On failure of the complementing condition and nonuniqueness in linear elastostatics

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Introduction

Consider a homogeneous, isotropic body composed of a compressible linearly elastic material and assume that the body is at equilibrium in a state of plane strain. The traction problem for such a body (in the absence of body forces and surface tractions) consists of finding a displacement $\mathbf{u} = (u_1, u_2)$ that satisfies (cf., e.g., Gurtin [4])

$$\mu \Delta \mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} = \mathbf{0} \text{ in } R, \tag{1}$$

$$[\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \lambda(\operatorname{div} \mathbf{u})\mathbf{I}]\mathbf{n} = \mathbf{0} \text{ on } \partial R. \tag{2}$$

Here $R \subset \mathbb{R}^2$ is a regular region, $\mathbf{n}$ the outward unit normal to the boundary, ∂R, and μ and λ are the (constant) Lamé moduli.

It is well-known that (1) and (2) have a unique solution, modulo an infinitesimal rigid deformation, provided that[1] $\mu \neq 0$, $\mu + \lambda \neq 0$, and $2\mu + \lambda \neq 0$.

The purpose of this note is to demonstrate that the above mentioned uniqueness result fails when $\mu = -\lambda$. In fact we show that (1) and (2) have an infinite number of linearly independent solutions (in spite of the ellipticity of the equations). The reason for this unusual (for an elliptic system) behavior is that the boundary conditions fail to satisfy the complementing (or

[1] Cf. Knops and Payne [7], pp. 63-65. The result is due to Muskhelisvili [10] and Hill [5].

Lopatinsky-Shapiro) conditions[2,3] (cf., e.g., Agmon, Douglis, and Nirenberg [1] or Morrey [9]).

We note that the problem considered is probably not in itself physically relevant since Poisson's ratio is infinite at the indicated value of the Lamé moduli. In spite of this we think this problem has interest other than as a mathematical peculiarity. The reason is that the complementing condition can fail in problems of physical interest.

In particular, Simpson and Spector [13] have shown that when a rectangular bar composed of Hadamard-Green material is compressed the complementing condition will eventually fail for the superimposed linear problem. In addition, results of Burgess and Levinson [2], Hill and Hutchinson [6], Sawyers and Rivlin [11], Simpson and Spector [12], Wesolowski [15], Wilkes [16], Young [17], and Zhong-Heng [18] and others can be used to show that failure of the complementing condition is not an uncommon occurence when one considers the compression and extension of elastic bars or plates.

We note that most of the above mentioned failures of the complementing condition are probably not physically observable

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[2]Thus the assertion of Marsden and Hughes [8], that the traction boundary condition always satisfies the complementing condition, is incorrect (without additional assumptions). An assumption ruling out infinite Poisson's ratios is (of course) very reasonable.

[3]This failure of the complementing condition at these values of the Lamé moduli was first noticed by Thompson [14].

since they do not occur until after the material has become infinitesimally unstable. However, this does not appear to always be true for bars in tension. In particular, it is possible that surface wrinkling [17] and microshear bands [3] may be precipitated by a failure of the complementing condition.

Nonuniqueness

Let $\mathcal{O} \subset \mathbb{R}^2$ be any open region such that the closure of $\mathcal{R}$ is contained in $\mathcal{O}$. We consider $\mathcal{O}$ to be a subset of the complex plane $\mathbb{C}$. If f is a (complex) analytic function on $\mathcal{O}$ we denote the real and imaginary parts of f by ϕ and ψ, respectively. Thus

$$f(z) = \phi(x,y) + i\,\psi(x,y) \quad \text{on} \quad \mathcal{O},$$

where $z = x + iy$.

Theorem. *Let $\mu = -\lambda$ and suppose that f is any analytic function. Then*

$$\underset{\sim}{u} = (\phi,\psi) \tag{3}$$

is a solution to (1), (2).

Proof. Equation (1) is clear since the real and imaginary parts of an analytic function are harmonic. To establish (2), we note that (2) is equivalent to $\underset{\sim}{S}\,\underset{\sim}{n} = \underset{\sim}{0}$, where

$$\underset{\sim}{S} = \mu \begin{pmatrix} \phi_x - \psi_y & \phi_y + \psi_x \\ \phi_y + \psi_x & \psi_y - \phi_x \end{pmatrix}.$$

It is now clear from the Cauchy-Riemann equations that $\underset{\sim}{S} = \underset{\sim}{0}$. Thus equation (2) is satisfied.

□

Remark. We note that we have found all solutions of equations (1) and (2). For if $\underset{\sim}{u} = (\phi,\psi)$ satisfies (1), (2) then ϕ and ψ are harmonic and hence C^{∞}. It follows that $\Delta \underset{\sim}{S} = \underset{\sim}{0}$ in R. Now it is easy to see that $\underset{\sim}{S}$ is singular if and only if it is the zero matrix. Thus $\underset{\sim}{S}\underset{\sim}{n} = \underset{\sim}{0}$ on ∂R is equivalent to $\underset{\sim}{S} = \underset{\sim}{0}$ on ∂R. By uniqueness for the Dirichlet problem we conclude that $\underset{\sim}{S} = \underset{\sim}{0}$ in R and hence that ϕ and ψ satisfy the Cauchy-Riemann equations in R. Thus $f \equiv \phi + i\psi$ is analytic.

It follows from the above remark that the stress field $\underset{\sim}{S}$ is identically zero independent of the displacement.

Acknowledgement

The authors would like to thank David Kinderlehrer, Larry Payne and Hans Weinberger for their helpful comments and Steve Passman for bringing our attention to the experiments on microshear band formation. This work was supported in part by the Institute for Mathematics and its Applications, University of Minnesota.

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