_T) which this gambler used? For example, suppose $\mathcal{S}$ is given by

$$(5) \qquad \mathcal{S}: \begin{array}{ll} \text{(i)} & (x,y) \succ (y,y) \\ \text{(ii)} & (x,y) \succ (y,x) . \end{array}$$

Choice (i) clearly implies $U(x) > U(y)$, where U is the utility used by the gambler. Given, then, that x is preferred to y, choice (ii) implies $P_H > P_T$, since the gambler chose to receive x, the preferred prize, on Heads. So $\mathcal{S}$ implies $P_H > P_T$. It implies no more than that since given P_H and P_T satisfying $P_H > P_T$ one can find a utility U such that (P,U) rationalizes $\mathcal{S}$. To see all this mathematically, one writes the <u>expected utility form</u> of the $\mathcal{S}$ in (5):

$$\text{(i')} \quad P_H U(x) + P_T U(y) > P_H U(y) + P_T U(y)$$

$$\text{(ii')} \quad P_H U(x) + P_T U(y) > P_H U(y) + P_T U(x) .$$

Then (i'), (ii') are equivalent, respectively, to

$$\text{(i'')} \quad [U(x) - U(y)] > 0$$

(ii") $(P_H - P_T)[U(x) - U(y)] > 0$,

where in (i") we have used the fact that $P_H \geq 0$. Clearly (i") and (ii") are equivalent to $P_H > P_T$. Summarizing, $\mathcal{S}$ reveals precisely that $P_H > P_T$.

In general we say that a set $\mathcal{S}$ of gamble-choices <u>reveals</u> some property (e.g. an inequality like $P_H > P_T$) about the relative likelihood of H and T if, whenever (P,U) rationalizes $\mathcal{S}$, then $P = (P_H, P_T)$ has that property, and conversely if P_H, P_T have that property then there is some U such that (P,U) rationalizes $\mathcal{S}$. A rigorous version of this definition is given in Section IV, but we now know enough to understand the examples given in the next section.

Our main interest now is: What can choice behavior (as represented by $\mathcal{S}$ reveal about the rationalizing probabilities (P_H, P_T)? This problem is a special case of revealed preference theory. In the classical revealed preference problem of Samuelson [6, 7], one observes market behavior (a consumer choosing a commodity bundle out of a budget set) and asks what that behavior reveals about the consumer's possible utility functions; i.e. if bundle x is chosen (cf. footnote page 2.1) from a budget which includes the bundle y , then one infers that $U(x) > U(y)$ for any U which "rationalizes" that consumer's choice. The general revealed preference question is: What does choice behavior reveal about possible rationalizations (cf. [4], [5])?

III. Examples

In this section we present several examples, increasing in complexity, of what can be revealed (about $P = (P_H, P_T)$), by a finite set of gamble-choices. Recall that, in particular, the gamble-choice

(1) $(x,y) \succ (z,w)$

reveals that P_H, P_T must satisfy: for some U,

(2) $P_H U(x) + P_T U(y) > P_H U(z) + P_T U(w)$.

We call (2) the expected-utility form of (1). In calculating with expected-utility forms the reader will find it much easier to omit the "U" and think of x, y, etc. interchangeably as prizes or as variables standing for the utilities of the corresponding prizes.

A. Rationally more probable than. From the example (II.5) it is evident that a gambler can reveal that event H is more probable than event T . Can the gambler further reveal information such as

(3) $P_H > \frac{3}{2} P_T$?

The reader may wish to try his own hand at answering this before proceeding.

Consider this set of gamble choices:

(4)
- i) $(x,v) \succ (w,x)$
- ii) $(w,y) \succ (y,x)$
- iii) $(x,x) \succ (s,v)$
- iv) $(t,y) \succ (y,x)$
- v) $(x,y) \succ (t,x)$
- vi) $(y,r) \succ (x,r)$

If one sums the expected utility forms of the choices (4, i-v) the result is.

(5) $$-P_H 2(U(y) - U(x)) + 3P_T(U(y) - U(x)) > 0 \ ;$$

and from (4,vi) we see that

(6) $$U(y) - U(x) > 0 \ .$$

Then (3) follows from (5) and (6).

Conversely, one can show that, for any $P_H > P_T \in [0,1]$ satisfying (3), there exists a $U: X \rightarrow \text{Reals}$ such that (U,P) rationalizes (4).

B. <u>Irrationally more probable than</u>. In subsection (A) we have seen that a gambler can reveal that event H is more than 3/2 times as probable as event T . Indeed, it is not hard to construct such examples when 3/2 is replaced by any nonnegative rational number. But what about irrational numbers?

Consider this set of gamble choices:

$$\text{(7)}\quad \begin{array}{rl} \text{i)} & (x,z) \succ (w,p) \\ \text{ii)} & (x,u) \succ (v,r) \\ \text{iii)} & (w,r) \succ (y,q) \\ \text{iv)} & (v,p) \succ (y,z) \\ \text{v)} & (q,y) \succ (u,x) \\ \text{vi)} & (y,x) \succ (x,x) \end{array}$$

Summing the expected utility forms of (7, i-iv) we get

$$\text{(8)}\qquad -P_H 2(U(y) - U(x)) + P_T(U(u) - U(q)) > 0$$

The expected utility form of (7, v) is:

$$\text{(9)}\qquad P_H(U(q) - U(u)) + P_T(U(y) - U(x)) > 0 \quad ,$$

while from (7, vi) we see that:

$$\text{(10)}\qquad U(y) - U(x) > 0 \quad .$$

Then from (8), (9), and (10) it follows that:

(11) $$P_H > \sqrt{2}\, P_T \quad .$$

Conversely, one can show that, for any P_H, $P_T \in [0,1]$ satisfying (11), there exists a $U: X \to$ Reals such that (U,P) rationalizes (7).

At first sight it may be surprising that just a finite number of revelations in a two-event world can express probability statements such as (11) involving irrational numbers. On the other hand, the expected utility function is quadratic in the "variables" P_H, P_T and $U(x)$ $(x \in X)$, so the fact that a square root appears is not as surprising as the fact, which will follow from our Theorem 1 , that if γ is <u>any</u> nonnegative algebraic number (i.e.if γ is a root of a polynomial with integer coefficients) then a finite set of gamble-choices can reveal

$$P_H > \gamma P_T \ .$$

C. <u>Unequalities</u>. The revelations so far have expressed statements of the form:

(13) $$P_H > \gamma P_T \ ,$$

for some real number γ . Of course it is not hard to express similar statements with the inequalities reversed. We can also express inequalities of the form:

$$(14) \qquad P_H \neq \frac{2}{3} P_T \quad .$$

Such an example can be obtained from (4,i) through (4,v) : for all P_H, P_T satisfying (14), there exists a U such that (U,P) rationalizes (4,i-v); and, conversely, for every (U,P) rationalizing (4,i-v), P satisfies (14).

D. Intervals. By combining gamble choices expressing:

$$(15) \qquad P_H < \frac{3}{2} P_T$$

with choices involving prizes different from those used for (15) and expressing:

$$(16) \qquad P_H > P_T \quad ,$$

our gambler can reveal (if we exclude the case $P_T = 0$) that:

$$(17) \qquad \frac{P_H}{P_T} \in (1, \frac{3}{2}) \quad .$$

Our gambler can also reveal (again excluding the case $P_T = 0$) that P_H/P_T belongs to a disjoint union of intervals. For example, let $\mathcal{S}$ denote a set of gamble-choices revealing (17), and let $\mathcal{S}'$ denote the "reverse" of $\mathcal{S}$,i.e. $(x,y) \succ (z,w) \in \mathcal{S}$ iff $(z,w) \succ (x,y) \in \mathcal{S}'$. Then $\mathcal{S}'$ reveals

$$P_H/P_T \in [0,1) \cup (\tfrac{3}{2}, \infty) \quad .$$

IV. Characterizing the Expressive Power of Gambling Revelations

We have seen in Section II examples of many probability statements expressed by choices between gambles. In this section we shall prove two theorems which together will characterize the set of all probability statements which can be expressed by gamble choices.

Rather than the general question of what $\mathcal{S}$ reveals about P_H and P_T, we will examine the slightly less general question of what $\mathcal{S}$ reveals about strictly positive values of P_H and P_T.*

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* The two cases we exclude, $P_H = 0$ and $P_T = 0$, can be analyzed by other methods since they involve no uncertainty. (If $P_H = 0$ then $P_T = 1$ and the question of finding a U such that (P,U) rationalizes a given $\mathcal{S}$ reduces to whether a partial order on a finite set can be extended to a total order. Necessary and sufficient conditions for this are well-known.) However, it is not clear whether our results extend to include either of the cases $P_H = 0$ or $P_T = 0$. For example, we do not know whether Theorem 1 is true if the value $P_T = 0$, i.e. $\rho = 0$, is not excluded (from the definition of $R(\mathcal{S})$ below). Note that if we wish to talk about the ratio P_H/P_T (see III.D as an example of how this simplifies things) we must exclude $P_T = 0$ unless we work in the extended (by ∞) real number system.

Accordingly, for the remainder of the paper we consider only positive probabilities P_H , P_T satisfying $P_H + P_T = 1$. We denote the ratio P_H/P_T by ρ . (Note that given $\rho > 0$, P_H and P_T are uniquely determined as $P_H = 1/(\rho+1)$ and $P_T = \rho/(\rho+1)$.)

For any set $\mathcal{S}$ of gamble choices we denote by $R(\mathcal{S})$ the set of all positive values $\rho = \frac{P_H}{P_T}$ for which there exists a $U: X \to$ Reals such that (U,P) rationalizes $\mathcal{S}$. Now we can give a rigorous version of what we meant in Section II by a set $\mathcal{S}$ of gamble-choices "revealing" a property of P_H and P_T . Namely, $\mathcal{S}$ reveals precisely the property $P_H/P_T \in R(\mathcal{S})$. Our aim in this Section is to characterize the sets $R(\mathcal{S})$ as $\mathcal{S}$ varies over all sets of gamble-choices on all finite sets X of prizes. Our result will be that the solution sets $R(\mathcal{S})$ are "polynomial sets," which we now define.

By an integer polynomial Q we mean a polynomial in ρ with integer coefficients; thus, for some positive integer d and some integers $a_0,\ldots,a_d$, Q is of the form:

$$(1) \qquad Q(\rho) = \sum_{k=0}^{d} a_k \rho^k$$

for all $\rho \in (0,\infty)$.

A set $\mathbb{A}$ of real numbers is called a polynomial set if there exist positive integers I, J and integer polynomials $Q_{ij}(\cdot)$ for each $i = 1,\ldots,I$ and $j = 1,\ldots,J$ such that:

(2) $$\mathbb{A} = \bigcap_{i=1}^{I} \bigcup_{j=1}^{J} \{\rho\colon \rho > 0 \ \& \ Q_{ij}(\rho) > 0\} .$$

For example, for the set $\mathbb{S}$ given in III.D, we have $R(\mathbb{S}) = \{\rho\colon 0 < \frac{1}{2}$ or $\rho > 1\}$, which is a polynomial set since it is equal to:

(3) $$\{\rho\colon \rho > 0 \ \& \ 1-2\rho > 0\} \cup \{\rho\colon \rho > 0 \ \& \ \rho-1 > 0\} .$$

Of course, polynomial sets in general are quite complicated, and it is not immediately clear that, given any polynomial set $\mathbb{A}$, we can find a set $\mathbb{S}$ of choices such that $R(\mathbb{S}) = \mathbb{A}$. (See the discussion at the end of III.B.)

Theorem 1. Let $\mathbb{A}$ be a polynomial set. Then there effectively exist a finite set X of prizes and a finite set $\mathbb{S}$ of gamble choices based on X, such that $R(\mathbb{S}) = \mathbb{A}$.

By "effectively exist" we mean that the existence proof is a constructive one: given $\mathbb{A}$ we will construct a specific X and $\mathbb{S}$ satisfying $R(\mathbb{S}) = \mathbb{A}$. Notice that the only relevant property of X is its cardinality; there is nothing special about the elements of X.

Proof of Theorem 1. For any set S of integer polynomial inequalities in the variables $\rho, z_1, \ldots, z_n$, we extend the notation and let $R(S)$ be the set of positive ρ such that, for some substition of real numbers for the $z_1, \ldots, z_n$, all the inequalities in S are simultaneously satisfied.

To prove the Theorem, we will first construct a set S_1 of integer polynomial inequalities with $R(S_1) = \mathbb{A}$. We will then construct sets S_2 and S_3 of inequalities, with $R(S_3) = R(S_2) = R(S_1)$. Finally we will show that the set S_3 of inequalities will correspond to the expected-utility form of a set $\mathbb{S}$ of gamble-choices on a set X of prizes, the correspondence being such that $R(\mathbb{S}) = R(S_3)$. Thus we will have proved $\mathbb{A} = R(\mathbb{S})$.

We will actually prove that the sets S_1, S_2 and S_3 of inequalities are "equivalent," a term which we now define. Suppose S is a set of inequalities in the variables ρ , $z_1,\ldots,z_n$, and suppose S' is a set of inequalities in the variables ρ , $z_1,\ldots,z_{n+m}$ $m \geq 0$. We say S and S' are <u>equivalent</u> if (a) given values $\bar{\rho}$, $\bar{z}_1,\ldots,\bar{z}_n$ of ρ , $z_1,\ldots,z_n$, respectively, which solve S , there are values $\bar{z}_{m+1},\ldots,\bar{z}_{n+m}$ of $z_{n+1},\ldots,z_{n+m}$, respectively, such that the values $\bar{\rho}$, $\bar{z}_1,\ldots,\bar{z}_{n+m}$ solve S' , and (b) given values $\bar{\rho}$, $\bar{z}_1,\ldots,\bar{z}_{n+m}$ of ρ , $z_1,\ldots,z_{n+m}$ which solve S' , the values $\bar{\rho}$, $\bar{z}_1,\ldots,\bar{z}_n$ solve S . We denote condition (a) above by $S \Rightarrow S'$ and condition (b) by $S' \Rightarrow S$. Clearly if S is equivalent to S' then $R(S) = R(S')$. (The converse is false.)

We will first consider the case $\mathbb{A} = \bigcup_{j=1}^{J} \{\rho\colon \rho > 0 \ \&\ Q_j(\rho) > 0\}$, where $Q_j(\rho)$ is an integer polynomial in ρ ; thus, for some d (independent of j), each $Q_j(\rho)$ is of the form:

$$(4) \qquad Q_j(\rho) = \sum_{i=0}^{d} \alpha_i^j \rho^i \qquad (j = 1,\ldots,J) ,$$

where the α_i^j are integers (perhaps zero or negative). Define new variables u_j, v_j for $1 \leqq j \leqq J$, and denote $u_j - v_j$ by x_j. Define the set S_1 to consist of the following $J + 1$ inequalities:

$$(5) \qquad S_1 : \quad \begin{array}{ll} \text{a)} & \sum_{j=1}^{J} Q_j(\rho) x_j > 0 \\ \text{b)} & x_j > 0 \qquad (j = 1,\ldots,J) \end{array} .$$

Let us prove that $R(S_1) = \mathbb{A}$. If $\bar{\rho} \in \mathbb{A}$, then $Q_k(\bar{\rho}) > 0$ for some k, so setting x_k very large and x_j small (yet positive) for all $j \neq k$ will solve S_1. Thus $R(S_1) \supseteq \mathbb{A}$. Conversely, if $\bar{\rho} \in R(S_1)$, then, by (5), $Q_j(\bar{\rho}) > 0$ for some $j = 1,\ldots,J$, so $\bar{\rho} \in A$. Thus $R(S_1) \subseteq \mathbb{A}$, so $R(S_1) = \mathbb{A}$.

Define new variables $\bar{u}_i$, $\bar{v}_i$ for $0 \leqq i \leqq d$. Denote $\bar{u}_i - \bar{v}_i$ by y_i. Let the $d + 1$ expressions $I_0,\ldots,I_d$ be defined by:

$$I_0 = \left(\sum_{j=1}^{J} \alpha_0^j x_j\right) + \rho y_1$$

$$(6) \qquad I_i = \left(\sum_{j=1}^{J} \alpha_i^j x_j\right) - y_i + \rho y_{i+1} \qquad (1 \leq i < d)$$

$$I_d = \left(\sum_{j=1}^{J} \alpha_d^j x_j\right) - y_d$$

Let S_2 consist of the $d + 1 + J$ inequalities:

$$(7) \qquad S_2 : \begin{cases} \text{(a)} \quad I_i > 0 \qquad (0 \leqq i \leqq d) \\ \text{(b)} \quad x_j > 0 \qquad (1 \leqq j \leqq J) \end{cases} .$$

Let us prove S_1 equivalent to S_2. To see that $S_2 \Rightarrow S_1$, note that

$$(8) \qquad \sum_{i=0}^{d} \rho^i I_i = \sum_{j=1}^{J} Q_j(\rho) x_j .$$

So if $x_1, \ldots, x_J > 0$ and $\rho > 0$ make each I_i positive, then the right hand side of (8), which is (5.i) must also be positive.

To see that $S_1 \Rightarrow S_2$, suppose $\rho > 0$ and $x_1, \ldots, x_J > 0$ solve S_1. We can rearrange the terms in (S_1, a) to get:

$$(9) \qquad \sum_{j=1}^{J} \alpha_0^j x_j + \sum_{j=1}^{J} \alpha_1^J \rho x_j + \ldots + \sum_{j=1}^{J} \alpha_d^j \rho^d x_j > 0 .$$

From this it follows that we can pick y_1 so that

$$(10) \qquad \sum_{j=0}^{J} \alpha_0^j x_j > -\rho y_1 > -\sum_{j=1}^{J} \alpha_1^j \rho x_j - \ldots - \sum_{j=1}^{J} \alpha_d^j \rho^d x_j .$$

Notice that the first inequality in (10) asserts $I_0 > 0$. Since $\rho > 0$ we can factor it out of the second inequality in (10) to obtain:

$$-y_1 > -\sum_{j=1}^{J} \alpha_1^j x_j - \dots - \sum_{j=1}^{J} \alpha_d^j \rho^{d-1} x_j \quad .$$

Thus we can choose y_2 so that

$$(11) \qquad \sum_{j=1}^{J} \alpha_1^j x_j - y_1 > -\rho y_2 > -\sum_{j=1}^{J} \alpha_2^j \rho x_j - \dots - \sum_{j=1}^{J} \alpha_d^j \rho^{d-1} x_j \quad .$$

Notice the first inequality in (11) is $I_1 > 0$. Now factor ρ out of the second inequality in (11) and continue according to this pattern, choosing $y_3, y_4, \dots, y_d$ satisfying $I_2 > 0, \dots, I_d > 0$. This shows $S_1 \Rightarrow S_2$, so S_1 and S_2 are equivalent. Hence $R(S_1) = R(S_2)$.

We now define S_3 . Fix some i with $0 \leqq i \leqq d$ and consider the expression I_i . For some positive integer K, I_i is of the form

$$(\sum_{k=1}^{K} z_k^i) + \rho y_{i+1}$$

where $y_{d+1} = 0$ and where each z^i is either equal to x_j or $-x_j$ for some j , or equal to y_i . Of course K depends on i . Define new variables $w_1^i, \dots, w_K^i$ and consider the set S_3^i of inequalities:

$$z_1^i + \rho(\bar{u}_{i+1} - w_1^i) > 0$$

(12) $\quad S_3^i : \quad z_k^i + \rho(w_{k-1}^i - w_k^i) > 0 \qquad 1 < k < K$

$$z_K^i + \rho(w_K^i - \bar{v}_{i+1}) > 0$$

We claim that S_3^i is equivalent to $\{I_i > 0\}$. To show $S_3^i \Rightarrow \{I_i > 0\}$ just add the inequalities in S_3^i and remember that $y_{i+1} = \bar{u}_{i+1} - \bar{v}_{i+1}$ (for $i = d$, define $\bar{u}_{d+1} = \bar{v}_{d+1}$) . The proof that $\{I_i > 0\} \Rightarrow S_3^i$ goes by elimination, similar to the above proof that $S_1 \Rightarrow S_2$. We omit the details.

Define the system S_3 to consist of the union of the S_3^i , $0 \leq i \leq d$, together with the inequalities $x_j > 0$ for $1 \leqq j \leqq J$. Clearly S_3 is equivalent to S_2 , so in particular $R(S_2) = R(S_3)$. Moreover, S_3 corresponds to the expected utility form (see Section II) of a set $\mathcal{S}$ of gamble-choices. Let us see why.

Define the set X of prizes to be all the variables u_j, v_j, $\bar{u}_i$, $\bar{v}_i$, and w_k^i . Note that each inequality in S_3 was of the form

(13) $\quad u - v + \rho(w - w') > 0$

where u, v, w and w' are in X . (For the inequality $x_j > 0$, for example, $u = u_j$, $v = v_j$ and $w = w'$.) Define $\mathcal{S}$ to be the set of all gamble-choices

(14) $\quad (u,w) \succ (v,w')$

such that the inequality (13) is in S_3 , and let $\mathfrak{J}$ be the set of expected-utility forms of the gamble choices (14) in $\mathfrak{S}$. Thus $\mathfrak{J}$ consists of the inequalities

(15) $$P_H U(u) + P_T U(w) > P_H U(v) + P_T U(w') ,$$

one for each $(u,w) \succ (v,w') \in \mathfrak{S}$.
Since we have assumed $P_H > 0$ and $\rho = P_T/P_H$, (15) is equivalent to

(16) $$U(u) - U(v) + \rho(U(w) - U(w')) > 0 .$$

Now it is clear that there exist real numbers $\rho > 0$ and u, v, w, w' solving (13) iff there are real numbers $\rho > 0$ and U(u), U(v), U(w), U(w'), solving (16). This implies in particular that $R(\mathfrak{S}) = R(S_3)$. We have proved $\mathbb{A} = R(S_1) = R(S_2) = R(S_3) = R(\mathfrak{S})$.

This completes the proof of Theorem 1 in case $\mathbb{A}$ is of the form $\cup_{j=1}^{J} \{\rho\colon \rho > 0 \ \& \ Q_j(\rho) > 0\}$. Now consider the general case,

$$\mathbb{A} = \cap_{\ell=1}^{L} \bigcup_{j=1}^{J} \{\rho\colon \rho > 0 \ \& \ Q_j^{\ell}(\rho) > 0\} .$$

By the preceding, for each ℓ , $1 \leqq \ell \leqq L$, we can find a set $\mathcal{S}^{\ell}$ of gamble choices such that

$$R(\mathcal{S}^{\ell}) = \bigcup_{j=1}^{J} \{\rho : \rho > 0 \ \& \ Q_j^{\ell}(\rho) > 0\} \ .$$

In constructing the $\mathcal{S}^{\ell}$ sets we will also take care that the set of prizes used in $\mathcal{S}^{\ell}$ is disjoint from the set of prizes used in $\mathcal{S}^{\ell'}$, for $\ell \neq \ell'$. Now define $\mathcal{S}$ to be the union of the sets $\mathcal{S}^{\ell}$, $1 \leq \ell \leq L$, and observe that

$$R(\mathcal{S}) = R(\bigcup_{\ell=1}^{L} \mathcal{S}^{\ell}) = \bigcap_{\ell=1}^{L} R(\mathcal{S}^{\ell}) = \mathbb{A} \quad ,$$

where the middle inequality holds since

$$R(\mathcal{J}^1 \cup \mathcal{J}^2) = R(\mathcal{J}') \cap R(\mathcal{J}^2)$$

whenever $\mathcal{J}'$, $\mathcal{J}^2$ are sets of gamble-choices based on disjoint sets of prizes. This completes the proof of the theorem.

We next prove a converse to Theorem 1.

<u>Theorem 2</u>. <u>Let $\mathcal{S}$ be a finite set of gamble-choices.</u>

<u>Then $R(\mathcal{S})$ is a polynomial set</u>.

This theorem will follow from the next two lemmas. We define a <u>weak polynomial set</u> almost exactly as we defined a polynomial set, except that we allow weak inequalities $\geqq$ as well as $>$. Thus a weak polynomial set $\mathbb{B}$ is a finite intersection of finite unions of sets of the form

$$\{\rho\colon Q(\rho) > 0 \;\;\&\; \rho > 0\} \text{ and } \{\rho\colon Q(\rho) \geqq 0 \;\;\&\; \rho > 0\}$$

where each Q is an integer polynomial.

Lemma 1. If $\mathcal{S}$ is a finite set of gamble-choices then $R(\mathcal{S})$ is a weak polynomial set.

Proof.* This is a special case of the first part of Tarski's decision method for elementary algebra (cf. Tarski [10], Kreisel-Krivine [3]). We will indicate why.

Let X be the set of prizes in gambles in $\mathcal{S}$, and suppose the elements of X are numbered thus:

$$X = \{x_1, x_2, \ldots, x_n\} .$$

Let S' denote the set of expected utility forms of the gamble-choices in $\mathcal{S}$, and let $\mathcal{J}$ denote the inequalities in S' divided by P_H, with ρ in place of P_T/P_H. Thus $\mathcal{J}$ is defined by

$$\text{"}(x,y) \succ (z,w)\text{"} \in \mathcal{S} \text{ iff } \text{"}U(x) + \rho U(y) > U(z) + \rho U(w)\text{"} \in \mathcal{J} .$$

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*Our original proof used results in [9]. H.J. Keisler pointed out the relevance of Tarski's Theorem.

Then $\mathcal{J}$ is a formula of elementary algebra (as defined in [10, Section 1]) in the free variables ρ , $U(x_1),\ldots,U(x_n)$. We denote $U(x_i)$ by z_i $(1 \leqq i \leqq n)$ and write $\mathcal{J}(\rho,z_1,\ldots,z_n)$ instead of $\mathcal{J}$, to emphasize the dependence of $\mathcal{J}$ on its free variables. Then $R(\mathcal{S})$ is, by definition, the set of $\rho > 0$ such that there exist $z_1,\ldots,z_n$ for which $\mathcal{J}(\rho,z_1,\ldots,z_n)$ is true, i.e. $\bar{\rho} \in R(\mathcal{S})$ iff the sentence (in Tarski's notation)

(17) $\quad \bar{\rho} > 0$ and $\exists z_1 \, \exists z_2 \ldots \exists z_n \, \mathcal{J}(\bar{\rho},z_1,\ldots,z_n)$

is true. By Tarski's Theorem [10, pg. 39, Theorem 31: take Φ to be (17)] there is a formula $\mathcal{R}(\rho)$ of elementary algebra with ρ as its only free variable and with no quantifiers, such that (17) is equivalent (as defined in [10, pg. 13]) to $\mathcal{R}(\rho)$. This implies, in particular, that

(18) $\quad \bar{\rho} \in R(\mathcal{S})$ iff $\mathcal{R}(\bar{\rho})$ is true.

Now it only remains to show that if $\mathcal{R}(\rho)$ is a formula of elementary algebra with no quantifiers then $\{\rho\colon \mathcal{R}(\rho)$ is true and $\rho > 0\}$ is a weak polynomial set. This follows from Tarski's definitions [10, Section 1]: The only symbols Tarski allows which are not in our definition of weak polynomial set are negation and equality, but these can be incorporated in polynomial inequalities in standard ways (e.g. $Q = 0$ iff $Q^2 \leqq 0$.) This completes the proof of the lemma.

<u>Lemma 2. If $\mathbb{B}$ is a weak polynomial set and $\mathbb{B}$ is an open subset of $\{\rho: \rho > 0\}$ then $\mathbb{B}$ is a polynomial set</u>.

<u>Proof</u>. Let $\mathbb{H}$ be the polynomial set obtained by changing each $\geqq$ in the representation of $\mathbb{B}$ to $>$. Let $K = \mathbb{B} \setminus \mathbb{H}$; if K is empty we're done. The set K is finite since each point in it is a zero of one of the (finitely many) polynomials in the representation of $\mathbb{B}$. If $\alpha \in K$ then since α is in the open set $\mathbb{B}$ there is an $\varepsilon > 0$ such that the interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ is in $\mathbb{B}$. Using the density of the rationals one can find an integer (even quadratic) polynomial Q_α with $\alpha \in \{\rho: Q_\alpha(\rho) > 0\} \subseteq (\alpha - \varepsilon, \alpha + \varepsilon)$. Then

$$\mathbb{H} \cup \bigcup_{\alpha \in K} \{\rho: Q_\alpha(\rho) > 0\} = \mathbb{B} \quad .$$

Since K is finite and $\mathbb{H}$ is a polynomial set, $\mathbb{B}$ is a polynomial set.

<u>Proof of Theorem 2</u>. Let $\mathbb{S}$ be a set of gamble-choices. Since the expected-utility forms of the gamble-choices in $\mathbb{S}$ involve <u>strict</u> inequalities, the set of values P_H, P_T, $U(x)$ $(x \in X)$, which solve the expected-utility forms, is an open set. This implies in particular that the set of (P_H, P_T) in $(0,1) \times (0,1)$, which appear in these solutions, is open, so $R(\mathbb{S})$ is open. Now apply Lemmas 1 and 2 to conclude that $R(\mathbb{S})$ is a polynomial set, Q.E.D.

(19) Remark: We remark without proof that if we had assumed a finite number n of events in our model, instead of taking $n = 2$ as we have done, the two theorems of this section would remain valid with the following changes:

(i) Define $R(\mathcal{S})$ to be the set of $(\rho_2,\ldots,\rho_n)$ with $\rho_i > 0$ for $1 < i \leqq n$ and such that there is $P = (P_1,\ldots,P_n)$, $P_i > 0$ for $1 \leq i \leq n$, and U with (P,U) rationalizing $\mathcal{S}$ such that $\rho_i = P_i/P_1$ for $1 < i \leqq n$. (The meaning of "rationalize" in this context should be clear.)

(ii) Define a polynomial set to be a finite intersection of a finite union of sets of the form

$$\{(\rho_2,\ldots,\rho_n)\colon \rho_i > 0 \text{ for } 1 < i \leqq n \text{ and } Q(\rho_2,\ldots,\rho_n) > 0\}$$

where Q ranges over polynomials in $n-1$ variables, with integer coefficients.

V. Characterizing the Family of Polynomial Sets

The theorems in Section IV have shown that if $\mathcal{F}$ denotes the family of sets $R(\mathbb{S})$, where $\mathbb{S}$ ranges over sets of gamble-choices on X and X ranges over all finite prize sets, then $\mathcal{F}$ equals the family of polynomial sets. In this section we will study the structure of $\mathcal{F}$. This will in particular provide an answer to the question posed about π in the first paragraph of the Introduction. The results of this section are surely well-known to number theorists and algebraic geometers, but, as we have been unable to provide a convenient reference, we have provided details.

To avoid trivialities we will speak only of <u>nonzero</u> integer polynomials, i.e. integer polynomials Q such that $Q(\rho) \neq 0$ for some ρ or, equivalently, such that some coefficient of Q is not zero. A real number $\bar{\rho}$ is a <u>root</u> of the polynomial $Q(\rho)$ if $Q(\bar{\rho}) = 0$. An <u>algebraic number</u> γ is a real number which is a root of a nonzero integer polynomial. We say a subset $\mathbb{B}$ of $\{\rho: \rho > 0\}$ is an <u>algebraic-interval set</u> if

$$\mathbb{B} = \bigcup_{k=1}^{K} (\gamma_k, \delta_k)$$

where, for $k=1,\ldots,K$, each of γ_k and δ_k is equal to an algebraic number except that δ_k may also equal $+\infty$.

<u>Proposition</u>: <u>The family $\mathcal{F}$ of polynomial sets equals the family of algebraic-interval sets.</u>

<u>Remark</u>: although we are restricting ourselves to subsets of $\{\rho: \rho > 0\}$, the Proposition generalizes to arbitrary subsets of the real numbers, using the same proof.

<u>Proof</u>: For this proof we will use the notation, for integer polynomials Q:

$$Q^+ = \{\rho: \rho > 0 \text{ and } Q(\rho) > 0\} .$$

Every nonzero integer polynomial has finitely many roots. From this it follows easily that if Q is an integer polynomial then

$$Q^+ = \bigcup_{k=1}^{K} (\gamma_k, \delta_k) ,$$

where each γ_k and δ_k is either zero or infinite or a root of Q. Thus Q^+ is an algebraic-interval set for every integer polynomial Q. Since the family of algebraic-interval sets is closed under finite unions and intersections, every polynomial set is an algebraic-interval set.

To prove that every algebraic-interval set is a polynomial set it suffices to prove that every set $\mathbb{A}$ of the form

(1) $\mathbb{A} = (0,\gamma)$ or $\mathbb{A} = (\gamma,\infty)$

is a polynomial set, since every algebraic-interval set can be formed by unions and intersections of sets $\mathbb{A}$ of the form (1).

Let γ be a positive algebraic number. We will prove that $(0,\gamma)$ is a polynomial set. By the Lemma which follows this proof, there is an integer polynomial Q and an $\varepsilon > 0$ such that

(2) $Q(\rho) > 0$ for $\rho \in (\gamma-\varepsilon,\gamma)$ and $Q(\rho) < 0$ for $\rho \in (\gamma,\gamma+\varepsilon)$.

Let α , β be rational numbers satisfying

(3) $$\gamma - \varepsilon < \alpha < \gamma < \beta < \gamma + \varepsilon \quad .$$

Let Q_α , Q_β be integer polynomials (they can even be chosen to be linear) such that

(4) $$Q_\alpha^+ = (0,\alpha) \quad \text{and} \quad Q_\beta^+ = (0,\beta) \quad .$$

Then, (2), (3) and (4) imply

(5) $$(0,\gamma) = (Q^+ \cap Q_\beta^+) \cup Q_\alpha^+ \quad .$$

Thus $(0,\gamma)$ is a polynomial set.

The proof that (γ,∞) is a polynomial set whenever γ is a positive algebraic number, is similar to the previous case. In fact one could, in the representation (5) of $(0,\gamma)$, re-place each polynomial Q with -Q and the result would be a representation of (γ,∞) .

The proof of the Proposition is complete.

The following lemma was used in the previous proof.

Lemma: If γ is an algebraic number then there is an integer polynomial Q and an $\varepsilon > 0$ such that

$$Q(\rho) > 0 \quad \text{for} \quad \rho \in (\gamma - \varepsilon, \gamma) \quad \text{and} \quad Q(\rho) < 0 \quad \text{for} \quad \rho \in (\gamma, \gamma + \varepsilon) .$$

Proof: Since γ is an algebraic number there is a nonzero integer polynomial $Q(\rho)$ with $Q(\gamma) = 0$. Let $\gamma_0, \gamma_1, \ldots, \gamma_K$ be the distinct (real and complex) roots of Q , with $\gamma = \gamma_0$. Then it is a consequence [1, pg. 103] of the Fundamental Theorem of Algebra that there are positive integers $i_0, \ldots, i_K$ (i_k is called the multiplicity of γ_k) such that for some nonzero integer c ,

$$Q(\rho) = c \prod_{k=0}^{K} (\rho - \gamma_k)^{i_k} .$$

We consider two cases, according to whether the multiplicity i_0 of the root γ_0 is odd or even.

Case I: i_0 is odd. Notice that since the γ_k are distinct,

$$\prod_{k=1}^{K} (\gamma_0 - \gamma_k)^{i_k} \neq 0 . \tag{6}$$

Since γ_0 is real and $P(\rho)$ is real for real ρ,

$$c \prod_{k=1}^{K} (\rho - \gamma_k)^{i_k} \text{ is real for real } \rho . \tag{7}$$

By (6) and (7) we can choose $\lambda \in \{+1, -1\}$ such that

$$\lambda c \prod_{k=1}^{K} (\gamma_0 - \gamma_k)^{i_k} < 0 .$$

By continuity there is some $\varepsilon > 0$ such that

$$\lambda c \prod_{k=1}^{K} (\rho - \gamma_k)^{i_k} < 0 \quad \text{for} \quad \rho \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) . \tag{8}$$

Since i_0 is odd,

$$(\rho - \gamma_0)^{i_0} < 0 \quad \text{for} \quad \rho \in (\gamma_0 - \varepsilon, \gamma_0) \tag{9}$$

$$\text{and} \quad (\rho - \gamma_0)^{i_0} > 0 \quad \text{for} \quad \rho \in (\gamma_0, \gamma_0 + \varepsilon) .$$

Combining (8) and (9) we see that

$$\lambda P(\rho) = \lambda c \prod_{k=0}^{K} (\rho - \gamma_k)^{i_k}$$

is positive on $(\gamma_0 - \varepsilon, \gamma_0)$ and negative on $(\gamma_0, \gamma_0 + \varepsilon)$. This completes the proof in Case I since $\gamma = \gamma_0$ and λQ is an integer polynomial.

<u>Case II</u>: i_0 is even. Since i_0 is a positive integer, this implies $i_0 \geqq 2$. Denote the derivative of $Q(\rho)$ by $Q'(\rho)$. We claim that γ_0 is a root of Q' with odd multiplicity $i_0 - 1$. This follows from [1, pg. 103] but we will give a proof here for completeness.

By repeated use of the product rule we get

$$(10) \qquad Q'(\rho) = \sum_{j=0}^{K} [i_j(\rho - \gamma_j)^{i_j - 1} \prod_{k \neq j} (\rho - \gamma_k)^{i_k}]$$

where the product is over all $k = 0,\ldots,K$ with $k \neq j$.
After rearranging, (10) becomes

$$(11) \qquad Q'(\rho) = (\rho - \gamma_0)^{i_0 - 1} Q_1'(\rho)$$

where Q_1' is defined by

$$(12) \qquad Q_1'(\rho) = i_0 \prod_{k=1}^{K} (\rho - \gamma_k)^{i_k}$$

$$+ (\rho - \gamma_0) \sum_{j=1}^{K} i_j (\rho - \gamma_j)^{i_j - 1} \prod_{k \neq 0,j} (\rho - \gamma_k)^{i_k} \; .$$

In (12), $\prod_{k \neq 0,j}$ means the product over all $k = 0,\ldots,K$ except $k = 0$ and $k = j$. Since the numbers $\gamma_0,\ldots,\gamma_K$ are distinct, $Q_1'(\gamma_0) \neq 0$. That and (11) imply that the multiplicity of the root γ_0 in Q' is i_0-1 .

Since Q' (the derivative of Q) is an integer polynomial, and the multiplicity of γ_0 in Q' is the odd number i_0-1, we have reduced the problem to that considered in Case I. This completes the proof of the Lemma.

(13) Remarks.

1. It is well-known [2] that the number π is transcendental, i.e. not algebraic. This implies that interval (π,∞) is not an algebraic-interval set, so by what we have proved above it is not a polynomial set. Thus by the theorems of Section IV, a gambler cannot reveal that tails is π times as likely as heads.

2. We mention without proof that there are polynomial sets which cannot be represented as $\{\rho: \rho > 0 \text{ and } Q(\rho) > 0\}$ for a single polynomial Q. The interval $(0, 2+\sqrt{2})$ is such a set, essentially because if Q is an integer polynomial that has $2+\sqrt{2}$ as a root with multiplicity k then Q has $2-\sqrt{2}$ as a root with multiplicity k.

3. As a final remark, we point out a simple consequence of the definition of Lebesgue measurability: if $\mathbb{M} \subseteq \{\rho: \rho > 0\}$ is a measurable set and $\varepsilon > 0$, then there is a set $\mathbb{A}$ which is a finite union of intervals with rational endpoints such that $\mathbb{A} \supseteq \mathbb{M}$ and $\mathbb{A} - \mathbb{M}$ has measure less than ε. Thus polynomial sets are "dense" in the family of measurable sets. Of course the previous sentence could be proved without all our

machinery, since only rational endpoints are needed to construct $\mathbb{A}$ in the previous argument.

VI. Open Problems

We have already mentioned (Section IV) the problem of extending our results to the case $\rho = 0$, and of interpreting and extending them to the case $\rho = \infty$ (i.e. $P_H = 0$).

There are two more interesting problems which we have not investigated.

The first* is to characterize the sets $R(\mathcal{S})$ where $\mathcal{S}$ ranges over the sets of gamble choices on a prize set X which give a total order on the set of all gambles on X, and X ranges over all finite sets. That is, consider only $\mathcal{S}$ satisfying: if x, y, z, w are (not necessarily distinct) prizes in X, and if $(x,y) \neq (z,w)$, then either

$$(x,y) \succ (z,w) \quad \text{or} \quad (z,w) \succ (x,y)$$

is in $\mathcal{S}$.

The second is a "dual" to the problem we have considered; namely, characterize the set of utility functions $U: X \to \mathbb{R}$ such that there exists a $P = (P_H, P_T)$ with (P,U) rationalizing $\mathcal{S}$, as $\mathcal{S}$ ranges over finite sets of gamble choices. The work of Scott [8] may be relevant here.

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*Suggested by David Schmeidler.

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