AN INVARIANCE PROPERTY FOR THE PROPAGATION OF HEAT AND SHEAR WAVES

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Abstract It is shown that the propagation of both shear waves in viscoelastic fluids and of heat waves has the same remarkable property; namely, the solution at an x, t point in one material is the same as the solution at an arbitrary shifted point in the x, t plane in a different material with a scaled relaxation function.

1. Introduction

In this note we study the propagation of heat when the temperature at the boundary is suddenly increased and Nunziato's constitutive equation [1]

$$C\frac{\partial\theta}{\partial t}+\int_0^{+\infty}E(s)\frac{\partial\theta}{\partial t}(\mathbf{x},t-s)\,ds=k\nabla^2\theta+\int_0^{+\infty}G(s)\nabla^2\theta(\mathbf{x},t-s)\,ds \tag{1}$$

is assumed to model the propagation of heat and give for this problem an invariance property.

In (1) θ is the temperature, C is the heat capacity, k is the conductivity, $s=t-\tau$ is the lapse time and τ is the past time, $E(s)$ and $G(s)$ are smooth, positive decreasing (to zero) functions, called relaxation functions.

In the last section we also give a similar invariance property for the propagation of shear waves in viscoelastic liquids.

2. The invariance property

The invariance property we are to derive is a generalization of a property found by Joseph, Narain and Riccius [2]. They show that the solution is invariant under a radial shift. Here we find invariance without radial shift. The solution at an x, t point in one material is the same as the solution at an arbitrary shifted point in the x, t plane in a different material with a scaled relaxation function. This can be expressed matematically by

$$\theta[x,t;C,E(s),k,G(s)] = \theta[x_0,t_0;C,(t/t_0)E(ts/t_0),\frac{t/t_0}{(x/x_0)^2}k,(\frac{t/t_0}{x/x_0})^2G(ts/t_0)] \quad . \tag{2}$$

Following Preziosi and Joseph [3] we find that the Laplace's transform of the solution is

$$\overline{\theta}(x,\omega) = \frac{\Theta}{\omega}exp[-x\sqrt{\frac{C+\overline{E}(\omega)}{k+\overline{G}(\omega)}\omega}] \quad , \tag{3}$$

$$\overline{G}(\omega) = \int_0^{+\infty} e^{-\omega t}G(t)\,dt \quad , \quad \overline{E}(\omega) = \int_0^{+\infty} e^{-\omega t}E(t)\,dt \quad .$$

The explicit solution of the problem is not needed here but is given in [4].

We next look at the point $\frac{x}{\phi_x}$ and $\frac{t}{\phi_t}$ with ϕ_x and ϕ_t positive real numbers for a material with conductivity

$$k_\phi = \frac{\phi_t}{\phi_x^2}k$$

and relaxation functions

$$E_\phi(s) = \phi_t E(\phi_t s) \quad , \quad G_\phi(s) = \frac{\phi_t^2}{\phi_x^2}G(\phi_t s) \quad .$$

The inverse transform of this shifted problem is

$$\theta[\frac{x}{\phi_x},\frac{t}{\phi_t};C,E_\phi(s),k_\phi,G_\phi(s)]=\frac{\Theta}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{d\omega}{\omega}exp[\omega\frac{t}{\phi_t}-\frac{x}{\phi_x}\sqrt{\frac{C+\overline{E}_\phi(\omega)}{k_\phi+\overline{G}_\phi(\omega)}\omega}] \quad .$$

The change of variable $\hat{\omega}=\omega/\phi_t$ allows us to write

$$\theta[\frac{x}{\phi_x},\frac{t}{\phi_t};C,E_\phi(s),k_\phi,G_\phi(s)]=\frac{\Theta}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{d\hat{\omega}}{\hat{\omega}}exp[\hat{\omega}t-x\sqrt{\frac{C+\overline{E}_\phi(\hat{\omega}\phi_t)}{\frac{\phi_x^2}{\phi_t}(k_\phi+\overline{G}_\phi(\hat{\omega}\phi_t))}\hat{\omega}}]$$

$$=u[x,t;C,\mathcal{L}^{-1}\{\overline{E}_\phi(\hat{\omega}\phi_t)\},\frac{\phi_x^2}{\phi_t}k_\phi,\mathcal{L}^{-1}\{\frac{\phi_x^2}{\phi_t}\overline{G}_\phi(\hat{\omega}\phi_t)\}]$$

But

$$\frac{\phi_x^2}{\phi_t}\overline{G}_\phi(\hat{\omega}\phi_t)=\int_0^{+\infty}\frac{\phi_x^2}{\phi_t}e^{-\hat{\omega}\phi_t s}G_\phi(s)\,ds=\int_0^{+\infty}e^{-\hat{\omega}\hat{s}}G_\phi(\hat{s}/\phi_t)\frac{\phi_x^2}{\phi_t^2}\,d\hat{s}=\mathcal{L}\{\frac{\phi_x^2}{\phi_t^2}G_\phi(\frac{s}{\phi_t})\} \quad ,$$

and at the same way $E_\phi(\hat{\omega}\phi_t)=\mathcal{L}\{\frac{1}{\phi_t}E_\phi(\frac{s}{\phi_t})\}$.

Thus we obtain

$$\theta[\frac{x}{\phi_x},\frac{t}{\phi_t};C,E_\phi(s),k_\phi,G_\phi(s)]=\theta[x,t;C,E(s),k,G(s)] \tag{4}$$

whenever

$$k_\phi=\frac{\phi_t}{\phi_x^2}k \quad , \quad E_\phi(s)=\phi_t E(\phi_t s) \quad , \quad G_\phi(s)=\frac{\phi_t^2}{\phi_x^2}G(\phi_t s) \quad . \tag{5}$$

Equation (2) now follows when $\phi_t=t/t_0$ and $\phi_x=x/x_0$.

3. Origin of effective conductivity, effective capacity, elastic conductivity and elastic capacity

From (5) it is evident that only the time scale enters the mechanism for the generation of an effective conductivity and effective capacity (associated with the relaxation of fast modes) and effective relaxation functions (associated with slower modes). This means that the modes with relaxation times much smaller than the time of observation of the phenomenon have decayed independently of the x-coordinate. We may introduce the effective relaxation functions $E_\mu(s)$, $G_\mu(s)$

$$E(s) = E_0(s) + E_\mu(s) \quad , \quad G(s) = G_0(s) + G_\mu(s) \tag{6}$$

where

$$E_0(s) \approx 0 \quad , \quad G_0(s) \approx 0 \quad , \quad s > \lambda_0$$

$$E_\mu(\lambda_0) \approx E_\mu(0) \quad , \quad G_\mu(\lambda_0) \approx G_\mu(0)$$

and $k = 0$. A relaxation function with these properties is drawn in fig. 1. The effective capacity and effective conductivity are then defined respectively by

$$C_0 = C + \int_0^{+\infty} E_0(s)\, ds \approx C + \int_0^{\lambda_0} E_0(s)\, ds \quad , \quad k_0 = \int_0^{+\infty} G_0(s)\, ds \approx \int_0^{\lambda_0} G_0(s)\, ds \quad . \tag{7}$$

Then, using (2) with $x_0 = x$ and $t_0 = 1$

$$\theta[x, t; C, E(s), 0, G(s)] = \theta[x, 1; C, tE(ts), 0, t^2 G(ts)] \quad .$$

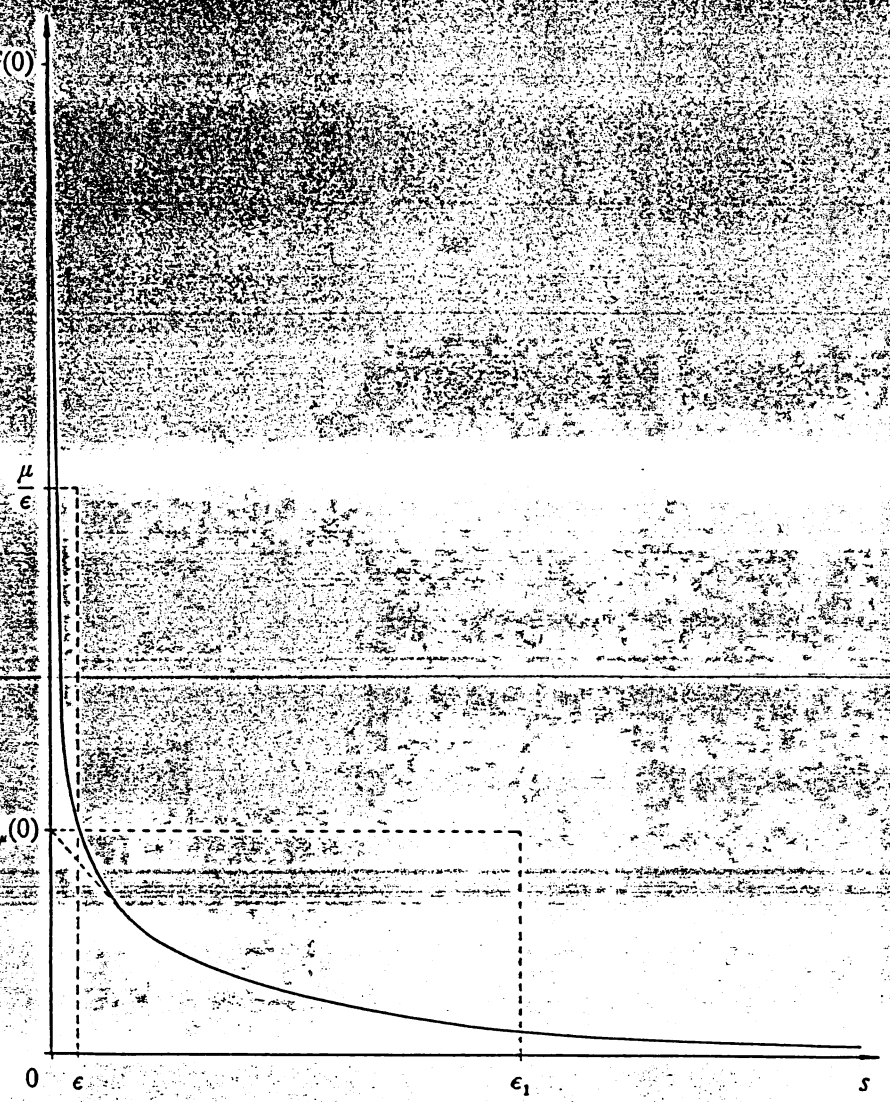

Figure 1. : Form of a relaxation function G(s) combining fast relaxing glassy response with an effective modulus $G_\mu(o)$.

It follows from (6) that

$$I = \int_0^{+\infty} e^{-\omega s} t^2 G_0(ts)\, ds = t \int_0^{+\infty} e^{-\omega \hat{s}/t} G_0(\hat{s})\, d\hat{s} \approx t \int_0^{\lambda_0} e^{-\omega \hat{s}/t} G_0(\hat{s})\, d\hat{s} \quad .$$

If now $t \gg \lambda_0$, $e^{-\omega \hat{s}/t} \approx 1$ in $[0, \lambda_0]$ and, using (7), we find that $I \approx tk_0$. In the same way $\mathcal{L}\{tE_0(ts)\} \approx C_0$. Thus we have

$$\theta[x,t;C,E_0(s)+E_\mu(s),0,G_0(s)+G_\mu(s)] \approx \theta[x,1;C_0,tE_\mu(ts),tk_0,t^2G_\mu(ts)] = \theta[x,t;C_0,E_\mu(s),k_0,G_\mu(s)] \quad .$$

This explains, for example, how an effective conductivity may arise from the relaxation of fast decaying modes.

It's usually believed [4] that a better model to describe heat conduction is Gurtin and Pipkin's equation [5]. In fact the presence of an effective conductivity in the theory of heat trasmission has exactly the same conceptual problem as pure diffusion; any disturbance is felt istantaneously throughout the material. An effective conductivity $k_0 \neq 0$ can be however a useful concept, even if, strictly speaking, $k_0 = 0$. In fact thermal energy is transported in solids by two different mechanism: by quantized electronic excitation, which are called free electrons, and by the quanta of lattice vibrations, which are called phonons. These quanta undergo collisions of dissipative nature, giving rise to thermal resistance in the medium. The relaxation time τ_0 is associated with the average communication time between these collisions for the commencement of resistive flow.

The magnitude of the relaxation time ranges between 10^{-11} for phonon–electron interactions to 10^{-13} for phonon–phonon and free electrons interactions [6]. We can ask: "What is the effect of the modes that have decayed at $t > 10^{-13}$sec on the subsequent transfer of heat?" The answer is that these decayed modes continue to play a role, producing diffusion with an effective conductivity k_0 associated with

the (possibly small) area under $G_0(s)$, $0 < s < \bar{s}$ where $\bar{s} = O(10^{-13}$ sec).

4. Invariance for the propagation of shear waves

We now consider a viscoelastic fluid for which the excess stress τ is determined by

$$\tau = 2\mu \mathbf{D}[\mathbf{u}(\mathbf{x}, t)] + \int_0^{+\infty} G(s)\mathbf{D}[\mathbf{u}(\mathbf{x}, t-s)]\, ds \quad , \tag{8}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity, $\mathbf{D}[\mathbf{u}] = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the rate of strain and μ is a positive constant called the Newtonian viscosity. The integral expression represents effects of relaxing elasticity. Equation (8) is a general linearization of every kind of constitutive model for homogeneous, isotropic fluids which depend on the history of the first spatial gradient of the deformation. It says that such fluids are completely characterized in motions which perturb rest or, more generally, which perturb rigid motions by a viscosity and a relaxation function.

Preziosi and Joseph [3] have studied for this constitutive equation the propagation of shear waves in a fluid occupying a semi–infinite region above a plate which undergoes a step increase of velocity from rest. The Laplace transform of their solution is a particular case of eq. (3) with $E(s) = 0$ and the obvious changes. Therefore the invariance property and the ideas on effective viscosity and effective rigidity still hold.

It can be argued that at very short times the molecular structure is frozen and the liquid does not flow. The wave istantaneously sees a material that looks like an organic glass. Then the stress associated with molecular forces between small molecules relaxes. The relaxation of larger structure, like those associated with macromolecules, is much slower and represented through an effective modulus $G_\mu(0)$. Eventually all elastic response decays and the fluid behaves diffusively with viscosity given through the area under the relaxation function $G(s)$.

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