SHELLINGS OF TILINGS

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0. Introduction

The notion of shellability originated in the context of polyhedral complexes and Combinatorial Topology. An abstraction of this concept for graded posets (i.e. partially ordered sets) was recently introduced by Björner and Wachs first in the finite case ([Bj_1] and [Bj-Wa]) and then in the infinite case ([Bj_2 , Rk. 4.21] and [W-W, Sect. 7]). Many posets arising in Combinatorics and in Convex Geometry were investigated and some proved to be shellable ([Bj_1], [Bj-Wa], [Sa], ...). A key achievement was the proof by Bruggesser and Mani that boundary complexes of convex polytopes are shellable ([B-M]).

I deal here with the case of polyhedral tilings of Euclidean spaces. I obtain a result parallel to Bruggesser and Mani's. Namely, if there is a convex parabolic (i.e. without asymptotes) function φ defined on $\mathbb{R}^d$ which reduces to a different affine fucntion on each d-face of a polyhedral tiling $\mathcal{T}$ of $\mathbb{R}^d$, then $\mathcal{T}$ is simply shellable. The existence of φ ensures that $\mathcal{T}$ is the projection of the boundary complex of some (d+1)-convex set with polyhedral faces.

I asked whether all polygonal tilings of the plane were shellable. This question was answered affirmatively by Björner and Stanley during a problem session on simplicial complexes at the Institute for Mathematics and its Applications (Univ. of Minnesota). They proved there independently that all regular CW-decompositions of the plane are extendably shellable. I have worked out in Section 3 a detailed version of their proof.

I am very grateful to A. Björner and R.P. Stanley for having communicated to me a sketch of their proof (Prop. 3.4 and Thm. 3.8). I also wish to thank the Institute for Mathematics and its Applications for their kind invitation to a workshop session in Combinatorics (March 88), during which my research received a major impulse.

1. Definitions

Let P be a poset (i.e. a partially ordered set). The *length* n of a finite totally ordered subset $\{a_o, ..., a_n\}$ of P ($a_o < a_1 < ... < a_n$) is its cardinality minus one; the (possibly infinite) least upper bound of these is the *length* l(P) of P. The *closed, lower* and *upper* intervals of P are respectively defined by $[a,b] = \{x \in P \mid a \leq x \leq b\}$, $(a] = \{x \in P \mid x \leq a\}$, $[a) = \{x \in P \mid a \leq x\}$ ($a,b \in P$, $a \leq b$). P is *bounded* if it has a minimum and a maximum element. The dual poset of P (defined on the same set, with the reverse order) is denoted by $\tilde{P}$, and $\hat{P}$ denotes the bounded poset obtained by adding to P a minimum element $\hat{0}$ and a maximum element $\hat{1}$. The *height* of $x \in P$ is the length of (x]. The *atoms* of a poset with a minimum element are its elements of height 1.

According to [W.-W., Sec. 7], a *recursive atom ordering* (in short, R.A.O.) of a poset P of finite length with a minimum element is defined by induction on the length :

(i) if l(P) = 1, then a R.A.O. of P is a well-ordering of its atoms;

(ii) if l(P) > 1, then a R.A.O. of P is a well-ordering, let be $\leq_R$, of its atoms set A such that for each $a \in A$, [a) has a R.A.O. that begins with the minimal elements of the subposet

$$H_a = [a) \cap \bigcup_{b \in A,\, b \leq_R a} [b)$$

(which are therefore atoms of [a)), and H_a is nonempty unless a is minimum for $\leq_R$. If P has a maximum element, then a *coatom* of P is an atom of $\tilde{P}$, and a *recursive coatom ordering* (in short, R.C.O.) of P is a R.A.O. of $\tilde{P}$. The R.C.O. is a combinatorial abstraction of the geometric concept of shellability originating in the theory of polyhedral complexes (see hereafter).

A *convex polytope* Q is the convex hull of a finite subset of an euclidean space $\mathbb{R}^n$. For the study of properties of convex polytopes, I refer to [Gr]. I denote by $\mathcal{F}(Q)$ the *face lattice* of Q, and by $\mathcal{B}(Q) = \mathcal{F}(Q) \setminus \{Q,\emptyset\}$ its *boundary complex.* A *polyhedral complex* $\mathcal{S}$ is a set of nonempty convex polytopes in some euclidean space $\mathbb{R}^n$ which contains the nonempty faces of its elements, and such that if $F,G \in \mathcal{S}$, then $F \cap G$ is a common face of F and G (possibly empty). The k-*faces* of $\mathcal{S}$ are its elements of dimension k, and their set is denoted by $\mathcal{S}^{(k)}$. The *dimension* of $\mathcal{S}$ is the maximum of the dimensions of its faces; $\mathcal{S}$ is called *pure* if all its maximal faces have the same dimension. $\mathcal{S}$ is *locally finite* if each point of $\cup\mathcal{S}$ has a neighborhood which meets only finitely many faces of $\mathcal{S}$. From now on, let $\mathcal{S}$ be pure d-dimensional. Then a *shelling (order)* of $\mathcal{S}$ is defined inductively on d by

(i) if d = 0, then a shelling of $\mathcal{S}$ is a well-ordering of it;

(ii) if $d > 0$, then a shelling order of $\mathcal{T}$ is a well-ordering $\leq_R$ of its d-faces such that for each $F \in \mathcal{T}^{(d)}$, the subcomplex

$$\mathcal{K}_F = \mathcal{B}(F) \cap \bigcup_{G \in P^{(d)},\, G <_R F} \mathcal{B}(G)$$

is pure (d-1)-dimensional if F is not minimum for $\leq_R$ and there is a shelling order of $\mathcal{B}(F)$ in which the elements of $\mathcal{K}_F$ (if any) come first.

In other words, a shelling of $\mathcal{T}$ is a R.C.O. of the poset obtained by ordering $\mathcal{T}$ by set-inclusion and adding a maximum element $\hat{1}$. Intuitively, this means that $\mathcal{T}$ can be obtained by pasting its d-faces successively in a suitable way. A *partial shelling* of $\mathcal{T}$ is a shelling of a pure d-dimensional subcomplex of $\mathcal{T}$. A shelling order $\leq_R$ is *simple* if $(\mathcal{T}^{(d)}, \leq_R)$ is either finite or isomorphic to $(\mathbb{N}, \leq)$. $\mathcal{T}$ is *shellable* if it has a shelling order; it is *simply extendably shellable* if each finite partial shelling of $\mathcal{T}$ comes first in some simple shelling of $\mathcal{T}$. A *polyhedral tiling* of $\mathbb{R}^d$ is a locally finite polyhedral complex $\mathcal{T}$ in $\mathbb{R}^d$ whose underlying set $\cup\mathcal{T}$ is $\mathbb{R}^d$.

Let X be a Hausdorff topological space. A *regular CW-decomposition* of X is a set $\mathcal{S}$ of subspaces σ of X which are each homeomorphic to some standard n-ball $\mathbb{B}^n$ $(n \geq 0)$, called *closed cells* (with boundary $\partial\sigma$) and such that

(i) $(\sigma \setminus \partial\sigma)_{\sigma \in \mathcal{S}}$ is a partition of X;

(ii) for each closed cell σ, $\partial\sigma$ is a union of finitely many closed cells (which have therefore dimension less than dim σ, by the Invariance of Domain);

(iii) a subset A of X is closed if and only if $A \cap \sigma$ is closed (in σ) for each $\sigma \in \mathcal{S}$.

A k-dimensional closed cell will also be called a *k-face*. The *dimension* of $\mathcal{S}$ is the (possibly infinite) least upper bound of the dimensions of its faces; $\mathcal{S}$ is *pure* if all its maximal faces have the same dimension; $\mathcal{S}$ is *shellable* if the poset obtained by ordering it by set-inclusion and adding a maximum element has a R.C.O.; $\mathcal{S}$ has the *intersection property* if for each $\sigma,\tau \in \mathcal{S}$, $\sigma \cap \tau$ either is empty or belongs to $\mathcal{S}$. So a locally finite polyhedral complex $\mathcal{T}$ is a regular CW-decomposition of $\cup\,\mathcal{T}$ with the intersection property. Regular CW-decompositions of a topological d-manifold M (e.g. $M = \mathbb{R}^d$) have nice combinatorial properties (cf. [Go_1, Corollary 5.1.12 and Remark 5.1.13.3]). In particular, they are pure d-dimensional and such that each (d-1)-face is contained in exactly two d-faces. Also, the (d-1)-faces and d-faces containing a given (d-2)-face are arranged cyclically. A CW-*subspace* of X is a subspace of X which is a union of closed cells. I call also CW-*ball* a CW-subspace which is homeomorphic to a ball, etc... Notice that if X is a manifold with boundary ∂X, then ∂X is a CW-subspace ([Go_1, Thm. 5.1.11]).

Given an euclidean space $\mathbb{R}^n$, I denote by $\mathbb{R}^{n*}$ its dual space (i.e. the space of linear functions on $\mathbb{R}^n$). If $a,b \in \mathbb{R}^n$, $a \neq b$, then $[a : b]$ stands for the segment $\{(1 - \lambda) a + \lambda b \mid \lambda \in [0,1]\}$ and $[a : b)$ for the half line $\{(1 - \lambda) a + \lambda b \mid \lambda \in [0, +\infty[\}$. A *cone* of $\mathbb{R}^n$ is a subset C such that for each $u \in C \setminus \{0\}$, $[0 : u) \subset C$. For $A,B \subset \mathbb{R}^n$, cone A denotes the convex cone spanned by A, $\mathbb{B}(A)$ its *barrier cone* $\{f \in \mathbb{R}^{n*} \mid \exists\, c \in \mathbb{R} : f(A) \subset]-\infty, c]\}$ and A_∞ its *asymptote cone* $\bigcap_{\varepsilon > 0} \overline{\{\lambda a \mid \lambda \in [0,\varepsilon],\, a \in A\}}$ (for the properties of these, see for instance [Go$_2$]); $A + B = \{a + b \mid a \in A,\, b \in B\}$. According to Bourbaki ([Bo; 17, p. 125]), a convex set $A \subset \mathbb{R}^n$ is *parabolic* if for each $u \in A_\infty \setminus \{0\}$ and each $x \in \mathbb{R}^n$, $\{x\} + [0 : u)$ meets A.

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a real function on $\mathbb{R}^d$. The *epigraph* of φ is $\Gamma_\varphi = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} \mid t \geq \varphi(x)\}$, so φ is convex if and only if Γ_φ is a convex set. If φ is convex, I call it *parabolic* if Γ_φ is parabolic. If $\mathcal{T}$ is a polyhedral tiling of $\mathbb{R}^d$, then φ is *strictly piecewise linear* (in short, strictly p.l.) *on* $\mathcal{T}$ if it is convex and reduces to (the restriction of) a different affine function on each d-face of $\mathcal{T}$.

A total order $\leq_L$, called *lexicographic order of coordinates,* is defined on $\mathbb{R}^n$ by

$$(a_1, \ldots, a_n) <_L (b_1, \ldots, b_n) \Leftrightarrow (\exists\, j \in \{1, \ldots, n\} : a_i = b_i \;\; (i < j),\; a_j < b_j).$$

The vector space of affine functions on $\mathbb{R}^n$ is denoted by $\mathbb{A}(\mathbb{R}^n)$. The *lexicographic order* on $\mathbb{A}(\mathbb{R}^n)$ is defined as follows :

$$\text{if} \qquad g : (x_1, \ldots, x_n) \to a_o + \sum_{i=1}^{n} a_i x_i,$$

$$h : (x_1, \ldots, x_n) \to b_o + \sum_{i=1}^{n} b_i x_i,$$

$$\text{then} \qquad (g \leq_L h) \Leftrightarrow ((a_o, \ldots, a_n) \leq_L (b_o, \ldots, b_n)).$$

A *curve* in $\mathbb{R}^2$ is a continuous map $\varphi : [0,1] \to \mathbb{R}^2$; $\varphi(0)$ is called its *origin* and $\varphi(1)$ its *extremity* ; it is *closed* if $\varphi(0) = \varphi(1)$; its is *simple* if $\varphi(x) \neq \varphi(y)$ whenever $x,y \in [0,1]$, $\{x,y\} \neq \{0,1\}$ and $x \neq y$.

Jordan Curve Theorem. If C is the range of a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus C$ has two (open) connected components. One of them, let A, is bounded and called the **interior** *of C; the other one, let E, is not bounded and called the* **exterior** *of C. Furthermore,* $C = \overline{A} \setminus A = \overline{E} \setminus E$.

Proof. See [Al, Ch. II, §1.1]. Alternatively, embed $\mathbb{R}^2$ in $\mathbb{S}^2$ by adding a point at infinity and then use [Sp, Ch.4, Sec. 7, Thm. 15]. ∎

Under the hypotheses of this theorem, C is said to *enclose* a subset X of $\mathbb{R}^2$ if $X \subset \overline{A}$. If φ is a closed curve in $\mathbb{R}^2$ with range C, and $p \in \mathbb{R}^2 \backslash C$, I denote by $\omega(\varphi,p)$ the *index of* p *relative to* φ (cf. [Al, Ch. II, §2.2 and 4.3] for a definition and a study of properties, or [Do, Ch. IV, Ex. 6.12.2 and 7.6.3], where it is called the winding number of φ at p). For a fixed p, it can be extended to an additive function on singular 1-chains of $\mathbb{R}^2\backslash\{p\}$ which cancels on boundaries. For a fixed φ, it is a locally constant function on $\mathbb{R}^2\backslash C$ which is zero outside a bounded subset. Moreover if φ is a simple closed curve then $\omega(\varphi,p)$ is equal to 0 if p is in the exterior of C and to ± 1 if p is in the interior of C. By an abuse of notations, I shall also call curve a subspace which is the range of a curve. So a (CW-)1-sphere C of $\mathbb{R}^2$ will be called a (CW-) simple closed curve, and I shall use the notation $\omega(C,p)$ when the mapping is clear from the context.

2. Tilings of $\mathbb{R}^d$

2.1. *Lemma. Let* A *be a convex subset of* $\mathbb{R}^d$. *Then* A *is parabolic iff for all* $f \in \mathbb{B}(A) \setminus \{0\}$ *and* $u \in A_\infty \setminus \{0\}$, $f(u) < 0$.

Proof. From the definition, A is parabolic iff for each $u \in A_\infty \setminus \{0\}$, A - [0 : u) is equal to $\mathbb{R}^d$. From a standard separation theorem, this is equivalent to say that no nonzero linear function is upper bounded on the convex set A - [0 : u). Now a linear function $f \in \mathbb{R}^{d*}$ is upper bounded on A - [0 : u) iff $f \in \mathbb{B}(A)$ and $f(u) \geq 0$. ∎

2.2. *Theorem (Bruggesser-Mani). Let* P *be a convex* d*-polytope in* $\mathbb{R}^d$. *Then* $\mathscr{B}$(P) *is a shellable polyhedral complex. Furthermore if* $u \in \mathring{P}$ *and* $v \in \mathbb{R}^d \setminus \{u\}$, *then there is a shelling order of* $\mathscr{B}$(P) *in which the* (d-1)*-faces of* P *whose affine hull meet* [u : v] *(if any) come first.*

Proof. If $v \in P$, the result is trivial (no affine hull of a (d-1)-face of P meets [u : v]), so we shall assume $v \in \mathbb{R}^d \setminus P$ in the sequel. The proof of [B-M, Prop. 2] can be used as such (remark that they denote there v by p and the line (u : v) by G), provided v and (u : v) are admissible with respect to P. If this is not the case, replace u by $u' \in \mathring{P}$ such that (u' : v) is admissible with respect to P, then v by $v' \in$ (u' : v) \P such that v' is admissible with respect to P and the affine hulls of (d-1)-faces of P which meet [u' : v'] are the same as those which meet [u' : v] (hence also [u : v]). ∎

2.3. *Corollary. Let* P *be a convex* d-*polytope in* $\mathbb{R}^d$, *and* $h_1, \ldots, h_n$ *a minimum number of affine functions such that* $P = \cap_{i=1}^{n} \{x \in \mathbb{R}^d \mid h_i(x) \leq 0\}$ *(so that the* (d-1)-*faces of* P *are* $F_i = P \cap \mathrm{Ker}\, h_i$ $(1 \leq i \leq n)$ *). Then there is a shelling order of* $\mathcal{B}(P)$ *in which the faces* F_i *such that* $h_i \geq_L 0$, *if any, come first.*

Proof. Let us denote by <a,b> the scalar product of $a,b \in \mathbb{R}^d$. There are $a^{(i)} \in \mathbb{R}^d$, $b^{(i)} \in \mathbb{R}$ $(1 \leq i \leq n)$ such that

$$h_i(x) = b^{(i)} + \langle a^{(i)}, x \rangle \quad (1 \leq i \leq n). \tag{1}$$

It is easy to construct an element z of $\mathbb{R}^d$ with positive coordinates such that

$$(b^{(i)} = 0,\ a^{(i)} <_L 0) \Rightarrow (\langle a^{(i)}, z \rangle < 0), \tag{2}$$

$$(b^{(i)} = 0,\ a^{(i)} >_L 0) \Rightarrow (\langle a^{(i)}, z \rangle > 0). \tag{3}$$

There is also $u \in \mathring{P}$ such that

$$h_i(u) - h_i(0) = \langle a^{(i)}, u \rangle \neq 0. \quad (1 \leq i \leq n) \tag{4}$$

Then for a positive real number ε small enough,

$$h_i <_L 0 \Rightarrow h_i(\varepsilon z) < 0, \tag{5}$$

$$h_i >_L 0 \Rightarrow h_i(\varepsilon z) > 0. \tag{6}$$

We can further impose, considering (4),

$$h_i(\varepsilon z) \neq h_i(u). \tag{7}$$

So, if we put $v = \varepsilon z$ and define λ_i by

$$h_i(\lambda_i u + (1 - \lambda_i)v) = 0, \tag{8}$$

we get

$$\lambda_i = \frac{h_i(v)}{h_i(v) - h_i(u)}, \tag{9}$$

$$\lambda_i(\lambda_i - 1) = \frac{h_i(v)\, h_i(u)}{(h_i(v) - h_i(u))^2} \tag{10}$$

and then, using further (5), (6) and $h_i(u) < 0$,

$$\begin{aligned}(h_i \geq_L 0) &\Leftrightarrow \lambda_i \in \,]0,1[\\ &\Leftrightarrow \mathrm{Ker}\, h_i \cap [u : v] \neq \varnothing.\end{aligned}$$

Then we apply Theorem 2.2. ∎

In what follows, $\mathcal{T}$ will be a polyhedral tiling of $\mathbb{R}^d$, $\varphi : \mathbb{R}^d \to \mathbb{R}$ a convex function strictly p.l. on $\mathcal{T}$, and for each $F \in \mathcal{T}^{(d)}$, h_F will be the affine function such that $\varphi_{|F} = h_{F|F}$. Also, f_F will be the linear function $h_F - h_F(0)$.

2.4. *Lemma.* $\varphi(x) = \mathrm{Max}\{h_F(x) \mid F \in \mathcal{T}^{(d)}\}$.

Proof. Let $x \in \mathbb{R}^d$. If $x \in G \in \mathcal{T}^{(d)}$, then $\varphi(x) = h_G(x)$. So it is enough to prove that for any $F \in \mathcal{T}^{(d)}$, $h_F(x) \leq \varphi(x)$. Let $y \in \mathring{F}$. Then there is $\mu \in \,]0,1[$ such that $\mu y + (1 - \mu)x \in F$.

Consequently,

$$\varphi(y) = h_F(y), \tag{11}$$

$$\varphi(\mu y + (1 - \mu)x) = h_F(\mu y + (1 - \mu)x). \tag{12}$$

Now by convexity of φ

$$\varphi(\mu y + (1 - \mu)x) \leq \mu\varphi(y) + (1 - \mu)\,\varphi(x), \tag{13}$$

hence, substituting (11) and (12) in (13) and using the affinity of h_F,

$$\mu\, h_F(y) + (1-\mu)\, h_F(x) \le \mu\, h_F(y) + (1-\mu)\, \varphi(x) \tag{14}$$

and finally

$$h_F(x) \le \varphi(x). \blacksquare \tag{15}$$

2.5. *Lemma. If* $F \in \mathcal{T}^{(d)}$ *and* $y \in \mathbb{R}^d \setminus F$, *then*

$$\varphi(y) > h_F(y).$$

Proof. Since the assertion is invariant by translation, we can assume throughout $0 \in \mathring{F}$. Since $y \notin F$, there is a (d-1)-face K of F which spans a hyperplane separating strictly y from F. That is, there is $g \in \mathbb{R}^{d*}$ such that

$$F \subset \{x \in \mathbb{R}^d \mid g(x) \le 1\},\ K = \{x \in F \mid g(x) = 1\},\ g(y) > 1. \tag{16}$$

Let G be the d-face of $\mathcal{T}$, other than F, which contains K. Since $h_F \neq h_G$, $\{x \in \mathbb{R}^d \mid h_F(x) = h_G(x)\}$ is either a hyperplane or $\varnothing$; but it contains $K = F \cap G$, so it is equal to $g^{-1}(\{1\})$. On the other hand, by Lemma 2.4,

$$h_{F|F} = \varphi_{|F} \ge h_{G|F}, \tag{17}$$

hence

$$F \subset \{x \in \mathbb{R}^d \mid h_F(x) \ge h_G(x)\} \tag{18}$$

and

$$\{x \in \mathbb{R}^d \mid h_F(x) < h_G(x)\} = g^{-1}(]1, +\infty[). \tag{19}$$

From there and Lemma 2.4 again follows that

$$\varphi(y) \ge h_G(y) > h_F(y). \blacksquare \tag{20}$$

2.6. *Lemma. Let* $M \in \mathcal{T} \cup \{\varnothing\}$, *and let* $K, F \in \mathcal{T}^{(d)}$ *such that* $M \subset K \cap F$. *Then*

$$h_K - h_F \in \text{cone}\{h_G - h_F \mid G \in \mathcal{T}^{(d)}, M \subset G \cap F \in \mathcal{T}^{(d-1)}\}.$$

Proof. Let us suppose that the result is not true, and try to reach a contradiction. Then, since $\{G \in \mathcal{T}^{(d)} \mid G \cap F \in \mathcal{T}^{(d-1)}\}$ is finite, Farkas Theorem gives $y_o \in \mathbb{R}^d$ and $t_o \in \mathbb{R}$ such that

$$(f_K - f_F)(y_o) + t_o(h_K(0) - h_F(0)) > 0 \tag{21}$$

and

$$(G \in \mathcal{T}^{(d)}, M \subset G \cap F \in \mathcal{T}^{(d-1)}) \Rightarrow (f_G - f_F)(y_o) + t_o(h_G(0) - h_F(O)) \leq 0. \tag{22}$$

Next we notice that for each $G \in \mathcal{T}^{(d)}$ such that $G \cap F \in \mathcal{T}^{(d-1)}$, the closed halfspace bounded by aff$(G \cap F)$ and containing F is $\{x \in \mathbb{R}^d \mid (h_G - h_F)(x) \leq 0\}$. Then

$$F = \{x \in \mathbb{R}^d \mid (G \in \mathcal{T}^{(d)}, G \cap F \in \mathcal{T}^{(d-1)}) \Rightarrow (h_G - h_F)(x) \leq 0\}. \tag{23}$$

Now we distinguish between six cases.

1^o. $M = \emptyset$, $t_o > 0$

Let $x_o = y_o / t_o$, so $h_K(x_o) > h_F(x_o)$ and

$$(h_G - h_F)(x_o) \leq 0 \quad (G \in \mathcal{T}^{(d)}, G \cap F \in \mathcal{T}^{(d-1)}). \tag{24}$$

The first of these inequalities shows, in view of Lemma 2.4, that $x_o \notin F$. Choose $x_1 \in \hat{F}$ and $\theta \in]0,1[$ such that

$$\theta x_o + (1 - \theta) x_1 \in F \setminus \hat{F}. \tag{25}$$

So there is $G_o \in \mathcal{T}^{(d)}$ such that

$$\theta x_o + (1 - \theta) x_1 \in G_o \cap F \in \mathcal{T}^{(d-1)}. \tag{26}$$

From $h_{G_o}(x_1) < h_F(x_1)$ (Lemma 2.5) and $h_{G_o}(x_o) \leq h_F(x_o)$ (see (24)), we deduce that

$$h_{G_o}(\theta x_o + (1 - \theta) x_1) < h_F(\theta x_o + (1 - \theta) x_1), \tag{27}$$

although both sides should be equal to $\varphi(\theta x_o + (1 - \theta) x_1)$ by (26).

2^{o}. $M = \emptyset$, $t_o < 0$

Let $x_o = y_o / t_o$, so

$$(h_G - h_F)(x_o) \geq 0 \quad (G \in \mathcal{T}^{(d)}, G \cap F \in \mathcal{T}^{(d-1)}). \tag{28}$$

Choose $x_1 \in \hat{F}$, hence by Lemma 2.5,

$$(h_G - h_F)(x_1) < 0 \quad (G \in \mathcal{T}^{(d)} \setminus \{F\}) \tag{29}$$

and considering (28)

$$(h_G - h_F)((1+\theta)x_1 - \theta x_o) < 0 \quad (G \in \mathcal{T}^{(d)}, G \cap F \in \mathcal{T}^{(d-1)}, \theta \geq 0). \tag{30}$$

Therefore, by (23) the half line $\{(1+\theta)x_1 - \theta x_o | \theta \geq 0\}$ would have to be contained in F.

3^{o}. $M = \emptyset$, $t_o = 0$

We have

$$(f_G - f_F)(y_o) \leq 0 \quad (G \in \mathcal{T}^{(d)}, G \cap F \in \mathcal{T}^{(d-1)}), \tag{31}$$

thus for any $x_1 \in \hat{F}$ (29) holds and with (31) leads to

$$(h_G - h_F)(x_1 + \theta y_o) < 0. \quad (G \in \mathcal{T}^{(d)}, G \cap F \in \mathcal{T}^{(d-1)}, \theta \geq 0). \tag{32}$$

Again, by (23), the half line $\{x_1 + \theta y_o | \theta \geq 0\}$ would have to be contained in F.

4^{o}. $M \neq \emptyset$, $t_o > 0$

Let $x_o = y_o / t_o$, and choose v in the relative interior of M. Consequently

$$(h_G - h_F)(x_o) \leq 0 \quad (G \in \mathcal{T}^{(d)}, M \subset G \cap F \in \mathcal{T}^{(d-1)}) \tag{33}$$

and

$$(h_K - h_F)(x_o) > 0. \tag{34}$$

For each $G \in \mathcal{T}^{(d)}$ such that $M \not\subset G$, we have $v \notin G$ hence, by Lemma 2.5, $(h_G - h_F)(v) < 0$. Thus there is $\varepsilon \in]0,1[$ such that

$$(h_G - h_F)((1-\varepsilon)v + \varepsilon x_o) < 0 \quad (G \in \mathcal{T}^{(d)}, G \cap F \in \mathcal{T}^{(d-1)}, M \not\subset G). \tag{35}$$

On the other hand,

$$(h_G - h_F)(v) = 0 \quad (G \in \mathcal{T}^{(d)}, M \subset G), \tag{36}$$

which gives together with (33) and (35)

$$(h_G - h_F)((1-\varepsilon)v + \varepsilon x_o) \leq 0 \quad (G \in \mathcal{T}^{(d)}, G \cap F \in \mathcal{T}^{(d-1)}). \tag{37}$$

Then by (23), $(1-\varepsilon)v + \varepsilon x_o \in F$. Now from $(h_K - h_F)(v) = 0$ and (34), we get

$$(h_K - h_F)((1-\varepsilon)v + \varepsilon x_o) > 0, \tag{38}$$

which brings a contradiction with Lemma 2.4.

5°. $M \neq \emptyset$, $t_o < 0$

Let again $x_o = y_o / t_o$, and consider v in the relative interior of M. We have

$$(h_K - h_F)(x_o) < 0 \tag{39}$$

and

$$(h_G - h_F)(x_o) \geq 0 \quad (G \in \mathcal{T}^{(d)}, M \subset G \cap F \in \mathcal{T}^{(d-1)}). \tag{40}$$

On the other hand, (36) still holds and gives together with (39) and (40)

$$(h_K - h_F)(2v - x_o) > 0, \tag{41}$$

$$(h_G - h_F)(2v - x_o) \leq 0 \quad (G \in \mathcal{T}^{(d)}, M \subset G \cap F \in \mathcal{T}^{(d-1)}). \tag{42}$$

Now we can use the same argument as in case 4°, replacing x_o by $2v - x_o$.

6°. $M \neq \varnothing, t_o = 0$

In this case

$$(f_K - f_F)(y_o) > 0 \tag{43}$$

and

$$(f_G - f_F)(y_o) \leq 0 \quad (G \in \mathcal{T}^{(d)}, M \subset G \cap F \in \mathcal{T}^{(d-1)}). \tag{44}$$

Then for any v in the relative interior of M, (36) holds hence

$$(h_K - h_F)(v + y_o) > 0 \tag{45}$$

and

$$(h_G - h_F)(v + y_o) \leq 0 \quad (G \in \mathcal{T}^{(d)}, M \subset G \cap F \in \mathcal{T}^{(d-1)}) \tag{46}$$

Now we can use the same argument as in case 4°, replacing x_o by $v + y_o$. ∎

2.7. *Lemma. If* φ *is parabolic, then for each* $a \in \mathbb{R}$, $\{F \in \mathcal{T}^{(d)} \mid h_F(0) \geq a\}$ *is finite.*

Proof. The standard norm of $\mathbb{R}^d$ and the dual norm of $\mathbb{R}^{d*}$ will be denoted by $\|\,.\,\|$. We suppose that there is an infinite sequence $F_n \in \mathcal{T}^{(d)}$ $(n \in \mathbb{N})$ with $h_{F_n}(0) \geq a$, and we show that this leads to a contradiction. For each n we choose $x_n \in \mathring{F}_n$.

Since every bounded subset of $\mathbb{R}^d$ meets only finitely many elements of $\mathcal{T}^{(d)}$,

$$\lim_{n \to \infty} \|x_n\| = \infty. \tag{47}$$

Let $M = \mathrm{Max}\{\varphi(x) \mid \|x\| \leq 1\}$ (this makes sense because the unit ball is compact and φ is convex hence continuous). Therefore for each $F \in \mathcal{T}^{(d)}$

$$\|f_F\| + h_F(0) = \mathrm{Max}\{h_F(x) \mid \|x\| \leq 1\} \leq M, \tag{48}$$

hence

$$\|f_{F_n}\| \leq M - a \quad (n \in \mathbb{N}). \tag{49}$$

Taking further into account

$$a \leq h_{F_n}(0) \leq \varphi(0) \quad (n \in \mathbb{N}), \tag{50}$$

we see that the sequences

$$\frac{x_n}{\| x_n \|}, \; f_{F_n} \text{ and } h_{F_n}(0)$$

are bounded. So we may assume, replacing x_n by a suitable subsequence if necessary, that they all converge. Let

$$\lim_{n\to\infty} \frac{x_n}{\| x_n \|} = u \neq 0, \tag{51}$$

$$\lim_{n\to\infty} h_{F_n} = f_o + b_o . \quad (f_o \in \mathbb{R}^{d*}, \; b_o \in \mathbb{R}) \tag{52}$$

Now from

$$\frac{\varphi(x_n)}{\| x_n \|} = \frac{h_{F_n}(x_n)}{\| x_n \|} = f_{F_n}\left(\frac{x_n}{\| x_n \|}\right) + \frac{h_{F_n}(0)}{\| x_n \|} \tag{53}$$

we get

$$\lim_{n\to\infty} \frac{\varphi(x_n)}{\| x_n \|} = f_o(u). \tag{54}$$

Since $(x_n, \varphi(x_n)) \in \Gamma_\varphi$ $(n \in \mathbb{N})$, it follows from (47), (51) and (54) that

$$(u, f_o(u)) \in (\Gamma_\varphi)_\infty \setminus \{0\}. \tag{55}$$

On the other hand, passing to the limit in the inequality

$$h_{F_n}(x) \leq \varphi(x) \leq t \quad ((x,t) \in \Gamma_\varphi) \tag{56}$$

yields

$$f_o(x) + b_o \leq t \quad ((x,t) \in \Gamma_\varphi), \tag{57}$$

so the linear function

$$k : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} : (x,t) \to f_o(x) - t$$

is bounded from above on Γ_φ. This brings a contradiction with Lemma 2.1, since $k((u, f_o(u))) = 0$. ∎

2.8. *Theorem. If there is a convex parabolic function* φ *which is strictly piecewise linear on the polyhedral tiling* $\mathcal{T}$ *of* $\mathbb{R}^d$, *then* $\mathcal{T}$ *admits a simple shelling order.*

Proof. We shall use the convex cone of $\mathbb{A}(\mathbb{R}^d)$

$$C = \{h \in \mathbb{A}(\mathbb{R}^d) \mid h \leq_L 0\}. \tag{58}$$

We consider the total order on $\mathcal{T}^{(d)}$

$$F \leq_R G \Leftrightarrow h_F \geq_L h_G, \tag{59}$$

and we verify that this is a simple shelling order.

We have first to prove that for every d-face F of $\mathcal{T}$ not minimum for $\leq_R$, $\mathcal{H}_F$ is pure (d-1)-dimensional. This is equivalent to show that if $M \in \mathcal{T} \cup \{\emptyset\}$ is such that $M \subset K \cap F$ with $K \in \mathcal{T}^{(d)}$ and $K <_R F$, then there is $G \in \mathcal{T}^{(d)}$ such that $M \subset F \cap G \in \mathcal{T}^{(d-1)}$ and $G <_R F$. Indeed if there is no such G then

$$\{h_G - h_F \mid G \in \mathcal{T}^{(d)}, M \subset F \cap G \in \mathcal{T}^{(d-1)}\} \subset C \tag{60}$$

and by Lemma 2.6 $h_K - h_F \in C$ whence $K \geq_R F$.

We have then to prove that there is a shelling order of $\mathcal{B}(F)$ in which the (d-1)-faces of $\mathcal{H}_F$ come first. We apply Corollary 2.3 to F, using equality (23) and the notice before it. We get a shelling order of $\mathcal{B}(F)$ in which the (d-1)-faces of the form $G \cap F$ with $G \in \mathcal{T}^{(d)}$, $h_G - h_F \geq_L 0$ come first.

Finally, the poset $(\mathcal{T}^{(d)}, \leq_R)$ is isomorphic to $\mathbb{N}$, because for each $F \in \mathcal{T}^{(d)}$ the set $\{G \in \mathcal{T}^{(d)} \mid G \leq_R F\}$ is finite as a consequence of Lemma 2.7. ∎

3. CW-decompositions of $\mathbb{R}^2$

Throughout $\mathcal{T}$ will denote a regular CW-decomposition of $\mathbb{R}^2$.

3.1. *Lemma. Let* C *be a* CW-*simple closed curve of* $\mathbb{R}^2$ *with interior* A *and exterior* E. *Then* $\overline{A}$ *and* $\overline{E}$ *are unions of 2-faces, and the edges of* C *are those edges of* $\mathcal{T}$ *which are contained in exactly one 2-face of* $\overline{A}$ *(equivalently,* $\overline{E}$*).*

Proof. By connectedness, A and E are unions of open cells. Thus $\overline{A}$ and $\overline{E}$ are CW-subspaces.

Let σ be a 1-open cell of $\mathcal{T}$, and G, G' the two 2-faces containing it. The corresponding open cells are $e = G \setminus \partial G$, $e' = G' \setminus \partial G'$. If $e, e' \subset A$ then $\sigma \subset \overline{e} \subset \overline{A}$. Furthermore σ is not contained in C, because any point of σ has a neighborhood, namely $\sigma \cup e \cup e'$, not meeting E. So $\sigma \subset A$. If $e, e' \subset E$, we deduce similarly $\sigma \subset E$. If $e \subset A$ and $e' \subset E$ then $\sigma \subset \overline{e} \cap \overline{e'} \subset \overline{A} \cap \overline{E} = C$.

On the other hand, if v is a vertex of $\mathcal{T}$ then either $v \in C$ and it is contained in an edge of C, or $v \in A$ and every edge containing v is contained in $\overline{A}$ by connectedness, or $v \in E$ and all edges containing v are contained in $\overline{E}$ by connectedness.

The lemma is a mere summing up of all of this. ∎

3.2. *Lemma. Let* B_1, B_2 *be two* CW-2-*balls of* $\mathbb{R}^2$ *such that* $B_1 \cap B_2 = \partial B_1 \cap \partial B_2$ *is a simple curve. Then* $B_1 \cup B_2$ *is a* CW-2-*ball.*

Proof. The basic tool of this proof is the fact that if A, A' are two 2-balls, then any homeomorphism $h : \partial A \to \partial A'$ can be extended to a homeomorphism $\overline{h} : A \to A'$ (identify both A, A' with the standard 2-ball $\mathbb{B}^2$ and use radial extension from the centre).

If $C = B_1 \cap B_2$ is closed, then $C = \partial B_1 = \partial B_2$. Since $B_1 \setminus \partial B_1$, $B_2 \setminus \partial B_2$ are two bounded connected components of $\mathbb{R}^2 \setminus C$, $B_1 = B_2$ and the result is trivial.

If C is not closed, we consider

$$\mathbb{B}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}, \tag{61}$$
$$\mathbb{B}^2_- = \{(x_1, x_2) \in \mathbb{B}^2 \mid x_1 \leq 0\}, \tag{62}$$
$$\mathbb{B}^2_+ = \{(x_1, x_2) \in \mathbb{B}^2 \mid x_1 \geq 0\}, \tag{63}$$
$$D = \mathbb{B}^2_- \cap \mathbb{B}^2_+, \partial_+ = \partial \mathbb{B}^2 \cap \mathbb{B}^2_+, \partial_- = \partial \mathbb{B}^2 \cap \mathbb{B}^2_- . \tag{64}$$

Take a homeomorphism $g_0 : C \to D$. Then extend it to a homeomorphism

$g : \partial B_1 \cup \partial B_2 \to \partial_+ \cup \partial_- \cup D$ such that $g(\partial B_1) = \partial_- \cup D$, $g(\partial B_2) = \partial_+ \cup D$. Extend further $g_{|\partial B_1}$ to a homeomorphism $\overline{g}_{|B_1} : B_1 \to \mathbb{B}^2_-$, and $g_{|\partial B_2}$ to a homeomorphism $\overline{g}_{|B_2} : B_2 \to \mathbb{B}^2_+$. We get the desired homeomorphism $\overline{g} : B_1 \cup B_2 \to \mathbb{B}^2$. ∎

3.3. *Lemma. Let* C_1, C_2 *be two* CW-*simple closed curves in* $\mathbb{R}^2$ *which meet in more than one point, with disjoint interiors* A_1, A_2. *Then there are two* CW-*simple closed curves* C', C'' *contained in* $C_1 \cup C_2$ *such that* C' *encloses all bounded connected components of* $\mathbb{R}^2 \setminus (C_1 \cup C_2)$ *and* C'' *encloses all of them but* A_2.

Proof. The proof is in three parts.

1. We consider two different points x, y of C_1 which divide it into two simple curves C_1', C_1'' with origin x and extremity y. We consider also a simple curve D with origin y and extremity x such that $D \cap \overline{A}_1 = \{x,y\}$ (fig. 1). We study the relations between A_1, the interior A_1' and exterior E_1' of $D \cup C_1'$, the interior A_1'' and exterior E_1'' of $D \cup C_1''$.

By additivity of the index, we have for every $p \in \mathbb{R}^2 \setminus (C_1 \cup D)$ (assimilating curves with singular 1-simplices) and a suitable orientation of C_1

$$\omega(D + C_1'', p) + \omega(C_1, p) = \omega(D + C_1', p). \tag{65}$$

Since A_1 is a connected subset of $\mathbb{R}^2 \setminus (D \cup C_1')$, it is contained either in A_1' or in E_1'; similarly, it is contained either in A_1'' or in E_1''. Considering (65) with $p \in A_1$, we then see that A_1 is contained either in A_1' or in A_1'', but not in both. So we can assume for instance

$$A_1 \subset A_1' \cap E_1''. \tag{66}$$

Consequently,

$$C_1'' \setminus \{x,y\} \subset \overline{A}_1 \setminus (D \cup C_1') \subset \overline{A}_1' \setminus (D \cup C_1') = A_1', \tag{67}$$

$$C_1' \setminus \{x,y\} \subset \overline{A}_1 \setminus (D \cup C_1'') \subset \overline{E}_1'' \setminus (D \cup C_1'') = E_1''. \tag{68}$$

Now considering (65) with $p \in A_1' \setminus C_1''$, we get

$$A_1' \subset A_1 \cup A_1'' \cup (C_1'' \setminus \{x,y\}). \tag{69}$$

Finally, since E'_1 is a non bounded connected subset of $\mathbb{R}^2 \setminus (D \cup C''_1)$ by (67), it has to be contained in E''_1 thus

$$A''_1 \subset A'_1 . \tag{70}$$

We can sum up (66), (67), (69) and (70) by saying that $\{A_1, A''_1, C''_1 \setminus \{x,y\}\}$ is a partition of A'_1.

2. Let us consider further a simple curve F with origin $u \in C''_1 \setminus \{x,y\}$ and extremity $v \in C'_1 \setminus \{x,y\}$, which does not meet A_1 (fig. 1). We show that F meets D. Let u' be the last point of F belonging to C''_1. If $u' \in D$, we are done. Otherwise, $u' \in A'_1$ by (67), and there is some point u" of F after u' satisfying $u'' \in A'_1 \setminus (A_1 \cup C''_1) \subset A''_1$. Now $v \in E''_1$ by (68), so F has to meet $D \cup C''_1$ at a point between u" and v. Therefore F meets D.

3. Since $C_1 \cap C_2$ is a compact CW-subspace, it has finitely many connected components $P_1, \ldots, P_k$ $(k \geq 1)$. Each P_i is either a CW-non closed simple curve with origin x_i and extremity y_i, or a point $x_i = y_i$ if $k \neq 1$. Furthermore, we can choose the indexes and orientations of the P_i's so that $C_1 = P_1 \cup Q_1 \cup P_2 \cup \ldots \cup P_k \cup Q_k$, with Q_i a CW-non closed simple curve which meets $C_1 \cap C_2$ only at its origin y_i and its extremity x_{i+1} (with the convention $x_{k+1} = x_1$). Now if R is any CW-simple curve contained in C_2 which meets C_1 at its endpoints x, y only, we must have $\{x,y\} = \{y_i, x_{i+1}\}$ for some $i \in \{1, \ldots, k\}$. Indeed otherwise there would be a point of $C_1 \cap C_2$ in each component of $C_1 \setminus \{x,y\}$, and each simple curve of C_2 joining them would have to meet R by part 2. So we can write $C_2 = P_1 \cup R_1 \cup P_2 \cup \ldots \cup P_k \cup R_k$, with R_i a CW-non closed simple curve which meets C_1 only at its origin y_i and extremity x_{i+1} (fig. 2).

In terms of singular 1-chains modulo boundaries, we have with suitable orientations of C_1 and C_2

$$C_1 \equiv P_1 + Q_1 + P_2 + \ldots + P_k + Q_k \ (\text{mod Im}\, \partial_2), \tag{71}$$

$$C_2 \equiv P_1 + R_1 + P_2 + \ldots + P_k + R_k \ (\text{mod Im}\, \partial_2). \tag{72}$$

Thus by the additivity properties of the index, for $p \in \mathbb{R}^2 \setminus (C_1 \cup C_2)$

$$\omega(C_1, p) - \omega(C_2, p) = \sum_{i=1}^{k} \omega(Q_i - R_i, p). \tag{73}$$

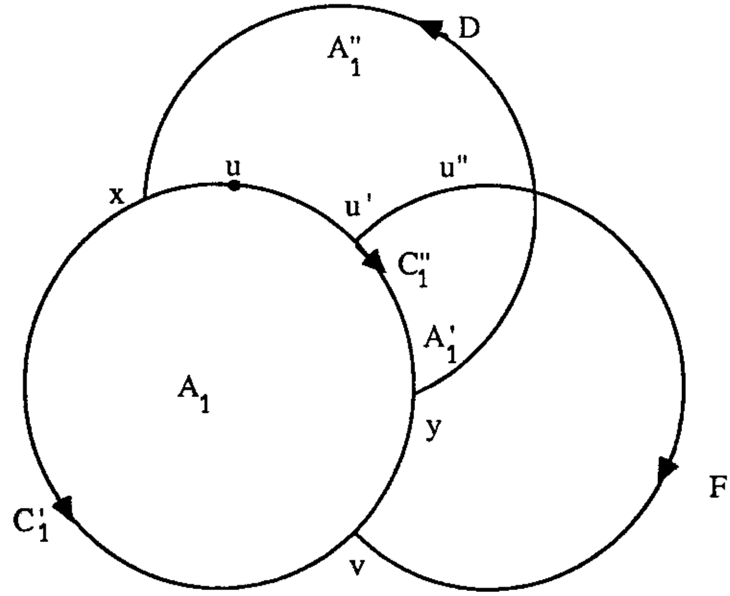

fig. 1

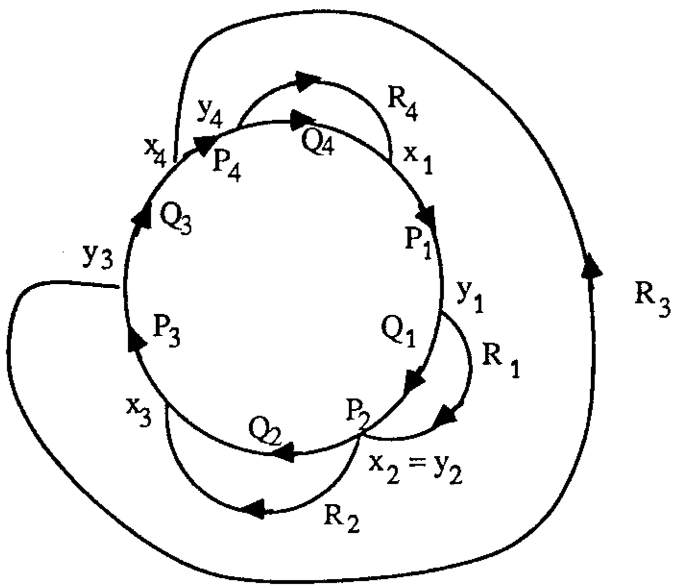

fig. 2

For each i, A_1 is contained either in the interior or in the exterior of $Q_i \cup R_i$. Then if we take $p \in A_1$ in (73), we see that there is some $j \in \{1, ..., k\}$ such that p hence A_1 is enclosed by $Q_j \cup R_j = C'$. Therefore C' encloses C_1 and by connectedness also R_1, ..., R_k (unless maybe when $k = 2$ and P_1, P_2 are points; in this case, assuming for instance $j = 2$ and $Q_2 \cup R_2$ does not enclose R_1, by part 1 either $R_1 \cup R_2$ or $R_1 \cup Q_2$ encloses the interior of $Q_2 \cup R_2$ (hence A_1 and C_1); in the first instance, we reach the contradiction $A_1 \subset A_2$, in the second instance we permute R_1, R_2). Thus the exterior of C' is a connected component of $\mathbb{R}^2 \setminus (C_1 \cup C_2)$, and all other components of $\mathbb{R}^2 \setminus (C_1 \cup C_2)$ are contained in the interior A' of C'.

Then let us put $C'' = (C_2 \setminus R_j) \cup Q_j$, so C' encloses C'' hence its interior A''. If we apply the considerations of part 1 to C'' and R_j, we see that $A' \setminus C''$ is partitioned into A_2 and A''. So C'' encloses all bounded connected components of $\mathbb{R}^2 \setminus (C_1 \cup C_2)$ but A_2. ∎

The following proposition was proved by Bing in the simplicial case ([Bi, Thm. 3]). His idea could be applied here as well, except that the CW-complex case requires to be somewhat more cautious.

3.4. *Proposition (Björner, Stanley). If* C *is a* CW-*simple closed curve with interior* A, *then* $\overline{A}$ *is extendably shellable.*

Proof. Let F_1, ..., F_k be any partial shelling of $\overline{A}$. We have to extend it to a shelling of $\overline{A}$. By repeated use of Lemma 3.2 we see that $B = F_1 \cup ... \cup F_k$ is a 2-ball. Then we induce on the number s of 2-faces of $\mathcal{T}$ which are enclosed by C but not contained in B. If $s = 0$, then $C = \partial B$ and the result is trivial.

If $s = 1$ then $\overline{A} = F_1 \cup ... \cup F_k \cup F_{k+1}$ with $F_{k+1} \in \mathcal{T}^{(2)}$. On the other hand C is the union of those edges of $\mathcal{T}$ which are contained in exactly one of the 2-faces F_1, ..., F_{k+1}, so $C = \overline{(\partial B \setminus \partial F_{k+1})} \cup \overline{(\partial F_{k+1} \setminus \partial B)}$. But since C is a simple closed curve, this ensures that $\partial B \cap \partial F_{k+1} = (F_1 \cup ... \cup F_k) \cap F_{k+1}$ is a simple curve (use the description of $C_1 = \partial B$, $C_2 = \partial F_{k+1}$ made in part 3 of Lemma 3.3 if $\partial B \cap \partial F_{k+1}$ is neither $\varnothing$ nor a point). So F_1, ..., F_{k+1} is a partial shelling.

If $s > 1$, and σ an edge of ∂B not contained in C, then the 2-face G of $\mathcal{T}$ containing σ and not contained in B is enclosed by C. We can apply Lemma 3.3 to ∂B and ∂G to get two CW-simple closed curves C', C'' contained in $\partial B \cup \partial G$ (hence enclosed by C) so that C' encloses at least F_1, ..., F_{k+1}, G, and C'' encloses exactly the 2-faces enclosed by C' except G.

We apply the induction hypothesis to F_1, ..., F_k and C'' to get a shelling F_1, ..., F_k, ..., F_t of the 2-faces of $\mathcal{T}$ enclosed by C''. Then we apply the induction hypothesis with $s = 1$ to show that F_1, ..., F_t, G is a shelling of the 2-faces of $\mathcal{T}$ enclosed by C'. A further application of the induction hypothesis to F_1, ..., F_t, G and C provides a shelling F_1, ..., F_k, ..., F_r of the 2-faces of $\mathcal{T}$ enclosed by C. ∎

3.5. *Corollary. If* C *is a* CW-*simple closed curve with interior* A, *then* $\overline{A}$ *is a* CW-2-*ball.* ∎

3.6. *Corollary. If* $\mathcal{S}$ *is a regular* CW-*decomposition of the* 2-*sphere* $\mathbb{S}^2$, $F_1, \ldots, F_k$ *a partial shelling of* $\mathcal{S}$, *and* G *a* 2-*face of* $\mathcal{S}$ *other than* $F_1, \ldots, F_k$, *then* $F_1, \ldots, F_k$ *can be extended to a shelling of* $\mathcal{S}$ *in which* G *comes last.*

Proof. Let $x \in G \setminus \partial G$, and identify $\mathbb{S}^2 \setminus \{x\}$ with $\mathbb{R}^2$. Then we can use the same argument as in the proof of Proposition 3.4, with $C = \partial G$. ∎

3.7. *Lemma. Any finite number of* 2-*faces* $F_1, \ldots, F_k$ *of* $\mathcal{T}$ *are enclosed by a* CW-*simple closed curve* C.

Proof. We induce on k. If k = 1, we take $C = \partial F_1$. If k > 1, we first construct a CW-simple closed curve C_o enclosing $F_1, \ldots, F_{k-1}$. If F_k has an edge in common with some F_i (i < k), then either F_k is already enclosed by C_o and we are finished, or we apply Lemma 3.3 to C_o and ∂F_k to get a CW-simple closed curve C enclosing both F_k and the interior of C_o. Now if F_k is any 2-face of $\mathcal{T}$ then there is a sequence $G_o, \ldots, G_r$ of 2-faces of $\mathcal{T}$ such that $G_o = F_{k-1}$, $G_r = F_k$ and G_{i-1}, G_i have at least an edge in common ($1 \le i \le r$). An application of the preceding argument to $G_1, \ldots, G_r$ successively leads to a CW-simple closed curve enclosing $F_1, \ldots, F_k$. ∎

3.8. *Theorem (Björner-Stanley). Every regular* CW-*decomposition* $\mathcal{T}$ *of* $\mathbb{R}^2$ *is simply extendably shellable.*

Proof. Let $F_1, \ldots, F_k$ be a partial shelling of $\mathcal{T}$. We put $k_o = k$. We construct successive increasing partial shellings $F_1, \ldots, F_{k_o}, \ldots, F_{k_1}, \ldots, F_{k_j}$ ($j \in \mathbb{N}$) of $\mathcal{T}$ inductively as follows.

If $F_1, \ldots, F_{k_j}$ is already constructed, then Lemma 3.7 provides a CW-simple closed curve C_j enclosing $F_1, \ldots, F_{k_j}$ as well as each 2-face of $\mathcal{T}$ sharing at least an edge with one of them. By Proposition 3.4 there is a shelling $F_1, \ldots, F_{k_j}, \ldots, F_{k_{j+1}}$ of the 2-faces of $\mathcal{T}$ enclosed by C_j.

We obtain this way a partial shelling $F_1, F_2, \ldots, F_{k_j}, \ldots$ of $\mathcal{T}$. Furthermore each 2-face G of $\mathcal{T}$ is among the F_i's. Indeed there is a sequence $G_o, \ldots, G_r$ of 2-faces of $\mathcal{T}$ such that $G_r = G$, $G_o = F_1$ and G_{i-1}, G_i have at least an edge in common ($1 \le i \le r$). So $G \in \{F_1, \ldots, F_{k_r}\}$. ∎

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