SUBJECTIVE PROBABILITY AND EXPECTED UTILITY WITHOUT ADDITIVITY

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ABSTRACT

An act maps states of nature to outcomes. Two acts f and g are comonotonic, by definition, if it never happens that $f(s) \succ f(t)$ and $g(t) \succ g(s)$ for some states of nature s and t. An axiom of comonotonic independence is introduced here. It weakens the von Neumann - Morgenstern axiom of independence as follows: If $f \succ g$ and if f, g and h are comonotonic then $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

If a nondegenerate, continuous and monotonic (state independent) weak order over acts satisfies comonotonic independence then it induces a unique non - (necessarily-) additive probability and a von Neumann-Morgenstern utility. Furthermore, expected utility with respect to the nonadditive probability, as defined here, represents the weak order over acts.

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1. INTRODUCTION

Bayesian statistics techniques are applicable when the information and uncertainty with respect to the parameters or hypotheses in question can be expressed by a probability distribution. This prior probability is also the focus of much of the criticism against the Bayesian school. My starting point is to join the critics in attacking a certain aspect of the prior probability: The probability attached to an uncertain event does not reflect the heuristic amount of information that led to the assignment of that probability. For example, when the information on the occurrence of two events is symmetric they are assigned equal prior probabilities. If the events are complementary the probabilities will be 1/2, independently of whether the symmetric information is meager or abundant.

There are two (unwritten?) rules for assigning prior probabilities to events in case of uncertainty. The first says that symmetric information with respect to the occurence of events results in equal probabilties. The second says that if the space is partitioned into k symmetric (i.e., equiprobable) events then the probability of each event is $1/k$. I agree with the first rule and object to the second. In the example above, if each of the symmetric and complementary uncertain events is assigned the index 3/7, the number, 1/7, $1/7 = 1 - (3/7 + 3/7)$, would indicate the decision maker's confidence in the probabiltiy assessment. Thus, allowing nonadditive (not necessarily additive) probabilities enables transmission or record of information that additive probabilities cannot represent.

The idea of nonadditive probabilities is not new. Nonadditive (objective) probabilties has been in use in physics for a long time (Fynman 1963). The nonadditivity describes the diffaction of elementary particles from mechanical behavior toward wave like behavior. Daniel Ellsberg (1961) presented his arguements against necessarily additive (subjective) probabilities with the help of the following "mind experiment": There are two urns with hundred black and red balls each. In urn I there are fifty balls of each color and there is no additional information about urn II. One ball is chosen at random from each urn. There are four events, denoted IR, IB, IIR, IIB, where IR denotes the event that the ball chosen from urn I is red, etc. On each of the events a bet is offered: $100 if the event occurs and zero if it does not. According to Ellsberg most decision makers are indifferent between betting on IR and betting IB and are similarly indifferent between bets on IIR and IIB. It may be that the majority are indifferent among all four bets. However, there is a nonnegligible proportion of decision makers who prefer every bet from urn I (IB or IR) to every bet from urn II, (IIB or IIR). These decision makers cannot represent their beliefs with respect to occurence of uncertain events through an additive probability.

The most compelling justification for representation of beliefs about uncertain events through additive prior probability has been suggested by Savage Building on previous works by Ramsey, de Finetti and von Neumann-Morgenstern, Savage suggested axioms for decision theory that lead to the criterion of maximization of expected utility. The expectation operation is carried out with respect to a prior probability derived uniquely from the decision maker's preferences over acts. The axiom violated by the preference of the select minority in the example above is the "sure thing principle," i.e., Savage's P2.

In this paper a simplified version of Savage's model is used. The simplification consists of the introduction of objective or physical probabilities. An act in this model assigns to each state an objective lottery over deterministic outcomes. The uncertainty concerns which state will occur. Such a model containing objective and subjective probabilities has been suggested by is Anscombe and Aumann (1963). They speak about roulette lotteries (objective) and horse lotterties (subjective). In the presentation here the version in Fishburn (1970) has been used. There can also be found the von Neumann-Morgenstern, (N-M), utility theorem used here.

The concept of objective probability is considered here as a physical concept like acceleration, momentum, or temperature; to construct a lottery with given objective probabilities (a roulette lottery) is a technical problem conceptually no different from building a thermometer. When a person has constructed a "perfect" die, he assigns a probability of 1/6 to each outcome. This probability is objective in the same sense as the temperature measured by the thermometer. Another person can check and verify the calibration of the thermometer. Similarly, he can verify the perfection of the die by measuring its dimensions, scanning it to verify uniform density, etc. ... Rolling the die many times is not necessarily the exclusive test for verification of objective probability.

On the other hand subjective or personal probability of an event is interpreted here as the number used in calculating the expectation (integral) of a random variable. This definition includes objective or physical probabilities as a special case where there is no doubt as to which number is to be used. This interpretation does not impose any restriction of additivity on probabilities, as long as it is possible to perform the expectation operation which is the subject of this work.

Subjective probability is derived from person's preferences over acts. In the Anscombe-Aumann type model usually five assumptions are imposed on preferences to define unique additive subjective probability and von Neumann-Morgenstern utility over outcomes. The first three assumptions are essentially von Neumann-Morgenstern's -- weak preorder, independence, and continuity -- and the fourth assumption is equivalent to Savage's P3, i.e., state-independence of preferences. The additional assumption is non-degeneracy: Without it uniqueness is not guaranteed.

The example quoted earlier can be embedded in such a model. There are four states: (IB, IIB), (IB, IIR), (IR, IIB), (IR, IIR). The outcomes are sums of dollars. The bet of $100 on IIB is an act which assigns the degenerate objective lottery of receiving $100 with probability one to each state in the event IIB and zero dollars with probability one to each state in the event IIR. The bet on IIR is similarly interpreted. Indifference between these two acts (bets) and the independence condition implies indifference between either of them and the act which assigns to each state the objective lottery of receiving $100 with probability 1/2 and receiving zero dollars with probability 1/2. In the spirit of the example this last act is indifferent to bet on IB (or IR), indifference not shared by the select minority.

Our first objective consists of restatement, or more specifically of weakening, of the independence condition such that the new assumption together with the other three assumptions can be consistently imposed on the preference relation over acts. In particular the special preferences of the example become admissible. It is obvious that the example's preferences between bets (acts) do not admit additive subjective probability. Do they define in some consistent way a unique nonadditive subjective probability, and if so, is there a way to define the expected utility maximization criterion for the nonadditive case?

An affirmative answer to this problem is presented in the last section. Thus the new model rationalizes nonadditive (personal) probabilities and admits the computation of expected utility with respect to these probabilities. It not only formally extends the additive model but also it makes the expected utility criterion applicable to cases where additive expected utility is not applicable.

Before turning to a precise and detailed presentation of the model, another heuristic observation is made. The nomenclature used in economics distinguishes between risk and uncertainty. Decisions in a risk situation are precisely the choices among roulette lotteries. The probabilities are objectively given; they are part of the data. For this case the economic theory went beyond von Neumann-Morgenstern utility and defined concepts of risk aversion, risk premium and certainty equivalence. Translating these concepts to the case of decisions under uncertainty we can speak about uncertainty aversion, uncertainty premium, and risk equivalence. Returning to the example, suppose that betting $100 on IIR is equivalent to betting $100 on a risky event with an (objective) probability of 3/7. Thus, the subjective probability of an event is its risk equivalent (P(IIR) = 3/7). In this example the number 1/7 computed earlier expresses the uncertainty premium in terms of risk. Note that nonadditive probability may not exhibit consistently either uncertainty aversion or uncertainty attraction. This is similar to the case of decisions in risk situations where von Neumann-Morgenstern utility (of money) may be neither concave nor convex.

2. AXIOMS AND BACKGROUND

Let X be a set and let Y be the set of distributions over X with finite supports

$$Y = \{ y: X \to [0,1] \mid y(x) \neq 0 \text{ for finitely many } x\text{'s in } X \text{ and } \sum_{x \in X} y(x) = 1 \}$$

For notational simplicity we identify X with the subset $\{ y \in Y \mid y(x) = 1$ for some x in $X \}$ of Y.

Let S be a set and let Σ be an algebra of subset of S. Both sets, X and S are assumed to be nonempty. Denote by L_0 the set of all Σ - measurable finite step functions from S to Y and denote by L_c the constant functions in L_0. Let L be a convex subset of Y^S which includes L_c. Note that Y can be considered a subset of some linear space, and Y^S, in turn, can then be considered as a subspace of the linear space of all functions from S to the first linear space. Whereas it is obvious how to perform convex combinations in Y

it should be stressed that convex combinations in Y^S are performed pointwise. I.e., for f and g in Y^S and α in $[0,1]$, $\alpha f + (1-\alpha)g = h$ where $h(s) = \alpha f(s) + (1-\alpha)g(s)$ on S.

In the neobayesian nomenclature elements of X are (deterministic) outcomes, elements of Y are random outcomes or (roulette) lotteries and elements of L are acts (or horse lotteries). Elements of S are states (of nature) and elements of Σ are events.

The primitive of a neobayesian decision model is a binary (preference) relation over L to be denoted by $\succsim$. Next are stated several properties (axioms) of the preference relation, which will be used in the sequel.

(i) Weak order. (a) For all f and g in L: $f \succsim g$ or $g \succsim f$. (b) For all f,g and h in L: If $f \succsim g$ and $g \succsim h$ then $f \succsim h$.

The relation $\succsim$ on L induces a relation also denoted by $\succsim$ on Y: $y \succsim z$ iff $y^S \succsim z^S$ where y^S denotes the constant function y on S (i.e. $\{y\}^S$). As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts, respectively, of $\succsim$.

Definition. Two acts f and g in Y^S are said to be comonotonic if for no s and t in S, $f(s) \succ f(t)$ and $g(t) \succ g(s)$.

A constant act f i.e., $f = y^S$ for some y in Y, and any act g are comonotonic. An act f statewise equivalent to a constant act i.e., $f(s) \sim y$ for all s in S, and any act g are comonotonic. If X is a set of numbers, then any two X-valued functions f and g are comonotonic iff $(f(s) - f(t))(g(s) - (g(t)) \geq 0$ for all s and t in S.

Clearly, IIR and IIB of the Introduction are not comonotonic. (Comonotonicity stands for common monotonicity.)

Next our new axiom for neobayesian decision theory is introduced.

(ii) Comonotonic independence. For all pairwise comonotonic acts f,g and h in L and for all α in $]0,1[$: $f \succ g$ implies $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha) h$.

Comonotonic independence is clearly a less restrictive condition than N-M's independence condition stated below.

(iii) Independence. for all f,g and h in L and for all α in $]0,1[$:

$f \succ g$ implies $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

(iv) Continuity. For all f, g and h in L: If $f \succ g$ and $g \succ h$ then there are α and β in $]0,1[$ such that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$.

Next, two versions of state-independence are introduced. The intuitive meaning of each of these conditions is that the preferences over random outcomes do not depend on the state that occurred. The first version is the one to be used here. The second version is stated for comparison since it is the common one in the literature.

(v) Montonicity. For all f and g in L: If $f(s) \succsim g(s)$ on S then $f \succsim g$.

(vi) Strict Monotonicity For all f and g in L , y and z in Y and E in Σ: If $f \succ g$, $f(s) = y$ on E and $g(s) = z$ on E, and $f(s) = g(s)$ on E^c, then $y \succ z$.

OBSERVATION If $L=L_0$ then (vi) and (i) imply (v).

Proof. If f and g are finite step functions then there is a finite chain $f=h_0, h_1, \ldots, h_k = g$ where each pair of consecutive functions h_{i-1}, h_i are constant on the set on which they differ. For this pair (vi) and (i) imply (v). Transitivity (i)b of $\succsim$ concludes the proof. Clearly (i) and (v) imply (vi).

For the sake of completeness we list as axiom:

(vii) Nondegeneracy. Not for all f and g in L $f \succsim g$.

Before presenting the von Neumann-Morgenstern theorem we point out that stating the axioms of (i) weak order, (iii) independece and (iv) continuity does not require that the preference relation $\succsim$ is defined on a set L containing L_c. Only the convexity of L is requied by (iii).

N-M THEOREM. Let M be a convex subset of some linear space, with a binary relation $\succsim$ defined on it. A necessary and sufficient condition for the relation $\succsim$ to satisfy (i) weak order, (iii) independence and (iv) continuity is the existence of an affine real valued function, say w, on M such that for all f and g in M : $f \succsim g$ iff $w(f) \geq w(g)$. (Affinity of w means that $w(\alpha f+(1-\alpha)g) = \alpha w(f) + (1-\alpha)w(g)$ for $0 < \alpha < 1$. Furthermore, an affine real valued function w' on M can replace w in the above statement iff there exist a positive number α and a number β such that $w'(f) = \alpha w(f) + \beta$ on M.

IMPLICATION. Suppose that a binary relation $\succsim$ on some convex subset L of Y^S with $L_c \subset L$ satisfies (i) weak order, (ii) comonotonic independence and (iv) continuity. Suppose also that there is a convex subset M of L with $L_c \subset M$ such that any two acts in M are comonotonic. Then by the N-M Theorem there is an affine function on M, to be denoted by J, which represents the binary relation $\succsim$ on M. I.e., for all f and g in M : $f \succsim g$ iff $J(f) \geq J(g)$. Clearly, if $M = L_c \equiv \{ y^S | y \in Y \}$ any two acts in M are comonotonic. Hence, if a function u is defined on Y by $u(y) = J(y^S)$ then u is affine and represents the induced preferences on Y. The affinity of u implies $u(y) = \Sigma_{x \in X} y(x)u(x)$.

When subjective probability enters into the calculation of exptected utility of an act, an integral with respect to finitely additive set function has to be defined. Denote by P a finitely additive probability measure on Σ and let a be a realvalued Σ-measurable function on S. For the special case where a is a finite step function, a can be uniquely represented by $\Sigma_{i=1}^{k} \alpha_i E_i^*$ where

$\alpha_2 > \alpha_2 > \dots > \alpha_k$ are the values that a obtains and E_i^* is the indicator function on S of $E_i \equiv \{ s \in S \mid a(s) = \alpha_i \}$ for $i=1,\dots,k$. Then

$$\int_S a dP = \sum_{i=1}^{k} P(E_i)\alpha_i .$$

The more general case where a is not finitely valued is treated as a special case of nonadditive probability.

ANSCOMBE-AUMANN THEOREM. Suppose that a preference relation $\succsim$ on $L=L_0$ satisfies (i) weak order, (iii) independence, (iv) continuity, (vi) strict monotonicty and (vii) nondegeneracy. Then there exist a unique finitely additive probability measure P on Σ and an affine real valued function u on Y such that for all f and g in L_0 : $f \succsim g$ iff

$$\int_S u(f(\cdot))dP \geq \int_S u(g(\cdot))dP.$$

Futhermore, if there exist P and u as above, then the preference relation they induce on L_0 satisfies conditions (i), (iii)), (iv), (vi) and (vii). Finally, the function u is unique up to a positive linear transformation.

3. THEOREMS

There are three apparent differences between the statement of the main result below and the A-A Theorem above:

1) Instead of strict monotonicity, monotonicity is used. It has been shown in the Observation that it does not make a difference. However for the forthcoming extention monotonicity is the natural condition. 2) Independence is replaced with comonotonic independence. 3) Finitely additive probability measure P is replaced with a nonadditive probability v. A real valued set function v on Σ is termed nonadditive probability if it satisfies the normalization conditions $v(\phi) = 0$ and $v(S) = 1$, and monotonicity i.e., for all E and G in Σ : $E \subset G$ implies $v(E) \leq v(G)$.

An additional difference is implicit in the definition of $\int_S a dv$ for v

nonadditive probability and $a = \Sigma_{i=1}^{k} \alpha_i E_i^*$ a finite step function with $\alpha_1 > \alpha_2 > \dots > \alpha_k$ and $(E_i)_{i=1}^{k}$ a partition of S. Let $\alpha_{k+1} = 0$ then define

$$\int_S a dv = \sum_{i=1}^{k} (\alpha_i - \alpha_{i+1})\, v(\textstyle\sum_{j=1}^{i} E_j)$$

For the special case of v additive the definition above coincides with the usual one mentioned in the previous section.

<u>THE THEOREM</u>. Suppose that a preference relation $\succsim$ on $L = L_0$ satisfies (i) weak order, (ii) canonotonic independence (iv) continuity, (v) monotonicity and (vii) nondegeneracy. Then there exist a unique nonadditive probability v on Σ an an affine realvalued function u on Y such that for all f and g in L_0 :

$$f \succsim g \quad \text{iff} \quad \int_S u(f(\cdot))dv \geq \int_S u(g(\cdot))dv.$$

Futhermore, if there exist v and u as above, u nonconstant, then the preference relation they induce on L_0 satisfies (i), (ii), (iv), (v) and (vii). Finally, the function u is unique up to positive linear transformations.

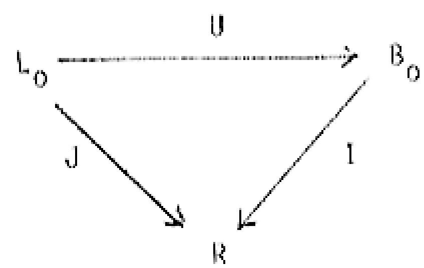
<u>PROOF</u> . From the Implication of the N-M Theorem we get a N-M utility u representing the preference relation $\succsim$ induced on Y. By nondegeneracty there are f^* and f_* in L_0 with $f^* \succ f_*$. Monotonicity, (v), implies existence of a state s in S such that $f^*(s) \equiv y^* \succ f_*(s) \equiv y_*$. Since u is given up to a positive linear transformation, suppose from now on $u(y^*) = 1$ and $u(y_*) = -1$. Denote $K = u(Y)$. Hence K is a convex subset of the real line including the interval $[-1,1]$.

For an arbitrary f in L_0 denote

$$M_f = \{\, \alpha f + (1-\alpha)y^S \mid y \in Y \text{ and } \alpha \in [0,1] \,\}$$

Thus M_f is the convex hull of the union of f and L_c. It is easy to see that any two acts in M_f are comonotonic. Hence, there is an affine real-valued function on M_f, which represents the preference relation $\succsim$ restricted to M_f. After rescaling, this function, J_f, satisfies $J_f(y^{*S}) = 1$ and $J_f(y_*^S) = -1$. Clearly, if $h \in M_f \cap M_g$ then $J_f(h) = J_g(h)$. So, defining $J(f) = J_f(f)$ for f in L_0, we get a real valued function on L_0 which represent the preferences $\succsim$ on L_0. It also satisfies for all y in Y : $J(y^S) = u(y)$. Let $B_0(K)$ denote the Σ-measurable, K-valued finite step functions on S. Let $U: L_0 \to B_0(K)$ be defined by $U(f)(s) = u(f(s))$ for s in S and f in L_0. The function U is onto, and if $U(f) = U(g)$ then, by monotonicity $f \sim g$, which in turn implies $J(f) = J(g)$.

We now define a real valued function I on $B_0(K)$. Given a in $B_0(K)$ let f in L_0 be such that $U(f) = a$. Then define $I(a) = J(f)$. I is well defined since as mentioned earlier J is constant over $U^{-1}(a)$.



We now have a realvalued function I on $B_0(K)$ which satisfies the following three conditions.

1) For all α in K: $I(\alpha S^*) = \alpha$.

2) For all pairwise comonotonic functions a,b and c in $B_0(K)$ and α in $[0,1]$: if $I(a) > I(b)$ then $I(\alpha a + (1-\alpha)c) > I(\alpha b + (1-\alpha)c)$.

3) If $a(s) \geq b(s)$ on S for a and b in $B_0(K)$, then $I(a) \geq I(b)$.

To see that 1) is satisfied let y in Y be such that $u(y) = \alpha$. Then $J(y^S) = \alpha$ and $U(y^S) = \alpha S^*$. Hence $I(\alpha S^*) = \alpha$. Similarly 2) is satisfied because comonotonicity is preserved by U and J represents $\succsim$ which satisfies comonotonic independence. Finally 3) holds because U preserves monotonicity.

The Corollary of Section 3 and the Remark following it in Schmeidler (1984), say that if a realvalued function I on $B_0(K)$ satisfies conditions 1), 2) and 3) then the nonadditive probability v on Σ defined by, $v(E) = I(E^*)$ satisfies for all a and b in $B_0(K)$:

$$* \qquad I(a) \geq I(b) \text{ iff } \int_S a\,dv \geq \int_S b\,dv.$$

Hence, for all f and g in L_0:

$$f \succsim g \text{ iff } \int_S U(f)\,dv \geq \int_S U(g)\,dv.$$

and the proof of the main part of the theorem is completed.

In order to prove the opposite direction note first that in Schmeidler (1984) is shown and referenced, that if I on $B_0(K)$ is defined by * then it satisfies conditions 1), 2) and 3). (Only 2) requires some proof.) Secondly, the assumptions of the opposite direction say that J is defined as a combination of U and I in the diagram. Hence the preference relation on L_0 induced by J satisfies all the required conditions. (U preserves monotonicity and commotonicity, and $\int_S a\,dv$ is a (sup) norm continuous function of a.)

Finally, uniqueness properties of the expected utility representation will be proved. Suppose that there exist an affine realvalued function u' on Y and a nonadditive probability v' on Σ s.t. for all f and g in L_0 :

$$** \qquad f \succsim g \text{ iff } \int_S u'(f(s))\,dv' \geq \int_S u'(g(s))\,dv'.$$

Note that monotonicity of v' can be derived instead of assumed. When considering ** for all f and g in L_c we immediately obtain from the uniqueness part of the N-M Theorem, that u' is a positive linear transformation of u. On the other hand it is obvious that the inequality in ** is preserved under positive linear transformations of the utility. Hence, in order to prove that $v' = v$ we may assume without loss of generality that $u' = u$. For an arbitrary E in Σ let f in L_0 be such that $U(f) = E^*$. (For example, $f(s) = y^*$ on E and $f(s) = y^*/2 + y_*/2$ on E^c. Then $\int_S U(f)dv = v(E)$ and $\int_S U(f)dv' = v'(E)$.) Let y in Y be such that $u(y) = v(E)$. (For example, $y = V(E)y^* + (1-v(E))(y^*/2 + y_*/2)$.) Then $f \sim y^S$ which in turn implies $u(y) = u'(y) = \int_S u'(y^S)dv' = v'(E)$. The last equality is implied by ** .

Q.E.D.

In order to extend the Theorem to more general acts we have to specify preciesly the set of acts L on which the extension holds and we have to correspondingly extend the definition of integral with respect to nonadditive probability. We start with the latter.

Denote by B the set of realvalued, bounded Σ-measurable functions on S. Given a in B and a nonadditive probabilty v on Σ we define.

$$\int_S a dv = \int_{-\infty}^{0} (v(a>\alpha)-1)d\alpha + \int_{0}^{\infty} v(a>\alpha)d\alpha$$

Each of the integrands above is monotonic, bonded and identically zero for $|\alpha| > \lambda$ for some number λ. A more detailed exposition and references appear in Schmeidler (1984). It should be mentioned here that this definition coincides, of course, with the one at the beginning of this section when a obtains finitely many values.

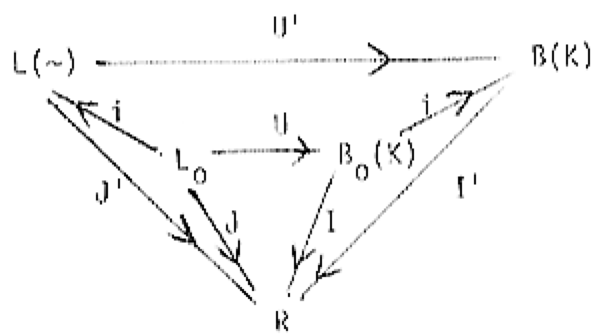
For the next definition existence of weak order $\succsim$ over L_c is presupposed. An act $f:S \to Y$ is said to be Σ-measurable if for all y in Y the sets $\{f(s) \succ y\}$ and $\{f(s) \succsim y\}$ belong to Σ. It is said to be bounded if there are y and z in Y such that $y \succsim f(s) \succsim z$ on S. The set of all Σ-measurable bounded acts in Y^S is dentoed by $L(\succsim)$. It clearly contains L_0.

<u>COROLLARY</u> (a) Suppose that a preference relation $\succsim$ over L_0

satisfies (i) weak order, (ii) comonotonic independence, (iv) continuity and (v) monotonicity. Then it has a unique extention to all of $L(\succsim)$ which satisfies the same conditions (over $L(\succsim)$).

(b) If the extended relation, to be also denoted by $\succsim$, is nondegenerate then there exist a unique nonadditive probabilty v on Σ and an affine realvalued function u (unique up to positive linear transformations) such that for all f and g in $L(\succsim)$: $f \succsim g$ iff $\int_S u(f(\cdot))dv \geq \int_S u(g(\cdot))dv$.

Proof: The case of degeneracy is obvious, so assume nondegenerate preferences. Consider the diagram below:



The inner triange is that of the proof of the Theorem. $B(K)$ is the set of K-valued, Σ-measurable, bounded functions on S, and i denotes identity. U' is the natural extention of U and is also onto. Because $B_0(K)$ is (sup) norm dense in $B(K)$ and I satisfies condtion 3), I' is the unique extention of I that satisfies on $B(K)$ the three conditions that I satisfies on $B_0(K)$.

The functional J' is defined on $L(\succsim)$ by: $J'(f) = I'(U'(f))$. Clearly J' extends J. Hence, the relation $\succsim$ on $L(\succsim)$ defined by: $f \succsim g$ iff $J'(f) \geq J'(g)$ extends the relation $\succsim$ on L_0, and satisfies the desired properties.

By the Corollary of section 3 in Sch. (1984) there exist a nonadditive probability v on Σ s.t. for all f and g in $L(\succsim)$: $I'(f) \geq I'(g)$ iff $\int_S U'(f)dv \geq \int_S U'(g)dv$.

Hence, the expected utility representation of the preference relation has been shown. To complete the proof of (b), uniqueness of v and uniqueness up to a positive linear transformation of u have to be established. However, it follows from the corresponding part of the Theorem. The uniqueness properties also imply that the extension of $\succsim$ from L_0 to $L(\succsim)$ is unique. Q.E.D.

Remark 1. Instead of first stating the Theorem for L_0 and then extending it to $L(\succsim)$, one can state directly the extended theorem. More precisely a preference relation on L, $L_0 \subset L \subset Y^S$ is defined s.t. in addition to the conditions (i), (ii), (iv), (v) and (vii) it satisfies $L = L(\succsim)$. Then it can be reprsented by expected utility with respect to nonadditive probability. However the first part of the Corollary shows that in this case the preference relation of $L(\succsim)$ is overspecified: The preferences over L_0 dictate those over $L(\succsim)$.

Remark 2. If Σ does not contain all subsets of S, and $\#X \geq 3$ then $L(\succsim)$ contains finite step functions that do not belong to L_0. Let y and z in Y be such that $y \sim z$ but $y \neq z$, and let $E \subset S$ but $E \notin \Sigma$. Define $f(s) = y$ on E and $f(s) = z$ on E^c. Clearly $f \notin L_0$. The condition $\#X \geq 3$ is required to guarantee existence of y and z as above.

Remark 3. It is an elementary exercise to show that under the conditions of the Theorem, v is additive iff $\succsim$ satisfies (iii) independence (instead of or in addition to (ii) comonotonic independence). Also an extention of an independent relation, as in the Corollary (a), is independent. Hence our results formally extend the additive theory.

We now introduce formally the concept of uncertainty aversion alluded to in the Introduction. A binary relation $\succsim$ on L is said to reveal uncertainty aversion if for any three acts f, g and h in L and any α in $[0,1]$: If $f \succsim h$ and $g \succsim h$ then $\alpha f + (1 - \alpha)g \succsim h$. Equivalently we may state: If $f \succsim g$ then $\alpha f + (1 - \alpha)g \succsim g$. For definition of strict risk aversion the conclusion should be a strict preference $\succ$. However some restrictions have to be imposed then on f and g. One such an obvious restriction in that f and g are not comonotonic. We will return to this question in a subsequent remark.

Intuitively, uncertainty aversion means that "smoothing" or averaging <u>utility</u> distributions makes the decision maker better of. Another way is to say that substituting objective mixing for subjective mixing makes the decision maker better of. The definition of uncertainity aversion may become more transparent when its full mathematical characterization is presented.

PROPOSITION. Suppose that $\succsim$ on $L = L(\succsim)$ is the extension of $\succsim$ on L_0 according to the Corollary. Let v be the derived nonadditive subjective probability and $I \equiv I'$ is the functional on B, $I(a) = \int_S a dv$. Then the following conditions are equivalent:

(i) $\succsim$ reveals uncertainty aversion.

(ii) For all a and b in B: $I(a + b) \geq I(a) + I(b)$.

(iii) For all a and b in B and for all α in $[0,1]$.

$$I(\alpha a + (1 - \alpha)b) \geq \alpha I(a) + (1 - \alpha)b .$$

(iv) For all a and b in B and for all α in $[0,1]$:

$$I(\alpha a + (1 - \alpha)b) \geq \min\{I(a), I(b)\}.$$

(v) For all α in R the set $\{ a \in B \mid I(a) \geq \alpha \}$ is convex.

(vi) There exists an $\bar{\alpha}$ in R s.t. the set $\{ a \in B \mid I(a) \geq \bar{\alpha} \}$ is convex.

(vii) For all a and b in B and for all α in $[0,1]$:

If $I(a) = I(b)$ then $I(\alpha a + (1 - \alpha)b) \geq I(a)$.

(viii) For all a and b in B: If $I(a) = I(b)$ then

$$I(a + b) \geq I(a) + I(b) .$$

(ix) v is convex. I.e., for all E and F in Σ:

$$v(E) + v(F) \leq v(EF) + v(E + F) .$$

(x) For all a in B: $I(a) = \min\{ \int_S a dp \mid p \in \text{core}(v) \}$, where $\text{core}(v) = \{ p:\Sigma \to R \mid p$ is additive, $p(S) = v(S)$ and for all E in Σ, $p(E) \geq v(E) \}$.

Proof. For any functional on B: (iii) implies (iv), (iv) implies (vii), (iv) is equivalent to (v) and (v) implies (vi). The positive homogeneity of degree one of I results in, (ii) equivalent to (iii) and (vii) equivalent to (viii). (vi) implies (v) because for all β in R, $(\beta = \alpha - \bar{\alpha})$, $I(a + \beta S^*) = I(a) + \beta$, and because adding βS^* preserves convexity.

(viii) implies (ix). Suppose, without loss of generality, that $v(E) \geq v(F)$. Then there is $\gamma \geq 1$ s.t. $v(E) = \gamma v(F)$. Since $I(E^*) = v(E) = \gamma v(F) = I(\gamma F^*)$, we have by (viii), $v(E) + \gamma v(F) \leq I(E^* + \gamma F^*)$. But $E^* + \gamma F^* = (EF)^* + (\gamma - 1)F^* + (E + F)^*$, which implies $I(E^* + \gamma F^*) = v(EF) + (\gamma - 1)v(F) + v(E + F)$. Inserting the last equality in the inequality above leads to the inequality in (ix). The equivalence of (ix), (x) and (ii) is stated as Proposition 3 in Sch. (1984).

Last but not least, (i) is equivalent to (iv). This is obvious when considering the mapping U' from the diagram in the proof of the Corollary.

Q.E.D.

The basic result of the proposition is the equivalence of (i), (iii), (iv), (ix) and (x). (iv) is quasiconcavity of I and it is the translation of (i) by U' from L to B. (iii) is concavity, which usually is a stronger assumption. Here I is concave iff it is quasiconcave. Concavity captures best the heuristic meaning of uncertainty aversion.

Remark 4. The Proposition holds if all the inequalities are strict (and in (i) it is strict uncertainty aversion). To do it precisely null or dummy events in

Σ have to be defined. An event E in Σ is termed dummy if for all F in Σ: $v(F + E) = v(F)$. In (ii)-(vii), in order to state strict inequality one has to assume that a and b' are not comonotonic for any b' which differs from b on a dummy set. To write strict inequality in (ix) one has to assume that $(E - F)^*$, $(EF)^*$ and $(F - E)^*$ are not dummies. In (x) a geometric condition on the core of v has to be assumed.

Remark 5. The point of view of this work is that if the information is too vague to be represented by an additive prior, it still may be represented by a nonadditive prior. Another possibility is to represent vague information by set of priors. Condition (x) and its equivalence to other conditions of the Proposition point out when the two approaches coincide.

Remark 6. The concept of uncertainty appeal can be defined by: $f \succsim g$ implies $f \succsim f + (1 - \alpha)g$. In the Proposition then all the inequalities have to be reversed and maxima have to replace minima. Obviously, additive probability or the independence axiom reveal uncertainty neutrality.

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