TEMPORAL AGGREGATION IN THE MULTIVARIATE REGRESSION MODEL*

by

John Geweke

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

ABSTRACT

The model considered in this paper is

$$y(t) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} x_j(t-s)\, b_j(s)\, ds + u(t), \quad E[x_j(s)\, u(t)] = 0, \tag{1}$$

where the time parameterization is continuous but the b_j's may be generalized as well as ordinary functions so that (1) subsumes discrete models and certain stochastic differential equations as well as the classical distributed lag model. Since economic variables are almost never observed continuously (1) is usually replaced with

$$Y(t) = \sum_{j=1}^{N} \sum_{s=-\infty}^{\infty} X_j(t-s)\, B_j(s) + U(t), \quad E[X_j(s)\, U(t)] = 0, \tag{2}$$

integer s and t, in which Y(t) and the $X_j(t)$ are regular samplings or averagings of y(t) and $x_j(t)$, respectively. The relation of the parameterization of this model to that of (1) is an important problem in the interpretation of applied studies, which must use (2). In this paper properties of the relationship are established and their implications for applied work are discussed.

Using projection operators in the frequency domain it is shown that if y and the x_j are taken to be realizations of mutually covariance stationary stochastic processes then

$$\underset{N\times 1}{B(t)} = \int_{-\infty}^{\infty} r_x(s)\, \underset{N\times 1}{b(t-s)}\, ds, \quad t \text{ integer},$$

where $r_x(s)$ is an N x N matrix determined by the covariance structure of the x_j. Since $r_x(s)$ is in general non-diagonal and as a function is not symmetric about s = 0, each $B_j(t)$ is a confounding of moving averages of all $b_i(s)$ which may exhibit systematic leading or lagging biases. A series of propositions proved using frequency domain analytic techniques demonstrates that very stringent assumptions about the b_j and x_j are necessary for r_x to be diagonal or for $B_j(t)$ to approximate $b_j(t)$ well for all j and t. Given weak restrictions on the x_j and b_j, however, it is shown that each B_j converges pointwise and in mean square to the b_j as the frequency of observation implicit in (2) increases.

Several examples of estimated $r_x(s)$ corresponding to actual economic time series for various levels of temporal aggregation are presented. These examples and the foregoing analysis suggest that if the x_j vary significantly between observations or are strongly correlated with one another, or if the b_j are not smooth, the inference about (1) from knowledge of (2) is likely to be difficult or impossible. Recommendations for the interpretation of the discrete model (2) in applied work are made.

Temporal Aggregation in the Multivariate Regression Model

1. Introduction

Consider the multivariate regression model

$$y(t) = \sum_{j=1}^{N} x_j * b_j(t) + u(t) \qquad (1)$$

where y is the dependent variable, the x_j's are independent variables, u is the residual and "*" denotes the convolution operator,

$$x_j * b_j(t) = \int_{-\infty}^{\infty} x_j(t-s)\, b_j\, (s)\, ds.$$

y, x_j (j = 1, ... N) and u are taken to be realizations of mutually convariance stationary stochastic processes with finite variances. In what follows we shall assume that the $x_j(t)$ are strictly exogenous: $E[x_j(t)\, u(s)] = 0$ for all j, s and t. The b_j are allowed to be generalized as well as ordinary functions.[1] (1) therefore subsumes special cases like

$$y(t) = \sum_{i=1}^{K} \sum_{j=1}^{N} x_j(t-s_i)\, b_j(s_i) + u(t) \qquad (2)$$

a "discrete time" model, and

$$y(t) = b\, \dot{x}(t) + u(t)$$

a stochastic differential equation.

With rare exceptions the nature of economic data and estimation procedures preclude using (1) directly in applied work. Since data are usually available only as discrete time series recorded at regular intervals (1) traditionally has been replaced by

$$Y(t) = \sum_{s=-\infty}^{\infty} \sum_{j=1}^{N} X_j(t-s)\ B_j(s) + U(t),\ t \text{ integer},$$
$$E[X_j\ (t)\cdot U(s)] = 0,\ \text{all } j \text{ and all integer } s \text{ and } t. \tag{3}$$

In this model $Y(t)$ and $X_j(t)$ may be point samplings of $y(t)$ and $x_j(t)$,

$$Y(t) = y(t),\quad X_j(t) = x_j(t),\quad t \text{ integer},$$

or unit averaged data,

$$Y(t) = \int_{t+a-1}^{t+a} y(r)\ dr,\quad X_j(t) = \int_{t+a-1}^{t+a} x_j(r)\ dr,\quad t \text{ integer},\ 0\le a\le 1.$$

Both the finite lengths of time series records and the application of tractable estimation procedures to (3) require that the $B_j(s)$ be functions of small numbers of parameters. In most cases this implies the need for replacement of (3) by a model with a finite number of parameters. This paper is concerned exclusively with the problem of replacement of (1) by a relationship between the variables as measured when all observations are made either as point samplings or as unit averages with the same frequency.

Sims ([9], [10]) has treated (1) when there is a single independent variable x. Using projection operators in the frequency domain he shows that

$$B(t) = r_x * b(t) = \int_{-\infty}^{\infty} r_x(t-s)\ b(s)\ ds \tag{4a}$$

$$r_x(t) = R_X^{-*} * R_x(t) = \sum_{s=-\infty}^{\infty} R_X^{-*}(s)\ R_x(t-s) \tag{4b}$$

(R_X^{-*} being the inverse under convolution of R_X, the autocovariance function of X) and derives some results about the form of r_x and the relation of B to b. In Section 2 it is shown that when N > 1 the relation of B to b is formally analagous to (4), but the multivariate aspect of the problem introduces two new complications which are important in the interpretation of discrete models: first, $B_j(t)$ is a linear combination of weighted averages of <u>all</u> $b_i(s)$; second, while B(t) is a moving average of b(s), the average is in general not symmetric about s = t as is the case in (4). The more frequently x and y are sampled, the less important are these problems: results proved in Section 3 show that (given some weak conditions) with sufficient temporal disaggregation the parameterization of (1) may be approximated arbitrarily well by that of (3). Based on these theoretical considerations Section 4 outlines conditions under which inferences about b based on assumptions like $B_i(t) = b_i(t)$ or $\sum_{t=-\infty}^{\infty} B_i(t) = \int_{-\infty}^{\infty} b_i(t)\,dt$ are apt to be misleading or mistaken. Experiments with actual time series reported in Section 5 suggest that increased temporal disaggregation within the range of conventional observational frequencies for macroeconomic data yields substantial improvements in the approximation of (1) by (3).

2. The Discrete Model and its Properties

In this section we consider the relation of the model (3) to (1) when Y and the X_j are all measured as either point or unit-averaged data. If the variables are unit averaged, then

$$\int_{t+a-1}^{t+a} y(s)\,ds = \sum_{j=1}^{N} b_j * \int_{t+a-1}^{t+a} x_j(s)\,ds + \int_{t+a-1}^{t+a} u(s)\,ds, \tag{5}$$

a model which meets all of our assumptions about (1). It is convenient to proceed assuming (1), taking note of those features of our results which depend on whether the basic model is (1) or (5).

In vector notation (3) is

$$Y(t) = \underset{1\times N}{X'} * \underset{N\times 1}{B(t)} + U(t); \quad E[X(t)U(s)] = 0, \text{ all integer t and s.}$$

Because X is strictly exogenous,

$$E[X(r)Y(t)] = E\{\sum_{s=-\infty}^{\infty} X(r)X(t-s)'B(s)\}, \text{all integer } t, r \text{ and } s$$

$$\Leftrightarrow \qquad R_{XY}(t) = R_X * B(t), \text{ all integer } t.$$

In the frequency domain,[2]

$$S_{XY}(\omega) = S_X(\omega)\tilde{B}(\omega), \ |\omega| \leq \pi.$$

So long as $S_X(\omega)$ is non-singular for all $\omega \in [-\pi,\pi]$,

$$\tilde{B}(\omega) = S_X(\omega)^{-1}S_{XY}(\omega) = [F[S_x](\omega)]^{-1}F[S_{xy}](\omega)$$

$$= [F[S_x](\omega)]^{-1}F[S_x\tilde{b}](\omega) = F\{[F[S_x](\omega)]^{-1}S_x\tilde{b}\}(\omega),$$

where F is the "folding operator,"[3]

$$F[S_x](\omega) = \sum_{k=-\infty}^{\infty} S_x(\omega+2\pi k).$$

B(t) is the inverse Fourier transform of

$$[F[S_x](\omega)]^{-1}S_x(\omega)\tilde{b}(\omega)$$

at integer t. Defining

$$\underset{N\times N}{\tilde{r}_x(\omega)} = [F[S_x](\omega)]^{-1}S_x(\omega), \tag{6a}$$

$$\tilde{B}(\omega) = F[\tilde{r}_x\tilde{b}]\ (\omega), \ |\omega| \leq \pi \tag{6b}$$

from which

$$r_x(t) = R_X^{-*} * R_x(t), \tag{7a}$$

$$B(t) = r_x * b(t), \ t \text{ integer}. \tag{7b}$$

R_X and R_x are the autocovariance matrices of X and x, respectively,

$$[R_X]_{ij}(t) = E[X_i(s)X_j(s+t)], \text{ integer } t \text{ and } s,$$

$$[R_x]_{ij}(t) = E[x_i(s)x_j(s+t)], \text{ real } t \text{ and } s.$$

R_X^{-*} is the inverse under convolution of R_X,

$$\sum_{k=1}^{N} \sum_{s=-\infty}^{\infty} [R_X^{-*}]_{ik}(t-s)[R_X]_{kj}(s) = \delta_{t,0}\cdot\delta_{i,j}$$

(δ = Kronecker δ).

For integer t

$$r_x(t) = \delta_{t,0}\cdot I_N$$

(7) differs from (4) in several important respects. $r_x(t)$ is a non-diagonal, non-symmetric matrix, and as a function it is not even symmetric about $t = 0$. Each $\tilde{B}_i(\omega)$ is a weighted average of all $\tilde{b}_j(\omega)$, and $B_i(t)$ is a convolution not just of $b_i(s)$, but of all $b_j(s)$, $j = 1, \ldots, N$. Figure 1 illustrates the relation of B to b for a variety of b's and possible covariance structures for x. In particular, these sketches show that (a) one sided b's do not imply one-sided B's, as in the univariate case; (b) because of the confounding of all b_j in each B_i (a problem which does not arise when N = 1) B_j may be far from zero even though $b_j \equiv 0$. We shall refer to this confounding as "contamination." Contamination is analagous to the classical omitted variables problem of econometrics: it arises whenever some of the $x_i(t)$ at integer t explain some variance of $x_j(s)$ ($i \neq j$ and s not an integer) which cannot be explained by $x_j(t)$ at integer t. In fact, the absence of such explanation is necessary and sufficient for the absence of contamination.

<u>Proposition 1.</u> Consider the regression relationships

$$x_j(t+r) = \sum_{i=1}^{N} \sum_{s=-\infty}^{\infty} c_{rij}(s)\, x_i(t-s) + u_j(t+r),$$

$$t \text{ integer},\ 0 \leq r < 1,\ j = 1, \ldots, N.$$

$r_x(t)$ is diagonal for all t if, and only if, $c_{rij} \equiv 0$ for $i \neq j$.

Proof. Write $C(r,s)$ with typical element $c_{rij}(s)$ and Fourier tranform $\tilde{C}(r,\omega)$.

$$\tilde{C}(r,\omega) = [F[S_x](\omega)]^{-1} \sum_{k=-\infty}^{\infty} e^{-ir(\omega+2\pi k)},\ S_x(\omega+2\pi k),\ -\infty < \omega < \infty,$$

where $\tilde{C}(r,\omega)$ is periodic in ω with period 2π. $c_{rij} \equiv 0$ for $i \neq j$ if and only if $\tilde{C}(r,\omega)$ is diagonal for all r and ω, and $r_x(t)$ is diagonal for all t if and only if $[F[S_x](\omega)]^{-1} S_x(\omega)$ is diagonal for all ω.

$$e^{ir\omega}\tilde{C}(r,\omega) = \sum_{k=-\infty}^{\infty} e^{-ir2\pi k}[F[S_x](\omega+2\pi k)]^{-1} S_x(\omega+2\pi k).$$

Necessity is clear: if $[F[S_x](\omega)]^{-1} S_x(\omega)$ is diagonal for all ω, $e^{ir\omega}\tilde{C}(r,\omega)$ must be also. But it is also true that

$$\int_0^1 \sum_{k=-\infty}^{\infty} e^{-ir2\pi k}[F[S_x](\omega+2\pi k)]^{-1} S_x(\omega+2\pi k)dr = \int_0^1 e^{ir\omega}\tilde{C}(r,\omega)$$

$$[F[S_x](\omega)]^{-1} S_x(\omega) = \int_0^1 e^{ir\omega}\tilde{C}(r,\omega)dr.$$

Hence if $\tilde{C}(r,\omega)$ is diagonal for all r and ω, $r_x(t)$ is diagonal.

Two specific cases may be identified in which "contamination" is no problem.[4]

Proposition 2. Suppose that, for all integer n, $t \in [0,1]$, and all i and j,

$$R_{x_{ij}}(n+t) = t \cdot R_{x_{ij}}(n+1) + (1-t) \cdot R_{x_{ij}}(n).$$

Then $r_x(t) = \text{diag}(\ r_0(t),\ \ldots\ ,\ r_0(t)\ )$ where

$$r_0(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}.$$

Proof. The assumptions are equivalent to

$$R_x(t) = R_X * \mathrm{diag}(\ r_0(t),\ \ldots\ ,\ r_0(t)\ )$$

whence

$$r_x(t) = R_X^{-*} * R_x(t) = \mathrm{diag}(\ r_0(t),\ \ldots\ ,\ r_0(t)\ ).$$

The Proposition 2 situation has the advantage that $r_x(t) = 0$ for $|t| \geq 1$, as well as $r_{x_{ij}}(t) = 0$ for $i \neq j$. It is worth noting that no x vector with one or more mean-square differentiable[5] components satisfies the conditions of Proposition 2.

Proposition 3. If $S_x(\omega)$ is non-singular in $(-\Pi, \Pi)$ but $S_x(\omega) = 0$ for $|\omega| > \Pi$, then $r_x(t) = \mathrm{diag}(\ \sin\pi t/\pi t,\ \ldots\ ,\ \sin\pi t/\pi t\ )$.

Proof. By explicit calculation,

$$\tilde{r}_{x_{ij}}(\omega) = \begin{cases} \delta_{ij}, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases} \text{ whence } r_{x_{ij}}(t) = \delta_{ij} \sin \pi t/\pi t.$$

Propositions 2 and 3 illustrate two types of x processes which guarantee that $r_x(t)$ is diagonal for all t. They differ in that the x processes of Proposition 2 are locally rough but imply r_x's with no side lobes, while those of Proposition 3 are so smooth that they are mean-square differentiable of all orders but imply r_x with significant side lobes. These results suggest the difficulty of establishing general conditions under which side lobes in $[r_x]_{ii}(t)$ are small. A weaker result which is directly applicable is the following.

Proposition 4. If Y and all of the X_j in (3) are unit averages, then

$$\int_{-\infty}^{\infty} [r_x]_{ij}(t)\, dt = \delta_{ij}.$$

Proof. From a well-known property of the Fourier transform,

$$\int_{-\infty}^{\infty} [r_x]_{ij}(t)\, dt = [\tilde{r}_x]_{ij}(0).$$

When Y and the X_j are unit averages, the multivariate regression model is (5). Denoting unit averaged $\tilde{y}$ and $\tilde{x}_j$ by $\tilde{\bar{y}}$ and $\tilde{\bar{x}}_j$, respectively, we apply the relations

$$\tilde{\bar{y}}(\omega) = D(\omega)\ \tilde{y}(\omega),\quad \tilde{\bar{x}}_j(\omega) = D(\omega)\ \tilde{x}_j(\omega)$$

where

$$D(\omega) = \frac{\sin(\omega/2)}{\omega/2}.$$

Hence

$$S_{\underline{x}}(\omega) = S_x(\omega)\ E^2(\omega)$$

where

$$E(\omega) = \mathrm{diag}(\ D(\omega),\ \ldots\ ,\ D(\omega)\ ).$$

Since $D(2\pi k) = \delta_{k,0}$, $F[S_{\underline{x}}](0) = S_x(0)$, whence

$$[\tilde{r}_x](0) = [S_x(0)]^{-1} S_x(0) = I_N.$$

Proposition 4 shows that when Y and the X_j are unit averaged, then r_x has certain desirable properties. Although $\int_{-\infty}^{\infty}[r_x]_{ij}(t)dt = 0$ for $i \neq j$, $[r_x]_{ij}$ can in principle show large oscillations between integer t (where it must be zero if $t \neq 0$) and there is no guarantee that $|B_i(t) - b_i(t)|$

is smaller when the X_j are unit averaged that when they are point data.

Fortunately it is possible to find reasonable sufficient conditions for $B_i(t) \doteq b_i(t)$. In deducing these conditions, as well as in subsequent sections, the following metric will be found very useful.

<u>Definition</u>. For any complex matrix A, $||A||$ is the square root of largest eigenvalue of A'A. We shall refer to $||A||$ as the "norm" of A.

Obviously $||A||$ is also the square root of largest eigenvalue of AA' and if A is Hermitian positive definite, $||A||$ is the largest eigenvalue of A. Its usefulness arises from the properties

(i) $||A|| > 0$, if $A \neq 0$, and $||0|| = 0$;

(ii) $||cA|| = |c| \cdot ||A||$;

(iii) $||A+B|| \leq ||A|| + ||B||$;

(iv) $||AB|| \leq ||A|| \cdot ||B||$.

We shall also make frequent use of the notation

$$\lambda_i(\omega) = i\text{'th largest eigenvalue of } S_x(\omega).$$

The following Lemma presents certain "regularity" conditions which the matrix S_x must satisfy in order to show that $||B(t) - b(t)||$ is small; these conditions will be assumed in several succeeding developments.

<u>Lemma 5</u>. Suppose the eigenvalues of $S_x(\omega)$ are positive for all ω such that $|\omega| < \omega^* \leq n\pi$, and

$$\sup_{\substack{i,j,|\omega|>\omega^* \\ i\neq j}} \left| \frac{S_{x_{ij}}(\omega)}{S_{x_{ii}}(\omega)} \right| = \frac{d}{N} \qquad , d < 1 \qquad (8)$$

Let $g_0(\omega) = \sup_{k\neq 0} ||g(\omega+2n\pi\, k)||$. Then

(i) For all $\omega\varepsilon[-\omega^*, \omega^*]$,

$$||[\sum_{m=-\infty}^{\infty} S_x(\omega+2n\pi\, m)]^{-1} \sum_{k\neq 0} S_x(\omega+2n\pi k)g(\omega+2n\pi k) \leq \frac{g_0(\omega)\sum_{k\neq 0}\lambda_1(\omega+2n\pi\, k)}{\lambda_N(\omega)} \qquad (9)$$

(ii) For all ω: $\omega^* < |\omega| \leq n\pi$,

$$||[\sum_{m=-\infty}^{\infty} S_x(\omega+2n\pi\, m)]^{-1} \sum_{k\neq 0} S_x(\omega+2n\pi\, k)g(\omega+2n\pi k)|| \qquad (10)$$

$$\leq \frac{g_0(\omega)(1+d)}{(1-d)} \sum_{k\neq 0} \sup_{i=1,\ldots,N} \left\{ \frac{S_{x_{ii}}(\omega+2n\pi\, k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2n\pi\, m)} \right\}$$

<u>Proof</u>. (10) cannot exceed

$$\sum_{k\neq 0}{}' \{||[\sum_{m=-\infty}^{\infty} S_x(\omega+2n\pi\, m)]^{-1} S_x(\omega+2n\pi k)g(\omega+2n\pi\, k)||\}$$

$$\leq g_0(\omega) \sum_{k\neq 0}{}' \{||[\sum_{m=-\infty}^{\infty} S_x(\omega+2n\pi m)]^{-1} S_x(\omega+2n\pi\, k)||\}$$

(i) The largest root of $[\sum_{m=-\infty}^{\infty} S_x(\omega+2n\pi\, m)]^{-1}$ is the inverse of the smallest root of $[\sum_{m=-\infty}^{\infty} S_x(\omega+2n\pi m)]$, which in turn is larger than $\lambda_N(\omega)$. Hence $||[\sum_{m=-\infty}^{\infty} S_x(\omega+2n\pi\, m)]^{-1}|| \leq 1/\lambda_N(\omega)$. (9) follows from properties (iii) and (iv) of the norm.

(ii) Define the metric $M(A) = \sup_{i,j} N|a_{ij}|$ for any square matrix $\underset{N\times N}{A}$ with typical element a_{ij}. Consider any matrices $\underset{N\times N}{A_1}$ and $\underset{N\times N}{A_2}$ with positive diagonal elements, for which

$A_1 = V_1(I + S_1)$, V_1 diagonal with $V_{1ii} = A_{1ii} (1\leq i\leq N)$ and $M(S_1) \leq d$;

$A_2 = V_2(I + S_2)$, V_2 diagonal with $V_{2ii} = A_{2ii}$ $(1\leq i\leq N)$ and $M(S_2) \leq d$.

Then $A_1 + A_2 = (V_1 + V_2)$ $(I + S_3)$, where

$$S_3 = (V_1 + V_2)^{-1}(V_1S_1 + V_2S_2) \qquad M(S_3) \le d.$$

The result obviously generalizes to any countable sequence of such matrices A_i with finite sum. We can therefore define

$$V_j = \mathrm{diag}(S_{x11}(\omega+2n\pi j), \ldots, S_{xNN}(\omega+2n\pi j)), \; V = \sum_{j=-\infty}^{\infty} V_j$$

and write

$$[\sum_{m=-\infty}^{\infty} S_x(\omega+2n\pi m)]^{-1} S_x(\omega+2n\pi k) = (I+S)^{-1}V^{-1}V_k(I+S_k)$$

where $M(S) \le d$, $M(S_k) \le d$. (10) is therefore bounded by

$$g_0(\omega) \; ||(I+S)^{-1}|| \sum_{k\neq 0} ||V^{-1}V_k||\cdot||(I+S_k)|| \le g_0(\omega)||(I+S)^{-1}||$$

$$\cdot \sup_{j\neq 0}||(I+S_j)|| \cdot \sum_{k\neq 0}||V^{-1}V_k|| \le g_0(\omega)\,\frac{(1+d)}{(1-d)} \sum_{k\neq 0} \sup_{i=1,\ldots,N} \left\{ \frac{S_{x_{ii}}(\omega+2n\pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2n\pi m)} \right\},$$

the latter inequality using the facts that $M(S) \le d \Rightarrow ||S|| \le d$ (see Fadeev and Fadeeva([1], p. 110)), and $||S|| \le d \Rightarrow ||(I+S)^{-1}|| \le \frac{1}{1-d}$ (see Noble ([7], p. 431)).

Lemma 5 could have been proved assuming only (8) with $\omega^* = 0$, or by assuming that $\lambda_1(\omega)/\lambda_N(\omega)$ is uniformly bounded.[6,7] The former alternative disallows x_j which are highly correlated at low frequencies, a condition violated in most applications. The latter precludes cases like

$$S_x(\omega) = \begin{bmatrix} \omega^{-2} & 0 \\ 0 & \omega^{-4} \end{bmatrix} \quad \text{for large } \omega,$$

in which the problem reduces to the univariate case and smallness of $||b(t) - B(t)||$ is easily shown given sufficient conditions on b; more generally, it would prevent consideration of cases in which some of the

x_j processes were k'th order mean-square differentiable but others were not. Any of these assumptions precludes S_x for which it is not possible to bound $|S_{x_{ij}}(\omega)/S_{x_{ii}}(\omega)|$.

Proposition 6. Suppose b is of bounded variation in an open interval including the point t_o. Define

$$\hat{b}(t_o) = 1/2 \lim_{\varepsilon\to 0} [b(t_o + \varepsilon) + b(t_o - \varepsilon)].$$

Let ε be some positive constant and suppose there exist ω_0 and ω^*, $\omega_0 \geq \omega^*$, such that

(i) $\quad || \int_{-\infty}^{\infty} |b(t)| dt || = A < \infty;$

(ii) $\quad \sup_{\substack{i,j,|\omega|>\omega^* \\ i\neq j}} \left| \frac{S_{x_{ij}}(\omega)}{S_{x_{ii}}(\omega)} \right| = \frac{d}{N},\ d < 1;$

(iii) $\quad \inf_{|\omega| < \omega^*} \lambda_N(\omega) \geq \frac{4\pi}{A\varepsilon} \int_{|\omega|>\pi} \lambda_1(\omega)d\omega;$

(iv) $\quad \int_{\omega^*}^{\omega_0} \sum_{k\neq 0} \sup_{i=1,\ldots,N} \frac{S_{x_{ii}}(\omega+2\pi m)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2\pi m)} d\omega \leq \frac{\pi(1-d)\varepsilon}{8A(1+d)}$

(v) $\quad \sup_{|\omega|>\omega_0} ||\tilde{b}(\omega)|| \leq \frac{\pi\varepsilon}{2N}$

(vi) $\quad ||\hat{b}(t_0) - \frac{1}{2\pi} \int_{|\omega|>\pi} \tilde{b}(\omega)e^{it\omega}d\omega|| \leq \frac{\varepsilon}{4}.$

Then

$$|| \hat{b}(t_0) - B(t_0) || \leq \varepsilon.$$

Proof. A standard result (given (i) and bounded variation of b near t_0) is

$$\tilde{b}(\omega) \equiv \lim_{T\to\infty} \int_{-T}^{T} b(t)e^{-i\omega t}dt \Rightarrow \hat{b}(t_0) = \lim_{W\to\infty} \int_{-W}^{W} \tilde{b}(\omega)e^{i\omega t_0}d\omega .$$

(See Titchmarsh ([11], p. 13).) Hence

$$\hat{b}(t_0) - B(t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{b}(\omega)e^{i\omega t_0}d\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{B}(\omega)e^{i\omega t_0}d\omega .$$

$$= \frac{1}{2\pi}\int_{|\omega|>\pi} \tilde{b}(\omega)e^{i\omega t_0}d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} (\tilde{b}(\omega) - \tilde{B}(\omega)e^{i\omega t_0}d\omega . \tag{11}$$

The norm of the first term is bounded by $\varepsilon/4$. The second is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [F[S_x](\omega)]^{-1}\{\sum_{k=-\infty}^{\infty} S_x(\omega+2\pi k)[\tilde{b}(\omega) - \tilde{b}(\omega+2\pi k)]\}e^{i\omega t_0}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [F[S_x](\omega)]^{-1} \sum_{k\neq 0} \{S_x(\omega+2\pi k)[\tilde{b}(\omega)-\tilde{b}(\omega+2\pi k)]\}e^{i\omega t_0}d\omega . \tag{12}$$

The integration in (12) may be divided into three parts: across $S_1 = [-\omega^*, \omega^*]$; $S_2 = \{\omega : \omega^* < |\omega| \leq \omega_0\}$; $S_3 = \{\omega : \omega_0 < |\omega| \leq \pi\}$. Across S_1, the norm of (12) is dominated by

$$\frac{2A}{2\pi}\int_{-\omega^*}^{\omega^*} \frac{\sum_{k\neq 0} \lambda_1(\omega+2\pi k)}{\lambda_N(\omega)} d\omega \leq \frac{A}{\pi \inf_{|\omega|\leq\omega^*} \lambda_N(\omega)} \int_{-\omega^*}^{\omega^*} \sum_{k\neq 0} \lambda_1(\omega+2\pi k)d\omega \leq \varepsilon/4;$$

across S_2, by

$$\frac{2A(1+d)}{2\pi(1-d)}\int_{S_2} \sum_{k\neq 0} \sup_{i=1,\ldots,N} \frac{S_{x_{ii}}(\omega+2\pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2\pi m)} d\omega \leq \varepsilon/4;$$

across S_3, by

$$\frac{1}{2\pi} \sup_{|\omega|>\omega_0} ||\tilde{b}(\omega)|| \int_{S_3} \sum_{k\neq 0} \sup_{i=1,\ldots,N} \frac{S_{x_{ii}}(\omega+2\pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2\pi m)} d\omega$$

$$\leq \frac{N}{2\pi} \sup_{|\omega|>\omega_0} ||\tilde{b}(\omega)|| \leq \varepsilon/4.$$

These conditions for $B(t) \doteq b(t)$ invlove the convariance structure of all x_i and both global and local properties of b. In a sense which is made exact in Lemma 5 the spectral density matrix of x must be "well conditioned" at all frequencies with variance concentrated in an interval which includes $\omega = 0$ and is a subset of $(-\pi, \pi)$. Unless $S_x(\omega)$ falls off drastically near π, condition (iii) of Proposition 6 requires that $|S_{x_{ii}}(\omega)|$ be less than 1/N for all $i \neq j$ in an interval (ω^*, ∞), $\omega^* << \pi$. Since this implies

$$\left| \frac{\sum_{k=-\infty}^{\infty} S_{x_{ij}}(\omega+2\pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2\pi m)} \right| < \frac{1}{N}, \quad |\omega| \varepsilon (\omega^*, \pi)$$

it is clear that if $\left| S_{X_{ij}}(\omega)/S_{X_{ii}}(\omega) \right|$ (which can be estimated) is not bounded by 1/N near π, the conditions of Proposition 6 cannot be met; however, since Proposition 6 is concerned with sufficiency, failure of $\left| S_{X_{ij}}(\omega)/S_{X_{ii}}(\omega) \right|$ to be bounded in this way does not preclude smallness of $||B(t) - \hat{b}(t)||$. The conditions on $\tilde{b}(\omega)$ prevent rapid oscillation in <u>any</u> $b_i(t)$, a "rapid" oscillation being one which is shorter than several periods. This smoothness condition ensures that the off-diagonal elements of r_x and the side lobes of the diagonal elements -- both of which oscillate with period 2 (see, e.g., Fig. 4) -- become unimportant when convoluted with b. These smoothness requirements are all relative to the frequency of observation which is implicit in the designation of the frequency π. This suggests that for any choice of ε, there might be some level of temporal disaggregation such that $||\hat{b}(t_0) - B(t_0)|| < \varepsilon$, an idea which is pursued in the next section.

3. Limiting Behavior of the Discrete Model

Consider models corresponding to (1) when x and y are recorded n times per period:

$$Y(\tfrac{t}{n}) = \sum_{s=-\infty}^{\infty} \frac{1}{n} \cdot {}^{n}B(\tfrac{s}{n}) \cdot X(\tfrac{t-s}{n}) + U(\tfrac{t}{n}), \text{ t integer.} \qquad (13)$$

It is clear that (for ordinary b) each coefficient of X in (13) will approach 0 with increasing n. If at the same time ${}^{n}B(\frac{s}{n}) \doteq b(\frac{s}{n})$ for large n, then one could make inferences about (1) from (13) when n is large in an obvious way. ${}^{n}B(\frac{s}{n})$ is a convolution of b with $r_x{}^n(t)$, where (from (6), with proper redifinition of π),

$$\tilde{r}_x{}^n(\omega) = [F_n[S_x](\omega)]^{-1} S_x(\omega),$$

$$F_n[S_x](\omega) = \sum_{k=-\infty}^{\infty} S_x(\omega+2n\pi k).$$

The investigation of the convergence of the parameters of (13) to the function b(t) of (1) is complicated by the fact that the set of parameters $S_n(t_1, t_2) = \{{}^{n}B(s): s\varepsilon[t_1,t_2]\}$ increases approximately in proportion with n. We must therefore be concerned not only with the local convergence problem, conditions under which $\lim_{n\to\infty} {}^{n}B(t) = b(t)$ for integer t, but also with global properties of ${}^{n}B$: how well does ${}^{n}B$ approximate b(t) <u>as a function</u> in the limit? The first question can be treated using Proposition 6.

<u>Corollary 7</u>. Suppose that b is an absolutely integrable function of bounded total variation. Define

$$\hat{b}(t) = \frac{1}{2} \lim_{\varepsilon\to 0}[b(t + \varepsilon) + b(t - \varepsilon)].$$

Suppose there exist $\omega^* > 0$ and $d < 1$ such that

(i) $\inf_{|\omega| \le \omega^*} \lambda_N(\omega) > 0$;

(ii) $\sup_{\substack{i,j,|\omega|>\omega^* \\ i \ne j}} |S_{x_{ij}}(\omega)/S_{x_{ii}}(\omega)| \le d/N$;

(iii) For all ω_0, $\inf_{|\omega| \le \omega_0} S_{x_{ii}}(\omega) > 0$, $i = 1, \ldots, N$.

Then

$$\lim_{n\to\infty} |\hat{b}(t) - {}^nB(t)|| = 0 \text{ for all integer } t.$$

Proof. Given any integer t, and $\varepsilon > 0$. Since

$${}^nB(t) = \frac{1}{2\pi} \int_{-n\pi}^{n\pi} [F_n[S_x](\omega)]^{-1} F_n[S_x \tilde{b}](\omega)\, e^{it\omega} d\omega,$$

it suffices to show the existence of ω_0 and n^* such that for all $n \ge n^*$,

(iii') $\inf_{|\omega| < \omega^*} \lambda_N(\omega) \ge \frac{4\pi}{A\varepsilon} \int_{|\omega| > n\pi} \lambda_1(\omega) d\omega$;

(iv') $\int_{\omega^*}^{\omega_0} \sum_{k \ne 0} \sup_{i=1,\ldots,N} \frac{S_{x_{ii}}(\omega+2n\pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2n\pi m)} d\omega \le \frac{\pi(1-d)\varepsilon}{8A(1+d)}$

(v') $\sup_{|\omega|>\omega_0} ||\tilde{b}(\omega)|| \le \frac{\pi\varepsilon}{2N}$;

(vi') $||\hat{b}(t) - \frac{1}{2\pi} \int_{|\omega|>n\pi} \tilde{b}(\omega) e^{it\omega} d\omega|| \le \frac{\varepsilon}{4}$.

(Note that conditions (i) and (ii) of Proposition 6 are satisfied by hypothesis.) The absolute integrability of b and the Riemann-Lebesgue Lemma guarantee the existence of ω_0 such that condition (v') is fulfilled. $\inf_{|\omega| < \omega^*} \lambda_N(\omega) > 0$, and the finite variance of the x_i implies

$$\int_{-\infty}^{\infty} \lambda_1(\omega) d\omega < \int_{-\infty}^{\infty} \sum_{i=1}^{N} \lambda_i(\omega) d\omega = \int_{-\infty}^{\infty} \mathrm{tr} S_x(\omega) d\omega = \sum_{i=1}^{N} \mathrm{var}(xi) < \infty;$$

hence there exists n_1 such that (iii') is satisfied for all $n \geq n_1$. Since $\inf_{|\omega| < \omega_0} S_{x_{ii}}(\omega) > 0$ but $\lim_{\nu\to\infty} \operatorname{ess\,sup}_{\omega > \nu} S_{x_{ii}}(\omega) = 0$, there exists n_2 such that (iv') is satisfied for all $n \geq n_2$. The basic convergence result for Fourier series applies to all $b_i(t)$ and insures that (vi') is met for all n exceeding some n_3. Defining $n^* = \sup(n_1, n_2, n_3)$, the result follows.

It is straightforward to show that if conditions (i) through (iii) obtain when x is not unit averaged, they are also valid for averaged x. Hence Corollary 7 applies when X and Y are both unit averaged.

The conditions on b are exactly those required to ensure the convergence of its Fourier transform. The requirements on x will be met if $\lambda_N(\omega)$ is strictly bounded away from zero on every finite interval and if for all $i \neq j$, $\lim_{\omega\to\infty} |S_{x_{ij}}(\omega)/S_{x_{ii}}(\omega)|=0$. In the univariate case these conditions reduce to $S_x(\omega)$ being bounded strictly away from zero on every finite interval.

Corollary 7 stipulates conditions under which the assumption $b_i(t) \doteq {}^nB_i(t)$, $i = 1, \ldots, N$, and given integer t, is valid for n sufficiently large. It does not address the question of how well the limiting shape of ${}^nB_i(t)$ approximates that of $b_i(t)$. One solution of the latter problem is provided by the next proposition.

<u>Proposition 8</u>. Given the assumptions of Corollary 7,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=-\infty}^{\infty} [{}^nB(\tfrac{t}{n}) - b(\tfrac{t}{n})]' \, [{}^nB(\tfrac{t}{n}) - b(\tfrac{t}{n})] = 0$$

<u>Proof</u>. The assumptions on b guarantee that it is square as well

as absolutely integrable. The Fourier transform of ${}^{n}B(\frac{s}{n}) - b(\frac{s}{n})$, s integer, is

$$[F_n[S_x](\omega)]^{-1}F_n[S_x\tilde{b}](\omega) - F_n[\tilde{b}](\omega)$$

$$=-[F_n[S_x](\omega)]^{-1}\sum_{k=-\infty}^{\infty} S_x(\omega+2n\pi k) \sum_{j\neq k} \tilde{b}(\omega+2n\pi j), \quad |\omega| < n\pi$$

It is easy to show that for any square integrable function f,

$$\frac{1}{n}\sum_{t=-\infty}^{\infty} f(\frac{t}{n})'f(\frac{t}{n}) = \int_{-n\pi}^{n\pi} \tilde{f}_n(\omega)'\tilde{f}_n(\omega)d\omega$$

where $\tilde{f}_n$ is the Fourier transform of $f(\frac{j}{n})$, $j=-\infty, \ldots, \infty$.

Taking note of the decomposition

$$\sum_{k=-\infty}^{\infty} S_x(\omega+2n\pi k) \sum_{j\neq k} b(\omega+2n\pi j)$$

$$=\sum_{k=-\infty}^{\infty} S_x(\omega+2n\pi k) \sum_{j\neq 0} \tilde{b}(\omega+2n\pi j) + \sum_{k\neq 0} S_x(\omega+2n\pi k)\tilde{b}(\omega)$$

$$- \sum_{k\neq 0} S_x(\omega+2n\pi k)\tilde{b}(\omega+2n\pi k)$$

it suffices to show

(a) $$\lim_{n\to\infty}\int_{-n\pi}^{n\pi} [\sum_{k=-\infty}^{\infty} S_x(\omega+2n\pi k) \sum_{j\neq 0} \tilde{b}(\omega+2n\pi j)]'[F_n[S_x](\omega)]^{-1'}$$
$$\cdot [F_n[S_x](\omega)]^{-1}\sum_{k=-\infty}^{\infty} S_x(\omega+2n\pi k) \sum_{j\neq 0} \tilde{b}(\omega+2n\pi j)d\omega = 0;$$

(b) $$\lim_{n\to\infty}\int_{-n\pi}^{n\pi} \tilde{b}(\omega)'[\sum_{k\neq 0} S_x(\omega+2n\pi k)]'[F_n[S_x](\omega)]^{-1'}$$
$$\cdot [F_n[S_x](\omega)]^{-1}[\sum_{k\neq 0} S_x(\omega+2n\pi k)]\tilde{b}(\omega)d\omega = 0;$$

(c) $$\lim_{n\to\infty}\int_{-n\pi}^{n\pi} \sum_{k\neq 0} S_x(\omega+2n\pi k) \tilde{b}(\omega+2n\pi k)]'[F_n[S_x](\omega)]^{-1'}$$
$$\cdot [F_n[S_x](\omega)]^{-1}[\sum_{k\neq 0} S_x(\omega+2n\pi k)\tilde{b}(\omega+2n\pi k)]d\omega = 0.$$

(a) states that $\lim_{n\to\infty}\int_{-n\pi}^{n\pi} [\sum_{j\neq 0} \tilde{b}(\omega+2n\pi j)]'[\sum_{k\neq 0} \tilde{b}(\omega+2n\pi k)]d\omega = 0$. Let $c_i^n(t)$ be the function defined by connecting the values of $b_i(\frac{s}{n})$ (s integer) with straight lines.

$$\int_{-\infty}^{\infty}|c_i^n(t) - b_i(t)|dt = \sum_{s=-\infty}^{\infty}\int_{s/n}^{(s+1)/n}|c_i^n(t) - b_i(t)|dt$$

$$\leq \frac{1}{n}\sum_{s=-\infty}^{\infty}\sup_{t_1,t_2\varepsilon[\frac{s}{n},\frac{s+1}{n}]}|b_i(t_1) - b_i(t_2)| \leq \frac{T_i}{n}$$

where T_i is the total variation in b_i. Hence

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}[c^n(t)-b(t)]'[c^n(t)-b(t)]dt = 0. \tag{14}$$

$$\tilde{c}^n(\omega) = D_n^{\,2}(\omega)\sum_{j=-\infty}^{\infty}\tilde{b}(\omega+2n\ \pi j);\text{ where } D_n(\omega) = \text{diag}(\frac{\sin(\omega/2n)}{\omega/2n},$$

$$\ldots,\frac{\sin(\omega/2n)}{\omega/2n}).$$

From (14) and the square integrability of b,

$$\lim_{n\to\infty}\int_{-n\pi}^{n\pi}[D_n^{\,2}(\omega)\sum_{j=-\infty}^{\infty}\tilde{b}(\omega+2n\pi\ j) - \tilde{b}(\omega)]'[D_n^{\,2}(\omega)\sum_{k=-\infty}^{\infty}\tilde{b}(\omega+2n\pi\ k)-\tilde{b}(\omega)]d\omega=0 \tag{15}$$

Now observe that

$$\int_{-n\pi}^{n\pi}[D_n^{\,2}(\omega)\tilde{b}(\omega) - \tilde{b}(\omega)]'[D_n^{\,2}(\omega)\tilde{b}(\omega) - \tilde{b}(\omega)]d\omega \tag{16}$$

$$\leq \int_{\sqrt{n}\ \pi}^{\sqrt{n}\ \pi}\tilde{b}(\omega)'[D_n^{\,2}(\omega)-I]'[D_n^{\,2}(\omega)-I]\tilde{b}(\omega)d\omega + \int_{|\omega|>\sqrt{n}\ \pi}\tilde{b}(\omega)'[D_n^{\,2}(\omega) - I]'[D_n^{\,2}(\omega)-I]$$

$$\cdot\tilde{b}(\omega)d\omega$$

The square integrability of b assures that the second term converges to 0 as $n\to\infty$. The first is bounded by

$$\int_{\infty}^{\infty}b(t)'b(t)dt\cdot\sup_{|\omega|<\sqrt{n}\ \pi}[D_n^{\,2}(\omega)-I]'[D_n^{\,2}(\omega)-I]$$

which also coverges to 0. (15) and the convergence of (16) to zero taken together imply

$$\lim_{n\to\infty}\int_{-n\pi}^{n\pi}[D_n^{\,2}(\omega)\sum_{j=-\infty}^{\infty}\tilde{b}(\omega+2n\pi\ j)-D_n^{\,2}(\omega)b(\omega)]'[D_n^{\,2}(\omega)\sum_{j=-\infty}^{\infty}\tilde{b}(\omega+2n\pi\ j)$$

$$-D_n^{\,2}(\omega)\tilde{b}(\omega)]d\omega = 0.$$

Since $\inf_{|\omega|\leq n\pi} ||D_n^{\,2}(\omega)|| = (\frac{2}{\pi})^2 > 0$, (a) follows.

Defining $S_n = \{\omega: \ \omega^* < |\omega| \leq n\pi\}$ and applying Lemma 5, expression (b) is dominated by

$$\int_{-\omega^*}^{\omega^*} \left[\frac{\sum_{k\neq 0} \lambda_1(\omega+2n\ \pi k)}{\lambda_N(\omega)}\right]^2 \tilde{b}(\omega)'\tilde{b}(\omega)d\omega \tag{17a}$$

$$+ \frac{1+d}{1-d} \int_{S_n} \left[\sum_{k\neq 0} \sup_{i=1,\ \dots,\ N} \frac{S_{x_{ii}}(\omega+2n\ \pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2n\pi\ m)}\right]^2 \tag{17b}$$

$$\cdot\ \tilde{b}(\omega)'\tilde{b}(\omega)d\omega$$

We wish to show that both (17a) and (17b) converge to 0 as $n\to\infty$. Choosing some $n_0 > \frac{\omega^*}{\pi}$, we see that for all $n \geq n_0$ the integrand in (17a) is bounded by the function

$$\frac{\sum_{k\neq 0} \lambda_1(\omega+2n\pi\ k)}{\inf_{|\omega|\leq\omega^*} \lambda_N(\omega)} \quad \tilde{b}(\omega)'\tilde{b}(\omega)$$

which is integrable over $[-\omega^*, \omega^*]$. The finite variance of x assures that $\lim_{n\to\infty} \sum_{k\neq 0} \lambda_1(\omega+2n\pi k) = 0$ a.e., and $\tilde{b}(\omega)'\tilde{b}(\omega)$ is uniformly bounded; the integrand therefore converges pointwise to zero for alomst all $\omega\varepsilon[-\omega^*, \omega^*]$. By the Lebesgue Convergence Theorem (Royden ([8], p. 88)), (17a) converges to zero. (17b) may be written

$$\frac{1+d}{1-d} \int_{|\omega|>\omega^*} \left[\sum_{k\neq 0} \sup_{i=1,\ \dots,\ N} \frac{S_{x_{ii}}(\omega+2n\pi\ k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2n\pi\ m)}\right]^2 \tilde{b}(\omega)'\tilde{b}(\omega)\chi_n(\omega)d\omega$$

where $\chi_n(\omega)$ is 1 for $|\omega|\leq n\pi$ and 0 elsewhere. The integrand is bounded by $N^2\tilde{b}(\omega)'\tilde{b}(\omega)$, which is integrable. Since $S_{x_{ii}}(\omega) > 0$ for all ω,

$$\sum_{k\neq 0} \sup_{i=1,\ldots,N} \left\{ \frac{S_{x_{ii}}(\omega+2n\pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2n\pi m)} \right\} < \sum_{k\neq 0} \sum_{i=1}^{N} \frac{S_{x_{ii}}(\omega+2n\pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2n\pi m)}$$

$$\xrightarrow[n\to\infty]{} 0 \text{ a.e.}$$

The Lebesgue Convergence Theorem therefore assures that (17b) converges to 0.

The proof of (c) is analagous. We define

$$\beta_n{}^* = \{\omega: \sum_{k=-\infty}^{\infty} \tilde{b}(\omega+2n\pi k)'\tilde{b}(\omega+2n\pi k) = 0\} \quad \text{and} \quad \tilde{b}_n(\omega) = \frac{\tilde{b}(\omega)}{\{F_n[\tilde{b}'\tilde{b}](\omega)\}^{1/2}} \text{ on } (-\infty, \infty)$$

and rewrite the expression as

$$\int_{[-n\pi, n\pi]-\beta_n{}^*} [\sum_{k\neq 0} S_x(\omega+2n\pi k)\tilde{b}_n(\omega+2n\pi k)]'[F_n[S_x](\omega)]^{-1}[F_n[S_x](\omega)]^{-1}[\sum_{j\neq 0} S_x(\omega+2n\pi j)$$

$$\cdot \tilde{b}_n(\omega+2n\pi j)]F_n[\tilde{b}'\tilde{b}]\ (\omega)d\omega$$

Taking note that $||\tilde{b}_n(\omega+2n\pi k)|| \leq 1$, this is dominated by

$$\int_{-\omega^*}^{\omega^*} \left[\frac{\sum_{k\neq 0} \lambda_1(\omega+2n\pi k)}{\lambda_N(\omega)}\right]^2 \sum_{m=-\infty}^{\infty} \tilde{b}(\omega+2n\pi m)'\tilde{b}(\omega+2n\pi m)\ d\omega \tag{18a}$$

$$+\left(\frac{1+d}{1-d}\right) \int_{S_n} \left[\sum_{k\neq 0} \sup_{i=1,\ldots,N} \left\{ \frac{S_{x_{ii}}(\omega+2n\pi k)}{\sum_{m=-\infty}^{\infty} S_{x_{ii}}(\omega+2n\pi m)} \right\}\right]^2 \tag{18b}$$

$$\cdot \sum_{m=-\infty}^{\infty} \tilde{b}(\omega+2n\pi m)'\tilde{b}(\omega+2n\pi m)d\omega$$

For $n \geq n_0 > \omega^*/\pi$ the integrand in (18a) is domin ated by the integrable function $\left[\frac{\sum_{k\neq 0} \lambda_1(\omega+2n\pi k)}{\lambda_N(\omega)}\right]^2 F[\tilde{b}'\tilde{b}](\omega)$.

Since $\sum_{k\neq 0} \lambda_1(\omega+2n\pi k)$ converges pointwise to zero for alomst all $\omega\varepsilon[-\omega^*, \omega^*]$ (18a) converges to zero by the Lebesgue Convergence Theorem. Similarly, (18b) also goes to zero.

As seen in the proof, the conditions on b in Proposition 8 ensure that it can be approximated arbitrarily well in the mean by a naive, linear interpolator if the points between which b is interpolated are chosen at sufficiently fine intervals -- this is the consequence of the total variation in b being finite. If in addition b is absolutely and square integrable, it fulfils all the requirements for convergence of ${}^{n}B$ to b. These conditions are fairly weak: indeed, it is difficult to imagine a regression model with an ordinary b which violates the conditions of Proposition 8.[8] The conditions on x in Corollary 7 and Proposition 8 are satisfied if if the eigenvalues of $S_x(\omega)$ are bounded strictly away from zero on every finite interval, and if in the regression relationships

$$x_i(t) = b_{ij} * x_j(t) + u_{ij}(t) \quad (i,j = 1, \ldots, N;\ i \neq j)$$

the b_{ij} are ordinary, absolutely integrable functions. Of all these requirements, the last is perhaps the most likely to be violated.

The limiting behavior of $\frac{1}{n} \sum_{t=-\infty}^{\infty} |{}^{n}B(\frac{t}{n}) - b(\frac{t}{n})|$ is not easily treated, but the following proposition may be useful.

<u>Proposition 9</u>. If $|| \int_{-\infty}^{\infty} |b(t)| dt || = A < \infty$, $S_x(0)$ is non-singular and $\lambda_1(\omega) = O(|\omega|^{-\alpha})$, $\alpha > 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=-\infty}^{\infty} {}^{n}B(\frac{t}{n}) = \int_{-\infty}^{\infty} b(s)\, ds.$$

<u>Proof</u>. The claim is equivalent to

$$\lim_{n\to\infty} \{[F_n[S_x](0)]^{-1} F_n[S_x \tilde{b}](0) - \tilde{b}(0)\} = 0$$

a sufficient condition for which is

$$\lim_{n\to\infty} || [F_n[S_x](0)]^{-1} \{F_n[S_x\tilde{b}](0) - F_n[S_x](0)\tilde{b}(0)\} || = 0$$

For large n, this expression is dominated by

$$\frac{\sum_{k\neq 0} \lambda_1(2n\pi k) || [\tilde{b}(0) - \tilde{b}(2n\pi k)] ||}{\lambda_N(0)}$$

$$\leq \frac{2A}{\lambda_N(0)} \sum_{k\neq 0} \lambda_1(2n\pi k) = \frac{2A}{\lambda_N(0)} (2n\pi)^{-\alpha} \sum_{k\neq 0} 0(|k|^{-\alpha}) \to 0.$$

When the data are unit averaged one may apply Proposition 4 in lieu of 9.

4. Interpretation of Discrete Models in Applied Work

Given the assumptions set forth in Section 1, the model (3) exists and the relation of its parameterization to that of (1) is

$$B(t) = \int_{-\infty}^{\infty} r_x(s)\ b(t-s)\ ds.$$

(6) and (7) imply that $r_x(s)$ is in general not a diagonal matrix and as a function of s is not symmetric about s = 0. Consequently $B_i(t)$ is a moving average not only of $b_i(s)$ but also of $b_j(s)$ $(j\neq i)$ and this average is not necessarily centered exactly about s = t. The "contamination" of $B_i(t)$ by $b_j(s)$ can be so strong that $B_i(t)$ is a confounding of all $b_j(s)$ which in no way resembles $b_i(t)$. Such a situation is exemplified in Figure 2, protraying an $r_x(s)$ estimated from actual data for which one off-diagonal element is as important as the on-diagonal ones. In circumstances like those of Figure 2 no inference about b(t) from knowledge of B(t) is possible. Even when certain $b_i(t) \equiv 0$, the corresponding $B_i(t)$ can be large in absolute value and have reasonable shapes: see Fig 1(d).

One interesting model in which inferences about b based on B are straightforward is

$$y(t) = x(t)'b + u(t) \tag{19}$$

a case subsumed by (1) when b is a vector multiple of Dirac delta functions. (7) then implies

$$Y(t) = X(t)'b + U(t),$$

a result which is perhaps intuitively obvious.[10]

If the basic model is not (19) but the apparently similar one

$$y(t) = \int_0^a x(t-s)'\, b(s)\, ds + u(t), \quad a < 1, \tag{20}$$

no such simple relation exists. Investigation often proceeds as if the discrete model corresponding to (19) were

$$Y(t) = X(t)' \int_0^a b(s)\, ds + U(t), \tag{21}$$

the assumption being that since "all effects die out within a period" (21) may be used in lieu of (20). We have seen that such a procedure is formally unjustified. Perusal of Figure 2 suggests that in applied contexts, unless a is small than inferences based on the assumption that (20) implies (21) can be wrong. The true discrete relation corresponding to (20) may be well approximated by

$$Y(t) = \sum_{s=-M_1}^{M_2} X(t-s)'\, B(s) + U(t) \tag{22}$$

with M_1 and M_2 small in absolute value, but because of contamination $B(0) \neq \int_0^a b(s)\, ds$. Misspecification of (22) by omission of leading and lagging terms does not imporve matters. This example shows that the considerations raised here apply not only to those models traditionally classified as "distributed lags" but also to situations where effects take less than one period.

Since hypotheses about model structure are usually concerned with b rather than B, it is important to recognize those circumstances in which inferences about the real time structure based on the observable one will be difficult. The results of Sections 2 and 3 in general and Proposition 6 in particular suggest three relations between the properties of the model (1) and the accuracy of the approximation $B_i(t) \doteq b_i(t)$.

First, $B_i(t)$ is close to $b_i(t)$ only if all components of b(s) are smooth in the sense made exact by conditions (v) and (vi) of Proposition 6. If any component of b(s) shows rapid oscillations then it cannot be assumed that $B_i(t) \doteq b_i(t)$ for any i. Furthermore, Corollary 7 implies that when $b_i(s)$ exhibits a discontinuity at s = t then the limiting approximation achieved with increased temporal disaggregation is

$$B_i(t) = \frac{1}{2}\lim_{\varepsilon\to 0} [b_i(t+\varepsilon) + b_i(t-\varepsilon)]. \tag{23}$$

This consideration is especially relevant in models with one-sided b_i's which are discontinuous at t = 0 and then decline: as in the univariate case (see Sims ([9], p. 554)) the corresponding B_i will not decline monotonically, because of (23). The reason this smoothness is required is that the off-diagonal elements of $r_x(s)$ and the side lobes of its on-diagonal terms oscillate about 0: this oscillation becomes unimportant only if b(s) is smooth. Any hypothesized b can, of course, be examined directly to see if it meets the various conditions imposed in developments 6, 7 and 8.

Second, the more highly correlated are the independent variables (in a sense made exact in Lemma 5) the greater is $|B_i(t) - b_i(t)|$ likely to be for all i and t. Lemma 5, the following discussion, and Appendix B indicate two alternative conditions on S_x which are sufficient for smallness of $|B_i(t) - b_i(t)|$. Both conditions involve the ratio of the largest to the smallest eigenvalue of $S_x(\omega)$. In the first condition,

$$\left|S_{x_{ij}}(\omega)/S_{x_{ii}}(\omega)\right| < 1/N, \text{ for all } \omega\varepsilon(\omega^*, \infty) \tag{24}$$

and the eigenvalue ratio must be bounded for all $\omega\varepsilon[-\omega^*, \omega^*]$.
Since (as shown in Section 2) (24) implies

$$\left|S_{X_{ij}}(\omega)/S_{X_{ii}}(\omega)\right| < 1/N, \ \omega\varepsilon(\omega^*,\pi),$$

it may be possible to reject the first condition by examination of the estimated spectral and cross-spectral densities of X. In either case, the bound on $||B(t) - b(t)||$ is linearly related to the bound on the eigenvalue ratio over the relevant range. If (24) does not obtain, this bound cannot be less than the ratio of the larges to smalles eigenvalue of the matrix

$$\text{cov }(x) = E\ xx' = E\ XX'$$

because (denoting the largest eigenvalue of a symmetric matrix by $||A||$ and the smalles by $((A))$),

$$\frac{||E\ xx'||}{((E\ xx'))} = \frac{\left|\left|\int_{-\infty}^{\infty} S_x(\omega)d\omega\right|\right|}{\left(\left(\int_{-\infty}^{\infty} S_x(\omega)d\omega\right)\right)} \leq \frac{\int_{-\infty}^{\infty} \left|\left|S_x(\omega)\right|\right| d\omega}{\int_{-\infty}^{\infty} ((S_x(\omega)\ ))d\omega} \leq \sup_{\omega} \frac{||S_x(\omega)||}{((S_x(\omega)\ ))} \ . \tag{25}$$

Even when (24) is true (25) will probably obtain when the supremum is taken over $\omega\varepsilon(-\omega^*, \omega^*)$ and $S_x(\omega)$ is concentrated near $\omega = 0$. Given a sample of T observations of X, this ratio may be approximated by the corresponding ratio for the matrix

$$\sum_{t=1}^{T} \underset{N\times 1}{(X^{(t)}} - \overline{X}) \cdot \underset{1\times N}{(X^{(t)}} - \overline{X})'.$$

When some subset of the N independent variables is highly correlated then off-diagonal elements of $r_x(s)$ can be large and contamination is apt to be severe. Results discussed in the next section suggest, e.g., that the estimable parameters of an annual or quarterly model in which several highly correlated interest rates appear as independent variables bear virtually no resemblance to the real time parameterization.

Third, for $|B_i(t) - b_i(t)|$ to be small all $x_i(t)$ must show substantial power at all low frequencies and exhibit little of their movement over periods of length two or less: this is condition (iii) of Proposition 6. The first requirement is satisfied by most economic time series.[11] The second depends on the variable and the frequency of observation, and if one has a choice in the level of temporal disaggregation of a model under consideration this criterion ought to be investigated by examining the eigenvalues of an estimated spectral density matrix of the independent variables. Even when no such choice exists, it may be worthwhile to estimate spectral densities of closely related, more frequently recorded series to see if the spectral shape of such variables precludes investigation of the hypotheses about b under consideration.

5. The Practical Benefits of Temporal Disaggregation

The three relations just outlined all involve the frequency of observation. In any model, if observations are not made often enough then the criteria of Proposition 6 cannot be met for any interesting ε; on the other hand, Corollary 7 and Proposition 8 assert that under reasonable conditions the real time model may be approximated by a discrete one with any desired degree of accuracy. The chief impediments to the application of the latter two developments are the impossibility of verifying their assumptions in any given case and the (related) difficulty that one must work with finite n.[12]

To provide insight into gains likely to be realized from temporal disaggregation in actual investigation, $r_x(s)$ for N = 2 was estimated using records of the 4 - 6 month prime commercial paper rate (PCPR) and the new issue rate for 3 month U.S. Treasury Bills (USTBR). The interest rate series were chosen because weekly, accurate observations are available and because it was hoped that their collinearity would illustrate contamination problems apt to be encountered in applied work.[13] Estimated spectral densities, eigenvalues and the ratios $\lambda_1(\omega)/\lambda_N(\omega)$, $|S_{x_{ij}}(\omega)/S_{x_{ii}}(\omega)|$ for selected frequencies are presented in Table 1.

In Figures 2 and 3 the estimated r_x matrices for n = 1 (corresponding to one year) and n = 4 (one quarter), respectively, are graphed. As one might suspect, when these two highly correlated series are observed only once a year the contamination is very severe: B_2 reflects a convolution with b_1 just as important in magnitude as that with b_2. (See Figure 2 and Table 2.) Temporal disaggregation to quarterly observations changes the shape of r_x in three ways. First, $r_x(s)$ is more concentrated about s = 0. Second, the contamination problem is greatly reduced: as reflected both in Figure 3 and Table 2, diagonal elements of r_x are larger relative to off-diagonal ones. Third, at the finer level of disaggregation elements of $r_x(s)$ do not show the pronounced shift (backward for the first row, forward for the second) of r_x for annual data.

The effect of using unit averaged data can be appreciated by comparing Figure 3 with Figure 4, which presents $r_x(s)$ for unit averaged data, n = 4. r_x in the latter case shows substantial imporvement over the former, its integral being given by Proposition 4.

It is a matter of conjecture whether these results are typical;[14]

the three improvements noted in r_x as annual data were replaced with quarterly are not a logical consequence of temporal dissaggregation which will necessarily be observed when any set of variables is observed more frequently. The results at least suggest that important gains can be realized from temporal disaggregation in the range of conventional measurement frequencies for macroeconomic data. In the case presented here contamination is so severe in the annual, discrete model that testing hypotheses under the assumption $B \doteq b$ is foolish. In the quarterly model contamination is present but not overwhelming: for some distributed lags reasonable inferences about some properties of b might be made on the basis of a good estimate of B. Contamination appears less a problem with unit averaged data and presumably further imporvements would be realized with monthly data.

Table 1

Values of $\lambda_1(\omega)/\lambda_N(\omega)$ and $\sup_{i,j;i\neq j} |S_{x_{ij}}/S_{x_{ii}}|$ at selected frequencies, USTBR and PCPR. 2π corresponds to the frequency 1 cycle per year.

Frequency	$\lambda_1(\omega)/\lambda_2(\omega)$	$\sup_{i,j;i\neq j} \lvert S_{x_{ij}}/S_{x_{ii}} \rvert$
0π	344.81	1.1007
0.5π	14.86	0.9866
2.0π	5.31	0.8056
3.5π	2.93	0.5467
5.0π	9.57	1.3139
8.0π	6.45	0.6624
13.0π	34.67	2.8585

Table 2

Values of $\int_{-\infty}^{\infty} r_x(t)\,dt$ for USTBR (x_1) and PCPR (x_2).

Annual X, point data

$$\begin{bmatrix} .4421 & .0030 \\ .4440 & .8243 \end{bmatrix}$$

Quarterly X, point data

$$\begin{bmatrix} .8867 & .0276 \\ .1264 & .9630 \end{bmatrix}$$

Quarterly x, Unit averaged data

$$\begin{bmatrix} 1.0000 & .0000 \\ .0000 & 1.0000 \end{bmatrix}$$

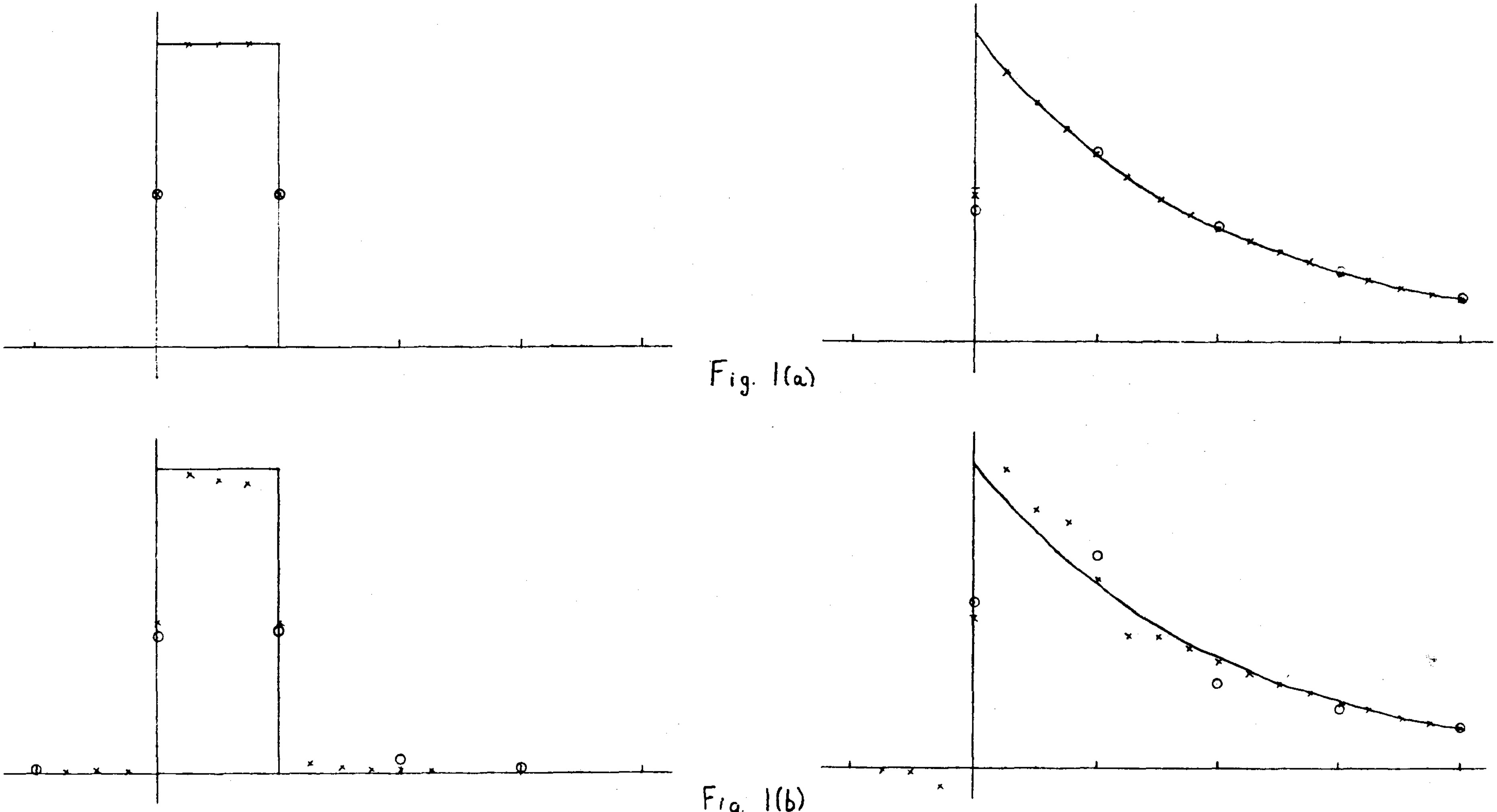

Figure 1. B_i corresponding to some b_i for $N = 2$. B_1 and b_1 are portrayed on the left, B_2 and b_2 on the right. The x's indicate B's corresponding to a frequency of observation fourtimes greater than that which generated the B's designated by the circles. In (a) and (b), $S_{x11}(\omega) = S_{x22}(\omega) = \inf(9, \omega^{-2})$; however in (a) x_1 and x_2 are uncorrelated, while in (b) their correlation is $\inf(.96, .96^{|\omega|})$. The essential difference between the (a) and (b) situations is the absence of the contamination in (a) which is present in (b). In (c) and (d) $S_{x11}(\omega) = S_{x22}(\omega) = \inf(1, \omega^{-2})$ and the correlation between x_1 and x_2 is $\inf(.96, .96^{|\omega|/8})$: the potential for contamination is greater than in (a) and (b). In addition, $b_1(t)$ is poorly behaved and b_2 shows strongly the effects of contamination. In 1.1(d), $b_2(t) = 0$.

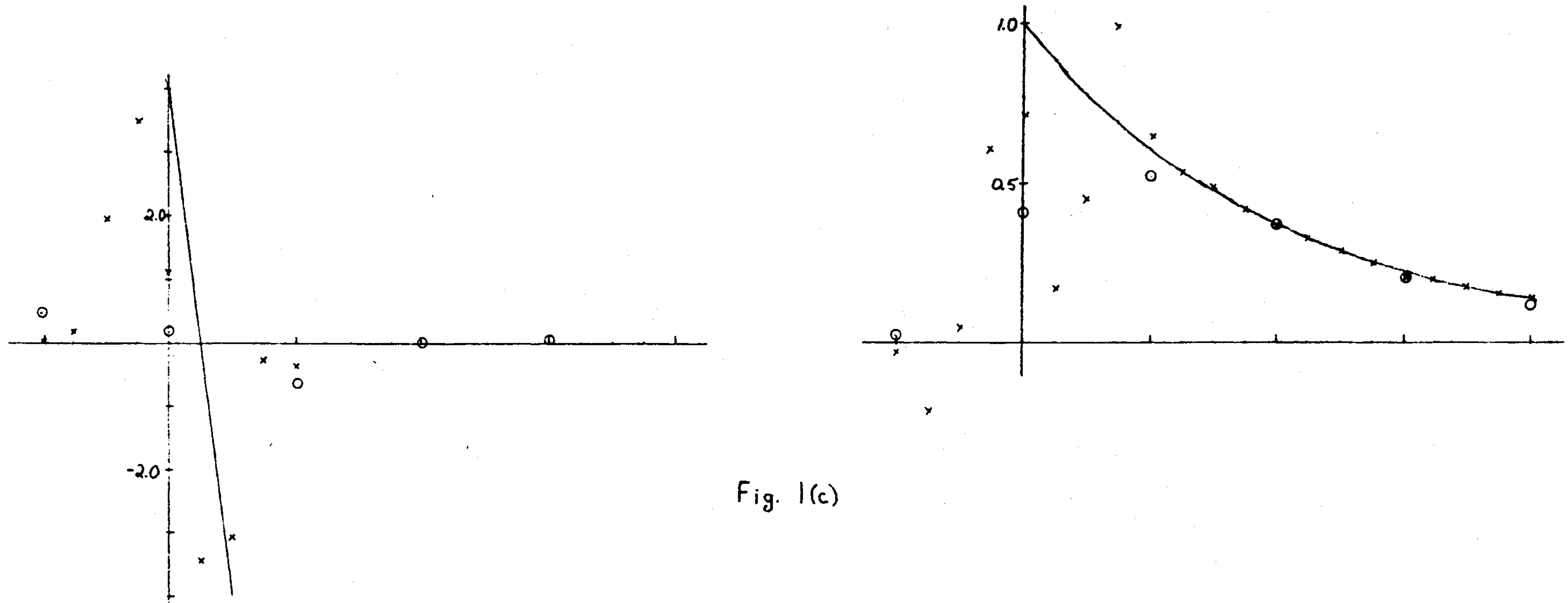

Fig. 1(c)

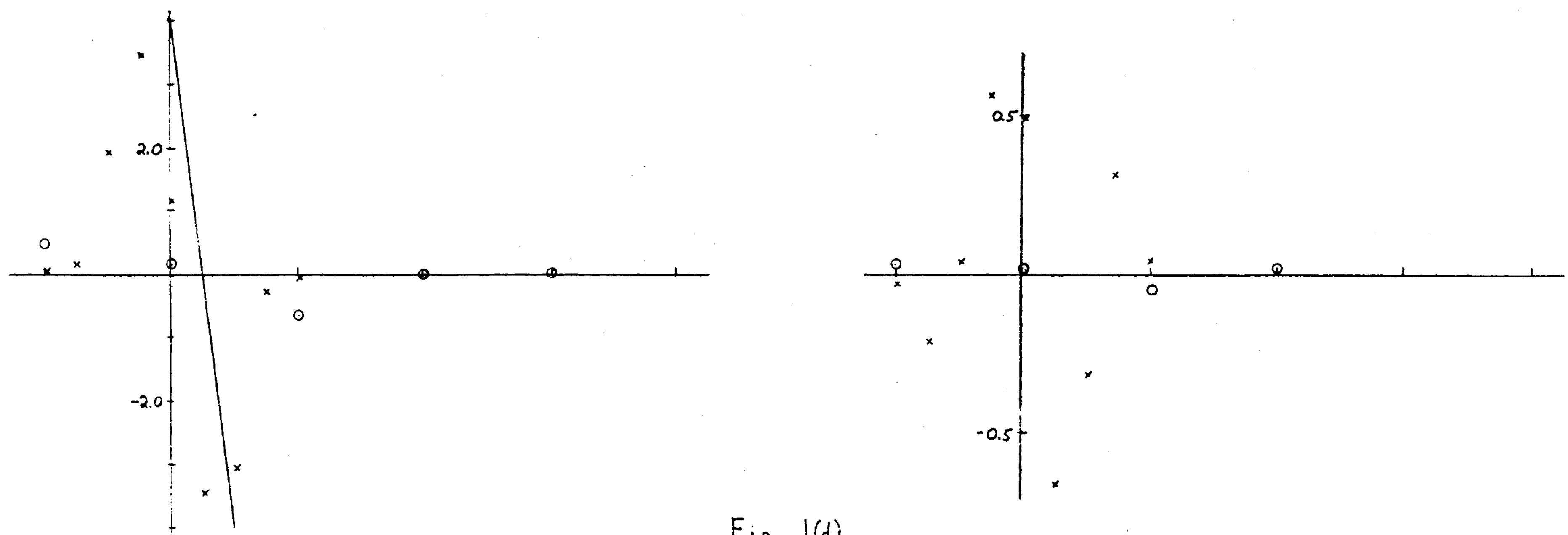

Fig. 1(d)

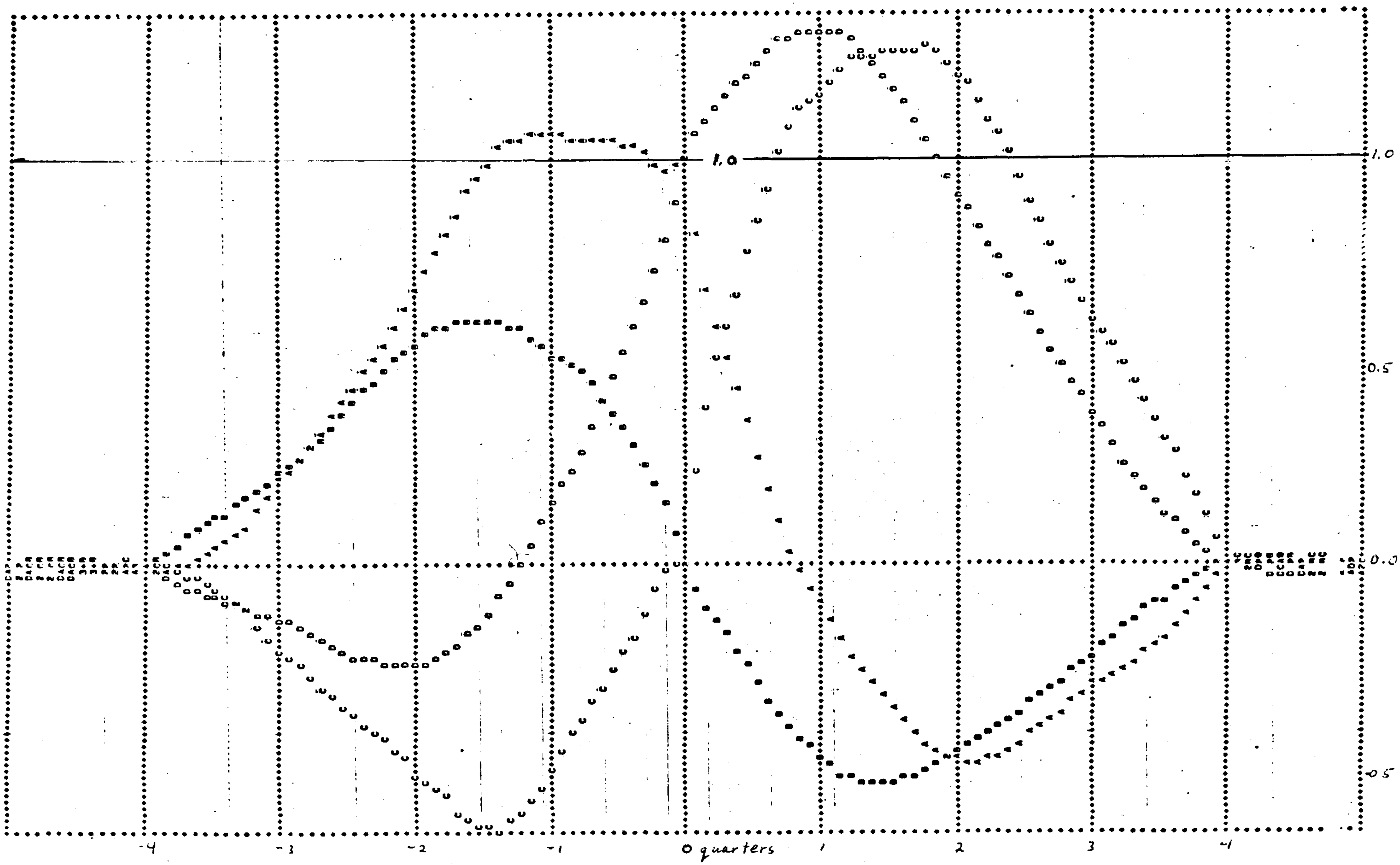

Figure 2. Estimated $r_x(t)$ for USTBR (x_1) and PCPR (x_2) when recorded as annual point data. $r_{x11}(t)$ = "A", $r_{x12}(t)$ = "B", $r_{x21}(t)$ = "C", $r_{x22}(t)$ = "D".

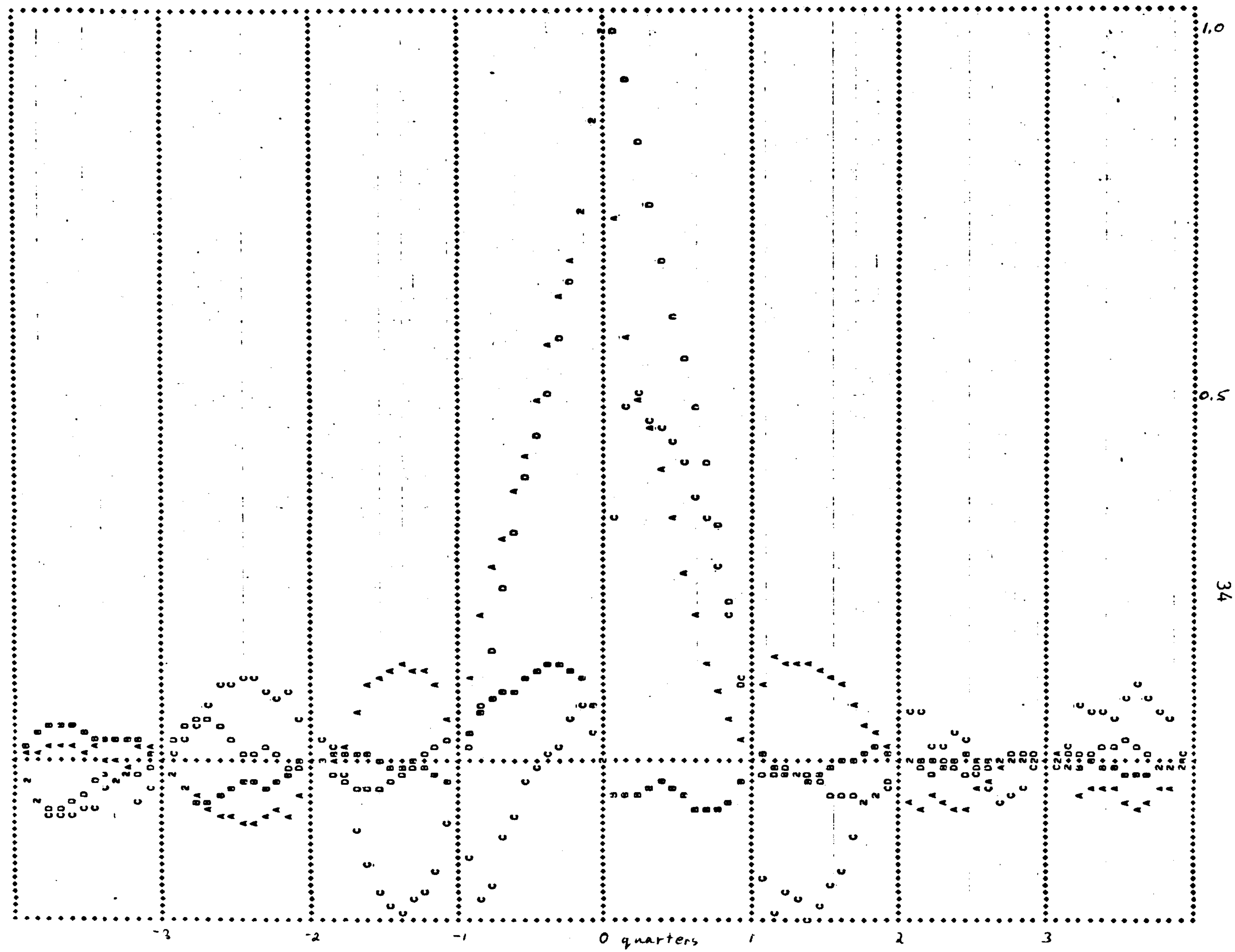

Figure 3. Estimated $r_x(t)$ for USTBR and PCPR when recorded as quarterly point data. Symbols are as in Fig. 2.

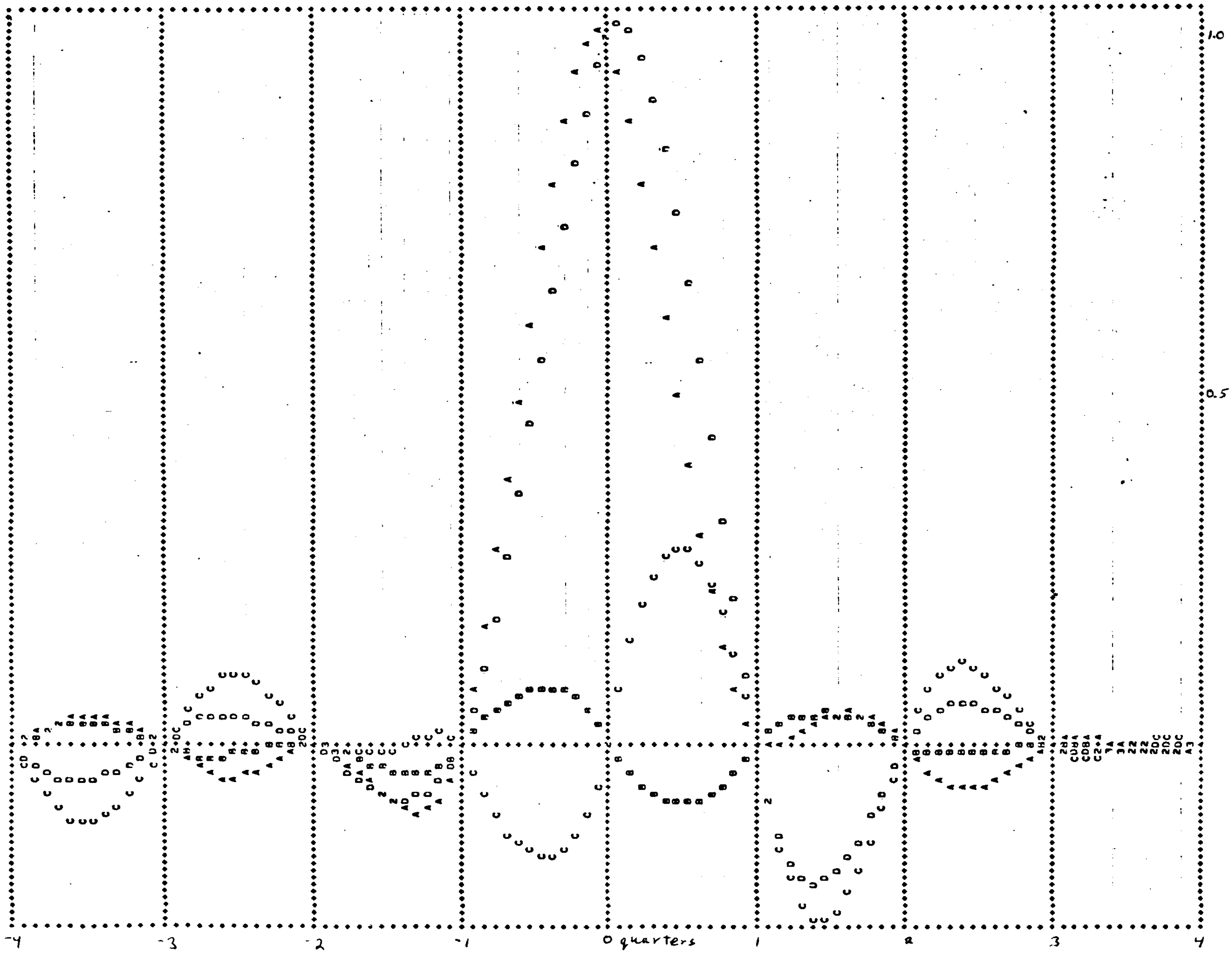

Figure 4. Estimated $r_x(t)$ for USTBR and PCPR when recorded as quarterly unit averaged data. Symbols are as in Fig. 2.

Footnotes

1. Generalized function are defined and discussed briefly by Jenkins and Watts ([5], p. 26). For a more elaborate introduction, Lighthill [6] is an excellent reference.

2. Unfortunately, there is no standard notation for frequency domain statistical analysis; the conventions used here are set forth in Appendix A.

3. See any standard reference on spectral analysis for a derivation of this result, e.g., Fishman ([2], pp. 50). We shall think of these functions as defined only on the interval $(-\pi, \pi)$, but they can be defined on the entire real line, in which case they are periodic.

4. Propositions 2 and 3 are the multivariate analogues of Propositions A and E proved by Sims [9].

5. Mean-square differentiability of a stochastic process is discussed by Hannan ([4], p. 6).

6. Developments 6 through 8 would also follow under either of these alternative assumptions. Corresponding versions of these Propositions for the latter assumption are presented in Appendix B.

7. Lemma 5 remains valid when the data are all unit average instead of point sampled, so long as (8) and (9) are defined as the appropriate limits when both numerator and demoninator are zero. $S_x(\omega)$ is pre- and post-multiplied by

$$\operatorname{diag}\left[\frac{\sin(\omega/2n)}{\omega/2n}, \ldots, \frac{\sin(\omega/2n)}{\omega/2n}\right]$$

If (10) is satisfied it is met <u>a fortiori</u> after this modification of S_x.

8. The most commonly arising case in which b is not a vector of ordinary function is $b(t) = b_1(t) + b_2(t)$ where b_1 is a vector of ordinary functions and $b_2(t)$ is a linear combination of Dirac delta dunctions which are zero for non-integer t. The underlying model may then be written

$$y(t) = \int_{-\infty}^{\infty} b_1(s)\, x(t-s)\, ds + \sum_{s=-\infty}^{\infty} \beta_2(s)\, x(t-s) + u(t).$$

Since $r_x(s)$ is I_N for s = 0 and 0 for s a non-zero integer, $B(s) = B_1(s) + \beta_2(s)$, and ${}^nB(s/n) = {}^nB(s/n) + \beta_2(s/n)$. If b_1 meets the conditions for b in Proposition 8, then ${}^nB_1$ converges in mean square to $b_1(s)$, but of course $n\beta_2(s/n)$ is either identically 0 or approaches infinity. If b_2 has delta function components relfecting impulses at $t \neq j/n$, j integer, ${}^nB$ will reflect serious contamination for all n.

If the basic model is a differential equation which can be cast in the form (1) with b having compenents which are linear combinations of first and higher order derivatives of the Dirac delta function, then (as Sims [10] has pointed out for univariate x) B does not in general look like the corresponding difference

8. continued

operator. The analysis of differential equations in the multivariate model is similar: as inspection of figures 2, 3, and 4 reveals, the B's are a hopeless confounding of multiples of derivatives of the elements of $r_x(s)$ at integer s, and the problem does not improve with increasing n.

9. See the discussion in Section 5, especially Table 1.

10. The result depends on X and Y being measured at exactly the same time, of course. If they are not -- say X is recorded in the middle of the quarter and Y and the end -- then (19) implies a two-sided lag distribution between X and Y whose relation to b depends on the shape of r_x.

11. Some evidence is provided by Granger [3].

12. Certain awkward problems arise if Corollary 7 and Proposition 8 are applied literally to some economic time series. Stocks like money and inventories are defined unambiguously at all points in time, but what about GNP at 2 A.M. (Eastern Standard Time) or housing starts on Sunday? In fact, of course, there are many economic variables which become conceptually difficult when temporally disaggregated beyond, say, weekly averages -- not to mention problems associated with actual measurement of these magnitudes. It is precisely in such cases that $\tilde{b}$ is periodic and b is a linear combination of delta functions. There is then a level of temporal

12. continued

disaggregation for which (1) and (3) are identical, but time series records are rarely detailed enough to aviod the problem in this way. What is relevant is how rapidly r_x improves over the range of conventinal observational frequencies, a question to which the experiments reported here are addressed.

13. 1042 observations of the two series, from July, 1951 through June 1971, were used to estimate 1664 equally spaced ordinates of the spectral density matrix. The same number of ordinates of $\tilde{r}_x(\omega)$ were then estimated using (6a) with the estimated S_x in place of S_x. The inverse Fourier transform of $\tilde{r}_x(\omega)$ yielded the series plotted in Figures 2 through 4. (It was implicitly assumed that $S_x(\omega) = 0$ for frequencies greater than 1 cycle/2 weeks.)

14. Experiments with PCPR and the narrowly defined money stock M1 yielded similar r_x.

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Appendix A

The following are notational conventions employed throughout. All assertions are proved in basic references on Fourier transforms (e.g., Titchmarsh ([11])) or spectral analysis (e.g., Fishman([2])).

For any absolutely integrable function f of bounded total variation, define its Fourier transform

$$\tilde{f}(\omega) = \lim_{T\to\infty} \int_{-T}^{T} f(t)\, e^{-it\omega} dt, \quad -\infty < \omega < \infty.$$

The inverse Fourier transform of $\tilde{f}$ is defined

$$\hat{f}(t) = \lim_{W\to\infty} \frac{1}{2\pi} \int_{-W}^{W} \tilde{f}(\omega)\, e^{it\omega} d\omega, \quad -\infty < t < \infty.$$

If f is continuous at t, $\hat{f}(t) = f(t)$; if not, $\hat{f}(t) = \frac{1}{2} \lim_{\varepsilon\to\infty} [f(t+\varepsilon) + f(t-\varepsilon)]$. (The existence of the limit is the consequence of the bounded variation of f in a neighborhood of t.)

For any function f(t) defined only for integer t such that $\sum_{t=-\infty}^{\infty} |f(t)| < \infty$, define its Fourier transform

$$\tilde{f}(\omega) = \lim_{T\to\infty} \sum_{t=-T}^{T} f(t)\, e^{-it\omega}, \quad -\pi \leq \omega \leq \pi.$$

The inverse Fourier transform of $\tilde{f}$ is defined

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(\omega)\, e^{it\omega} d\omega, \quad t \text{ integer},$$

and $\hat{f}(t) = f(t)$.

The complex conjugate of any complex number a is denoted $\bar{a}$. (In a few instances, $\bar{x}$ and $\bar{y}$ have been used to denote the unit averages of realizations of stochastic processes x and y; this alternative use is, hopefully, clear from context.) If x is a vector of complex numbers, then $\underset{N\times 1}{x'} = (\overline{x}_1, \ldots, \overline{x}_N)$.

If x is a covariance stationary stochastic process with mean 0, we denote its Fourier (or Cramér) representation $\tilde{x}(\omega)$. $\tilde{x}(\omega)$ is a differential operator with the properties

(1) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega) e^{it\omega} d\omega$;

(2) $E\, \overline{\tilde{x}}_i(\omega)\, \tilde{x}_j(\nu) = S_{xij}(\omega)$ if $\omega = \nu$ and 0 otherwise.

Likewise if x and y are two such processes,

$$E\, \overline{\tilde{x}}_i(\omega)\, \tilde{y}_j(\nu) = S_{xyij}(\omega) \text{ if } \omega = \nu \text{ and 0 otherwise.}$$

Appendix B

Proofs of the convergence results of Chapter 1 assuming only that $\lambda_1(\omega)/\lambda_N(\omega)$ is uniformly bounded --

Lemma 5'. Suppose the eigenvalues of $S_x(\omega)$ are all positive for all ω. Then

$$\left|\left|[\sum_{m=-\infty}^{\infty} S_x(\omega+2\pi m)]^{-1}[\sum_{k\neq 0} S_x(\omega+2\pi k)]\right|\right| \leq \frac{\sum_{k\neq 0} \lambda_1(\omega+2\pi k)}{\sum_{m=-\infty}^{\infty} \lambda_N(\omega+2\pi m)}$$

Proof. The largest eigenvalue of $[\sum_{m=-\infty}^{\infty} S_x(\omega+2\pi m)]^{-1}$ is the inverse of the smallest eigenvalue of $\sum_{m=-\infty}^{\infty} S_x(+2\pi m)$, and $\leq \sum_{m=-\infty}^{\infty} \lambda_N(\omega+2\pi m)$.

Hence

$$\left|\left|[\sum_{m=-\infty}^{\infty} S_x(\omega+2\pi m)]^{-1}\right|\right| \leq [\sum_{m=-\infty}^{\infty} \lambda_N(\omega+2\pi m)]^{-1}.$$

$||\sum_{k\neq 0} S_x(\omega+2\pi k)|| \leq \sum_{k\neq 0} \lambda_1(\omega+2\pi k)$; the result follows from properties (iii) and (iv) of the norm.

Proposition 6'. Suppose b is bounded variation in an open interval including in the point t_0. Define

$$\hat{b}(t_0) = \frac{1}{2} \lim_{\varepsilon\to 0}[b(t_0+\varepsilon) + b(t_0-\varepsilon)].$$

Let ε be some positive constant and suppose there exists $\omega_0 \varepsilon (0, \pi)$ such that

(i) $||\int_{-\infty}^{\infty} |b(t)|dt|| = A < \infty$;

(ii) $\lambda_1(\omega)/\lambda_N(\omega) \leq d < \infty$ almost everywhere;

(iii) $\inf_{\omega\leq\omega_0} \lambda_1(\omega) \geq \frac{2dA}{\pi\varepsilon} \int_{|\omega|>\pi} \lambda_1(\omega)\, d\omega$;

(iv) $\sup_{|\omega|>\omega_0} ||\tilde{b}(\omega)|| \leq \frac{\pi\varepsilon}{2d}$;

(v) $||\hat{b}(t_0) - \frac{1}{2\pi} \int_{|\omega|\leq\pi} \tilde{b}(\omega)e^{it_0\omega}d\omega|| \leq \varepsilon/4$, t_0 integer.

Then

$$||b(t_0) - B(t_0)|| < \varepsilon.$$

<u>Proof</u>. $||b(t_0) - B(t_0)|| \leq ||\int_{|\omega|>\pi} \tilde{b}(\omega)\ e^{i\omega t_0}\ d\omega|| + \frac{1}{2\pi}||\int_{-\pi}^{\pi} \tilde{b}(\omega) - \tilde{B}(\omega))$

$$e^{i\omega t_0}\ d\omega|| \qquad \text{(B.1)}$$

By (v) the first term of (B.1) is bounded by $\varepsilon/4$. The second is the norm of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [F[S_x](\omega)]^{-1}\{\sum_{k=-\infty}^{\infty} S_x(\omega+2\pi k)[\tilde{b}(\omega) - \tilde{b}(\omega+2\pi k)]\}e^{i\omega t_0}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [F[S_x](\omega)]^{-1}\sum_{k\neq 0} S_x(\omega+2\pi k)\tilde{b}(\omega)e^{i\omega t_0}d\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} [F[S_x](\omega)]^{-1} \cdot \sum_{k\neq 0} S_x(\omega+2\pi k)\tilde{b}(\omega+2\pi k)e^{i\omega t_0}d\omega \qquad \text{(B.2)}$$

Applying Lemma 5' the norm of the first term of (B.2) is bounded by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sum_{k\neq 0} \lambda_1(\omega+2\pi k)}{\sum_{m=-\infty}^{\infty} \lambda_N(\omega+2\pi m)} ||\hat{b}(\omega)||d\omega \quad \leq \frac{d}{2\pi}\int_{-\pi}^{\pi} \frac{\sum_{k\neq 0} \lambda_1(\omega+2\pi k)}{\sum_{m=-\infty}^{\infty} \lambda_1(\omega+2\pi m)} ||\tilde{b}(\omega)||d\omega$$

$$\leq \frac{d}{2\pi}[\sup_{|\omega|\leq\omega_0} |\tilde{b}(\omega)|] \cdot [\inf_{|\omega|>\omega_0} \lambda_1(\omega)]^{-1} \int_{|\omega|>\pi} \lambda_1(\omega)d\omega + \frac{d}{2\pi} \sup_{|\omega|>\omega_0}$$

$$||\tilde{b}(\omega)|| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

The norm of the second term of (B.2) is bounded by

$$\frac{d}{2\pi} \int_{\pi}^{\pi} \frac{\sum_{k\neq 0} \lambda_1(\omega+2\pi k)\tilde{b}(\omega+2\pi k)}{\sum_{m=-\infty}^{\infty} \lambda_1(\omega+2\pi m)} \equiv \frac{d}{2\pi} \sup_{|\omega|>\pi} |\tilde{b}(\omega)| \leq \frac{\varepsilon}{4}$$

<u>Corollary 7'</u>. Suppose that b is an absolutely integrable function of bounded total variation. Define

$$\hat{b}(t) = \frac{1}{2}\lim_{\varepsilon \to 0}[b(t+\varepsilon) + b(t-\varepsilon)].$$

Suppose

(i) for all $\omega^* > 0$, $\inf_{|\omega|<\omega^*} \lambda_1(\omega) > 0$;

(ii) $\lambda_1(\omega)/\lambda_N(\omega) \leq d < \infty$ a.e.

Then

$$\lim_{n\to\infty} ||\hat{b}(t) - {}^nB(t)|| = 0 \text{ for all integer } t.$$

Proof. Given any integer t, and $\varepsilon >$. It suffices to show the existence of ω_0 and n^* such that for all $n \geq n^*$,

(iii') $\inf_{|\omega|\leq\omega_0} \lambda_1(\omega) \geq \frac{2dA}{\pi\varepsilon} \int_{|\omega|>n\pi} \lambda_1(\omega)\, d\omega$;

(iv') $\sup_{|\omega|>\omega_0} ||\tilde{b}(\omega)|| \leq \pi\varepsilon/2d$;

(v') $||\hat{b}(t) - \frac{1}{2\pi} \int_{|\omega|<n\pi} \tilde{b}(\omega)\, e^{it\omega} d\omega|| \leq \varepsilon/4$.

Just as in the proof of Corollary 7, such existance is the consequence of the Riemann-Lebesgue Lemma, finite variance of x and the basic convergence result for Fourier series.

Proposition 8'. Suppose the b_i are square and absolutely integrable functions of bounded total variation, $\sum_{t=-\infty}^{\infty} |R_{x_i}(t)| < \infty$ for all i, and the roots of $S_x(\omega)$ are all positive a.e. Let there exist $d < \infty$ such that $\lambda_1(\omega)/\lambda_N(\omega) \leq d$ a.e. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=-\infty}^{\infty} [{}^nB(\tfrac{t}{n}) - b(\tfrac{t}{n})]' [{}^nB(\tfrac{t}{n}) - b(\tfrac{t}{n})] = 0.$$

Proof. Proceed as in the proof of Proposition 8. (a) converges to 0 exactly as before, and use Lemma 5' to bound (b) and (c) by

$$\int_{-n\pi}^{n\pi} \left[\frac{\sum\limits_{k\neq 0} \lambda_1(\omega+2n\pi k)}{\sum\limits_{m=-\infty}^{\infty} \lambda_N(\omega+2n\pi m)} \right]^2 \tilde{b}(\omega)'\tilde{b}(\omega)d\omega \qquad \text{(B.3)}$$

and

$$\int_{-n\pi}^{n\pi} \left[\frac{\sum\limits_{k\neq 0} \lambda_1(\omega+2n\pi k)}{\sum\limits_{m=-\infty}^{\infty} \lambda_N(\omega+2n\pi m)} \right]^2 \sum_{m=-\infty}^{\infty} \tilde{b}(\omega+2n\pi m)'\tilde{b}(\omega+2n\pi m)d\omega \qquad \text{(B.4)}$$

respectively. ((B.3) is the analogue of (17), (B.4) the analogue of (18).) Observing that

$$\frac{\sum\limits_{k\neq 0} \lambda_1(\omega+2n\pi k)}{\sum\limits_{m=-\infty}^{\infty} \lambda_N(\omega+2n\pi m)} \leq \frac{d \sum\limits_{k\neq 0} \lambda_1(\omega+2n\pi k)}{\sum\limits_{m=-\infty}^{\infty} \lambda_1(\omega+2n\pi m)} \leq 1 \text{ a.e.}$$

and

$$\lim_{n\to\infty} \frac{\sum\limits_{k\neq 0} \lambda_1(\omega+2n\pi k)}{\sum\limits_{m=-\infty}^{\infty} \lambda_1(\omega+2n\pi m)} = 0 \text{ a.e.,}$$

one can use the Lebesgue Convergence Theorem exactly as in the proof of Proposition 8 to prove that (B.3) and (B.4) converge to 0.