High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order

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- Regge limit in a conformal theory.
- High-energy scattering and Wilson lines.
- Evolution equation for color dipoles.
- Leading order: BK equation.
- Non-linear evolution equation in the NLO.
- $\mathcal{N} = 4$: study of 2-dim conformal invariance at high energies
- NLO BK kernel in $\mathcal{N} = 4$.
- NLO amplitude in $\mathcal{N} = 4$ SYM
- Conclusions.
Conformal four-point amplitude

Analog of QCD photon-photon scattering:

\[ A(x, y, x', y') = (x - y)^4 (x' - y')^4 N_c^2 \langle \mathcal{O}(x) \mathcal{O}^\dagger(y) \mathcal{O}(x') \mathcal{O}^\dagger(y') \rangle \]

\[ \mathcal{O} = \text{Tr}\{Z^2\} \quad (Z = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)) - \text{chiral primary operator} \]

In a conformal theory the amplitude is a function of two conformal ratios

\[ A = F(R, R') \]

\[ R = \frac{(x - y)^2 (x' - y')^2}{(x - x')^2 (y - y')^2}, \quad R' = \frac{(x - y)^2 (x' - y')^2}{(x - y')^2 (x' - y)^2} \]

At large \( N_c \)

\[ A(x, y, x', y') = A(g^2 N_c) \quad \quad g^2 N_c = \lambda - 't \text{ Hooft coupling} \]
Conformal four-point amplitude

Analog of QCD photon-photon scattering:

\[ A(x, y, x', y') = (x - y)^4(x' - y')^4 N_c^2 \langle O(x) O^\dagger(y) O(x') O^\dagger(y') \rangle \]

\[ O = \text{Tr}\{Z^2\} \ (Z = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)) - \text{chiral primary operator} \]

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At large \( N_c \)

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Our goal is the resummation of \((\lambda \ln s)^n\) at large energies in the next-to-leading approximation

\[ (\lambda \ln s)^n (c_n^{\text{LO}} + c_n^{\text{NLO}} \lambda) \]
Regge limit in the coordinate space

Regge limit: $x_+ \to \rho x_+, \; x'_+ \to \rho x'_+, \; y_\to \rho' y_-, \; y'_\to \rho' y_- \quad \rho, \rho' \to \infty$

Full 4-dim conformal group: $A = F(R, r)$

$$
R = \frac{(x - y)^2(x' - y')^2}{(x - x')^2(y - y')^2} \to \frac{\rho^2 \rho'^2 x_+ x'_+ y_- y'_-}{(x - x')^2(y - y')^2} \to \infty
$$

$$
r = \frac{[(x - y)^2(x' - y')^2 - (x' - y)^2(x - y')^2]^2}{(x - x')^2(y - y')^2(x - y)^2(x' - y')^2}
$$

$$
\to \frac{[(x' - y')_\perp^2 x_+ y_- + x'_+ y'_-(x - y)_\perp^2 + x_+ y'_-(x' - y)_\perp^2 + x'_+ y_-(x - y')_\perp^2]^2}{(x - x')^2(y - y')^2 x_+ x'_+ y_- y'_-}
$$
4-dim conformal group versus $SL(2, C)$

Regge limit: $x_+ \to \rho x_+, \ x'_+ \to \rho' x'_+, \ y_- \to \rho' y_-, \ y'_- \to \rho' y_- \quad \rho, \rho' \to \infty$

Regge limit symmetry: 2-dim conformal group $SL(2, C)$ formed from $P_1, P_2, M^{12}, D, K_1$ and $K_2$ which leave the plane $(0, 0, z_\perp)$ invariant.
Pomeron in a conformal theory

\[ A(x, y; x', y') \xrightarrow{s \to \infty} \frac{i}{2} \int d\nu \ f_+(\omega(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2} \]

L. Cornalba (2007)

\[ f_+(\omega) = \frac{e^{i\pi\omega} - 1}{\sin \pi\omega} \text{ - signature factor} \]

\( \Omega(r, \nu) \) - solution of the eqn \((\Box_{H_3} + \nu^2 + 1)\Omega(r, \nu) = 0. \)

Explicit form:

\[ \Omega(r, \nu) = \frac{\nu^2}{\pi^3} \int d^2 z \left( \frac{\kappa^2}{(2\kappa \cdot \zeta)^2} \right)^{\frac{1}{2}+i\nu} \left( \frac{\kappa'^2}{(2\kappa' \cdot \zeta)^2} \right)^{\frac{1}{2}-i\nu} \]

\[ \zeta = p_1 + \frac{z_\perp^2}{s} p_2 + z_\perp, \quad p_1^2 = p_2^2 = 0, \quad 2(p_1, p_2) = s \]

\[ \kappa = \frac{1}{2x_+}(p_1 - \frac{x^2}{s} p_2 + x_\perp) - \frac{1}{2y_+}(p_1 - \frac{y^2}{s} p_2 + y_\perp), \quad \kappa^2 \kappa'^2 = \frac{1}{R} \]

\[ \kappa' = \frac{1}{2x'_-}(p_1 - \frac{x'^2}{s} p_2 + x'_\perp) - \frac{1}{2y'_-}(p_1 - \frac{y'^2}{s} p_2 + y'_\perp), \quad 4(\kappa \cdot \kappa')^2 = \frac{r}{R} \]

The dynamics is described by \( \omega(\lambda, \nu) \) and \( F(\lambda, \nu) \).
Pomeron in the conformal theory

\[ A(x, y; x', y') \xrightarrow{s \to \infty} \frac{i}{2} \int d\nu f_+ (\omega(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2} \]

Pomeron intercept \( \omega(\nu, \lambda) \) is known in two limits:

1. \( \lambda \to 0 : \quad \omega(\nu, \lambda) = \frac{\lambda}{\pi} \chi(\nu) + \lambda^2 \omega_1(\nu) + ... \)

\[ \chi(\nu) = 2\psi(1) - \psi\left(\frac{1}{2} + i\nu\right) - \psi\left(\frac{1}{2} - i\nu\right) \] - BFKL intercept,

\( \omega_1(\nu) \) - NLO BFKL intercept \quad Lipatov, Kotikov (2000)

2. \( \lambda \to \infty : \quad \text{AdS/CFT} \quad \Rightarrow \quad \omega(\nu, \lambda) = 2 - \frac{\nu^2 + 4}{2\sqrt{\lambda}} + ... \)

2 = graviton spin, next term -  \quad Brower, Polchinski, Strassler, Tan (2006)

Cornalba, Costa, Penedones (2007)
The function $F(\nu, \lambda)$ in two limits:

1. $\lambda \to 0$:
   \[ F(\nu, \lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + \ldots \]
   \[ F_0(\nu) = \frac{\pi \sinh \pi \nu}{4\nu \cosh^3 \pi \nu} \]
   Coralba, Costa, Penedones (2007)
   
   $F_1(\nu) = $ see below

2. $\lambda \to \infty$:
   \[ \text{AdS/CFT} \Rightarrow \omega(\nu, \lambda) = \pi^3 \nu^2 \frac{1 + \nu^2}{\sinh^2 \pi \nu} + \ldots \]
   Coralba (2007)
Pomeron in the conformal theory

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Cornalba, Costa, Penedones (2007)

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L.Cornalba (2007)

Leningrad (LNPI) school:
A good paper is a paper where high-order Feynman diagrams are calculated.
Pomeron in the conformal theory

\[ A(x, y; x', y') \xrightarrow{s \to \infty} \frac{i}{2} \int d\nu f_+(\omega(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2} \]

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   \[ F_0(\nu) = \frac{\pi \sinh \frac{\pi \nu}{4\nu \cosh^2 \frac{\pi \nu}{2}}}{4\nu \cosh^2 \frac{\pi \nu}{2}} \]
   Cornalba, Costa, Penedones (2007)

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   L. Cornalba (2007)

Leningrad (LNPI) school:
A good paper is a paper where high-order Feynman diagrams are calculated.

Moscow (ITEP) school: Do not calculate Feynman diagrams - think instead!
Pomeron in the conformal theory

\[ A(x, y; x', y') \xrightarrow{s \to \infty} \frac{i}{2} \int d\nu \, f_+(\omega(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2} \]

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   L.Cornalba (2007)

We calculate \( F_1(\nu) \) (and confirm \( \omega_1(\nu) \)) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)
Light-cone expansion and DGLAP evolution in the NLO

\[ \mu^2 \text{ - factorization scale (normalization point)} \]

\[ k_{\perp}^2 > \mu^2 \text{ - coefficient functions} \]

\[ k_{\perp}^2 < \mu^2 \text{ - matrix elements of light-ray operators (normalized at } \mu^2) \]
Light-cone expansion and DGLAP evolution in the NLO

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OPE in light-ray operators

\[
T\{j_\mu(x)j_\nu(y)\} = \frac{x_\xi}{2\pi^2x^4} \left[ 1 + \frac{\alpha_s}{\pi} (\ln x^2 \mu^2 + C) \right] \bar{\psi}(x) \gamma_\mu \gamma_\xi \gamma_\nu [x, y] \psi(y) + O(\frac{1}{x^2})
\]

\[ [x, y] \equiv Pe^{i\xi} \int_0^1 du (x-y)^\mu A_\mu (ux + (1-u)y) \] - gauge link
Light-cone expansion and DGLAP evolution in the NLO

\[ k_\perp^2 > \mu^2 \quad \Rightarrow \quad k_\perp^2 < \mu^2 \]

\( \mu^2 \) - factorization scale (normalization point)

\( k_\perp^2 > \mu^2 \) - coefficient functions

\( k_\perp^2 < \mu^2 \) - matrix elements of light-ray operators (normalized at \( \mu^2 \))

Renorm-group equation for light-ray operators \( \Rightarrow \) DGLAP evolution of parton densities

\[ (x - y)^2 = 0 \]

\[ \mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x, y]\psi(y) = K_{\text{LO}} \bar{\psi}(x)[x, y]\psi(y) + \alpha_s K_{\text{NLO}} \bar{\psi}(x)[x, y]\psi(y) \]
Expansion of the amplitude in color dipoles in the NLO

The high-energy operator expansion is

\[
T\{\hat{O}(x)\hat{O}(y)\} = \int d^2z_1 d^2z_2 \ I^{LO}(z_1, z_2) \text{Tr}\{\hat{U}_\eta z_1 \hat{U}_\eta^\dagger z_2\}
\]

\[
+ \int d^2z_1 d^2z_2 d^2z_3 \ I^{NLO}(z_1, z_2, z_3) [\frac{1}{N_c} \text{Tr}\{T^n \hat{U}_\eta z_1 \hat{U}_\eta^\dagger z_3 T^n \hat{U}_\eta z_3 \hat{U}_\eta^\dagger z_2\} - \text{Tr}\{\hat{U}_\eta z_1 \hat{U}_\eta^\dagger z_2\}]
\]

In the leading order - conf. invariant impact factor

\[
I_{LO} = \frac{x_+^{-2} y_+^{-2}}{\pi^2 Z_1^2 Z_2^2}, \quad Z_i \equiv \frac{(x - z_i)_+^2}{x_+} - \frac{(y - z_i)_+^2}{y_+}
\]
Expansion of the amplitude in color dipoles in the NLO

\[ U_\eta^x = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} du \, p_1^\mu A_\mu^\eta (u p_1 + x_\perp) \right] \]

\[ A_\mu^\eta (x) = \int \frac{d^4 k}{(2\pi)^4} \theta (e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu (k) \]

η - rapidity factorization scale

Rapidity \( Y > \eta \) - coefficient function ("impact factor")
Rapidity \( Y < \eta \) - matrix elements of (light-like) Wilson lines with rapidity divergence cut by \( \eta \)
Each path is weighted with the gauge factor $P e^{i g} \int dx_\mu A^\mu$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
Spectator frame: propagation in the shock-wave background.

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$[x \rightarrow z$: free propagation$]$
Each path is weighted with the gauge factor $P e^{i g \int d\nu A_{\mu}}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$[x \rightarrow z$: free propagation$] \times$

$[U^{ab}(z_{\perp})$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$] \times$
Each path is weighted with the gauge factor $P e^{i g \int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$$[x \rightarrow z: \text{free propagation}] \times [U^{ab}(z_\perp) - \text{instantaneous interaction with the } \eta < \eta_2 \text{ shock wave}] \times [z \rightarrow y: \text{free propagation}]$$
The NLO impact factor is not Möbius invariant ⇒ the color dipole with the cutoff \( \eta \) is not invariant.

However, if we define a composite operator \((a - \text{analog of } \mu^{-2} \text{ for usual OPE})\)

\[
\begin{align*}
[\text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\}]^{\text{conf}} &= \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\} \\
+ \frac{\lambda}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\text{Tr}\{T^n \hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_3} T^n \hat{U}^\eta_{z_3} \hat{U}^{\dagger \eta}_{z_2}\} - N_c \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\}] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\lambda^2)
\end{align*}
\]

the impact factor becomes conformal in the NLO.
Operator expansion in conformal dipoles

\[
T\{\hat{O}(x)\hat{O}(y)\} = \int d^2z_1 d^2z_2 \ I^{\text{LO}}(z_1, z_2) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}^{\text{conf}}
+ \int d^2z_1 d^2z_2 d^2z_3 \ I^{\text{NLO}}(z_1, z_2, z_3) \left[ \frac{1}{N_c} \text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}\right]
\]

\[
I^{\text{NLO}} = - I^{\text{LO}} \frac{\lambda}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \ln \frac{z_{12}^2 e^{2\eta a s^2}}{z_{13}^2 z_{23}^2} Z_3^2 - i\pi + 2C \right]
\]

The new NLO impact factor is conformally invariant
\[
\Rightarrow \quad \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}^{\text{conf}} \text{ is Möbius invariant}
\]

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.
To get the evolution equation, consider the dipole with the rapidities up to $\eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to $\eta_2$).
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$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes$$
Leading order: BK equation

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \ldots \quad \Rightarrow
\]

\[
\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}
\]

\[
U_{zx}^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow \quad (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s (\eta_1 - \eta_2) (U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}
\]

⇒ Evolution equation is non-linear
Non linear evolution equation

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_{\perp})\hat{U}^\dagger(y_{\perp})\} \]
Non linear evolution equation

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BK equation

\[ \frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]

Non-linear evolution equation

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LLA for DIS in pQCD $\Rightarrow$ BFKL

(LLA: $\alpha_s \ll 1$, $\alpha_s \eta \sim 1$)
Non-linear evolution equation

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_{\perp}) \hat{U}^\dagger(y_{\perp})\} \]

BK equation

\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z (x - y)^2}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\}
\]


LLA for DIS in pQCD \(\Rightarrow\) BFKL

(LLA: \(\alpha_s \ll 1, \alpha_s \eta \sim 1\))

LLA for DIS in sQCD \(\Rightarrow\) BK eqn

(LLA: \(\alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s A^{1/3} \sim 1\))

(s for semiclassical)
Conformal invariance of the dipole kernel

Formally, a light-like Wilson line

\[ [\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\} \]

is invariant under inversion (with respect to the point with \( x^- = 0 \)).
Conformal invariance of the dipole kernel

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Indeed,

\((x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow\) after the inversion \( x_\perp \rightarrow x_\perp / x_\perp^2 \) and \( x^+ \rightarrow x^+ / x_\perp^2 \)
Conformal invariance of the dipole kernel

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\[
[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \rightarrow \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} \frac{dx^+}{x_\perp^2} A_+\left(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\right) \right\} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]
\]
Conformal invariance of the dipole kernel

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[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}
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Indeed,

\[(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow \text{after the inversion } x_\perp \to x_\perp/x_\perp^2 \text{ and } x^+ \to x^+/x_\perp^2 \Rightarrow\]

\[
[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \to \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} \frac{dx^+}{x_\perp^2} A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}) \right\} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]
\]

⇒ The dipole kernel is invariant under the inversion \( V(x_\perp) = U(x_\perp/x_\perp^2) \)

\[
\frac{d}{d\eta} \text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^4} \frac{(x - y)^2 z^4}{(x - z)^2 (z - y)^2} \left[ \text{Tr}\{V_x V_z^\dagger\} \text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\} \right]
\]
Conformal invariance of the dipole kernel

**SL(2,C) for Wilson lines**

\[
\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)
\]

\[
[\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,
\]

\[
[\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z})
\]

\[
z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z, \bar{z})
\]
Conformal invariance of the dipole kernel

$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$

$[\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$

$[\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z\partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z})$

$z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z, \bar{z})$

Conformal invariance of the evolution kernel

$$\frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2} \int dz \ K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\} \text{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \begin{bmatrix} x^2 \partial_x + y^2 \partial_y + z^2 \partial_z \end{bmatrix} K(x, y, z) = 0$$
Conformal invariance of the dipole kernel

**SL(2,C) for Wilson lines**

\[ \hat{S}_- \equiv \frac{i}{2} (K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2} (D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2} (P^1 - iP^2) \]

\[ [\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2} [\hat{S}_+, \hat{S}_-] = \hat{S}_0, \]

\[ [\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z}) \]

\[ z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z, \bar{z}) \]

Conformal invariance of the evolution kernel

\[ \frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2} \int dz \ K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}]\text{Tr}\{U_x U_y^\dagger\}] \]

\[ \Rightarrow \left[ x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right] K(x, y, z) = 0 \]

In the leading order - OK. In the NLO - ?
Non-linear evolution equation in the NLO

\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} =
\int \frac{d^2 z}{2\pi^2} \left( \alpha_s \frac{(x - y)^2}{(x - z)^2(z - y)^2} + \alpha_s^2 K_{NLO}(x, y, z) \right) \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_z U_y^\dagger\} \right] +
\alpha_s^2 \int d^2 z d^2 z' \left( K_4(x, y, z, z') \{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x, y, z, z') \{U_x, U_{z'}^\dagger, U_{z'}^\dagger, U_z, U_z^\dagger, U_y^\dagger\} \right)
\]

\(K_{NLO}\) is the next-to-leading order correction to the dipole kernel and \(K_4\) and \(K_6\) are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.
Definition of the NLO kernel

Operator equation

\[
\frac{d}{d \eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} = \alpha_s K_{LO} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + \alpha_s^2 K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + O(\alpha_s^3)
\]
Operator equation

\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + O(\alpha_s^3) \]

\[ \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + O(\alpha_s^3) \]
Definition of the NLO kernel

Operator equation

\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3) \]

\[ \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3) \]

We calculate the “matrix element” of the r.h.s. in the shock-wave background

\[ \langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3) \]
Definition of the NLO kernel

Operator equation

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + O(\alpha_s^3)
\]

\[
\alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + O(\alpha_s^3)
\]

We calculate the “matrix element” of the r.h.s. in the shock-wave background

\[
\langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} \rangle + O(\alpha_s^3)
\]

Subtraction of the (LO) contribution

\[
\Rightarrow \left[ \frac{1}{v} \right]_+ \text{ prescription in the integrals over Feynman parameter } v
\]

Typical integral

\[
\int_0^1 dv \frac{1}{(k-p)_\perp^2 v + p_\perp^2 (1-v) \left[ \frac{1}{v} \right]_+} = \frac{1}{p_\perp^2} \ln \frac{(k-p)_\perp^2}{p_\perp^2}
\]
Gluon part of the NLO BK kernel: diagrams

I. Balitsky (JLAB & ODU)

High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order

Shifmania

15 May 09

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Diagrams for 1→3 dipoles transition
Diagrams for $1 \rightarrow 3$ dipoles transition

(XXXI)  

(XXXII)  

(XXXIII)  

(XXXIV)
"Running coupling" diagrams

I. Balitsky (JLAB & ODU)

High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order

Shifmania
15 May 09
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1 → 2 dipole transition diagrams

(a) x_1 a x_2 b k' k d c
(b) x_1 a x_2 b k' k d c
(c) x_1 a x_2 b k' k d c
(d) x_1 a x_2 b k' k d c
(e) x_1 a x_2 b k' k d c
(f) x_1 a x_2 b k' k d c
(g) x_1 a x_2 b k' k d c
(h) x_1 a x_2 b k' k d c
(i) x_1 a x_2 b k' k d c
(j) x_1 a x_2 b k' k d c
Scalar and gluino loops

I. Balitsky (JLAB & ODU)  High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order
Evolution equation for color dipole in $\mathcal{N} = 4$ (G. Chirilli and I.B.)

\[
\frac{d}{d\eta} \text{Tr}\{ \hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta} \} = \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[ \frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
\times \left[ \text{Tr}\{ T^a \hat{U}_{z_1}^\eta \hat{T}^a_{z_3} T^a \hat{U}_{z_2}^{\dagger \eta} \} - N_c \text{Tr}\{ \hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta} \} \right] \\
- \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
\times \text{Tr}\{ [T^a, T^b] \hat{U}_{z_1}^\eta T^a_{z_1} T^b_{z_2} \hat{U}_{z_2}^{\dagger \eta} + T^b T^a \hat{U}_{z_1}^\eta [T^b', T^a'] \hat{U}_{z_2}^{\dagger \eta} \} (\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'}
\]

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.
\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\}
= \frac{\alpha_s}{\pi^2} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[ \frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\}
\times \left[ \text{Tr}\{T^a \hat{U}_z^\eta \hat{U}_z^{\dagger \eta} T^a \hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\} - N_c \text{Tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\} \right]
- \frac{\alpha_s^2}{4\pi^4} \int d^2z_3 d^2z_4 \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[ 1 + \frac{z_{13}^2 z_{24}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}
\times \text{Tr}\{[T^a, T^b] \hat{U}_z^\eta T^{a'} T^{b'} \hat{U}_z^{\dagger \eta} + T^b T^a \hat{U}_z^\eta [T^{b'}, T^{a'}] \hat{U}_z^{\dagger \eta}\} (\hat{U}_z^\eta)^{aa'} (\hat{U}_z^\eta - \hat{U}_z^{\dagger \eta})^{bb'}
\]

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff \(\alpha < \sigma = e^{2\eta}\) in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant.
Evolution equation for composite conformal dipoles in $\mathcal{N} = 4$

$$\frac{d}{d\eta} \left[ \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\} \right]^{\text{conf}}$$

$$= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 - \frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} \right] \left[ \text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger \eta} T^a \hat{U}_{z_3}^{\dagger \eta} \hat{U}_{z_2}^{\dagger \eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\} \right]^{\text{conf}}$$

$$- \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2}{z_{13}^2 z_{24}^2 z_{34}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{23}^2 z_{24}^2 z_{34}^2} + \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\}$$

$$\times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^a \hat{U}_{z_2}^{\dagger \eta} + T^b T^a \hat{U}_{z_2}^{\dagger \eta} [T^b, T^a'] \hat{U}_{z_2}^{\dagger \eta} \} \left[ (\hat{U}_{z_3}^\eta)^{a a'} (\hat{U}_{z_4}^\eta)^{b b'} - (z_4 \to z_3) \right]$$

Now Möbius invariant!
To find $A(x, y; x', y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$
\hat{U}_n(x, y) = 1 - \frac{1}{N_c^2 - 1} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\}
$$

Conformal dipole operator in the BFKL approximation

$$
\hat{U}_{\text{conf}}^n(z_1, z_2) = \hat{U}_n(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2} \int d^2z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} [\hat{U}_n(z_1, z_3) + \hat{U}_n(z_2, z_3) - \hat{U}_n(z_1, z_2)]
$$
To find \( A(x, y; x', y') \) we need the linearized (NLO BFKL) equation. With two-gluon accuracy

\[
\hat{U}^\eta(x, y) = 1 - \frac{1}{N_c^2 - 1} \text{Tr}\{\hat{U}_x^\eta \hat{U}_y^{\eta\dagger}\}
\]

Conformal dipole operator in the BFKL approximation

\[
\hat{U}_\text{conf}^\eta(z_1, z_2) = \hat{U}^\eta(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2} \int d^2 z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} [\hat{U}^\eta(z_1, z_3) + \hat{U}^\eta(z_2, z_3) - \hat{U}^\eta(z_1, z_2)]
\]

NLO BFKL

\[
\frac{d}{d\eta} \hat{U}_\text{conf}^\eta(z_1, z_2)
\]

\[
= \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 - \frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} \right] [\hat{U}_\text{conf}^\eta(z_1, z_3) + \hat{U}_\text{conf}^\eta(z_2, z_3) - \hat{U}_\text{conf}^\eta(z_1, z_2)]
\]

\[
+ \frac{\alpha_s^2 N_c^2}{8\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{23}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{23}^2} + \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \right\} \hat{U}_\text{conf}^\eta(z_3, z_4)
\]

\[
+ \frac{3\alpha_s^2 N_c^2}{2\pi^3} \zeta(3) \hat{U}_\text{conf}^\eta(z_1, z_2)
\]
Pomeron intercept

Pomeron intercept = the eigenvalue of the NLO BFKL equation

\[ \omega(n, \nu) = \frac{\alpha_s}{\pi} N_c \left[ \chi(n, \frac{1}{2} + i\nu) + \frac{\alpha_s N_c}{4\pi} \delta(n, \frac{1}{2} + i\nu) \right], \]

\[ \delta(n, \gamma) = 6\zeta(3) - \frac{\pi^2}{3} \chi(n, \gamma) - \chi''(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \]

where

\[ \chi(n, \gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1 - \gamma + \frac{n}{2}) \]

\[ \Phi(n, \gamma) = \int_0^1 \frac{dt}{1 + t} t^{\gamma - 1 + \frac{n}{2}} \left\{ \frac{\pi^2}{12} - \frac{1}{2} \psi' \left( \frac{n + 1}{2} \right) - \text{Li}_2(t) - \text{Li}_2(-t) \right. \]

\[ - \left. \left( \psi(n + 1) - \psi(1) + \ln(1 + t) + \sum_{k=1}^{\infty} \frac{(-t)^k}{k + n} \right) \ln t - \sum_{k=1}^{\infty} \frac{t^k}{(k + n)^2} [1 - (-1)^k] \right\} \]
Pomeron intercept

Pomeron intercept = the eigenvalue of the NLO BFKL equation

\[ \omega(n, \nu) = \frac{\alpha_s}{\pi} N_c \left[ \chi(n, \frac{1}{2} + i\nu) + \frac{\alpha_s N_c}{4\pi} \delta(n, \frac{1}{2} + i\nu) \right], \]

\[ \delta(n, \gamma) = 6\zeta(3) - \frac{\pi^2}{3} \chi(n, \gamma) - \chi''(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \]

where

\[ \chi(n, \gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1 - \gamma + \frac{n}{2}) \]

\[ \Phi(n, \gamma) = \int_0^1 \frac{dt}{1 + t} t^{\gamma-1+\frac{n}{2}} \left\{ \frac{\pi^2}{12} - \frac{1}{2} \psi' \left( \frac{n + 1}{2} \right) - \text{Li}_2(t) - \text{Li}_2(-t) \right. \]

\[ - \left( \psi(n + 1) - \psi(1) + \ln(1 + t) + \sum_{k=1}^{\infty} \frac{(-t)^k}{k + n} \right) \ln t - \sum_{k=1}^{\infty} \frac{t^k}{(k + n)^2} \left[ 1 - (-1)^k \right] \]

Agrees with \( j \to 1 \) asymptotics of 3-loop splitting functions

Assembling NLO $F(\nu)$

Impact factor should not scale with energy \( \Rightarrow a = \frac{x+y}{(x-y)^2} \) (analog of $\mu^2 = Q^2$ in DIS)

\[
(x - y)^4 T\{\hat{O}(x)\hat{O}(y)\} = \frac{(x - y)^4}{\pi^2} \int d^2z_1d^2z_2 \frac{(x+y_+)^{-2}}{\mathcal{Z}_1^2 \mathcal{Z}_2^2} \left\{ \hat{U}^{\text{conf}} \right\}
\]

\[
- \frac{\lambda}{4\pi^2} \int \frac{d^2z_3}{z_1^2 z_3^2} \frac{z_{12}^2}{z_3^2} \left[ \ln \frac{x+y+z_{12}^2 e^{2\eta}}{(x-y)^2 z_1^2 z_3^2} - i\pi \right] \left[ \hat{U}^{\text{conf}}(z_1, z_3) + \hat{U}^{\text{conf}}(z_2, z_3) - \hat{U}^{\text{conf}}(z_1, z_2) \right]
\]

The projection onto the conformal eigenfunctions \( \left( \frac{z_{12}^2}{z_1^2 z_2^2} \right)^\gamma \) \( (\gamma = \frac{1}{2} + i\nu) \)
Assembling NLO $F(\nu)$

Impact factor should not scale with energy ⇒ $a = \frac{x+y+}{(x-y)^2}$ (analog of $\mu^2 = Q^2$ in DIS)

$$(x - y)^4 T\{\hat{O}(x)\hat{O}(y)\} = \frac{(x - y)^4}{\pi^2} \int d^2 z_1 d^2 z_2 \frac{(x+y+)^{-2}}{Z_1^2 Z_2^2} \left\{ \hat{U}^{\text{conf}} \right. - \frac{\lambda}{4\pi^2} \int \frac{d^2 z_3 z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \ln \frac{x+y+z_{12}^2 e^{2\eta}}{(x-y)^2 z_{13}^2 z_{23}^2 z_3^2} - i\pi \right] \hat{U}^{\text{conf}}(z_1, z_3) + \hat{U}^{\text{conf}}(z_2, z_3) - \hat{U}^{\text{conf}}(z_1, z_2) \left\}$$

The projection onto the conformal eigenfunctions $\left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^\gamma$ ($\gamma = \frac{1}{2} + iv$)

$$\int d z_1 d z_2 (x - y)^4 T\{\hat{O}(x)\hat{O}(y)\} \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^\gamma = \left( \frac{\kappa^2}{(2\kappa \cdot \zeta_0)^2} \right)^\gamma [I^A_{\text{LO}}(\gamma) + I^A_{\text{NLO}}(\gamma)] \hat{U}(z_0, \gamma),$$

$$\hat{U}(z_0, \gamma) = \int d^2 z_1 d^2 z_2 \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^\gamma \hat{U}(z_1, z_2)$$

$$I^A_{\text{LO}}(\gamma) = \frac{\Gamma^2(1 - \gamma)}{\Gamma(2 - 2\gamma)} \Gamma(1 + \gamma) \Gamma(2 - \gamma)$$

$$I^A_{\text{NLO}}(\gamma) = \frac{\lambda}{16\pi^2} I^A_{\text{LO}} \left[ \chi^2(\gamma) - \psi'(\gamma) - \psi'(1 - \gamma) - 2 + \frac{\pi^2}{3} - F(\gamma) - F(1 - \gamma) \right]$$
Assembling NLO $F(\nu)$

Similarly

$$
\int dz_1 dz_2 (x' - y')^4 T\{\hat{O}(x')\hat{O}(y')\} \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^{1-\gamma} = \left(\frac{\kappa^2}{(2\kappa \cdot \zeta_0)^2}\right)^{1-\gamma} [I^A_{\text{LO}}(\gamma) + I^A_{\text{NLO}}(\gamma)] \hat{V}(z_0, \gamma)
$$

$$
\hat{V}(z_0, \gamma) = \int d^2 z_1 d^2 z_2 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^{\gamma} \hat{V}(z_1, z_2)
$$

$$
I^B_{\text{LO}}(\gamma) = \frac{\Gamma^2(1 + \gamma)}{\Gamma(2 + 2\gamma)} \Gamma(1 + \gamma) \Gamma(2 - \gamma)
$$

$$
I^B_{\text{NLO}}(\gamma) = \frac{\lambda}{16\pi^2} I^B_{\text{LO}} \left[\chi^2(\gamma) - \psi'(\gamma) - \psi'(1 - \gamma) - 2 + \frac{\pi^2}{3} - F(\gamma) - F(1 - \gamma)\right]
$$
The last ingredient is the amplitude of scattering of two conformal dipoles 
\( \gamma \equiv \frac{1}{2} + i\nu \)

\[
\langle \hat{U}(z_0, \gamma) \hat{V}(z'_0, \gamma) \rangle = \delta(\nu - \nu')\delta(z - z') \ R^{\frac{1}{2} \omega(\nu)} \ [A_{\text{LO}}(\gamma) + A_{\text{NLO}}(\gamma)]
\]

\[
A_{\text{LO}}(\gamma) = \frac{\Gamma(-\gamma)\Gamma(\gamma - 1)}{\Gamma(1 + \gamma)\Gamma(2 - \gamma)}, \quad A_{\text{NLO}}(\gamma) = -\frac{\lambda}{4\pi^2}A_{\text{LO}} \left[ \frac{\chi(\gamma)}{\gamma(1 - \gamma)} - \frac{\pi^2}{3} \right]
\]

The result is \( F(\nu) = F_{\text{LO}}(\nu) + \lambda F_{\text{NLO}}(\nu) + O(\lambda^2) \)

\[
F_{\text{LO}}(\nu) = I_{\text{LO}}^A(\nu)A_{\text{LO}}(\nu)I_{\text{LO}}^B(\nu),
\]
\[
F_{\text{NLO}}(\nu) = I_{\text{NLO}}^A(\nu)A_{\text{LO}}(\nu)I_{\text{LO}}^B + I_{\text{LO}}^A(\nu)A_{\text{NLO}}(\nu)I_{\text{LO}}^B + I_{\text{NLO}}^A(\nu)A_{\text{LO}}(\nu)I_{\text{NLO}}^B(\nu)
\]
The NLO evolution of composite “conformal” dipoles in QCD is given by

\[
\frac{d}{d\eta} \text{tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2 z \left( \left[\text{tr}\{U_x U_z^\dagger\}\text{tr}\{U_z U_y^\dagger\} - N_c \text{tr}\{U_x U_y^\dagger\}\right] 
\times \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( b \ln(x - y)^2 \mu^2 + b \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] 
+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z'}{(z - z')^4} \left\{ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x - y)^2(z - z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{X'^2 Y^2} \right\} 
\times \left[ \text{tr}\{U_x U_z^\dagger\}\text{tr}\{U_z U_{z'}^\dagger\}\left\{U_{z'} U_y^\dagger\right\} - \text{tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z' \rightarrow z) \right] 
+ \frac{(x - y)^2(z - z')^2}{X^2 Y'^2} \left[ 2 \ln \frac{(x - y)^2(z - z')^2}{X'^2 Y^2} + \left( 1 + \frac{(x - y)^2(z - z')^2}{X^2 Y'^2 - X'^2 Y^2} \right) \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right] 
\times \left[ \text{tr}\{U_x U_z^\dagger\}\text{tr}\{U_z U_{z'}^\dagger\}\left\{U_{z'} U_y^\dagger\right\} - \text{tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z' \rightarrow z) \right] \right\} \right)
\]

\[ b = \frac{11}{3} N_c - \frac{2}{3} n_f, \quad X \equiv x - z, \quad Y \equiv y - z, \quad X' \equiv x - z', \quad Y \equiv y - z' \]

**K_{NLO BK}** = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N} = 4$) part

Linearized $K_{NLO BK}$ reproduces the known result for the forward NLO BFKL kernel.
High-energy operator expansion in Wilson lines works at the NLO level.
Conclusions

- High-energy operator expansion in Wilson lines works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL eigenvalues.
- The correlation function of four $Z^2$ operators is calculated at the NLO order.
Happy Birthday!