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EXACT SOLUTIONS OF HYDRODYNAMIC TYPE EQUATIONS HAVING INFINITELY MANY CONSERVED DENSITIES

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Abstract. We give a scheme to construct exact solutions of hydrodynamic type equations which possess an infinite number of conserved densities. The solutions are given in an implicit (hodograph) form which is expressed only in terms of the conserved densities.

There has been significant progress in the study of hydrodynamic type equations having infinitely many symmetries [1–5], which take the following form,

$$\frac{\partial W}{\partial T_n} = A_n \frac{\partial W}{\partial X}, \quad \text{for} \quad n = 1, 2, \ldots, \tag{1}$$

where $W = (W_1, \ldots, W_N)^t \in \mathbf{R}^N$ and A_n is an $N \times N$ matrix function of W only. Here we assume $A_1 = I = N \times N$ identity matrix, that is, the systems (1) are translationally invariant. By the term symmetries, we mean that (1) are compatible, i.e. $\partial^2 W/\partial T_n \partial T_m = \partial^2 W/\partial T_m \partial T_n$ (the flows commute). The systems (1) include many interesting equations such as,

a) N-reduction of Benney's moment equations (the dispersionless KP hierarchy [6]),
b) g-gap averaged equation of the KdV equation ($N = 2g + 1$) [7],
c) gas-dynamical equations with additional symmetries [3,8].

The purpose of this letter is to construct exact solutions of (1) in an implicit (hodograph) form which is a point transformation of the dependent and independent variables. The hodograph solutions of hydrodynamic type equations (1) have been found first by Tsarev [2], and later by Gibbons and Kodama [9, 10]. They presented the schemes for constructing the solutions, but did not give any explicit formula for the solutions. In this letter, we derive the explicit and simple formulae for the hodograph solutions which are given only in terms of the conserved densities. This may give a meaning of the integrability of (1), which is usually defined as the existence of infinitely many conserved densities (or symmetries). Throughout this letter, we assume that (1) has infinitely many conserved densities, say $\{H_n(W)\}_{n=1}^{\infty}$. Consequently, (1) may be written in Hamiltonian form (if there exists),

$$A_n \frac{\partial W}{\partial X} = \mathcal{J}(\nabla H_n)^t, \quad \text{for} \quad n = 1, 2, \ldots, \tag{2}$$

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where $\mathcal{J}$ is a (nondegenerate) Hamiltonian operator (see e.g. [3]) and $\nabla H_n = (\partial H_n/\partial W_1, \cdots, \partial H_n/\partial W_N)$. For given systems (1), it may not be easy to find the Hamiltonian structure in (2), and sometimes there is no Hamiltonian formalism. In our discussion, we only need the following relations among H_n's (instead of the explicit form (2)),

$$\frac{\partial H_n}{\partial T_m} = \frac{\partial H_m}{\partial T_n} \quad , \quad \text{for} \quad n, m = 1, 2, \ldots \tag{3}$$

which may be derived from the compatibility conditions for (1). We also assume that the first N conserved densities are functionally independent, that is, the Jacobian does not vanish

$$\frac{\partial(H_1, \cdots, H_N)}{\partial(W_1, \cdots, W_N)} \neq 0 \,, \tag{4}$$

so that $\dim(\text{Span}\{\nabla H_i\}_{i=1}^{\infty}) = \dim(\text{Span}\{A_i\}_{i=1}^{\infty}) = N$. This can hold at least locally. It should be noted that these assumptions are not very restrictive; in particular, they are all satisfied for the examples mentioned earlier.

Under these assumptions (3) and (4), we first note that for each $l \geq 1$ the matrix A_{N+l} in the $(N + l)$ th flow lies in the span of $\{A_i\}_{i=1}^{N}$, i.e.

$$A_{N+l} = \sum_{i=1}^{N} \mu_l^i \, A_i \tag{5}$$

where $\mu_l^i = \mu_l^i(W)$ are scalar functions of $W \in \mathbf{R}^N$. (This follows from the remark after (4).) Equivalently, in terms of H_n's, we have

$$\nabla H_{N+l} = \sum_{i=1}^{N} \mu_l^i \nabla H_i \,. \tag{6}$$

This can be obtained from (5) and (3) with $\partial H_n/\partial X = \partial H_1/\partial T_n$ (giving $\nabla H_n = \nabla H_1 A_n$). Because of (4), the higher conserved densities H_{N+l} are expressible in terms of the first N densities, $\{H_i\}_{i=1}^{N}$, i.e. $H_{N+l} = K_l(H_1, \cdots, H_N)$. Therefore we obtain, from (6),

$$\mu_l^i = \frac{\partial K_l}{\partial H_i} \,, \quad \text{for} \begin{cases} i = 1, \cdots, N \\ l = 1, 2, \cdots \end{cases} . \tag{7}$$

We now, as in [10], show that the function μ_l^i gives a hodograph solution of (1). From (5), the $(N + l)$th flow in (1) can be written by

$$\frac{\partial W}{\partial T_{N+l}} = \sum_{i=1}^{N} \mu_l^i \frac{\partial W}{\partial T_i} \quad , \quad \forall\, l \geq 1, \tag{8}$$

which implies that W is a constant vector along the characteristics,

$$\frac{dT_{N+l}}{-1} = \frac{dT_i}{\mu_l^i(W)} \quad , \quad \text{for} \quad i = 1, \cdots, N. \tag{9}$$

The integrals of (9) are straight lines given by

$$T_i^0 = T_i + \mu_l^i(W) T_{N+l} \ , \tag{10}$$

where T_i^0 determine the initial positions of the characteristics in the space $(T_1, \cdots, T_N)$ at $T_{N+l} = 0$ for all $l \geq 1$. Then the solution of (1) can be written in the form,

$$W(T_1, T_2, \cdots) = W^0(T_1^0, \cdots, T_N^0), \tag{11}$$

where W^0 is the initial function W at $T_{N+l} = 0$ for all $l \geq 1$, i.e. $W^0(T_1, \cdots, T_N) = W(T_1, \cdots, T_N, 0, \cdots)$, and $W^0(T_1^0, \cdots, T_N^0)$ satisfies the first N equations in (1),

$$\frac{\partial W^0}{\partial T_n^0} = A_n(W^0) \frac{\partial W^0}{\partial X^0} \quad , \quad \text{for} \ n = 1, \cdots, N. \tag{12}$$

Since the systems (1) are translationally invariant under $T_{N+l} \longrightarrow T_{N+l} + C_l$ with arbitrary constant C_l, the hodograph solutions of (12) can be obtained from (10), that is,

$$T_i^0 = \mu_l^i(W) C_l \quad , \quad \forall \, l \geq 1 \, . \tag{13}$$

Consequently, we obtain from (7) and (13) the hodograph solutions in terms of the conserved densities,

$$T_i^0 = \frac{\partial}{\partial H_i} \sum_{l=1}^{\infty} C_l \ K_l(H_1, \cdots, H_N), \quad \text{for} \ \ i = 1, \cdots N, \tag{14}$$

where $H_{N+l} = K_l(H_1, \cdots, H_N)$. Note that from (14), we have the hodograph equations of (3),

$$\frac{\partial T_n^0}{\partial H_m} = \frac{\partial T_m^0}{\partial H_n}, \quad \text{for} \quad n, m = 1, \cdots, N \, . \tag{15}$$

Thus the conserved densities are directly connected to the solutions of (1), and the existence of infinitely many integrals derives a large class of solutions. It is also interesting to note that the formula (14) may be useful to derive a general form of the conserved densities (see the example discussed below). In a future communication, we will study the class of solutions given by (14), which obviously includes shock waves and simple waves. It may be also interesting to discuss some geometrical meaning of (14).

We now apply the result (14) to the following well-known examples: (Hereafter we omit the zeros in the superscript in (12) and (14).)

i) For $N = 2$ we study the equations of gas dynamics [3],

$$\frac{\partial}{\partial T_2}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} W_2 & W_1 \\ f(W_1) & W_2 \end{pmatrix} \frac{\partial}{\partial X}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} , \tag{16}$$

where W_1 and W_2 represent the density and the velocity, respectively, and the function $f(W_1)$ is related to the pressure $P(W_1)$ according to the equation $f(W_1) = P'(W_1)/W_1$. Infinitely many conserved densities of (16) are found explicitly in [3]; and they are

$$\left.\begin{aligned} H_1 &= W_1 \ , \ H_2 = W_1 W_2 \ , \\ H_3 &= \frac{1}{2}\ W_1 W_2^2 + F(W_1) \\ &= \frac{1}{2}\ H_1^{-1} H_2^2 + F(H_1) \ , \\ H_4 &= \frac{1}{6}\ W_1 W_2^3 + W_2 F(W_1) \\ &= \frac{1}{6} H_1^{-2} H_2^3 + H_1^{-1} H_2 F(H_1) \end{aligned}\right\} \tag{17}$$

where $F''(W_1) = d^2F/dW_1^2 = f(W_1)$. In this case, the hodograph solutions given by (14) are

$$\left.\begin{aligned} T_1 = X &= \frac{\partial}{\partial H_1}(C_1 H_3 + C_2 H_4 + \cdots) \\ &= C_1 \left(-\frac{1}{2} W_2^2 + F'(W_1)\right) + C_2 \left(-\frac{1}{3} W_2^3 - \frac{W_2}{W_1} F(W_1) + W_2 F'(W_1)\right) \\ &\qquad + \cdots , \\ T_2 &= \frac{\partial}{\partial H_2}(C_1 H_3 + C_2 H_4 + \cdots) \\ &= C_1\ W_2 + C_2 \left(\frac{1}{2} W_2^2 + \frac{1}{W_1} F(W_1)\right) + \cdots , \end{aligned}\right\} \tag{18}$$

where C_l's are arbitrary constants. For this example, our result (14) can be easily and directly verified from the hodograph equations of (16): For the point transformation, $X = X(W_1, W_2)$ and $T_2 = T_2(W_1, W_2)$, we have

$$\begin{pmatrix} -\frac{\partial X}{\partial W_2} \\ \frac{\partial X}{\partial W_1} \end{pmatrix} = \begin{pmatrix} W_2 & W_1 \\ f(W_1) & W_2 \end{pmatrix} \begin{pmatrix} \frac{\partial T_2}{\partial W_2} \\ -\frac{\partial T_2}{\partial W_1} \end{pmatrix} , \tag{19}$$

where we have used $\partial W_1/\partial T_2 = -J \partial X/\partial W_2$, $\partial W_1/\partial X = J\ \partial T_2/\partial W_2$, $\partial W_2/\partial T_2 = J\ \partial X/\partial W_1$ and $\partial W_2/\partial X = -J\ \partial T_2/\partial W_1$, with the Jacobian $J = \partial(W_1, W_2)/\partial(X, T_2)$.

Using the change of variables $H_1 = W_1$ and $H_2 = W_1 W_2$, we obtain

$$
(20) \qquad \left.\begin{aligned}
\frac{\partial X}{\partial H_2} &= \frac{\partial T_2}{\partial H_1} \quad , \\
\frac{\partial X}{\partial H_1} &= -2\frac{H_2}{H_1}\frac{\partial T_2}{\partial H_1} - \frac{H_2^2}{H_1^2}\frac{\partial T_2}{\partial H_2} + H_1 f(H_1)\frac{\partial T_2}{\partial H_2} \ .
\end{aligned}\right\}
$$

(Note that the first equation in (20) gives (15).) The equation (20a) implies that there exists a function $K(H_1, H_2)$ satisfying $X = T_1 = \partial K/\partial H_1$ and $T_2 = \partial K/\partial H_2$. One may also show from (20b) that $\partial K/\partial T_2$ is a total derivative with respect to X, i.e. K is a conserved density. This result maybe useful to derive a general form of the conserved density. As an example, we take the Born-Infeld equation in the Riemann invariant form [12],

$$
(21) \qquad \frac{\partial}{\partial T_2}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_2 & 0 \\ 0 & V_1 \end{pmatrix} \frac{\partial}{\partial X}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}
$$

which can be transformed into the gas dynamic equation (16) with $f(W_1) = W_1^{-3}$ and $V_1 = W_2 + W_1^{-1}, V_2 = W_2 - W_1^{-1}$ [13]. From the hodograph equations of (21)which give a wave equation for T_2, we find

$$
(22) \qquad \left.\begin{aligned}
T_2 &= \phi(V_1) + \psi(V_2) \ , \\
T_1 &= X = \Phi(V_1) - V_1\phi(V_1) + \Psi(V_2) - V_2\psi(V_2),
\end{aligned}\right\}
$$

where $\phi(V_1)$ and $\psi(V_2)$ are arbitrary functions of V_1 and V_2, respectively, and $\Phi' = \phi, \Psi' = \psi$. Then using (14), $T_i = \partial K/\partial H_i (i = 1, 2)$, we obtain $H(V_1, V_2) = K(H_1, H_2)$.

$$
(23) \qquad H(V_1, V_2) = 2(\Phi(V_1) + \Psi(V_2))/(V_1 - V_2).
$$

(This, of course, can also be derived from the equation for the conserved density, $\partial H/\partial T_2 = \partial G/\partial X$ with some function $G(V_1, V_2)$.)

ii) For $N = 3$, we study the Lax reduction of Benney's moment equations [6, 9, 10];

$$
(24) \qquad \left.\begin{aligned}
\frac{\partial}{\partial T_2}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ -2W_1 & 0 & 1 \\ -W_2 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial X}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}, \\
\frac{\partial}{\partial T_3}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} &= \begin{pmatrix} -W_1 & 0 & 1 \\ -W_2 & -W_1 & 0 \\ 0 & -W_2 & W_1 \end{pmatrix} \frac{\partial}{\partial X}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}.
\end{aligned}\right\}.
$$

The conserved densities of the systems (24) satisfying (3) are

$$\left.\begin{aligned}
&H_1 = W_1\ ,\ H_2 = W_2,\ H_3 = -\frac{1}{2}W_1^2 + W_3 \\
&H_4 = 0\ , \\
&H_5 = -\frac{1}{2}W_1^3 + W_1W_3 + \frac{1}{2}W_2^2 = H_1H_3 + \frac{1}{2}H_2^2\ , \\
&H_6 = -W_1^2W_2 + W_2W_3 = -\frac{1}{2}H_1^2H_2 + H_2H_3\ , \\
&\cdots\ .
\end{aligned}\right\} \tag{25}$$

We then obtain

$$\left.\begin{aligned}
T_1 = X &= \frac{\partial}{\partial H_1}(C_1H_5 + C_2H_6 + \cdots) \\
&= C_1\left(-\frac{1}{2}W_1^2 + W_3\right) - C_2W_1W_2 + \cdots\ , \\
T_2 &= \frac{\partial}{\partial H_2}(C_1H_5 + C_2H_6 + \cdots) \\
&= C_1W_2 + C_2(-W_1^2 + W_3) + \cdots\ , \\
T_3 &= \frac{\partial}{\partial H_3}(C_1H_5 + C_2H_6 + \cdots) \\
&= C_1W_1 + C_2W_2 + \cdots\ .
\end{aligned}\right\} \tag{26}$$

Note that T_3 consists of the conserved densities themselves. Furthermore, one can show [6] that the hodograph solutions (26), $T_i = T_i^0(W_1, \cdots, W_3),\ i = 1, 2, 3$, satisfy

$$\frac{\partial T_3^0}{\partial T_2} = \frac{\partial T_1^0}{\partial X}\ ,\quad \frac{\partial T_3^0}{\partial T_3} = \frac{\partial T_2^0}{\partial X}\ . \tag{27}$$

Thus, T_1^0 and T_2^0 are the fluxes for T_2- and T_3- flows, respectively. (This is true for any Lax reduction of Benney's Equation [9–11].)

In a later publication, we will study the class of solutions derived from (14), and the applications of our results to physically interesting initial (as well as boundary) value problems.

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