Symmetry of Constant Mean Curvature Hypersurfaces in Hyperbolic Space.

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Introduction

In a recent paper, M. Do Carmo and B. Lawson studied hypersurfaces M of constant mean curvature in hyperbolic space |2|. They use the Alexandrov reflection technique to study M given the asymptotic boundary $\partial_\infty M$. For example, one of their theorems says M is a horosphere when $\partial_\infty M$ reduces to a point. They also prove a Bernstein type theorem for minimal graphs.

In this paper we shall extend their results to other boundary conditions. We prove an embedded M , of constant mean curvature, with $\partial_\infty M$ a subset of a codimension one sphere S , either is invariant by reflection in the hyperbolic hyperplane containing S or is a hypersphere. In the former case M is a "bigraph" over H : it meets any geodesic orthogonal to H either not at all or transversaly in two points (one on each side of H) or tangentially on H .

As a corollary of this, when $\partial_\infty(M)$ equals two points p and q , then M is a hypersurface of revolution about the geodesic joining p to q.

We also consider minimal immersed hypersurfaces M H^n with M regular at ∞. When $\partial_\infty M$ consists of two disjoint spheres S_1, S_2 we prove M is a catenoid or M is the two hyperbolic planes containing S_1 and S_2.

The principal techniques we use to obtain these results are the Alexandrov reflection principle and R. Schoen's adaptation of this to complete minimal surfaces |4|.

I. **Definition and Notation.**

When we refer to plane, distance, line, etc. we always mean the hyperbolic object in H^n. We work with the Poincare model of H^n: H^n is the interior of the unit ball in R^n. The asymptotic boundary of H^n is identified with the boundary of the unit ball and denoted by $S(\infty)$. Given $A \subset H^n$, we denote by $\partial_\infty A$ the set of accumulation points of A in $S(\infty)$ and call it the asymptotic boundary of A . When the context is clear, we will omit the subscript ∞ .

Fix a hyperplane in H^n. We have two natural coordinate systems. First, one can use the geodesics orthogonal to P_0 to give each point coordinates (x,t) where $x \in P_0$ and t is the distance from x to (x,t). This system does not suit our purposes since translation along one geodesic orthogonal to P_0 does not leave invariant another such geodesic. Also this does not extend to a coordinate system on $S(\infty)$.

Instead we shall use the latitude-longitude system. More precisely, choose cordinates in P_0 and let γ be the geodesic orthogonal to an origin $0 \in P_0$. Let γ_t be the 1-parameter group of isometries of H^n which along γ is translation by a distance t and such that the curves $t \to \gamma_t(x)$ are orthogonal to P_0 for each $x \in P_0$ (a positive sense along γ is chosen once and for all). Then each point of H^n has coordinates (x,t) where $x \in P_0$ and $\gamma_t(x) = (x,t)$.

Denote by P_t the plane $\gamma_t(P_0)$. We refer to P_t as a horizontal plane and the curve $t \to \gamma_t(x)$ as the vertical curve through x . Notice that for each s the reflection of H^n through the plane P_s is given by the formula $(x,t) \to (x, 2s - t)$.

Let $S_t = \partial_\infty P_t$. Then the coordinate system (x,t) extends to a coordinate system on $S(\infty)$ where each point (except the two limit points of γ) has a unique coordinate (x,t), $x \in S_0$, $t \in R$. By a Moebius transformation we can send γ to the north pole - south pole geodesic and P_0 to the equatorial plane. Then the coordinates on $S(\infty)$ are the usual latitude-longitude coordinates.

We say $A \subset H^n$ is a graph over P_s if the vertical projection of A to P_s is injective, and A has locally bounded slope if the vertical field $v = (0,1)$ is not tangent to A at any interior point of A .

We say A is <u>above</u> B, $A > B$, if whenever a vertical curve meets both A and B, then every point of A (on this vertical) is above every point of B. These notions extend directly to $S(\infty)$ with respect to the horizontals S_t and the vertical curves.

For $A \subset H^n$ and $s \in R$, let $A_s+ = \{(x,t) \in A / t > s\}$ and similarly let A_s- be the set of points of A below P_s. Let $A_s^*+ = \{(x, 2s-t)/(x,t) \in A_s+\}$. Also let H_s+(resp. H_s-) be the set of all points above P_s (resp. below P_s).

II. The Main Result.

Now let M be a complete properly embedded hypersurface in H^n of constant mean curvature. Our main result is:

<u>Theorem 2.1. If $\partial_\infty M \subset S_o = \partial_\infty(P_o)$, then M is a hypersphere (in which case $\partial_\infty M = S_o$) or M is invariant under the reflection through P_o and M is a bigraph over P_o (i.e. M_o+ is a graph over P_o of locally bounded slope and M_o- as well).</u>

<u>Remarks</u>

1. It would be interesting to have examples of M as in 2.1, having prescribed boundary. for example, given 3 (or n) points on the equator, does there exist a constant mean curvature M with boundary these points?

2. Suppose $\partial_\infty M = S_o$ and M satisfies 2.1. Then is M a hypersphere? Do Carmo and Lawson claim this in theorem B of [2], however their proof assumes S_o is the homological boundary of M. Imagine a torus in H^3, touching $S(\infty)$ along the equator. Perhaps such a surface can have constant mean curvature H (it could not be of revolution but we can not even rule out the possibility $H > 1$).

We shall use the following version of the <u>maximum principle</u> [4]:

1. Let M_1, M_2 be connected complete hypersurfaces of constant mean curvatures

C_1, C_2. Suppose M_1 and M_2 are tangent at an interior point x and their mean curvature vectors both point in the vertical up direction. If M_1 is above M_2 in a neighborhood of x and $C_2 > C_1$, then $M_1 = M_2$.

2. Assume x is an interior point of ∂M_1 and ∂M_2, ∂M_1, ∂M_2 are tangent at x and M_1, M_2 as well. Also suppose the mean curvature vectors of M_1, M_2 point in the vertical up direction at x. If M_1 is above M_2 in a neighborhood of x and $C_2 > C_1$, then $M_1 = M_2$.

<u>Proof of 2.1</u>. Let C and C' be the connected components of $H^n - M$ and assume the mean curvature vector X of M points into C. Clearly we can suppose $X \neq 0$ since $X = 0$ easily implies $M = P_o$. So there are points of M above or below P_o; assume above for the moment. For t sufficiently large, P_t is disjoint from M, so there is a largest $T>0$ such that $P_T \cap M \neq \phi$.

Let $J = \{t \in [0,T] \,/\, M_t+$ is a graph of locally bounded slope over P_t, $M_t^*+ > M_t-$, and for $s > t$, X points into H_s- at each point of $M \cap P_s\}$. We have $T \in J$ and if $t \in J$, $t < s < T$, then $s \in J$. We shall prove $0 \in J$ by showing J is open and closed in $[0,T]$.

First we see why J is <u>closed</u>. Suppose $(t,T] \subset J$. If M_t+ is not a graph then two points of M_t+ are on the same vertical, so there is an s, $t < s \leq T$ and $x \in P_o$, such that (x, t) and (x,s) are both in M. We choose s so there are no other points of M on the vertical L joining (x,t) to (x,s). Now M is a graph in a neighborhood of (x,s), never vertical, and X points into H_s- at (x,s). This implies $L \subset C$ and M is tangent to P_t at (x,t) and below P_t in a neighborhood of (x,t). The reason for the latter property is the vertical curves meeting M in a neighborhood of (x,s), descend to fill a neighborhood of (x,t) in P_t. So if any point of M near (x,t) were strictly above P_t we would not have $M_\tau+$ a graph for some τ slightly larger than t. This violates $(t,T] \subset J$.

Now since M is entirely below P_t in a neighborhood of (x,t), X must point into H_t- at (x,t). But $L \subset C$ and X points into C along M so this is a contradiction, and M_t+ is a graph over P_t. M_t+ has locally bounded slope as well. Also $M_t^*+ > M_t-$ since were this not the case, it would already fail to be true for s slightly larger than t.

Finally, the mean curvature vector X points into H_t- at each point of $M \cap P_t$, since at $(x,t) \in M$, M is either entirely below P_t at (x,t) (in which case X points down) or (x,t) is an accumulation point of points of M above P_t, and then X points into H_t- by continuity. Notice that our convention allows X to be tangent to P_t and point into H_t-. So we have proved J is closed.

Next we show J is <u>open</u>. Let $[t,T] \subset J$ with $t>0$. Let $(x,t) \in M$ and let $D \subset M$ be a disc containing (x,t). Notice that M is not vertical at (x,t), for if this were so, consider the half discs D_t^*+ and D_t-. They have the same boundary, they are tangent at (x,t) and their mean curvature vectors are the same at (x,t). Moreoever, D_t^*+ is not vertical at points strictly below P_t and $D_t^*+ > D_t-$ hence they do not cross at (x,t). So by the maximum principle, $D_t^*+ = D_t-$ near (x,t) and by analytic continuation $M_t^*+ = M_t-$. But $t > 0$ so this contradicts $\partial M \subset S_0$.

This proves M is a graph in a neighborhood U of P_t and not vertical in U. This implies X points down in U as well (for $s \in (0,t)$, the part of M between P_s and P_t is compact). It remains to verify that $M_s^*+ > M_s-$ for s near t. This is done exactly as in [4] so we just sketch the argument here. Since $M \cap U$ is a graph, we have $M_s^*+ - V > M_s-$ for V a neighborhood of P_t, $V \subset U$, and s near t. Also $M_s+ - V$ is compact and its image under reflection through P_t is disjoint from M_t- so by continuity, for s near t, we have $(M_s^*+ - V) > M_s-$. This means $M_s^*+ > M_s-$ for s near t. This argument uses the fact that M_t+ is strictly above M_t- whenever $(t,T] \subset J$ and $t \neq 0$.

If not, there would be a largest $t_o > t$ for which this fails. Then $M^*_{t_o}+$ and $M_{t_o}-$ are tangent at some point q and their mean curvature vectors have the same direction at q (M separates into two connected components and if X points into C then the mean curvature vector of $M^*_{t_o}+$ also points into C). Thus M would be invariant under reflection through P_{t_o} by the maximum principle and this violates the hypothesis on $\partial_\infty M$. Thus we have proved $0 \in J$; in particular, X points into H_o- at each point of $M \cap P_o$.

Suppose there are points of M strictly below P_o. Then the same reasoning used to prove $J = [0,T]$, shows M_o- is a graph of bounded slope and X points into H_o+ at each point of $M \cap P_o$ (just turn H^n over). Hence X is horizontal along $M \cap P_o$, or equivalently M is vertical at each $x \in M \cap P_o$. Let $x \in M \cap P_o$ and D be a disc in M containing x. Since $D^*_o+ > D_o-$ and D is vertical at x, not vertical at each $y \in D \cap P_s$, $s>0$, we conclude, as before, $D^* = D$ and hence $M^* = M$ by analytic continuation. Thus when M has points on both sides of P_o, we know M is a bigraph and invariant by reflection through P_o.

To complete the proof of theorem 2.1, we must show that if $M \subset H_o+$ then M is a hypersphere. First we show the mean curvature H of M must be between 0 and 1: $0 < H < 1$. Consider the family L_τ of horospheres tangent to $S(\infty)$ at the south pole, the parameter τ chosen so that L_o = the south pole and $L_\infty = S(\infty)$. For small positive τ, L_τ is disjoint from M so there is a smallest τ such that $L_\tau \cap M \neq \phi$. Let x be such an intersection point, clearly $x \in P_s \cap M$, $s>0$ so X points into H_s- at x. Now the mean curvature vector Y of L_τ at x also points into H_s- and M and L_τ are tangent at x so X and Y have the same direction at x. Therefore H is less than the mean curvature of L_τ, which is one, and we have strict inequality by the maximum principle. Let N_o be the hypersphere of H^n with boundary ∂P_o and with mean curvature vector pointing down and of length H. Foliate H^n by the hyperspheres $\gamma_t N_o = N_t$.

We claim $M = N_o$. If not, there are points of M above or below N_o. Suppose there are points below. Then there is a smallest negative t such that $N_t \cap M \neq \phi$. N_t is tangent to M, on one side of M, and their mean curvature vectors have the same orientation. So by the maximum principle, $M = N_t$, which contradicts $\partial M \subset S_o$. A similar argument works if there are points of M above N_o. Q.E.D.

Corollary 2.2. In addition to the hypothesis of 1.1, assume $\partial_\infty M$ consists of two distinct points p and q. Then M is a hypersurface of revolution about the geodesic Γ joining p to q. More precisely. Let Q be any hyperplane orthogonal to Γ. Then M is transverse to Q and $M \cap Q$ is a round sphere centered at Γ Q.

Remark. These hypersurfaces of revolution have been classified by Hsiang [3], and in this paper he obtains a special case of corallary 2.2.

Proof of 2.2. It follows immediately from 2.1 that M is transverse to Q except possibly at $y = \Gamma \cap Q$, and that $M \cap Q$ is either the empty set, or $\{y\}$, or a round sphere centered at y. It cannot be empty since Q separates p and q. If $M \cap Q = \{y\}$, then M is tangent to Q at y and lies on one side of Q near y. It follows that hyperplanes Q' close to Q, on the other side, do not meet M. This is a contradiction so $M \cap Q$ is a round sphere.

III. Minimal Hypersurfaces of H^n.

Let M be a minimal hypersurface of H^n. We say that M is regular at ∞ if the asymptotic boundary B of M is a C^2 codimension one submanifold of $S(\infty)$ and $\overline{M} = M$ B is of class C^1 on B. M. Anderson has proved that any C^2 codimension one submanifold $B \subset S(\infty)$ bounds a minimal $M \subset H^n$ [1]. We do not know if one has boundary regularity as in the euclidean category. In this section we adapt the work of R. Schoen to our context to obtain information about M given B.

<u>Theorem 3.1.</u> <u>Let</u> $B \subset S(\infty)$ <u>be a</u> C^2 <u>codimension one immersed boundary, not necessarily connected. Assume</u> B_0+ <u>is a graph of locally bounded slope and</u> $B_0^*+ > B_0-$. <u>Let</u> M <u>be a minimal hypersurface immersed in</u> H^n <u>with</u> $\partial M = B$ <u>and</u> M <u>regular at</u> ∞. <u>Then</u> M_0+ <u>is a graph of locally bounded slope and</u> $M_0^*+ > M_0-$.

<u>Proof:</u> First we remark that M is orthogonal to $S(\infty)$ along B. We see this as follows. Let $x \in B$ and S_1, $S_2 \subset S(\infty)$ be round codimension one spheres passing through x such that each S_i is tangent to B at x and S_i and B are on one side of each other at x, and S_1 and S_2 are on opposite sides of B at x (This is where we need B to be C^2). Let D_1, D_2 be the disks on $S(\infty)$ with boundary S_1, S_2. Choose the S_k small enough so that int D_k is disjoint from B for $k = 1,2$. Let $y \in$ int D_k and $S(y)$ be a small round sphere centered at y. Let $H(y)$ be the hyperbolic plane of H^n with boundary $S(y)$; $H(y)$ is a hemisphere orthogonal to $S(\infty)$. For $S(y)$ small, $H(y)$ is disjoint from M. As $S(y)$ grows to become S_k, $H(y)$ stays disjoint from M since if this were not so we would have $M = H(y)$ by the maximum principle. Thus M is forced between the two hyperbolic planes H_1, H_2 with boundaries S_1, S_2, and is orthogonal to B.

Let $T > 0$ be the largest T such that $P_T \quad B \neq \phi$. Let $J = \{t \in [0,T] \,/ M_t+$ is a graph of locally bounded slope and $M_t^*+ > M_t-\}$. As in the proof of 2.1, we see that J is open and closed in $[0,T]$ hence $0 \in J$ and 3.1 is proved. We will not go through the details, however some comments are in order. M need not be embedded to make the argument since the maximum principle applies without worrying about local orientations. Also M orthogonal to $S(\infty)$

along B implies M is a graph of locally bounded slope in a neighborhood of B_o+. One can prove, as in theorem 2 of |4|, that if B is embedded and $B_o^*+ = B_o-$ then M is embedded and $M^* = M$.

<u>Theorem 3.2.</u> <u>Let S_1, S_2 be disjoint round spheres in $S(\infty)$ and let M be a connected minimal hypersurface in H^n with $\partial_\infty M = S_1 \cup S_2$ and M regular at ∞. Then M is a catenoid.</u>

<u>Remark.</u> By a conformal transformation of $S(\infty)$ we can make S_1 and S_2 horizontal (just send the geodesic joining the centers of S_1 and S_2, to the north pole-south pole geodesic). The radius R of this annular region between S_1 and S_2 is a conformal invariant. If R is small enough then there exists a catenoid with boundary $S_1 \cup S_2$.

<u>Proof of 3.2.</u> We can assume S_1, S_2 are horizontal and symmetric with respect to the equator. Then by 3.1, M is also symmetric with respect to the equatorial plane. It suffices to show M is a surface of revolution about the geodesic γ joining the north and south poles. Let P be a hyperplane containing γ; we need show M is invariant by reflection in P. Rotate H^n by $\pi/2$ so that P becomes horizontal. Then $B = S_1 \cup S_2$ satisfies the hypothesis of 3.1 from above and below P so M is invariant by reflection through P.

<u>Theorem 3.3.</u> <u>Let $B \subset S(\infty)$ be a C^2 codimension one submanifold and suppose B is a graph of locally bounded slope over S_o. Let M be a constant mean curvature hypersurface embedded in H^n with the homological boundary of M equal to B. Then M is a graph over P_o. If M is minimal and immersed with $\partial_\infty M = B$, then M is a graph as well and M is unique.</u>

<u>Remark.</u> The existence of a minimal M with $\partial_\infty M = B$ is proved in [1]. When M is minimal and the homological boundary of M equals B, then unicity is proved in [2].

Proof of 3.3. Suppose M is not a graph. Let γ be a vertical curve such that M intersects γ in at least two points p and q and the distance T between p and q is maximum with respect to this property. Then $\gamma_T M$ and M are tangent at one of the points p,q and have the same mean curvature vectors. Thus $\gamma_T M = M$ which contradicts $\partial_\infty M$ a graph.

Bibliography

1. M. Anderson, Complete minimal varieties in hyperbolic space (to appear).

2. M. Do Carmo and B. Lawson, Alexandrov-Bernstein type theorems in hyperbolic space (to appear).

3. Wu-yi Hsiang, On generalization of theorems of A.D. Alexandrov and C. Delauney on hypersurfaces of constant mean curvature, Duke Math. Journal, Vol. 49, 1982, pp. 485-496.

4. R. Schoen, Uniqueness, symmetry, and embeddedness of minimal surfaces (to appear).