A NOTE ON THE ELASTICITY OF DERIVED DEMAND UNDER DECREASING RETURNS

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Assaf Razin

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

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I. Introduction

In his classical book, The Theory of Wages, Hicks [4] analyzes Marshall's four rules for the things on which the elasticity of derived demand depends. The smaller the elasticity, the stronger may be expected to be the trade-union power in the industry. Hicks' analysis is confined, however, to the assumption of "constant returns". This assumption may be inappropriate for short run analyses, where some factors of production are temporarily immobile. If one is interested, therefore, in the short run properties of the demand for factors of production, the assumption of "decreasing returns" may be more plausible. The purpose of this note is to derive the general formula and to discuss the properties of a derived demand for a factor of production under conditions of decreasing returns to scale.

II. The Model

Consider a competitive industry in which a product X is being made by the cooperation of two factors, L and K, under conditions of decreasing returns to scale. If the elasticity of demand for the product (in absolute values) is η and the

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elasticity of supply of K is e, how is λ, the elasticity of demand for L (in absolute values) determined?

To analyze this problem it is convenient to transform the decreasing-returns-to-scale production function $X = F(L,K)$ into a constant-returns-to-scale function G, $G(L,K,T) = TF\left(\frac{L}{T},\frac{K}{T}\right)$, by adding a fictitious factor T. This fransformation was first introduced in Berglas and Razin [2]. If we set T=1, profits will be equal to the (shadow) price of T, P_T. The prices of L,K are denoted by P_L, P_K respectively; the price of X is P.

III. The Elasticity of Derived Demand

Consider first the case where the supply of K is perfectly elastic (i.e., $e = \infty$), while the price of X is a function of X, $P = P(X)$. The model can be written as

(1) $P_L a_L + P_K a_K + P_T a_T = P(X)$

(2) $a_T X = 1$ (or T=1) ,

where a_i is the minimizing cost (for given prices P_L, P_K, P_T) input-output coefficient of factor i which is a function of factor prices

(3) $a_i = a_i(P_L, P_K, P_T)$, $i = K,L,T$.

Equation (1) has cost per unit of output plus profits per unit of output on the l.h.s. and the price of the product on the r.h.s. Since P_L and P_K are given, equations (1) and (2) (by using (3)) have two endogenous variables, P_T and X, to solve for.

By differentiating (1)-(2) totally we get

(4) $a_T dP_T - \frac{\partial P(X)}{\partial X} dX = -a_L dP_L$

(5) $X \frac{\partial a_T}{\partial P_T} dP_T + a_T dX = -X \frac{\partial a_T}{\partial P_L} dP_L$

where use has been made of the condition for cost minimization, $\sum_i P_i da_i = 0$, $i = L,K,T$. The elasticities of the input-output coefficients can be written in terms of Allen's partial elasticities of substitution σ_{ij} 1/:

(6) $\frac{P_j}{a_i} \frac{\partial a_i}{\partial P_j} = \alpha_j \sigma_{ij}$, $\sum_{j=L,K,T} \alpha_j = 1$, $\sum_{j=L,K,T} \alpha_j \sigma_{ij} = 0$, $i = L,K,T$.

where α_j is the relative share of the cost of factor j in the total revenue.

Solving (4)-(5) and converting into elasticities by using (6), we get

(7) $\frac{P_L}{X} \frac{DX}{DP_L} = (\alpha_L) \frac{\eta(\sigma_{TT} - \sigma_{TL})}{\eta - \sigma_{TT}}$

(8) $\frac{P_L dP_T}{P_T dP_L} = \left(\frac{\alpha_L}{\alpha_T}\right) \cdot \frac{\sigma_{TL} - \eta}{\eta - \sigma_{TT}}$

Consider the limiting case where the product demand is perfectly elastic. In this case we get from (7)-(8):

(7a) $\frac{P_L}{X} \frac{dX}{dP_L} = \alpha_L (\sigma_{TT} - \sigma_{TL})$

(8a) $\frac{P_L}{P_T} \frac{dP_T}{dP_L} = -\frac{\alpha_L}{\alpha_T}$

Equation (7a) can be used to define an inferior input. An inferior input is an input whose quantity decreases when the price of the product increases, while factor prices are kept constant. Equivalently (see Syrquin [6]) an inferior input can be defined as a factor such that production increases

1/ See Allen [1], page 504. For a detailed discussion of the relationships among different concepts of elasticities of substitution which were used in the literature, the reader may consult Sato and Koizumi [5].

when the factor price increases, holding constant other factor prices and the product price. From (7a) L is an inferior input if $\sigma_{TT} - \sigma_{TL} > 0$. Similarly, L is a <u>normal</u> input if $\sigma_{TT} - \sigma_{TL} < 0$, whereas L is a <u>neutral</u> input if $\sigma_{TT} = \sigma_{TL}$. Equation (8a) conforms with the well-known result that profits must decline when the price of one factor increases while other prices are held constant. Note that when demand for the product is less than perfectly elastic the change in profits as a result of the increase in P_L (equation (7)) cannot a priori be signed. There is an opposing effect to a profits decrease since the price of the product will also rise.

An economic interpretation of the own elasticity of substitution σ_{TT} can now be given. Treat parametrically the price of the product and consider the elasticity of supply of X, $\varepsilon = (\partial X/\partial P)(P/X)$. From production theory it is known that $\partial L/\partial P = -\partial X/\partial P_L$ and $\partial K/\partial P = -\partial X/\partial P_K$. We will derive ε by utilizing these conditions, the equalities of marginal productivities and price ratios, equation (7a) and a (7a)-like equation for $(\partial K/\partial P_K)(PK/K)$ as follows:

$$\varepsilon = \frac{P}{X}\left[\frac{\partial X}{\partial L}\frac{\partial L}{\partial P} + \frac{\partial X}{\partial K}\frac{\partial K}{\partial P}\right]$$

$$= -\alpha_L(\sigma_{TT} - \sigma_{TL}) - \alpha_K(\sigma_{TT} - \sigma_{TK})$$

$$= -\sigma_{TT}$$

where in the last equation use has been made of (6). Therefore, <u>$-\sigma_{TT}$ is equal to the elasticity of supply of the product.</u>[1/]

1/ Note that in the case of constant returns to scale $-\sigma_{TT} = \infty$.

We can now proceed to derive the elasticity of the derived demand for L as follows:

$$(9) \quad \lambda = (-)\,\frac{P_L}{L}\,\frac{dL}{dP_L} = -\,\frac{P_L}{a_L X}\,\frac{da_L X}{dP_L}$$

$$= -\left[\frac{P_L}{a_L}\,\frac{\partial a_L}{\partial P_L} + \frac{P_L}{a_L}\,\frac{\partial a_L}{\partial P_T}\,\frac{dP_T}{dP_L} + \frac{P_L}{X}\,\frac{dX}{dP_L}\right]$$

Substituting equations (6)-(8) into (9) we get

$$(10) \quad \lambda = \alpha_L \cdot \frac{\eta(\sigma_{LL} - 2\sigma_{LT} + \sigma_{TT}) - (\sigma_{TT}\sigma_{LL} - \sigma_{TL}^{2})}{\sigma_{TT} - \eta}$$

The denominator of (10) is clearly negative. The concavity of the production function implies that the term in the first parentheses, in the numerator of (10), is negative, while the term in the second parentheses is positive, so that the demand for L obviously declines when its price rises. We are now in a position to verify the validity of Marshall's "fourth rule" for conditions of decreasing returns to scale. We have to show that the more elastic the demand for the product the more elastic is the derived demand for the factor. Differentiating (10) with respect to η, we get

$$(11) \quad \frac{\partial\lambda}{\partial\eta} = \alpha_L\,\frac{(\sigma_{TT} - \sigma_{LT})^2}{(\sigma_{TT} - \eta)^2} \geq 0 \ .$$

Equation (11) proves the above statement, except in the case of a <u>neutral</u> input where λ is independent of η. The reason for the result in the exceptional case is obvious. When L is a neutral input a change in its price will not affect production of X, thus, there will be no change in the equilibrium price of X. In this case, therefore, the elasticity of the derived demand for the factor is determined solely by the technology of production.[1]

IV. Homogeneous Production Functions

It may be instructive to consider the example where the production function is <u>homogeneous</u> of degree $1-\alpha_T$. Then, it is easy to see that since $\left(\frac{1}{T}\right)^{1-\alpha_T} F(K,L) = F\left(\frac{K}{T}, \frac{L}{T}\right)$ the transformation $G = TF\left(\frac{K}{T}, \frac{L}{T}\right)$ takes a simple form , $G = T^{\alpha_T} F(K,L)$.[2] The partial elasticities of substitution of T and other factors reduce in this case to

$$(12) \quad \sigma_{TT} = -\frac{1-\alpha_T}{\alpha_T}, \quad \sigma_{Ti} = 1 \text{ for } i = L,K.$$

To investigate the response of output and profits to an increase in P_L we substitute (12) into (7)-(8) as follows:

$$(7a) \quad \frac{P_L}{X}\frac{dX}{dP_L} = -\alpha_L \frac{\eta}{\alpha_T \eta + (1-\alpha_T)}$$

$$(8a) \quad \frac{P_L}{P_T}\frac{dP_T}{dP_L} = \alpha_L \frac{1-\eta}{\alpha_T \eta + (1-\alpha_T)}$$

Therefore, in the case of homothetic production functions output will always decrease when P_L increases; whereas profits will rise or fall when P_L increases according to whether the elasticity of the demand for the product is smaller or greater than unity.

Substituting (12) into (10) we obtain the derived demand's elasticity for the case of a homogeneous production function as

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1/ From (10), when L is a neutral input (i.e., $\sigma_{TT} = \sigma_{TL}$) we get $\lambda = -\alpha_L(\sigma_{LL} - \sigma_{TT})$.

2/ See Berglas and Razin [3] for a general equilibrium analysis in this case.

follows:

$$(10a)\quad \lambda = -\alpha_L \frac{\sigma_{LL}(1-\alpha_T(1-\eta))-\eta+\alpha_T(1-\eta)}{1-\alpha_T(1-\eta)} = \alpha_K\sigma_{LK} + \alpha_T + \alpha_L \frac{\eta-\alpha_T(1-\eta)}{1-\alpha_T(1-\eta)}$$

where, in the last equation, use has been made of (6).

The special case in which returns to scale are constant, i.e., $\alpha_T = 0$, yields the Hicks-Allen formula for the elasticity of derived demand[1/]:

$$(10b)\quad \lambda = \alpha_K\sigma_{LK} + \alpha_L\eta$$

Suppose we measure the degree of decreasing returns by α_T. What is the effect of decreasing returns on the elasticity of derived demand? To give an answer to this question we differentiate (10a) with respect to α_T holding α_L constant (therefore, having $\partial\alpha_K/\partial\alpha_T = -1$), to get

$$(13)\quad \frac{\partial\lambda}{\partial\alpha_T} = -\alpha_L \frac{(1-\eta)^2}{[1-\alpha_T(1-\eta)]^2} - \sigma_{LK}$$

Therefore, if the factors are substitutes in production ($\sigma_{LK} > 0$), for a given relative share of the factor L, α_L, the smaller the degree of decreasing returns to scale, α_T, the more elastic is the derived demand, λ. When $\sigma_{LK} < 0$, however, the sign of $\partial\lambda/\partial\alpha_T$ cannot a priori be determined. Also, when α_K is held constant, the results of the effect of a change in α_T compensated by an opposite change in α_L on λ are ambiguous.

V. Distributive Share

It may be of interest to analyze, for the case of decreasing returns, the effect of a change in the price of the factor on

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1/ See Hicks [4], page 244 when $e = \infty$, and Allen [1] page 373.

the distributive share of this factor. Consider the distributive share of the factor L

$$(14)\quad \alpha_L = \frac{a_L(P_L, P_T)P_L}{P(X)}$$

Differentiating logarithmically the right-hand side of (14), using (6)-(7) and (8), we get

$$(15)\quad \frac{P_L}{\alpha_L}\frac{\partial\alpha_L}{\partial P_L} = \frac{P_L\,\partial a_L}{a_L\,\partial P_L} + \frac{P_T}{a_L}\frac{\partial a_L}{\partial P_T}\frac{P_L}{P_T}\frac{\partial P_T}{\partial P_L} + 1 - \frac{X}{P}\frac{\partial P}{\partial X}\frac{P_L}{X}\frac{\partial X}{\partial P_L}$$

$$= \alpha_L\sigma_{LL} + (\alpha_L\sigma_{TL})\frac{\sigma_{TL}-\eta}{\eta-\sigma_{TT}} + 1 + \alpha_L\frac{\sigma_{TT}-\sigma_{TL}}{\eta-\sigma_{TT}}$$

In the limiting case when the demand for the product is infinitely elastic (i.e., $\eta = \infty$), equation (15) reduces to

$$(15a)\quad \frac{P_L}{\alpha_L}\frac{\partial\alpha_L}{\partial P_L} = \alpha_L(\sigma_{LL} - \sigma_{TL}) + 1$$

$$= \alpha_K(1-\sigma_{LK}) + (1-\alpha_K)(1-\sigma_{LT})$$

where, in the last equation, use has been made of (6).

When the production function is <u>homogeneous</u>, (i.e., $\sigma_{LT} = 1$) (15a) reduces to

$$(15b)\quad \frac{P_L}{\alpha_L}\frac{\partial\alpha_L}{\partial P_L} = \alpha_K(1-\sigma_{LK}) \Rightarrow \frac{\partial\alpha_L}{\partial P_L} \gtreqless 0 \quad \text{as} \quad 1 \gtreqless \sigma_{LK}\ .$$

Equation (15b) is a generalization of Hicks' condition for the change in the factor's distributive share, when the price of the factor changes, for a <u>homogeneous</u> production function.[1/] Obviously if the production function is of a Cobb-Douglas form, $\sigma_{LK} = \sigma_{LT} = 1$, we get $\partial\alpha_L/\partial P_L = 0$.

1/ Hicks' condition, derived for the case of a linear homogeneous production function may be found in [4] chapter 6.

VI. Inelastic Factor's Supply

The analysis of the effect of a positive supply elasticity of K on the derived demand for L is rather complex. To simplify the exposition we will assume a perfectly elastic demand for the product. When $e < \infty$, the price of K, P_K becomes an endogenous variable. We can write $P_K = P_K(K)$ and add to the model, which is given by equations (1)-(2), a third equation:

$$(16) \quad a_K X = K \ .$$

Differentiating totally (1)-(2) and (16), and using (as before) the cost minimization condition we get

$$(17) \quad \begin{pmatrix} a_T & 0 & a_K \dfrac{\partial P_K}{\partial K} \\ X \dfrac{\partial a_T}{\partial P_T} & a_T & X \dfrac{\partial a_T}{\partial P_K} \dfrac{\partial P_K}{\partial K} \\ X \dfrac{\partial a_K}{\partial P_T} & a_K & X \dfrac{\partial a_K}{\partial P_K} \dfrac{\partial P_K}{\partial K} - 1 \end{pmatrix} \begin{pmatrix} dP_T \\ dX \\ dK \end{pmatrix} = \begin{pmatrix} -a_L \\ -X \dfrac{\partial a_T}{\partial P_L} \\ -X \dfrac{\partial a_K}{\partial P_L} \end{pmatrix} dP_L$$

From (17) we can solve for dP_T/dP_L, dX/dP_L and dK/dP_L and substitute into (18):

$$(18) \quad \lambda = -\frac{P_L}{a_L X}\frac{da_L X}{dP_L} = -\left[\frac{P_L}{a_L}\frac{\partial a_L}{\partial P_L} + \frac{\partial a_L}{\partial P_K}\frac{\partial P_K}{\partial K}\frac{dK}{dP_L} + \frac{\partial a_L}{\partial P_T}\frac{dP_T}{dP_L} + \frac{P_L}{X}\frac{dX}{dP_L}\right]$$

We get, when converting into elasticities:

$$(19) \quad \lambda = \frac{\alpha_L \ \alpha_K}{[e-\alpha_K(\sigma_{KK}-2\sigma_{TK}+\sigma_{TT})]}\Big[\frac{-e}{\alpha_K}(\sigma_{LL}-2\sigma_{TL}+\sigma_{TT}) + \sigma_{LL}(\sigma_{KK}-2\sigma_{TK}+\sigma_{TT})$$

$$+ \sigma_{KK}(\sigma_{TT}-2\sigma_{TL}) - 2\sigma_{TT}\sigma_{KL} + 2\sigma_{KT}\sigma_{TL} + 2\sigma_{TK}\sigma_{KL} - \sigma_{KT}^2 - (\sigma_{KL}-\sigma_{TL})^2\Big]$$

Differentiating (17) with respect to e we get

$$(20) \quad \frac{\partial\lambda}{\partial e} = \frac{\alpha_L \alpha_K [(\sigma_{KL}-\sigma_{TL})^2 - (\sigma_{TT}-\sigma_{KT})^2]^2}{[e - \alpha_K(\sigma_{KK}-2\sigma_{TK}+\sigma_{TT})]} > 0$$

Therefore, we proved that the larger the elasticity of supply of the cooperative factor K, the larger is the elasticity of the derived demand for L. This is a proof, for the case of decreasing returns to scale, of Marshall's "third rule".

VII. Concluding Remarks

Marshall's "first rule" with regard to the effect of the degree of substitution between factors on the elasticity of derived demand can be generalized to the case of decreasing returns to scale. Let σ_{KL} measure the degree of factors substitution one can express σ_{LL} in equation (10) or (19), in terms of cross elasticities (using (6)) to get

$$(21)\quad \frac{\partial\lambda}{\partial\sigma_{KL}} > 0 \ .$$

As already shown by Hicks [4], Marshall's "second rule" with regard to the effect of the magnitude of the factor's distributive share on the elasticity of derived demand for the factor cannot apriori be determined even in the case of constant returns to scale. Under decreasing returns to scale this ambiguity also prevails.

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