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A METHOD FOR SOLVING THE DISPERSIONLESS KP HIERARCHY AND ITS EXACT SOLUTIONS II

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Abstract. We give an algebraic method to construct an infinite number of exact solutions of the KP hierarchy in terms of the hodograph transform. These solutions are obtained from a unique decomposition of the matrices in the higher commuting flows. The method presented here can be also applied to any hydrodynamic type equations with additional symmetries.

1. This paper is an extension of the previous paper [1] concerning a solution method of the dispersionless KP equation,

$$(U_{T_3} - UU_X)_X = U_{T_2T_2} \tag{1}$$

where $U = U(T_1, T_2, \cdots)$ and $T_1 = X$. We constructed several solutions of (1) using reductions to finitely many dependent variables and the hodograph transform. However, we discussed only the first few reductions ($N \leq 2$). In this letter, we study the general reduction and give an explicit scheme for constructing exact solutions of the reduced equations of the dispersionless KP hierarchy. The scheme presented here is totally algebraic. The main result in this letter is that in the case of N dependent variables (N-reduction), these solutions are obtained from a unique decomposition of the higher flows into the first N independent flows.

2. The dispersionless KP hierarchy can be derived from the following scheme [2]: Let k be an asymptotic series with respect to a parameter P;

$$k = P + \frac{U_1}{P} + \frac{U_2}{P^2} + \dots \ , \tag{2}$$

where $U_i = U_i(T_1, T_2, \cdots)$ with infinitely many independent variables T_n. Consider the following evolution equations for $k = k(U_i; P)$ with respect to T_n's,

$$\frac{\partial k}{\partial T_n} = \{Q_n, k\} := \frac{\partial Q_n}{\partial P}\frac{\partial k}{\partial X} - \frac{\partial Q_n}{\partial X}\frac{\partial k}{\partial P} \ , \tag{3}$$

where Q_n is the part, polynomial in P, of k^n/n; we denote $Q_n := (k^n/n)_+ = nth$ degree polynomial in P. It should be noted that (3) are Benney's moment equations [3]. Now the equations in (3) are compatible, i.e. $\partial^2 k/\partial T_n \partial T_m = \partial^2 k/\partial T_m \partial T_n$, and give the dispersionless KP hierarchy.

$$\frac{\partial Q_n}{\partial T_m} - \frac{\partial Q_m}{\partial T_n} + \{Q_n, Q_m\} = 0 \ . \tag{4}$$

The dispersionless KP equation (1) is obtained from (4) with $n = 2$ and $m = 3$. It should be also noted that (3) gives an infinite number of equations for $\{U_i\}_{i=1}^{\infty}$.

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3. We now, as in [1], reduce (3) into a set of equations with finitely many dependent variables, say $\{W_i\}_{i=1}^N$ and call it N-reduction, that is, $k = k(W_1, \cdots, W_N; P)$, so the reduced equations are

$$\frac{\partial W}{\partial T_n} = A_n \frac{\partial W}{\partial X} \quad , \quad \text{for} \quad n = 1, 2, \ldots \tag{5}$$

where $W \in \mathbf{R}^N, A_1 = I = N \times N$ identity matrix and $A_n = A_n(W)$ is an $N \times N$ matrix function of W. There are several examples of reductions:

Example 1 (Lax reduction [4]). This reduction is given by

$$\frac{1}{N+1}\left(k^{N+1}\right)_+ = \frac{1}{N+1}k^{N+1} = \frac{1}{N+1}P^{N+1} + W_1 P^{N-1} + \cdots + W_N, \tag{6}$$

which includes the dispersionless KdV equation ($N = 1$), $W_{1,T_3} = W_1 W_{1,X}$, the dispersionless Boussinesq equation ($N = 2$),

$$\frac{\partial}{\partial T_2}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -W_1 & 0 \end{pmatrix} \frac{\partial}{\partial X}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \tag{7}$$

and higher order dispersionless Lax equations ($N \geq 3$).

Example 2 (Zakharov reduction [5]). This reduction is given by

$$k = P + \sum_{i=1}^{M} \frac{h_i}{P - V_i} \tag{8}$$

which can be obtained from the classical limit of the M component vector nonlinear Schrödinger equation. For $M = 1$ [1, 6], the reduction gives the classical shallow water equations [7],

$$\frac{\partial}{\partial T_2}\begin{pmatrix} h_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} V_1 & h_1 \\ 1 & V_1 \end{pmatrix} \frac{\partial}{\partial X}\begin{pmatrix} h_1 \\ V_1 \end{pmatrix}. \tag{9}$$

Many other reductions can be found, but there seems to be no systematic method for classifying them [8].

3. Let us discuss some of the properties of the reduced equations (5). Here we assume that the matrix A_2 is diagonalizable.

PROPOSITION 1. *The system (5) may be reduced to Riemann invariant form.*

Proof. From (3) with the reduction $k = k(W; P)$ and $W \in \mathbf{R}^N$, we have, for $n = 2$,

$$\nabla k (A_2 - PI) = -\nabla Q_2 \frac{\partial k}{\partial P} , \tag{10}$$

where $\nabla k = (\partial k/\partial W_1, \cdots, \partial k/\partial W_N)$. From (10), we see that the eigenvalues of A_2, say $P_1, \cdots, P_N$, must be roots of $\partial k/\partial P = 0$, and the left eigenvectors are given by $\nabla k_i = \nabla k(W; P_i)$. That is, the characteristic polynomial of A_2 must divide the function $\partial k/\partial P = 0$. Then the Riemann invariants are given by $k_i = k(W; P_i)$, and

$$\frac{\partial k_i}{\partial T_2} = P_i \frac{\partial k_i}{\partial X} \quad , \text{ for } i = 1, \cdots, N, \tag{11}$$

and for the T_n-flow we have, similarly

$$\frac{\partial k_i}{\partial T_n} = v_i^n \frac{\partial k_i}{\partial X} \quad , \text{ for } \begin{cases} i = 1, \cdots, N \, , \\ n = 1, 2, \cdots \, , \end{cases} \tag{12}$$

where the characteristic speeds v_i^n are given by the polynomial $\partial Q_n/\partial P$ at $P = P_i$, i.e.

$$v_i^n = v^n(P_i) = \frac{\partial Q_n}{\partial P}\Big|_{P=P_i} \qquad \square \tag{13}$$

REMARK. It may be shown that the polynomials $v^n(P)$ satisfy the following recurrence

$$v^{n+1} = Pv^n + H_1 v^{n-1} + \cdots + H_{n-1} v^1 \, , \tag{14}$$

where the H_n are the conserved densities of (3) which can be obtained by the inverse of $k(P)$;

$$P = k - \frac{H_1}{k} - \frac{H_2}{k^2} - \cdots \quad . \tag{15}$$

This is frequently useful for computations [9].

PROPOSITION 2. *The matrix A_n is given by the polynomial $v^n(P) = \partial Q_n/\partial P$ with A_2 substituted for P, i.e. $A_n = v^n(A_2)$.*

Proof. Define a matrix K consisting of the left eigenvectors ∇k_i, i.e.

$$K = \begin{pmatrix} \nabla k_1 \\ \vdots \\ \nabla k_N \end{pmatrix} = (K_{ij}) = \left(\frac{\partial k_i}{\partial W_j} \right) \, . \tag{16}$$

Then we have $KA_2 = DK$ with the diagonal matrix $D = \text{diag }(P_1, \cdots, P_N)$, and using $\nabla k_i A_n = v_i^n \nabla k_i$, we obtain

$$\begin{aligned} A_n &= K^{-1} \begin{pmatrix} v_1^n & & O \\ & \ddots & \\ O & & v_N^n \end{pmatrix} K \\ &= K^{-1} v^n(D) K = v^n(K^{-1}DK) = v^n(A_2) \qquad \square \end{aligned} \tag{17}$$

This gives a direct proof that the A_n all commute. From (17), we note that A_n is an $(n-1)st$ degree polynomial in A_2. Thus we obtain:

PROPOSITION 3. *For $l \geq 1$, any matrix A_{N+l} lies in the span of $\{A_i\}_{i=1}^N$, i.e.*

$$A_{N+l} = \sum_{i=1}^{N} \mu_l^i \, A_i \ , \tag{18}$$

where $\mu_l^i = \mu_l^i(W)$ are scalar functions of $\{W_j\}_{j=1}^N$.

Proof. Use the Cayley-Hamilton theorem; A_2 has only N linearly independent powers, specifically if $F_N(P) = \det(PI - A_2)$, then $F_N(A_2) = 0$ □

We note that the decomposition (18) may be rewritten in the following form of the polynomials $v^n(P) = \partial Q_n / \partial P$,

$$v^{N+l}(P) \equiv \sum_{i=1}^{N} \mu_l^i(W) v^i(P) \ , \quad \mod F_N(P). \tag{19}$$

This formula, together with the recurrent formula (14), is more useful for calculating the coefficients $\{\mu_l^i(W)\}_{i=1}^N$.

We now state our main result:

THEOREM. *The solution of (5) can be given in the form,*

$$W(T_1, T_2, \cdots) = W^0(T_1^0, T_2^0, \cdots, T_N^0), \tag{20}$$

where $W^0(T_1, \cdots, T_N) = W(T_1, \cdots, T_N, 0, \cdots)$, and

$$T_i^0 = T_i + \sum_{l=1}^{\infty} \mu_l^i T_{N+l} \tag{21}$$

with the same scalar functions $\mu_l^i(W)$ defined in Proposition 3.

Proof. From (18), the $(N + l)$th flow can be written as

$$\frac{\partial W}{\partial T_{N+l}} = \sum_{i=1}^{N} \mu_l^i \frac{\partial W}{\partial T_i} \quad , \quad \forall \, l \geq 1 \ . \tag{22}$$

This implies that W is a constant vector along the characteristic, which are straight lines, given by

$$\frac{dT_{N+l}}{-1} = \frac{dT_i}{\mu_l^i(W)} \quad , \text{ for } i = 1, \cdots, N. \tag{23}$$

The integrals of (23) are

$$T_i^0 = T_i + \mu_l^i(W) T_{N+l}\ , \quad \forall\, l \geq 1\ , \tag{24}$$

which gives (21) on taking a sum over l. Here T_i^0 gives the initial position of the characteristics at $T_{N+l} = 0$ for all $l \geq 1$ □

We note that the vector function $W^0(T_1^0, \cdots, T_N^0)$ in (20) satisfies

$$\frac{\partial W^0}{\partial T_n^0} = A_n(W^0) \frac{\partial W^0}{\partial X^0}\ , \quad \text{for } n = 1, \cdots, N\ . \tag{25}$$

From (5), (21) and (25), one can verify that the functions $\mu_l^i(W)$ satisfy the following systems,

$$\sum_{i=1}^{N} \left(A_i \frac{\partial \mu_l^i}{\partial T_n} - A_i A_n \frac{\partial \mu_l^i}{\partial X} \right) = 0\ , \quad \forall\, l \geq 1\ , \tag{26}$$

or equivalently,

$$\sum_{i=1}^{N} \left(v^i(P) \frac{\partial \mu_l^i}{\partial T_n} - v^i(P) v^n(P) \frac{\partial \mu_l^i}{\partial X} \right) \equiv 0, \qquad \text{mod } F_N(P). \tag{27}$$

As a Corollary of the Theorem, one can construct exact implicit (hodograph) solutions of (25), using the translational invariance of (5), i.e. $T_{N+l} \to T_{N+l} + C_l$ where C_l is an arbitrary constant:

Corollary. *The hodograph solutions of (25) are*

$$T_i^0 = \sum_{l=1}^{\infty} C_l \mu_l^i(W)\ , \quad \textit{for } i = 1, \cdots, N. \tag{28}$$

Thus the coefficients μ_l^i in the decomposition of the higher flows (18) give a solution of (5). This statement can be applied to any hydrodynamic type equations with additional symmetries [10]. Namely, by decomposing the matrix A_m of the additional symmetry into the first M independent matrices, say $A_1, \cdots, A_M$, $A_m = \sum_{i=1}^{M} \alpha_m^i A_i$, the hodograph solutions are given by $T_i^0 = \alpha_m^i(W)$. The decomposition is unique provided dim $(\text{Span}\{A_i\}_{i=1}^{\infty}) = M \quad (M \leq N$ for $N \times N$ commuting matrices).

A formula equivalent to (18) was obtained by Tsarev [10] in somewhat different way. His formula for the hodograph solutions is

$$X + \lambda_i(W) T = \nu_i(W), \quad \text{for } i = 1, \cdots, N, \tag{29}$$

where $\lambda_i(W)$ and $\nu_i(W)$ are the eigenvalues of two commuting diagonalizable Hamiltonian systems of hydrodynamic type. Although the formulas (18) (or (19)) and (29) are formally equivalent, it seems that calculation with (29), that is, in terms of the eigenvalues of matrices (λ_i and ν_i), is harder than the calculation with (18) or (19) based on the symmetric functions of eigenvalues ($v^n(P)$). In particular, for the Lax reductions, the method using (19) yields a much more effective way of constructing solutions in terms of conservation laws (see [8] and a future communication [9]).

4. We apply the theorem to an example and show how to construct its exact solutions. As a typical example, we take an $N = 2$ reduction, i.e. $W \in \mathbf{R}^2$. In this case, the matrix A_3 in the T_3-flow is given by

$$\begin{aligned} A_3 &= v^3(A_2) = A_2^2 + U_1 I \\ &= (trA_2)A_2 + (U_1 - \det A_2)I \end{aligned} \tag{30}$$

where we have used $A_2^2 = (trA_2)A_2 - (\det A_2)I$ (Cayley-Hamilton). Therefore we obtain the solution [1],

$$\left.\begin{aligned} X^0 &= U_1 - \det A_2 \ , \\ T_2^0 &= trA_2 \ . \end{aligned}\right\} \tag{31}$$

From A_4 in the T_4-flow, $A_4 = v^4(A_2) = A_2^3 + 2U_1A_2 + U_2 I$, which can be decomposed into

$$A_4 = \{2U_1 + (trA_2)^2 - \det A_2\}A_2 + \{U_2 - trA_2 \det A_2\}I, \tag{32}$$

we obtain

$$\left.\begin{aligned} X^0 &= U_2 - trA_2 \det A_2 \ , \\ T_2^0 &= 2U_1 + (trA_2)^2 - \det A_2 \ , \end{aligned}\right\} \tag{33}$$

and so on. (Note that the characteristic polynomial $F_2(P) = P^2 - trA_2P + \det A_2$.)

Before ending this letter, we discuss the equations (26) which the hodograph solutions satisfy. For the case of an $N = 2$ reduction, the equations (26) give, after setting $T_i^0 = \mu_1^i(W)$,

$$\left.\begin{aligned} \frac{\partial X^0}{\partial T_2} &= -\det A_2 \frac{\partial T_2^0}{\partial X} \ , \\ \frac{\partial T_2^0}{\partial T_2} &= \frac{\partial X^0}{\partial X} + trA_2 \frac{\partial T_2^0}{\partial X} . \end{aligned}\right\} \tag{34}$$

It is interesting to note that if $trA_2 = 0$, which corresponds to the Lax reduction, then T_2^0 and X^0 give a conserved density and flux, respectively (see [8] for 3×3 case). This fact can be generalized to the case of the Lax reduction with arbitrary N using the recurrence (14). This will be discussed in a future communication [9].

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