Preface, Abstract, Keywords

This survey report sketches some basic aspects on numerical analysis of systems of (ordinary) stochastic differential equations (SDEs) at a fairly introductory level, meant to be written for both engineers and mathematicians on the edge towards the 21st century. SDEs are often met in Natural and Environmental Sciences, Financial Markets and Marketing when “uncertainty” in modeling or “erratic behavior” of the environment plays an essential role. Mostly, the analytical solutions are not known, and one has to resort to numerical techniques.

Starting from the well-known Ito Formula (as stochastic counterpart of deterministic chain rule and as the link between continuous and discrete time stochastic dynamical systems) we arrive at stochastic Taylor expansions of functionals of SDEs (Wagner-Platen Expansion). As in deterministic analysis, the latter formulas are essential for the systematic construction of stochastic-numerical methods and the investigation of the local behavior of their approximating trajectories. A comprehensive toolbox for numerical schemes to build a zoo of stochastic numerical methods is presented thereafter.

Starting from these main tools, we present a fairly general main principle for the qualitative analysis of stochastic numerical approximations constructed by the zoo of methods. This principle is carried by the interplay between the key concepts of D-invariance, consistency, contractivity, stability and convergence of stochastic difference methods along adapted discretizations. The entire report relies on that interplay in the $p$-th mean sense and culminates in an adequateness diagram for approximation theory, having a similar meaning as the Lax-Richtmeyer equivalence theorem in deterministic numerical analysis. From there we explain the main convergence results in $p$-th mean, strong $p$-th mean, double $L^p$-sense and weak sense, and how convergence rates carry over between these concepts.

So we arrive at estimates of nonlinear contractivity and stability exponents, the class of balanced implicit methods, preservation of boundary conditions, nonnegativity, asymptotically exact numerical methods, stochastic and mean square stability, A, AN, B, BN-stability of some drift-implicit Theta methods, control on error propagation by $p$-th mean contractive dynamics, weak approximation of H"older-continuous and convex functionals, We also discuss implicit splitting techniques by Petersen-Schurz, Artemiev's Rosenbrock schemes, Doss-approach, Gaines representation of Mil'shtein methods, Ozaki's local linearization techniques, stochastic waveform iterations of Schneider-Schurz, multistep methods of Horvath-Bokor, Denk-
Hersch method for highly oscillating systems, Predictor-Corrector Methods, Talay-Tubaro extrapolations, Newton's asymptotically efficient method, exact methods of Cambanis and Hu, Clark-Cameron result and convergence order bounds. The presented theory is illustrated by numerous examples. Many of these results are, to the best of the author's knowledge, not found in any other monograph.

The whole report is organized in 10 chapters running from the analytical background knowledge up to some practical implementation issues touching possible simplifications, linear stochastic partial differential equations, random number generation, variable step size algorithms, variance reduction and statistical estimations. The report ends in an up-to-date, nearly exhaustive reference list until December 1999.

We are aiming at giving some overview on foundations rather than a comprehensive volume, all details and all proofs. The entire volume is written as simply as possible so that the interested reader can grasp the key ideas and gain a mathematical feeling for the possible range of potential applications. It is also meant to be an updated small reference manual, can guide your course preparations in this area and be understood as an unique supplement to existing literature. The report is based on numerous lecture series which the author has given over the last decade on nearly 5 continents. We also plan to continue this report with a larger special volume in the nearest future. Thus, any comments would be helpful, addressed directly to the author.

**Primary Classifications (1991 Math Subject Classification):** 65U05, 60H10

**Keywords:**

Numerical Analysis, Ordinary Stochastic Differential Equations
Numerical Methods, Main Principles of Stochastic Numerical Analysis
Stochastic Kantorovich-Lax-Richtmeyer Principles
D-Invariance, Consistency, Contractivity, Stability, Convergence
Stochastic Taylor Expansions for SDEs, Stochastic Difference Methods
Stochastic Numerical Integration, Applications of Ito Formula
Qualitative Properties of Discrete Stochastic Dynamical Systems
Asymptotic Behavior of Discrete Stochastic Dynamical Systems
Asymptotically Exact Numerical Methods, Adequate Numerical Methods
Characteristic Exponents, Variable Step Sizes
Applications of Stochastic Numerical Analysis, Implementation Issues