Non Abelian Vortices in $\mathcal{N} = 1^*$ and their gravity duals

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Introduction

A non-Abelian vortex is a flux tube soliton (Abrikosov-Nielsen-Olesen's style) with some internal orientational $\mathbb{CP}^{N-1}$ degree of freedom, which corresponds to a colour+flavour rotation

The topic of the talk is to discuss these objects in the Higgs vacuum of $\mathcal{N}=1^*$ mass deformation of $\mathcal{N}=4$ theory, and then in the large N_c IIB string dual

Theoretical setting: $\mathcal{N}=1^*$

Let us start with $\mathcal{N}=4$ $SU(N_c)$ SYM (which contains an $\mathcal{N}=1$ vector multiplet and 3 adjoint chiral multiplets, $\Phi_{1,2,3}$), with the superpotential

$$W=\frac{1}{g_{YM}^2}\mathrm{Tr}([\Phi_1,\Phi_2]\Phi_3)$$

The bosonic degrees of freedom are a gauge field and 3 complex scalars in the adjoint representation. There are also 4 Weyl fermions, also in the adjoint.

The $\mathcal{N}=1^*$ is a mass deformation with the superpotential

$$\Delta W=\frac{1}{g_{YM}^2}\sum\frac{1}{2}m\mathrm{Tr}(\Phi_i^2)$$

The potential

$$V_F = \mathrm{Tr}\left(w_1.w_1^\dagger + w_2.w_2^\dagger + w_3.w_3^\dagger \right) \,, \qquad w_i = \epsilon_{ijk}\Phi_j\Phi_k + m_i\Phi_i \,,$$

$$V_D = \frac{1}{4}\mathrm{Tr}\left([\Phi_1^\dagger, \Phi_1] + [\Phi_2^\dagger, \Phi_2] + [\Phi_3^\dagger, \Phi_3] \right)^2 \,.$$

In order for this to vanish we have to choose the VEVs as $SU(2)$ representations

$$\Phi_k = imJ_k$$

For large N_c really a lot of discrete vacua, totally or partially Higgsed/Confined/Obliqued confined, with and without mass gap

The Higgs Vacuum

We will concentrate on a weakly coupled vacuum, which is completely Higgsed (for $m_k \neq 0$)

This vacuum is given by the $N_c \times N_c$ $SU(2)$ irreducible representation.

For example, for $N_c = 4$:

$$\Phi_1 = \frac{im}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{im}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -i2 & 0 \\ 0 & i2 & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix},$$

$$\Phi_3 = \frac{im}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Colour Flavour Locking

$\mathcal{N}=4$ has an $SU(4)\approx SO(6)$ global symmetry
The mass deformation, if we keep all the 3 masses $m_k=m$, breaks this to $SO(3)$
In the Higgs vacuum, for generic N_c, we can find a combination of the global and of some gauge generators which is unbroken

$$U_F=\exp(T_j a_j), \qquad W_C=\exp(iJ_l a_l)$$

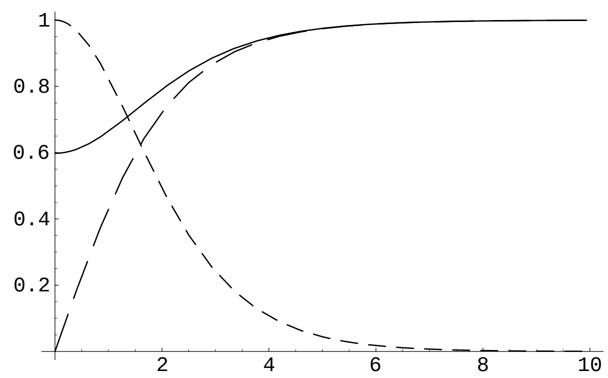
$$T_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; T_2=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; T_3=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\vec{\Phi}\rightarrow U_F\,\vec{\Phi}\,, \qquad \Phi_i\rightarrow W_C\,\Phi_i\,W_C^\dagger\,.$$

Vortex $N_c = 2$

$$\Phi_1 = \frac{im}{2}\psi_1(r)\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{im}{2}\psi_1(r)\begin{pmatrix} 0 & -ie^{i\varphi} \\ ie^{-i\varphi} & 0 \end{pmatrix}, \qquad \Phi_3 = \frac{im}{2}\kappa_1(r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$A_x = \frac{-y}{r^2}(1 - f(r))Y, \qquad A_y = \frac{x}{r^2}(1 - f(r))Y, \quad Y = \frac{1}{2}\sigma_3.$$



κ_1 (solid), ψ_1 (long dashes), f (short dashes).

Vortex $N_c = 4, k = 1$

$$\Phi_1 = \frac{mi}{2}\begin{pmatrix} 0 & \sqrt{3}\psi_1 & 0 & 0 \\ \sqrt{3}\psi_1 & 0 & 2\psi_2 & 0 \\ 0 & 2\psi_2 & 0 & \sqrt{3}\psi_3 e^{i\varphi} \\ 0 & 0 & \sqrt{3}\psi_3 e^{-i\varphi} & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{mi}{2}\begin{pmatrix} 0 & -i\sqrt{3}\psi_1 & 0 & 0 \\ i\sqrt{3}\psi_1 & 0 & -i2\psi_2 & 0 \\ 0 & i2\psi_2 & 0 & -i\sqrt{3}\psi_3 e^{i\varphi} \\ 0 & 0 & i\sqrt{3}\psi_3 e^{-i\varphi} & 0 \end{pmatrix},$$

$$\Phi_3 = \frac{mi}{2}\begin{pmatrix} 3\kappa_1 - 2\kappa_3 & 0 & 0 & 0 \\ 0 & \kappa_2 + 2\kappa_3 & 0 & 0 \\ 0 & 0 & -\kappa_2 + 2\kappa_3 & 0 \\ 0 & 0 & 0 & -3\kappa_1 - 2\kappa_3 \end{pmatrix}.$$

$$A_x = \frac{-y}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right), \qquad A_y = \frac{x}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right),$$

$$Y_1 = \frac{1}{4}\mathrm{Diag}(1,1,1,-3), \qquad \lambda_1 = \frac{1}{\sqrt{12}}\mathrm{Diag}(1,1,-2,0), \qquad \lambda_2 = \frac{1}{2}\mathrm{Diag}(1,-1,0,0).$$

$$\exp(2\pi i Y_1) = \mathrm{Diag}(e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2})$$

$$f(0) = 1, \qquad g_\ell(0) = 0, \qquad f(\infty) = g_\ell(\infty) = 0\,.$$

Vortex $N_c = 4, k = 2$

$$\Phi_1 = \frac{mi}{2}\begin{pmatrix} 0 & \sqrt{3}\psi_1 & 0 & 0 \\ \sqrt{3}\psi_1 & 0 & 2\psi_2 e^{i\varphi} & 0 \\ 0 & 2\psi_2 e^{-i\varphi} & 0 & \sqrt{3}\psi_3 \\ 0 & 0 & \sqrt{3}\psi_3 & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{mi}{2}\begin{pmatrix} 0 & -i\sqrt{3}\psi_1 & 0 & 0 \\ i\sqrt{3}\psi_1 & 0 & -i2\psi_2 e^{i\varphi} & 0 \\ 0 & i2\psi_2 e^{-i\varphi} & 0 & -i\sqrt{3}\psi_3 \\ 0 & 0 & i\sqrt{3}\psi_3 & 0 \end{pmatrix},$$

$$Phi_3 = \frac{mi}{2}\begin{pmatrix} 3\kappa_1 - 2\kappa_3 & 0 & 0 & 0 \\ 0 & \kappa_2 + 2\kappa_3 & 0 & 0 \\ 0 & 0 & -\kappa_2 + 2\kappa_3 & 0 \\ 0 & 0 & 0 & -3\kappa_1 - 2\kappa_3 \end{pmatrix}.$$

$$A_x = \frac{-y}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right), \qquad A_y = \frac{x}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right),$$

$$Y_2 = \frac{1}{2}\mathrm{Diag}(1,1,-1,-1), \qquad \lambda_1 = \frac{1}{2}\mathrm{Diag}(1,-1,0,0), \qquad \lambda_2 = \frac{1}{2}\mathrm{Diag}(0,0,1,-1).$$

$$\exp(2\pi i Y_2) = \mathrm{Diag}(-1,-1,-1,-1)$$

$$f(0) = 1, \qquad g_\ell(0) = 0, \qquad f(\infty) = g_\ell(\infty) = 0\,.$$

Generic N_c, k

$$Y_k = \mathrm{Diag}\left(\underbrace{\frac{k}{N_c}, \cdots, \frac{k}{N_c}}_{N_c - k\,\text{elements}}, \;\; -\frac{N_c - k}{N_c}, \cdots, -\frac{N_c - k}{N_c} \right),$$

$$\Phi_{1,2}(r, \varphi) = e^{iY_k \varphi}\, \Phi_{1,2}(r, \varphi = 0)\, e^{-iY_k \varphi}$$

$3(N_c - 1)$ profile functions $(\psi_i, \kappa_i, f, g_i)$

Numerical solution found for $2 \leq N_c \leq 6$

Tensions

In the limit

$$m_3 << m_1 = m_2 = m$$

the theory is almost $\mathcal{N} = 2$ supersymmetric
The k-string becomes BPS objects with tension:

$$T^{BPS}_{N_c,k} = 2\pi \frac{m m_3}{g^2_{\rm YM}} k(N_c - k)\,.$$

What happens if we extrapolate this formula to $m = m_3$?? Values of $T_{N_c,k}/T^{\rm BPS}_{N_c,k}$ for $2 \leq N_c \leq 6$ and different k:

N_c	2	3	4	5	6
$k=1$	0.894	0.926	0.943	0.954	0.961
$k=2$			0.944	0.954	0.962
$k=3$					0.962

Tension Ratios

For $N_c = 4$ we find the following numerical result,

$$\frac{T_{N_c=4,\,k=2}}{T_{N_c=4,\,k=1}} = 1.334$$

while the prediction from Casimir scaling is $4/3$. For $N_c = 5$ we find

$$\frac{T_{N_c=5,\,k=2}}{T_{N_c=5,\,k=1}} = 1.501$$

while the Casimir scaling prediction is $3/2$. Finally, for $N_c = 6$:

$$\frac{T_{N_c,\,k=2}}{T_{N_c=6,\,k=1}} = 1.6008, \qquad \frac{T_{N_c=6,\,k=3}}{T_{N_c=6,\,k=1}} = 1.801,$$

while the Casimir scaling values are $8/5$ and $9/5$.

Effective world-sheet theory

For all N_c, k, the vortex moduli space is:

$$\frac{SO(3)_{C+F}}{U(1)_{C+F}} = \mathbb{CP}^1 = S^2$$

Switch on a (z,t) dependent C+F rotation on the vortex world-sheet:

$$\vec{\Phi} \to U_F(z,t) \cdot \Big(W_C(z,t)\, \vec{\Phi}\, W_C^\dagger(z,t) \Big) ,$$

$$A_{x,y} \to W_C\, A_{x,y}\, W_C^\dagger , \qquad W_C\, J_3\, W_C^\dagger = \vec{n}(z,t) \cdot \vec{J} .$$

Ansatz for the gauge field along the vortex:

$$A_s = -\left(\vec{n} \times \partial_s \vec{n}\right)^a \, J^a \rho(r)$$

Effective world-sheet theory

Bosonic $\mathbb{CP}^1$ sigma model:

$$S_{1+1} = \int dz\, dt \left(B_{N_c,k} (\partial_s \vec{n})^2 - k(N_c - k) \frac{\theta_{3+1}}{8\pi} \epsilon^{sr}\, \epsilon^{abc}\, n^a \partial_s n^b \partial_r n^c \right) ,$$

where $B_{N_c,k}$ can be calculated numerically for each N_c, k

$$\theta_{1+1} = k(N_c - k)\theta_{3+1} \,.$$

The String Dual of the Higgs Vacuum

Constructed by Polchinski and Strassler by considering an appropriate deformation of the $AdS_5 \times S^5$ background.
The $\mathcal{N} = 1^*$ deformation is achieved by switching on a three form

$$G_3 = F_3 - \tau H_3$$

where

$$\tau = \frac{i}{g_s} + \frac{C_0}{2\pi}$$

The N_c D3 branes on which the parent $\mathcal{N} = 4$ theory lives polarizes (by Myers dielectric effect) in a spherical D5 brane with N_c units of D3 charge on the top of it

String vs Gauge Theory quantities

$$4\pi g_s = g_{\rm YM}^2\,, \qquad \frac{R_{\rm AdS}}{\sqrt{\alpha'}} = (4\pi g_s N_c)^{1/4} \gg 1$$

In the Higgs vacuum, with $m_1 = m_2 = m_3 = m$, the metric in the interior matches onto the geometry generated by a D5-brane wrapped on an S^2 carrying N_c units of D3-charge.
The six transverse directions are denoted as $w^{1,2,3}$, $y^{1,2,3}$
The D3 branes spread out along the w^i directions with $y^i = 0$ and the resulting D5-brane wraps around sphere of radius:

$$r_0 = \pi\alpha' m N_c.$$

The $D5$ shell thickness is:

$$\rho_c = (\alpha' m)\sqrt{g_s N_c \pi}\,, \qquad \rho_c << r_0$$

Metric, Dilaton and B_2

Coordinates: $(x_\mu, \vec{y}, \vec{w})$

$$ds^2_{\rm string} = Z_x^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + Z_y^{1/2}(dy^2 + y^2 d\Omega_y^2 + dw^2) + Z_\Omega^{1/2} w^2 d\Omega_w^2 ,$$

$$Z_x = Z_y = \frac{R^4_{\rm AdS}}{\rho_+^2 \rho_-^2} , \qquad Z_\Omega = \frac{R^4_{\rm AdS}\, \rho_-^2}{\rho_+^2 (\rho_-^2 + \rho_c^2)^2} , \qquad \rho_\pm = \sqrt{y^2 + (w \pm r_0)^2} .$$

$$e^{2\Phi} = g_s^2 \frac{\rho_-^2}{\rho_-^2 + \rho_c^2} , \qquad C_0 = \theta_{3+1} = 0 .$$

$$B_2 = - \frac{\alpha' \pi N_c}{1 + \rho_-^2/\rho_c^2} \, \sin\theta_w \, d\theta_w \wedge d\phi_w \, .$$

Probes D1 branes: $k = 1$ vortex

$$S_{\rm DBI} = \frac{1}{2\pi\alpha'} \int d^2\xi \left\{ e^{-\Phi} \sqrt{(-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}))} \right\} .$$

Let us consider a D1-brane oriented in the x_0, x_1 directions and the embedding $(\xi_0, \xi_1) = (x_0, x_1)$, so that the pullback of the metric onto the world-sheet is:

$$G_{00} = -Z_x^{-1/2} + \dots , \qquad G_{11} = Z_x^{-1/2} + \dots$$

This DBI is minimized when :

$$w = \frac{r_0 + \sqrt{r_0^2 - 2\rho_c^2}}{2} \approx r_0 - \frac{\rho_c^2}{2r_0}, \qquad y = 0.$$

Probes D1 branes: $k = 1$ vortex

Tension:

$$T_{\rm D1} = \frac{N_c m^2}{2 g_s} = \frac{2\pi N_c m^2}{g_{\rm YM}^2}\,.$$

Let us then consider an arbitrary dependence of $\vec{n}_w$ on the world-sheet coordinates (x_0, x_1) and introduce this into the DBI action.

$$\mathcal{L}_{\rm kin} = \frac{N_c}{4 g_s}(\partial_s \vec{n}_w)^2 = \frac{\pi N_c}{g_{\rm YM}^2}(\partial_s \vec{n}_w)^2\,.$$

Probes D1 branes: θ term

$$S_{CS} = \frac{1}{2\pi\alpha'} \int \, [C_2 + C_0 \, B_2]$$

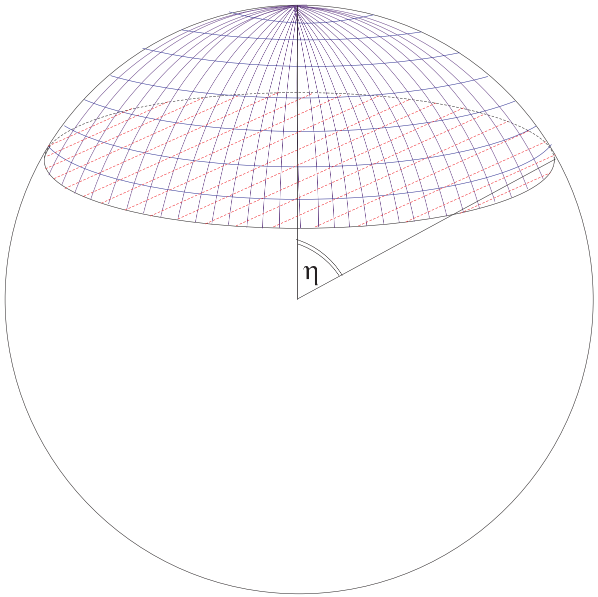
There is a subtlety: the contribution to Wess-Zumino terms from the pullback of B_2 have to be omitted.

$$S_{CS} = \frac{1}{2\pi\alpha'} \int C_2 = \frac{1}{2\pi\alpha'} \frac{\theta_{3+1}}{2\pi} \int B_2 \big|_{\theta_{3+1}=0} =$$

$$= \frac{\theta_{1+1}}{8\pi} \, \epsilon^{sr} \; \vec{n}_w \cdot (\partial_s \vec{n}_w \times \partial_r \vec{n}_w) \; , \qquad \theta_{1+1} = N_c \theta_{3+1} \, .$$

Probes D3 branes with flux: k strings

At large N, k (k/N fixed), a good description of the bound state of k D1 branes is given by a D3 with k units of flux on top of it



The minimal energy configuration for the probe D3 brane in the $\vec{w}$ space is given by the red disc and the blue "polar cap" shown in the figure.

The polar cap

We put all the flux on the top of the polar cap:

$$F_2 = \frac{k}{(\cos \bar{\eta}_k - 1)} \sin\theta_w (d\theta_w \wedge d\phi_w)$$

Then the DBI action reads:

$$S_{\rm cap} = \frac{1}{(2\pi)^3 \alpha'^2} \int d^4\xi \left\{ e^{-\Phi} \sqrt{(-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}))} \right\} .$$

At

$$|\vec{w}| = r_0 \,, \qquad (1 - \cos\bar{\eta}_k) = \frac{2k}{N_c}$$

the tension of the polar cap vanishes exactly.

The disc

The disc lies for most of its extension at $|\vec{w}| - r_0 \gg \rho_c$. In this limit the relevant metric (at $\vec{y} = 0$) is

$$ds^2\Big|_{w-r_0\gg\rho_c} = \frac{w^2 - r_0^2}{R_{\rm AdS}^2} dx_\mu dx_\nu \eta^{\mu\nu} + \frac{R_{\rm AdS}^2}{w^2 - r_0^2}(dw^2 + w^2(d\theta_w^2 + \sin^2\theta_w d\phi_w^2))\,;$$

the dilaton is simply $e^\Phi = g_s$. The B_2 field is small in this region. The warp factors from the two different subspaces cancel out and then the resulting DBI action is equivalent to the one for a membrane in flat space.

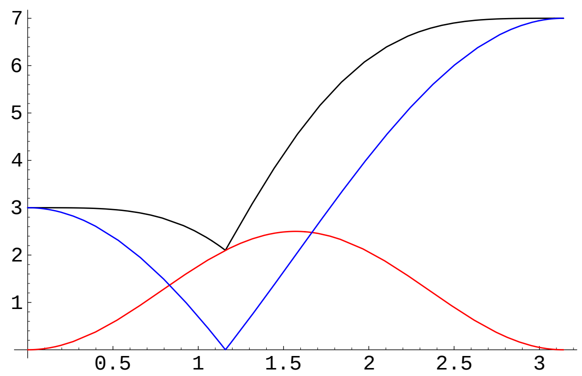
$$T_{\rm D3} = \frac{m^2}{2g_s} k(N_c - k)\,.$$

Some checks

1) The tension for a spherically symmetric configuration is bigger:

$$\frac{m^2}{2g_s} N_c(N_c - k)$$

2) A rather interesting cross-check of the picture above results when one determines the tension of the same kind of configuration (a polar cap with a disc glued) with generic values of $\bar{\eta}_k$.



Energy as a function of $\bar{\eta}_k$ for $N = 10$, $k = 3$.

The θ term

As for the D1 brane, it comes from the Chern-Simons coupling in the D3-brane theory:

$$S_{\rm cap}^{\rm CS} = \frac{1}{(2\pi)^3\alpha'^2}\int_{\rm cap} C_2 \wedge (2\pi\alpha' F_2)\,.$$

The only non-zero contribution comes from the polar cap, since it has a non-zero F_2,

$$F_2 = \frac{N_c}{2}\sin\theta_w (d\theta_w \wedge d\phi_w)\,.$$

The result is the same as in field theory

$$\mathcal{L}_\theta = -\frac{k(N_c-k)\,\theta_{3+1}}{8\pi}\,\epsilon^{sr}\vec{n}_w\cdot(\partial_s\vec{n}_w\times\partial_r\vec{n}_w)\,.$$

Confining vacuum

S-duality: maps the Higgs vacuum to the confining one, maps the D5 background to a NS5 background

Tension of a probe F string:

$$T_{\mathrm{F1}} = \frac{m^2 g_s N_c}{2},$$

Effective world-sheet theory for F string:

$$S_{\mathrm{F1}} = \int d^2x \left(\frac{g_s N_c}{4} (\partial_s \vec{n}_w)^2 \right).$$

From S-duality the tension of the k-string:

$$T^{\text{confining}}_{N_c,k} = m^2 \frac{g^2_{\mathrm{YM}}}{8\pi} k(N_c - k).$$

Comments on the world-sheet theory

A $\mathbb{CP}^1 = S^2$ sigma model:

$$S_{1+1} = \int dz\, dt \left(B_{N_c,k} (\partial_s \vec{n})^2 - \frac{\theta_{1+1}^{N_c,k}}{8\pi} \epsilon^{sr} \epsilon_{abc} n^a \partial_s n^b \partial_r n^c \right) ,$$

For $\theta_{1+1} = 0, \pi$ this model is exactly solvable:
1) For $\theta_{1+1} = 0$ there is mass gap and the spectrum consists of an $SO(3)$ triplet (with a known S-matrix)
2) For $\theta_{1+1} = \pi$ the theory flows to an infrared conformal point, there is no mass gap and the spectrum consists of a deconfined $SO(3)$ doublet.

Summary

1. The tension of the k-vortex in the Higgs vacuum is found to be proportional to $k(N_c - k)$. The world-sheet theory for the k-vortex is a $\mathbb{CP}^1$ sigma model with $\theta_{1+1} = k(N_c - k)\theta_{3+1}$.

2. S-duality at $\theta_{3+1} = 0$ maps the Higgs vacuum to the confining one, so the formula for the tension can be applied also for the confining vacuum in the large N_c limit. Also, the effective world-sheet theory is still a $\mathbb{CP}^1$ sigma model.