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IMA Preprint Series # 474

February 1989

HIGHER ORDER NONLINEAR DEGENERATE PARABOLIC EQUATIONS*

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Abstract. This paper is concerned with nonlinear degenerate parabolic equations of the form $u_t + (-1)^{m-1}D(f(u)D^{2m+1}u) = 0$ with $f(u) \sim |u|^n$ $(n \geq 1)$ near $u = 0$ and $D = \partial/\partial x$. Under appropriate boundary conditions it is shown that there exists a weak solution u . Some of the main results of the paper are that $u \geq 0$ if $u_0 \geq 0$, and that the support of $u(\cdot, t)$ (when $u_0 \geq 0$) increases with t (for the last property we require that $n \geq 2$ and $m = 1$).

§1. **Introduction.** In this paper we consider higher order nonlinear degenerate parabolic equations of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(f(u)\,\frac{\partial^3 u}{\partial x^3}\right) = 0 \tag{1.1}$$

and, more generally,

$$\frac{\partial u}{\partial t} + (-1)^{m-1}\frac{\partial}{\partial x}\left(f(u)\,\frac{\partial^{2m+1} u}{\partial x^{2m+1}}\right) = 0 \tag{1.2}$$

where

$$f(u) = |u|^n f_0(u), \qquad f_0(u) > 0 \tag{1.3}$$

and n is a real number, $n \geq 1$.

Equation (1.1) arises in modeling the motion of viscous droplets spreading over a solid surface: in [3] [4] [5] and [9] the authors take $f(u) = |u|^3$ (but also $f(u) = |u|^3 + \beta|u|$ in [4], and $f(u) = |u|^3 + \beta u^2$ in [9]); further, since they assume, on physical grounds, that $u \geq 0$, they replace $|u|$ by u. Equation (1.2) with $m = 2$ and $f(u) = |u|^n, n = 3$ appears in recent work by King [6] [7] [8] and Tayler and King [13] in a model of oxidation of silicon in semiconductor devices. Some explicit solutions and heuristic asymptotic analysis with respect to n, for (1.2) with $f(u) = |u|^n$, is given in a recent work by Smyth and Hill [11].

In this paper we shall consider first (1.1) in a bounded interval in the x–space, with appropriate boundary conditions, and we shall prove in §§2,3 the existence of a weak solution. We next establish (in §4) the remarkable phenomena that

(1.4) if the initial data are ≥ 0
then the solution is ≥ 0.

*This paper is partially supported by National Science Foundation Grant DMS–86–12880 and U.S. Army Grant No. DAJA–86–C–0040.

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It is well known that this positivity result is false for solutions of the linear equation $u_t + \partial^4 u/\partial x^4 = 0$. In the process of proving (1.4) we establish some estimates. These estimates provide additional regularity of the weak solution and, in particular, allow us to assert that the weak solution satisfies:

$$\iint u\phi_t = \iint f(u)u_{xx}\phi_{xx} + \iint f'(u)u_x u_{xx}\phi_x$$

for any test function ϕ.

In §5 we show that if $n \geq 4$ then the support of the solution $u(\cdot,t)$ increases with t. A slightly weaker result is established in §6 in case $2 \leq n < 4$.

Finally, in §7 we extend all the results of §§2–4 to equations of the form (1.2) with $m \geq 2$.

§2. The approximating problems. In §§2–5 we study the equation

$$u_t + (f(u)u_{xxx})_x = 0 \quad \text{in} \quad Q_{T_0} \equiv \Omega \times (0, T_0) \tag{2.1}$$

where $T_0 > 0, \Omega$ is a bounded interval, say

$$\Omega = \{-a < x < a\}.$$

with initial conditions

$$u(x,0) = u_0(x) \ , \quad u_0 \in H^1(\Omega) \tag{2.2}$$

and boundary conditions

$$u_x = u_{xxx} = 0 \quad \text{on} \quad x = \pm a\ . \tag{2.3}$$

We assume that

$$f(u) = |u|^n f_0(u), \qquad f_0 \in C^{1+\alpha}(\mathbf{R}^1),\ f_0 > 0 \tag{2.4}$$

where $\alpha \in (0,1)$, and take

$$n > 1\ ; \tag{2.5}$$

the case $n = 1$ will be considered at the end of Section 4.

Since (2.1) is degenerate at $u = 0$, we begin by approximating it by a family of non–degenerate diffusions:

$$u_t + ((f(u) + \epsilon)u_{xxx})_x = 0 \quad \text{in} \quad Q_{T_0}\ , \tag{2.6}$$

where $\epsilon > 0$.

We also approximate u_0 in the $H^1(\Omega)$–norm by $C^{4+\alpha}$ functions $u_{0\epsilon}$ satisfying (2.3), and replace (2.2) by

$$u(x,0) = u_{0\epsilon}(x) . \tag{2.7}$$

Using the parabolic Schauder estimates [1] [2] [12] one can prove that (2.6), (2.7), (2.3) has a unique solution in a small time interval, say in Q_σ for some small $\sigma > 0$. The derivatives

$$u_t, u_x, u_{xx}, u_{xxx}, u_{xxxx}$$

are all Hölder continuous in $\overline{Q_\sigma}$. Later on we shall prove an a priori Hölder estimate for the solution u_ϵ of (2.6), (2.7), (2.3) in every domain Q_σ, independently of σ. This allows us to extend the solution u_ϵ step–by–step to all of Q_{T_0}.

We shall now assume that u_ϵ is a solution in Q_σ, for some $0 < \sigma < T_0$ and derive various estimates.

Setting $u = u_\epsilon$, we begin with

$$\begin{aligned} &\int_\Omega [u_x(x,t+h)^2 - u_x(x,t)^2]\, dx \\ &= \int_\Omega [u_x(x,t+h) + u_x(x,t)][u_x(x,t+h) - u_x(x,t)]\, dx \\ &= -\int_\Omega [u_{xx}(x,t+h) + u_{xx}(x,t)][u(x,t+h) - u(x,t)]\, dx \end{aligned}$$

since $u_x = 0$ on the boundary. Dividing by h and letting $h \to 0$ we get, for any $0 < t_1 < t_2 < \sigma$,

$$-\int_{t_1}^{t_2}\int_\Omega u_t\, u_{xx}\, dxdt = \frac{1}{2}\left[\int_\Omega u_x(x,t)^2\, dx]\right]_{t=t_1}^{t=t_2} .$$

Multiplying (2.6) by $u_{\epsilon,xx}$ and integrating over Q_T $(0 < T < \sigma)$ and using the last identity, we get

$$\frac{1}{2}\int_\Omega u_{\epsilon,x}^2(x,T)\, dx + \int_0^T\int_\Omega (f(u_\epsilon)+\epsilon)\, u_{\epsilon,xxx}^2\, dxdt = \frac{1}{2}\int_\Omega u_{0\epsilon,x}^2\, dx . \tag{2.8}$$

Hence

$$\int_\Omega u_{\epsilon,x}^2(x,T)\, dx \le \int_\Omega u_{0\epsilon,x}^2\, dx . \tag{2.9}$$

Integrating (2.6) over Ω_T we also have

$$\int_\Omega u_{\epsilon,x}(x,T)\,dx = \int_\Omega u_{0\epsilon,x}\,dx\ . \tag{2.10}$$

Notice that

$$\int_\Omega u_{0\epsilon,x}^2 \le (1+\eta(\epsilon))\int_\Omega u_{0,x}^2 \qquad (\eta(\epsilon)\to 0 \text{ if } \epsilon\to 0). \tag{2.11}$$

Hence from (2.9), (2.10) we deduce, by the Poincare inequality, that

$$|u_\epsilon(x,t)| \le A \qquad \text{in} \qquad Q_\sigma \tag{2.12}$$

where A is a constant independent of ϵ, σ.

From (2.9), (2.11) and Sobolev's inequality we also deduce that

$$|u_\epsilon(x_2,t) - u_\epsilon(x_1,t)| \le K|x_2 - x_1|^{1/2} \qquad \text{in } Q_\sigma \tag{2.13}$$

where K a constant independent of ϵ, σ.

Setting

$$h_\epsilon = (f(u_\epsilon)+\epsilon)u_{\epsilon,xxx} \tag{2.14}$$

we see from (2.8), (2.12) that

$$|h_\epsilon|_{L^2(Q_\sigma)} \le A_1, \qquad A_1 \quad \text{independent of } \sigma, \epsilon. \tag{2.15}$$

LEMMA 2.1. *There exists a constant M independent of σ, ϵ such that*

$$|u_\epsilon(x,t_2) - u_\epsilon(x,t_1)| \le M|t_2 - t_1|^{1/8} \tag{2.16}$$

for all $x\in\Omega$, t_1 and t_2 in $(0,\sigma)$.

Proof. We suppose that

$$|u_\epsilon(x_0,t_2) - u_\epsilon(x_0,t_1)| > M|t_2 - t_1|^{1/8}$$

for some x_0 and t_2, t_1 and derive an upper bound for M which is independent of σ, ϵ; for simplicity we suppose that $u_\epsilon(x_0,t_2) > u_\epsilon(x_0,t_1)$ and that $t_2 > t_1$; thus

$$u_\epsilon(x_0,t_2) - u_\epsilon(x_0,t_1) > M|t_2 - t_1|^\beta\ , \quad 0 < t_1 < t_2 < \sigma \tag{2.17}$$

where $\beta = 1/8$.

We shall use the relation

$$\iint u_\epsilon \phi_t = -\iint h_\epsilon \phi_x \tag{2.18}$$

which is valid for any "reasonable" test–function. Since $u_{\epsilon,t}$ is continuous in $\overline{Q}_\sigma$ and $h_\epsilon = 0$ on the lateral boundary, we may take any ϕ such that

$$\phi \in Lip(\overline{Q}_\sigma),\ \phi = 0 \ \text{ near } t = 0 \text{ and near } t = \sigma;$$

ϕ need not vanish on the lateral boundary. We shall construct a test function ϕ of the form

$$\phi(x,t) = \xi(x)\theta_\delta(t) \tag{2.19}$$

where ξ and θ_δ are defined as follows:

Definition of ξ.

$$\xi(x) = \xi_0\left(\frac{x - x_0}{\frac{M^2}{16K^2}(t_2 - t_1)^{2\beta}}\right)$$

where M is from (2.17) and K from (2.13), and $\xi_0(x) = \xi_0(-x)$, $\xi_0 \in C_0^\infty$, $\xi_0(x) = 1$ if $0 \le x < \frac{1}{2}$, $\xi_0(x) = 0$ if $x \ge 1$ and $\xi_0'(x) \le 0$ if $x \ge 0$. Thus

$$\xi(x) = \begin{cases} 0 & \text{if } |x - x_0| \ge \frac{M^2}{16K^2}(t_2 - t_1)^{2\beta} \\ 1 & \text{if } |x - x_0| \le \frac{1}{2}\frac{M^2}{16K^2}(t_2 - t_1)^{2\beta}. \end{cases} \tag{2.20}$$

Definition of θ_δ. We take

$$\theta_\delta(t) = \int_{-\infty}^{t} \theta_\delta'(s)\, ds$$

where

$$\theta_\delta'(t) = \begin{cases} 1/\delta & \text{if } \ |t - t_2| < \delta \\ -1/\delta & \text{if } \ |t - t_1| < \delta \\ 0 & \text{elsewhere,} \end{cases}$$

where $\delta < \frac{1}{2}(t_2 - t_1)$. Notice that θ_δ is Lipschitz continuous and $|\theta_\delta| \le 1$; $\theta_\delta = 0$ near $t = 0$ and near $t = \sigma$, if δ is small enough.

Inserting (2.19) into (2.18) we get

$$\iint u_\epsilon \xi(x)\theta'_\delta(t) = -\iint h_\epsilon \xi'(x)\theta_\delta(t). \tag{2.21}$$

The left–hand side satisfies

$$\iint u_\epsilon \xi \theta'_\delta(t) \longrightarrow \int \xi(x)(u_\epsilon(x,t_2) - u_\epsilon(x,t_1))\,dx \quad \text{as } \delta \longrightarrow 0.$$

We shall estimate the last expression from below. In view of (2.20) we only need to consider values of x such that

$$|x - x_0| \le \frac{M^2}{16K^2}(t_2 - t_1)^{2\beta}. \tag{2.22}$$

For such values,

$$\begin{aligned} u_\epsilon(x,t_2) - u_\epsilon(x,t_1) &= [u_\epsilon(x,t_2) - u_\epsilon(x_0,t_2)] \\ &\quad + [u_\epsilon(x_0,t_2) - u_\epsilon(x_0,t_1)] + [u_\epsilon(x_0,t_1) - u_\epsilon(x,t_1)] \\ &\ge -2K|x - x_0|^{1/2} + M(t_2 - t_1)^\beta \qquad \text{by (2.13), (2.17),} \\ &\ge \frac{M}{2}(t_2 - t_1)^\beta \qquad \text{by (2.22).} \end{aligned}$$

Hence, if we assume that the set $\{\xi = 1\}$ is included in Ω (otherwise, very minor modifications are necessary),

$$\int \xi(x)(u_\epsilon(x,t_2) - u_\epsilon(x,t_1))\,dx \ge \frac{M}{2}(t_2 - t_1)^\beta \, \frac{M^2}{16K^2}(t_2 - t_1)^{2\beta}.$$

On the other hand, the right–hand side of (2.21) is bounded from above by

$$\left| \iint h_\epsilon \xi'(x)\theta_\delta \right| \le \frac{C_1}{\frac{M^2}{16K^2}(t_2 - t_1)^{2\beta}} \left(\iint h_\epsilon^2 \right)^{1/2} \frac{\sqrt{2}M}{4K}(t_2 - t_1)^\beta (t_2 - t_1 + 2\delta)^{1/2}.$$

We thus conclude, after letting $\delta \to 0$, that

$$M^3(t_2 - t_1)^{3\beta} \le C_2 \frac{1}{M}(t_2 - t_1)^{\frac{1}{2} - \beta}$$

where C_2 is a constant independent of ϵ, M and σ. Since $\beta = \frac{1}{8}$, we find that $M \le C_2^{1/4}$, and the lemma follows.

From Lemma 2.1 and (2.13), (2.12), it follows that there is an upper bound on the $C^{\frac{1}{2},\frac{1}{8}}_{x,t}$ –norm of u_ϵ in Q_σ, which is independent of σ, ϵ. This a priori bound allows us to conclude that u_ϵ can be extended step–by–step to a solution of (2.6), (2.7), (2.3) in all of Q_{T_0}, and that

(2.23) $\quad \{u_\epsilon\}$ is a uniformly bounded and equi–continuous family in $\overline{Q}_{T_0}$.

§3. Existence of weak solution. By (2.23), every sequence $\epsilon \to 0$ has a subsequence such that

(3.1) $$u_\epsilon \longrightarrow u \qquad \text{uniformly in } \overline{Q}_{T_0}.$$

THEOREM 3.1. *Any function u obtained as in (3.1) satisfies the following properties:*

(3.2) $$u \in C(\overline{Q}_{T_0}), \quad \textit{in fact } u \textit{ is uniformly Hölder continuous (exponent } \frac{1}{2}) \textit{ in } x \textit{ and Hölder continuous (exponent } \frac{1}{8}) \textit{ in } t,$$

(3.3) $$u_t, u_x, u_{xx}, u_{xxx}, u_{xxxx} \quad \textit{belong to } C(P)$$

where $P = \overline{Q}_{T_0} \setminus (\{u = 0\} \cup \{t = 0\})$, and

(3.4) $$f(u)u_{xxx} \in L^2(P);$$

u satisfies (2.1) in the following sense:

(3.5) $$\iint\limits_{Q_{T_0}} u\phi_t + \iint\limits_{P} f(u)u_{xxx}\phi_x = 0$$

for all $\phi \in Lip\,(\overline{Q}_{T_0}), \phi = 0$ near $t = 0$ and near $t = T_0$,

(3.6) $$u(x,0) = u_0(x), \quad x \in \overline{\Omega},$$

(3.7) $$u_x(\cdot,t) \to u_{0x} \quad \textit{strongly in } L^2(\Omega) \textit{ as } t \to 0,$$

and

(3.8) $\quad$ *u satisfies (2.3) at all points of the lateral boundary where $u \neq 0$.*

Proof. The assertions (3.2), (3.6) are obvious. For ϕ as asserted in (3.5) we have

$$\iint_{Q_{T_0}} u_\epsilon \phi_t + \iint_{Q_{T_0}} f(u_\epsilon) u_{\epsilon,xxx} \phi_x + \epsilon \iint_{Q} u_{\epsilon,xxx} \phi_x = 0. \tag{3.9}$$

From (2.8), $\epsilon \iint u^2_{\epsilon,xxx} \le C$; hence, by Hölder's inequality,

$$\epsilon \iint u_{\epsilon,xxx} \phi_x \to 0 \qquad \text{if} \quad \epsilon \to 0. \tag{3.10}$$

From (2.15) it follows that, for a subsequence,

$$h_\epsilon \to h \quad \text{weakly in } L^2(Q_{T_0}). \tag{3.11}$$

Next, by regularity theory of uniformly parabolic equations and the uniformly Hölder continuity of the u_ϵ we deduce that

$$u_{\epsilon,t}, u_{\epsilon,x}, u_{\epsilon,xx}, u_{\epsilon,xxx}, u_{\epsilon,xxxx} \quad \text{are uniformly convergent in any compact subset of } P. \tag{3.12}$$

It follows that

$$f(u) u_{xxx} = h \quad \text{on} \quad P, \tag{3.13}$$

that (3.3), (3.8) hold and (recalling (3.11)) that (3.4) holds; further, for any $\delta > 0$,

$$\iint_{|u|>\delta} f(u_\epsilon) u_{\epsilon,xxx} \phi_x \longrightarrow \iint_{|u|>\delta} f(u) u_{xxx} \phi_x \,. \tag{3.14}$$

On the other hand, if ϵ is sufficiently small, depending on δ, then by (2.4)

$$\Big| \iint_{|u|\le\delta} f(u_\epsilon) u_{\epsilon,xxx} \phi_x \Big| \le C\delta^{n/2} \left\{ \iint f(u_\epsilon) u^2_{\epsilon,xxx} \right\}^{1/2} \le C_1 \delta^{n/2} \tag{3.15}$$

where (2.8) has been used in the last inequality.

To prove (3.7) notice that from $u_{0\epsilon} \to u_0$ in $H^1(\Omega)$ and (2.9) we get

$$\limsup_{t\to 0} \int_\Omega u_x^2(x,t)\, dx \le \int_\Omega u_{0x}^2\, dx.$$

Since also

$$u_x(\cdot, t) \to u_{0x} \quad \text{weakly in } L^2(\Omega)$$

as $t \to 0$, the assertion (3.7) follows.

Taking $\epsilon \to 0$ in (3.9) and using (3.10), (3.14), (3.15) we deduce, since δ is arbitrary, that (3.5) is satisfied, and the proof of the theorem is complete.

REMARK 3.1. Since $u_t + h_x = 0$ in the sense of weak derivatives in Q_{T_0}, we have:

$$u_t \in L^2(0, T_0; H^{-1}(\Omega)).$$

REMARK 3.2. From (2.10) we deduce that

$$\int_\Omega u(x,t)\, dx = \text{const.} = \int_\Omega u_0(x)\, dx. \tag{3.16}$$

REMARK 3.3. If ϕ is smooth and $\phi_x = 0$ on the lateral boundary then from (3.9) we obtain

$$\begin{aligned} \iint_{Q_T} u_\epsilon \phi_t + \epsilon \iint_{Q_T} u_{\epsilon,xxx}\phi_x = \iint_{Q_T} f(u_\epsilon)u_{\epsilon,xx}\phi_{xx} \\ + \iint_{Q_T} f'(u_\epsilon)u_{\epsilon,x}u_{\epsilon,xx}\phi_x \end{aligned} \tag{3.17}$$

This equation will be used in §4.

REMARK 3.4. Let $u_0 \geq 0$. In general the solution of (2.6) may take negative values. Let, for example, $\phi_\epsilon(x)$ be the solution of

$$\begin{gathered} (|\phi|^n + \epsilon)\phi''' = h(x) \qquad \text{where } h'(0) = 1 \\ \phi(0) = \phi'(0) = 0, \quad \phi''(0) = 1. \end{gathered}$$

If we take $u_0 = \phi_\epsilon$ then $u_0 \geq 0$ near $x = 0$ and the solution u_ϵ of (2.6) with $f(u) = |u|^n$ satisfies

$$\frac{\partial u_\epsilon(0,0)}{\partial t} = -\left(\left(\left|\phi_\epsilon(x)\right|^n + \epsilon\right)\phi_\epsilon'''(x)\right)'\Big|_{x=0} = -1$$

Thus $u_\epsilon(0,t)$ takes negative values near $t = 0$. In §4. we shall prove, however, that if $u_0 \geq 0$ then $u \geq 0$.

Definition 3.1. The solution u satisfying the assertions of Theorem 3.1 will be called a *weak solution.*

This concept is very weak; it includes stationary solution with compact support of the form

$$(x-b)^+(c-x)^+, \qquad -a < b < c < a.$$

Such solutions will be excluded in §4 when we shall prove that for u_0 satisfying a certain positivity condition, the weak solution constructed in Theorem 3.1 satisfies: $u_{xx} \in L^2(Q_{T_0})$.

§4. **Nonnegative solutions.** In this section we assume, in addition to $u_0 \in H^1(\Omega)$, that

$$u_0 \geq 0 \tag{4.1}$$

and prove that the weak solution $u(x,t)$ constructed in Theorem 3.1 satisfies:

$$u(x,t) \geq 0 \quad \text{a.e.} \tag{4.2}$$

Under some additional positivity assumptions on u_0 (depending on n) we shall prove additional positivity and regularity properties for u.

We introduce the functions

$$g_\epsilon(s) = -\int_s^A \frac{dr}{f(r)+\epsilon}\,, \quad G_\epsilon(s) = -\int_s^A g_\epsilon(r)\,dr \tag{4.3}$$

where $A > \max|u_\epsilon|$ for all small ϵ. Then

$$G_\epsilon'(s) = g_\epsilon(s), \qquad G_\epsilon''(s) = g_\epsilon'(s) = \frac{1}{f(s)+\epsilon}\,, \tag{4.4}$$

$$g_\epsilon(s) \leq 0,\ G_\epsilon(s) \geq 0 \quad \text{if} \quad s \leq A, \tag{4.5}$$

$$G_\epsilon(s) \leq G_0(s) \quad \text{for all } s \tag{4.6}$$

where $G_0 = \lim\limits_{\epsilon\to 0} G_\epsilon$ and, for $0 < s \leq A$,

$$G_0(s) = \begin{cases} A_0 + O(s^{2-n}) & \text{if } 1 < n < 2,\ A_0 > 0, \\ C_2 \log\frac{1}{s} + O(1) & \text{if } n = 2,\ C_2 > 0, \\ C_1\, s^{2-n} + R(s) & \text{if } n > 2, \\ \qquad C_1 > 0, \qquad R(s) = \begin{cases} O(s^{3-n}) & \text{if } n > 3 \\ O(\log\frac{1}{s}) & \text{if } n = 3 \\ O(1) & \text{if } n < 3\,; \end{cases} \end{cases} \tag{4.7}$$

the constants are positive and depend on $f_0(0)$, where f_0 is the function appearing in (2.4).

Denote by $\tilde{G}_0(s)$ the function $G_0(s)$ corresponding to $f(u) = |u|^n$, i.e.,

$$(4.8) \qquad \tilde{G}_0(s) = \begin{cases} \dfrac{A^{2-n}}{2-n} + \dfrac{s\,A^{1-n}}{n-1} - \dfrac{s^{2-n}}{(2-n)(n-1)} & \text{if } 1 < n < 2 \\ \log\dfrac{A}{s} + \dfrac{s}{A} - 1 & \text{if } n = 2 \\ \dfrac{s^{2-n}}{(n-2)(n-1)} - \dfrac{A^{2-n}}{n-2} + \dfrac{sA^{1-n}}{n-1} & \text{if } n > 2 \end{cases}$$

Then, for $0 \le s \le A$,

$$(4.9) \qquad c_1\tilde{G}_0(s) \le G_0(s) \le c_2\tilde{G}_0(s), \qquad c_1 > 0;$$

indeed this follows from (see (2.4))

$$k_1|s|^n \le f(s) \le k_2|s|^n \qquad (k_1 > 0,\ 0 \le s \le A).$$

From (4.8), (4.9) we deduce, in particular, that

$$(4.10) \qquad G_0(0) = \begin{cases} \infty & \text{if } n \ge 2 \\ A_0 & \text{if } 1 < n < 2. \end{cases}$$

If we formally multiply (2.1) by $G_0(u)$, integrate over Q_T, and use the relations

$$(4.11) \qquad g_0'(s) = G_0''(s) = \frac{1}{f(s)}\,,$$

we obtain

$$(4.12) \qquad \int_\Omega G(u(x,T))\,dx + \int_0^T\int_\Omega u_{xx}^2\,dxdt = \int_\Omega G(u_0(x))\,dx\,.$$

In order to proceed rigorously we assume, in addition to (4.1), that

$$(4.13) \qquad \begin{aligned} &\int_\Omega \big|\log u_0\big| < \infty \quad \text{if} \quad n = 2,, \\ &\int_\Omega u_0^{2-n}dx < \infty \quad \text{if} \quad 2 < n < 4, \\ &u_0 > 0 \quad \text{in} \quad \overline{\Omega} \quad \text{if} \quad n \ge 4. \end{aligned}$$

Remark 4.1. If $n \ge 4$ then the conditions $u_0 \in H^1(\Omega)$ and $\int |u_0|^{2-n} < \infty$ imply that $u_0 \ne 0$ in $\overline{\Omega}$; see the proof of Theorem 4.1 (iii) below. If $2 \le n < 4$ then (4.13) implies that the set of zeros of u_0 must have zero measure; if $1 < n < 2$ then u_0 may have compact support.

THEOREM 4.1 (NONNEGATIVITY). *Under the assumption (4.1), (4.13),*

(i) *if $1 < n < 2$ then the solution u is ≥ 0 in Q_{T_0},*
(ii) *if $2 \leq n < 4$ then again $u \geq 0$; further, the set $\{u = 0\}$ has zero measure and, in fact,*

$$\int_\Omega \big|\log u(x,t)\big|\, dx \leq C < \infty \qquad \forall\, t \in [0, T_0] \quad \text{if } n = 2, \tag{4.14}$$

$$\int_\Omega u(x,t)^{2-n}\, dx \leq C < \infty \qquad \forall\, t \in [0, T_0] \quad \text{if } 2 < n < 4; \tag{4.15}$$

(iii) *if $n \geq 4$ then $u > 0$ in $\overline{Q}_{T_0}$; such a solution is unique.*

From (iii) it follows that, when $u_0 > 0$ and $n \geq 4$, the weak solution u is a classical solution and all the derivatives

$$u_t, u_x, u_{xx}, u_{xxx}, u_{xxxx}$$

are continuous in $\overline{Q}_{T_0} \setminus \{t = 0\}$.

Proof of Theorem 4.1. We can choose the smooth approximation $u_{0\epsilon}$ of u_0 such that $u_{0\epsilon} \geq u_0$. Then, from (4.1), (4.13) and (4.9) we have

$$\int_\Omega G_\epsilon(u_{0\epsilon}(x))\, dx \leq C, \qquad C \text{ independent of } \epsilon. \tag{4.16}$$

Multiplying equation (2.6) by $g_\epsilon(u_\epsilon)$ and integrating over Q_T, $\;T \in (0, T_0)$, we get, after performing an integration by parts and using the boundary condition for u_ϵ and (4.4),

$$\int_\Omega G_\epsilon(u_\epsilon(x,T))\, dx + \int_0^T \int_\Omega u_{\epsilon,xx}^2 dx dt = \int_\Omega G_\epsilon(u_{0\epsilon}(x))\, dx. \tag{4.17}$$

Consequently, by (4.16),

$$\int_\Omega G_\epsilon(u_\epsilon(x,T))\, dx \leq C\ , \quad G_\epsilon(u_\epsilon) \geq 0 \tag{4.18}$$

and

(4.19)
$$\iint_{Q_{T_0}} u_{\epsilon,xx}^2 \; dxdt \le C.$$

We proceed to prove that

(4.20)
$$u \ge 0 \qquad \text{in} \qquad Q_{T_0}.$$

If this is not true then there is a point $(x_0, t_0) \in Q_{T_0}$ such that $u(x_0, t_0) < 0$. Since $u_\epsilon \to u$ uniformly, there exist $\delta > 0$ and $\epsilon_0 > 0$ such that

$$u_\epsilon(x, t_0) < -\delta \quad \text{if} \quad |x - x_0| < \delta, \; x \in \Omega, \; \epsilon < \epsilon_0.$$

But for such x,

$$G_\epsilon(u_\epsilon(x, t_0)) = -\int_{u_\epsilon(x,t_0)}^{A} g_\epsilon(s)\; ds \ge -\int_{-\delta}^{0} g_\epsilon(s)\; ds$$

$$\longrightarrow -\int_{-\delta}^{0} g_0(s)\; ds \qquad \text{as} \qquad \epsilon \to 0,$$

by the monotone convergence theorem where $g_0(s) = \lim_{\epsilon \to 0} g_\epsilon(s)$, and the integral on the right–hand side is equal to $+\infty$ for $n \ge 1$ since $g_0(s) = -\infty$ if $s < 0$, by (4.3). It follows that

$$\lim_{\epsilon \to 0} \int G_\epsilon(u_\epsilon(x, t_0))\; dx = \infty,$$

a contradiction to (4.18).

Having proved (4.20), we now specialize to $n \ge 2$ and prove that, for each $T \in (0, T_0)$,

(4.21) the set $\{u(\cdot, T) = 0\}$ has measure zero

If the assertion (4.21) is not true then for some $t_0 \in (0, T_0)$ the set $E = \{u(\cdot, t_0) = 0\}$ has positive measure. Since $u_\epsilon \to u$ uniformly, there exists a modulus of continuity $\sigma(\epsilon)$ such that

$$u_\epsilon(x, t_0) < \sigma(\epsilon) \qquad \text{for all } x \in E.$$

Now, for any $x \in E$ and for any $\delta > 0$,

$$G_\epsilon(u_\epsilon(x, t_0)) \ge -\int_{\sigma(\epsilon)}^{A} g_\epsilon(s)\; ds \ge -\int_{\delta}^{A} g_\epsilon(s)\; ds \to -\int_{\delta}^{A} g_0(s)\; ds$$

if ϵ is small enough (so that $\sigma(\epsilon) < \delta$), and

$$\int_\delta^A g_0(s)\, ds \geq \begin{cases} c\,\delta^{2-n} & \text{if} \quad n > 2 \qquad (c > 0) \\ c\, \log \dfrac{1}{\delta} & \text{if} \quad n = 2 \qquad (c > 0). \end{cases}$$

Hence

$$\varlimsup_{\epsilon \to 0} \int G_\epsilon(u_\epsilon(x, t_0))\, dx \geq \begin{cases} c\,\delta^{2-n} \text{ (meas } E) \to \infty & \text{if} \quad n > 2 \\ c\, \log \dfrac{1}{\delta} \text{ (meas } E) \to \infty & \text{if} \quad n = 2 \end{cases}$$

if $\delta \to 0$, a contradiction to (4.18).

Let $n \geq 2$. At the points (x, t) where $u(x, t) > 0$,

$$G_\epsilon(u_\epsilon(x,t)) \longrightarrow G_0(u(x,t)), \tag{4.22}$$

where G_0 satisfies (4.8), (4.9). Since the set $\{u(\cdot, t) = 0\}$ has measure zero for any t, it follows that, for any t, (4.22) holds for almost all x. From (4.18) and Fatou's lemma we then deduce that

$$\int_\Omega G_0(u(x,t))\, dx \leq C,$$

which, in view of (4.8), (4.9), yields the assertions (4.14), (4.15), for all $n \geq 2$.

In order to complete the proof of Theorem 4.1 it remains to prove (iii). If u is not positive everywhere in $\overline{Q}_{T_0}$ then there exists a point (x_0, t_0) in Q_{T_0} such that $u(x_0, t_0) = 0$. By the Hölder continuity of u,

$$u(x, t_0) < K|x - x_0|^{1/2}$$

and thus

$$\int_\Omega u(x, t_0)^{2-n} dx \geq c \int_\Omega |x - x|^{(2-n)/2} dx = \infty \qquad \text{if} \quad n \geq 4,$$

which is a contradiction to (4.15) (which was proved for all $n > 2$).

To prove uniqueness of positive solutions (if $n \geq 4$) suppose v is another positive solution. Then for any $0 < T < T_0$,

$$0 < C_1 \leq u(x,t),\ v(x,t) \leq C_2 \quad \text{for all } x \in \Omega,\ 0 \leq t \leq T. \tag{4.23}$$

Set $w = u - v$. Subtracting the differential equations for u, v and multiplying by w_{xx}, and then integrating over $\Omega \times (t_0, t)$ and letting $t_0 \to 0$, we obtain

$$\frac{1}{2} \int_\Omega w_x(x,t)^2 dx + \int_0^t \int_\Omega (f(u) u_{xxx} - f(v) v_{xxx}) w_{xxx} = 0;$$

here we have used the fact that

$$u_x(\cdot,t) \to u_{0x} \quad \text{strongly in } L^2(\Omega) \text{ as } t \to 0,$$

and the same for v. Writing

$$f(u)u_{xxx} - f(v)v_{xxx} = f(u)w_{xxx} + (f(u) - f(v))v_{xxx}$$

and noting that

$$|f(u) - f(v)| \le C_3|w|,$$

we get

$$\sup_{0<\tau<t} \frac{1}{2} \int_\Omega w_x(x,\tau)^2 \, dx + C_4 \int_0^t \int_\Omega w_{xxx}^2 \le C_3 \int_0^t \int_\Omega \left| w v_{xxx} w_{xxx} \right| .$$

Since the right–hand side is bounded by

$$\frac{1}{2} C_4 \int_0^t \int_\Omega w_{xxx}^2 + C_5 \int_0^t \int_\Omega w^2 v_{xxx}^2,$$

we get

$$\sup_{0<\tau<t} \int_\Omega w_x(x,\tau)^2 \, dx + \int_0^t \int_\Omega w_{xxx}^2 \le C_6 \int_0^t \int_\Omega w^2 v_{xxx}^2 . \tag{4.24}$$

From (2.8) with $\epsilon \to 0$ and (4.23) we see that

$$\int_0^T \int_\Omega v_{xxx}^2 \qquad \text{is finite.} \tag{4.25}$$

Next

$$\int_\Omega u(x,t) \, dx = \int_\Omega v(x,t) \, dx = \int_\Omega u_0 \, dx,$$

so that $\int_\Omega w(x,t) \, dx = 0$ and, by Poincare's inequality,

$$\sup_{\Omega \times (0,t)} w^2 \le C_7 \sup_{0<\tau<t} \int_\Omega w_x^2(x,\tau) \, dx. \tag{4.26}$$

Using (4.25), (4.26) in (4.24) we get

$$\sup_{0<\tau<t} \int_\Omega w_x(x,\tau)^2 dx \le C_8 \Big(\int_0^t \int_\Omega v_{xxx}^2 \Big) \sup_{0<\tau<t} \int_\Omega w_x(x,\tau)^2 \, dx$$

and then, for small t, $w_x \equiv 0$. This yields the asserted uniqueness

THEOREM 4.2. *Let u_0 be as in Theorem 4.1 and let $n > 1$. Then the solution u satisfies:*

$$u_x \in L^2(0, T_0; H_0^1(\Omega)), \tag{4.27}$$

and (2.1) holds in the following sense:

$$\iint\limits_{Q_{T_0}} u\phi_t = \iint\limits_{Q_{T_0}} f(u)u_{xx}\phi_{xx} + \iint\limits_{Q_{T_0}} f'(u)u_x u_{xx}\phi_x \tag{4.28}$$

for all $\phi \in C^2(\overline{Q}_{T_0})$ with $\phi = 0$ near $t = 0$ and near $t = T_0$, and $\phi_x = 0$ on $\partial\Omega \times (0, T_0)$.

REMARK 4.2. (4.27) implies that $u_x = 0$ on $\partial\Omega \times (0, T_0)$ for almost all t ($u_{xxx} = 0$ on $\partial\Omega \times (0, T_0)$ where $u \neq 0$).

REMARK 4.3. Notice that in view of (4.27) and the continuity of u, all the integrals in (4.28) make sense.

Proof. The assertion (4.27) follows from (4.17). To prove (4.28) we shall let $\epsilon \to 0$ in (3.17). But first we establish:

LEMMA 4.3. *As $\epsilon \to 0$*

$$u_{\epsilon,x} \to u_x \quad \text{in} \quad L^2(Q_{T_0}) \quad \text{strongly.} \tag{4.29}$$

Proof. From (4.19) it follows that

$$u_{\epsilon,x} \to u_x \quad \text{weakly in} \quad L^2(0, T_0; H_0^1(\Omega)). \tag{4.30}$$

Next, recalling that $u_{\epsilon,t} = -h_{\epsilon,x}$, where h_ϵ is defined by (2.14) and satisfies (2.15), we have

$$u_{\epsilon,xt} = -h_{\epsilon,xx}$$

in the distribution sense, and thus

$$u_{\epsilon,xt} \quad \text{are uniformly bounded in } L^2(0, T_0; H^{-2}(\Omega)). \tag{4.31}$$

We shall now use a compactness lemma of Lions [10; p. 58]:

Let E_0, E and E_1 be reflexive Banach spaces such that $E_0 \subset E \subset E_1$, the imbedding $E_0 \to E$ is compact and the imbedding $E \to E_1$ is continuous. Assume also that $1 < p_0, p_1 < \infty$. If $\{v_k\}$ is a bounded sequence in $L^{p_0}(0, T_0; E_0)$ and $\{dv_k/dt\}$ is a bounded sequence in $L^{p_1}(0, T_0; E_1)$, then there exists a subsequence of $\{v_k\}$ which converges strongly both in $L^{p_0}(0, T_0; E)$ and in $C([0, T_0]; E_1)$.

Taking $E_0 = H_0^1(\Omega), E = L^2(\Omega), E_1 = H^{-2}(\Omega)$, $p_0 = p_1 = 2$ and $v_k = u_{\epsilon_k,x}$, the assertion (4.29) then follows (using (4.30), (4.31)).

Having proved Lemma 4.3, we now let $\epsilon \to 0$ in (3.17); using the uniform convergence $u_\epsilon \to u$ and (4.29), (4.30), the relation (4.28) follows.

From (4.17) we see that

$$t \to \int_\Omega G_\epsilon(u_\epsilon(x,t))\, dx \qquad \text{is monotone decreasing.}$$

Taking $\epsilon \to 0$ we get:

COROLLARY 4.4. *The function*

$$t \longrightarrow \int_\Omega G_0(u(x,t))\, dx$$

is monotone decreasing.

If, in particular, $f(u) = |u|^n$ then the function $G_0(s)$ is given by (4.8) for $s > 0$; since $\int_\Omega u(x,t)\, dx$ is constant, we conclude that

$$\int_\Omega u(x,t)^{2-n} dx \qquad \text{increases in } t \text{ if } 1 < n < 2$$

$$\text{and decreases in } t \text{ if } n > 2,$$

$$\int_\Omega \log \frac{1}{u(x,t)}\, dx \quad \text{decreases in } t \text{ if } n = 2.$$

COROLLARY 4.5. *Let u_0 be as in Theorem 4.1 and suppose $n \geq \frac{8}{3}$. Then the set*

$$\{t \in (0, T_0); \exists\, x \in \overline{\Omega} \quad \text{with} \quad u(x,t) = 0\}$$

has zero measure and, consequently, the boundary condition $u_{xxx} = 0$ holds for almost all t.

Proof. From (4.27) it follows that

$$u_{xx}(\cdot, t_0) \in L^2(\Omega)\,, \quad u_x(\pm a, t_0) = 0 \tag{4.32}$$

for almost all t_0. Thus it suffices to show that (4.32) implies $u(x, t_0) \neq 0$ for all $x \in \overline{\Omega}$. Suppose $u(x_0, t_0) = 0$ for some $x_0 \in \overline{\Omega}$. Since (4.32) implies $u(\cdot, t_0) \in C^{1,1/2}$, and $u(\cdot, t_0) \geq$

0, we have that $u_x(x_0,t_0)=0$ if $x_0 \in \text{int } \Omega$; also (by (4.32)) $u_x(x_0,t_0)=0$ if $x_0 \in \partial\Omega$. It follows that

$$u(x,t_0) \le C|x-x_0|^{3/2}$$

where C is a constant (depending on t_0). Consequently

$$\int u(x,t_0)^{2-n}\, dx \ge c \int |x-x_0|^{3(2-n)/2}\, dx = \infty \quad (c>0)$$

if $n \ge 8/3$, a contradiction to (4.15).

We conclude this section by considering the case where $u_0(x) \ge 0$ without the additional condition (4.13). If we define

$$\tilde{u}_{0\delta}(x) = u_0(x) + \delta$$

and denote by $\tilde{u}_\delta(x,t)$ the solution u constructed in Theorem 3.1 for the initial data $\tilde{u}_{0\delta}$, which then satisfies all the properties asserted in Theorems 4.1, 4.2, then $\tilde{u}_\delta$ satisfies the estimates

$$\int_\Omega |\tilde{u}_{\delta,x}|^2\, dx \le C\ , \quad |\tilde{u}_\delta| \le A\ , \quad \iint_{Q_{T_0}} f(\tilde{u}_\delta)\tilde{u}_{\delta,xxx}^2\, dxdt \le C,$$

$$|\tilde{u}_\delta(x_1,t_1) - \tilde{u}_\delta(x_2,t_2)| \le K(|x_1-x_2|^{1/2} + |t_1-t_2|^{1/8})$$

with constants C, A, K independent of δ. Taking a subsequence

$$\tilde{u}_\delta \longrightarrow u$$

we obtain:

THEOREM 4.6. *For any $u_0 \ge 0$ there exists a weak solution in the sense of Theorem 3.1 such that $u \ge 0$.*

REMARK 4.4. Most of the results of the previous sections remain true if $n=1$, although some of the arguments require minor modifications. In particular, Lemma 2.1, Theorem 3.1 and Theorem 4.1 remain valid if $n=1$. Furthermore, Theorem 4.2 still holds provided that we use the positive approximations of Section 6 in order to prove (4.28).

§5. Expansion of the support. In this section we continue to assume that

$$u_0 \in H^1(\Omega), \quad u_0 \ge 0 \tag{5.1}$$

and assume also that $n \ge 4$. We consider the weak solution u constructed in Theorem 4.6; then $u = \lim_{\delta \to 0} \tilde{u}_\delta$ where $\tilde{u}_\delta$ is the classical positive solution of (2.1), (2.3) with initial data

$$\tilde{u}_\delta(x,0) = u_0(x) + \delta \quad , \quad \delta > 0.$$

THEOREM 5.1. *The support of the function $t \longrightarrow u(\cdot,t)$ is increasing with t.*

Proof. Set $v = \tilde{u}_\delta$. Let $\xi(x)$ be a smooth nonnegative function such that

(5.2) $$\xi'(\pm a) = 0\ ,$$

(5.3) $$\int_{-a}^{a} \xi(x) u_0^{2-n}(x)\, dx \le C < \infty\ .$$

Introduce also the function

$$G_0(s) = \int_A^s d\tau \int_A^\tau \frac{d\tau_1}{f(\tau_1)}\quad ,\ \text{as in §4},$$

where $A > \max \tilde{u}_\delta$. Multiplying the equation (2.1) for $u = \tilde{u}_\delta = v$ by $\xi G_0'(\tilde{u}_\delta)$ and integrating over Q_T, we get

(5.4) $$\begin{aligned} &\int_\Omega \xi(x) G_0(v(x,T))\, dx - \int_\Omega \xi(x) G_0(v_0(x))\, dx \\ &- \iint_{Q_T} [v_{xxx} v_x \xi + v_{xxx} h(v) \xi']\, dx dt = 0 \end{aligned}$$

where

$$h(v) = -f(v) \int_v^A \frac{ds}{f(s)}\ .$$

Also

(5.5) $$-\iint_{Q_T} v_{xxx} v_x \xi = \iint_{Q_T} [v_{xx}^2 \xi + v_x v_{xx} \xi']$$

since $v_x = 0$ on $x = \pm a$, and

(5.6) $$\iint_{Q_T} v_{xxx} h(v) \xi' = -\iint_{Q_T} [h'(v) v_x v_{xx} \xi' + h(v) v_{xx} \xi'']$$

by (5.2)

Substituting (5.5), (5.6) into (5.4) and using the relations

$$h(v) \sim C_1 v,\quad h'(v) \sim C_2 \qquad \text{for}\quad v \text{ near } 0 \qquad (C_1 > 0,\ C_2 > 0)$$

where C_1, C_2 depend on $f_0(0)$, we deduce that, since $v < A$,

$$\int_{-a}^{a} \xi(x)v(x,T)^{2-n} + \iint_{Q_T} v_{xx}^2 \xi \leq C \iint_{Q_T} [|\xi' v_x v_{xx}| + |v_{xx}\xi''|] + C \,. \tag{5.7}$$

We now choose ξ to have the form $\xi = \zeta^s$ where ζ is is smooth nonnegative function and $s \geq 4$. Then

$$|\xi'| \leq C\xi^{s-1} \,, \quad |\xi''| \leq C\zeta^{s-2} \,.$$

Hence

$$\iint |\xi' v_x \; v_{xx}| \leq C \left\{ \iint \zeta^s v_{xx}^2 \quad \iint \zeta^{s-2} v_x^2 \right\}^{1/2}$$
$$\leq C_1 \left\{ \iint \xi v_{xx}^2 \right\}^{1/2} \,,$$

since $\iint v_x^2$ is bounded (independently of δ), and

$$\iint |v_{xx}\xi''| \leq C \iint \zeta^{s-2} |v_{xx}| \leq C \left\{ \iint \zeta^s v_{xx}^2 \iint \zeta^{s-4} \right\}^{1/2}$$
$$\leq C_1 \left\{ \iint \xi v_{xx}^2 \right\}^{1/2}$$

since ζ^{s-4} is bounded (recalling that $s \geq 4$). Substituting these estimates in (5.7) we conclude that

$$\int_{-a}^{a} \xi(x)v(x,T)^{2-n} dx \leq C' \tag{5.8}$$

where C' is a constant independent of T, δ.

Suppose $u_0(x) > 0$ in an interval $\{\lambda \leq x \leq \mu\}$ and choose $\zeta(x)$ smooth in $\mathbf{R}^1$, positive in $\lambda < x < \mu$ and vanishing on $\{-\infty < x \leq \lambda\} \cup \{\mu \leq x < \infty\}$. Then the function $\xi = \zeta^4$ satisfies (5.2), (5.3) and consequently (5.8) must hold and, in particular,

$$\int_{\lambda+\epsilon}^{\mu-\epsilon} v(x,T)^{2-n} \, dx \leq C''(\epsilon) \qquad \forall \epsilon > 0 \; (v = \tilde{u}_\delta)$$

where $C''(\epsilon)$ is a constant independent on T and δ. Letting $\delta \to 0$ we get

$$\int_{\lambda+\epsilon}^{\mu-\epsilon} u(x,T)^{2-n} \, dx \leq C''(\epsilon) \,.$$

Since $u(\cdot,T) \in C^{1/2}$ and $n \geq 4$, this inequality implies that $u(x,T) > 0$ if $\lambda + \epsilon < x < \mu - \epsilon$ (cf. the proof of Theorem 4.1 (iii)). Recalling that ϵ is arbitrary, it follows that $u(x,T) > 0$ for all x in Ω for which $u_0(x) > 0$. This is also true if $x = \pm a$, by choosing $\zeta(\pm a) > 0$, $\zeta'(\pm a) = 0$ in the above proof. Hence the support of $u(\cdot,T)$ contains the support of $u_0(\cdot)$.

Similarly one can show that if $0 < t_1 < t_2 < T_0$ then the support of $u(\cdot,t_2)$ contains the support of $u(\cdot,t_1)$.

§6. Approximation by positive $\mathbf{u}_\epsilon$. We can construct a weak solution using also other approximations to $f(s)$ and u_0. In this section we shall use the approximations

$$f_\epsilon(s) = \frac{s^4 f(s)}{\epsilon f(s) + s^4}\,, \tag{6.1}$$

$$u_{0\epsilon}(x) = u_0(x) + \epsilon^\theta \qquad (0 < \theta < \frac{1}{2}) \tag{6.2}$$

(this choice of θ is needed in the proof of (6.7) below) in order to show that, if $2 \leq n < 4$, the resulting weak solution, u, is such that the "weak support" of $t \to u(\cdot,t)$ is monotone increasing.

The solution u_ϵ of the approximating system satisfies

$$u_{\epsilon,t} + (f_\epsilon(u_\epsilon)u_{\epsilon,xxx})_x = 0, \tag{6.3}$$

$$u_\epsilon(x,0) = u_{0\epsilon}(x). \tag{6.4}$$

Since $\lim_{s\to 0} \dfrac{f_\epsilon(s)}{s^4} = \dfrac{1}{\epsilon}$ if $1 < n < 4$ while $f_\epsilon(s)$ has the form (2.4) if $n \geq 4$,

$$\lim_{s\to 0} \frac{f_\epsilon(s)}{s^4} = \begin{cases} \dfrac{f_0(0)}{\epsilon f_0(0) + 1} & \text{if } n = 4 \\ 0 & \text{if } n > 4 \end{cases}$$

and $u_{0\epsilon}(x) > 0$, Theorem 4.1 (iii) implies that there exists a unique positive (and smooth) solution u_ϵ of (6.3), (6.4), (2.3) for all $t > 0$. Let u be any limit of a subsequence of u_ϵ, $\epsilon \to 0$. One can easily modify the arguments in §§2–5 to show that u is a weak solution satisfying all the properties derived above. Let us for instance establish (4.18), (4.19). To do this we use the functions

$$g_\epsilon(s) = -\int_s^A \frac{dr}{f_\epsilon(r)}\,, \quad G_\epsilon(r) = -\int_s^A g_\epsilon(r)\,dr$$

and establish, analogously to (4.17), that

$$\int_\Omega G_\epsilon(u_\epsilon(x,T))\,dx + \int_0^T\int_\Omega u_{\epsilon,xx}^2\,dxdt = \int_\Omega G_\epsilon(u_0(x) + \epsilon^\theta)\,dx. \tag{6.5}$$

We compute

$$G''_\epsilon(s) - G''(s) = \frac{1}{f_\epsilon(s)} - \frac{1}{f(s)} = \frac{\epsilon}{s^4}$$

and consequently

(6.6)
$$G_\epsilon(s) - G(s) = \epsilon \int\limits_s^A \frac{r-s}{r^4} dv = \epsilon \left(\frac{1}{6s^2} + \frac{s}{3A^3} - \frac{1}{2A^2} \right).$$

It follows that

(6.7)
$$|G_\epsilon(u_0 + \epsilon^\theta) - G(u_0 + \epsilon^\theta)| \le C\epsilon^{1-2\theta} \to 0 \quad \text{if} \quad \epsilon \to 0$$

and therefore

(6.8)
$$\int\limits_\Omega G_\epsilon(u_0 + \epsilon^\theta)\, dx \longrightarrow \int\limits_\Omega G(u_0)\, dx \, .$$

From (6.5) and (6.8) we obtain (4.18), (4.19), upon which Theorem 4.1 is based.

The fact that the u_ϵ are positive smooth functions will enable us to extend the proof of Theorem 5.1 to the case $2 \le n < 4$:

THEOREM 6.1. *If $2 \le n < 4$ and $u_0 \ge 0$, then any weak solution obtained by the approximations (6.1)–(6.4) has the following property:*

if $u_0(x) > 0$ in an interval $\omega \subset \Omega$, then
$u(\cdot, t) > 0$ a.e. on ω, for all $0 < t < T_0$.

Proof. The proof is based on extending the ideas which occur in the proof of Theorem 5.1. Take $\zeta \in C^2(\overline{\Omega})$, supp ζ in ω, $\zeta \ge 0, (\zeta^4)' = 0$ on $\partial\Omega$, and let $\xi = \zeta^4$. Multiplying (6.3) by $\xi(x) g_\epsilon(u_\epsilon(x,t))$ and integrating, we obtain after several integrations by parts,

(6.9)
$$\int\limits_\Omega \xi(x) G_\epsilon(u_\epsilon(x,T))\, dx + \int\limits_0^T \int\limits_\Omega \xi u^2_{\epsilon,xx} = R + \int \xi(x) G_\epsilon(u_0(x) + \epsilon^\theta)\, dx,$$

where

$$R = -\iint \xi' u_{\epsilon,xx} u_{\epsilon,x} - \iint \xi' u_{\epsilon,xx} u_{\epsilon,x} (1 + f'_\epsilon(u_\epsilon) g_\epsilon(u_\epsilon))$$
$$- \iint \xi'' u_{\epsilon,xx} f_\epsilon(u_\epsilon) g_\epsilon(u_\epsilon).$$

We easily estimate, if $0 < s < A$,

$$|g_\epsilon(s)| \le K\,\frac{\epsilon s^n + s^4}{s^{n+3}}\ ,$$
$$f_\epsilon(s) \le K \frac{s^{n+4}}{\epsilon s^n + s^4}.$$

Also, by explicitly computing $f'_\epsilon(s)$ we find that

$$|f'_\epsilon(s)| \le K_2 \frac{s^{n+3}}{\epsilon s^n + s^4}.$$

Hence

$$|f_\epsilon(u_\epsilon) g_\epsilon(u_\epsilon)| \le K_3 |u_\epsilon|,$$
$$|f'_\epsilon(u_\epsilon) g_\epsilon(u_\epsilon)| \le K_3.$$

Using these estimates we can estimate $|R|$ from above:

$$|R| \le C \iint \zeta^2 |u_{\epsilon,xx}| \cdot \zeta |u_{\epsilon,x}| + C \iint \zeta^2 |u_{\epsilon,xx}| \cdot u_\epsilon.$$

If we estimate the right–hand side by the Schwarz inequality and use the result in (6.9), we get,

$$\text{(6.10)}\qquad \begin{aligned} \int_\Omega \zeta^4 G_\epsilon(u_\epsilon(x,T))\ dx + \int_0^T \int_\Omega \zeta^4 u^2_{\epsilon,xx} &\le C \int \zeta^4 G_\epsilon(u_0 + \epsilon^\theta)\ dx \\ &+ C \int_0^T \int_\Omega (\zeta^2 u_{\epsilon,x} + u_\epsilon^2). \end{aligned}$$

Letting $\epsilon \to 0$ and noting (cf. (6.7), (6.8)) that

$$\int \zeta^4 G_\epsilon(u_0 + \epsilon^\theta) \longrightarrow \int \zeta^4 G(u_0)$$

and that the last term on the right–hand side of (6.10) is bounded independently of ϵ, we conclude that

$$\text{(6.11)}\qquad \int_\Omega \zeta^4 G(u_0)\ dx < \infty \ \text{ implies } \ \int_\Omega \zeta^4 G(u(\cdot, t) < \infty$$

for all $0 < t < T_0$. But this property can be used to establish Theorem 6.1, by the same arguments used in the proof of Theorem 4.1 for $2 \le n < 4$.

§7. Equations of order ≥ 6. In this section we shall extend some of the results of §§2–4 to equation (1.2) with $m \geq 2$; $f(u)$ is assumed to satisfy (2.4). We shall take the initial condition (2.2) with

$$u_0 \in H^m(\Omega) \tag{7.1}$$

and the boundary conditions

$$Du = D^3u = D^5u = \cdots = D^{2m+1}u = 0 \quad \text{on} \quad \partial\Omega \times (0, T_0) \tag{7.2}$$

where $D = \frac{\partial}{\partial x}$. We begin by introducing the approximating equations

$$u_t + (-1)^{m-1} D((f(u) + \epsilon) D^{2m+1} u) = 0 \tag{7.3}$$

with the boundary conditions (7.2) and the initial conditions

$$u(x, 0) = u_{0\epsilon}(x) \tag{7.4}$$

where $u_{0\epsilon}$ are smooth ($C^{2m+2+\alpha}$) and satisfy (7.2), and $u_{0\epsilon} \to u_0$ in $H^m(\Omega)$ as $\epsilon \to 0$

Denote the solution of (7.2)–(7.4) by u_ϵ. Multiplying (7.3) by $D^{2m}u_\epsilon$ and integrating over Q_T, we get

$$\frac{1}{2}\int_\Omega |D^m u_\epsilon(x,T)|^2 + \iint_{Q_T} (f(u_\epsilon) + \epsilon)|D^{2m+1}u_\epsilon|^2 = \frac{1}{2}\int_\Omega |D^m u_{0\epsilon}(x)|^2 \; ; \tag{7.5}$$

hence

$$\int_\Omega |D^m u_\epsilon(x,T)|^2 \leq C \, , \tag{7.6}$$

$$\iint_{Q_{T_0}} f(u_\epsilon)|D^{2m+1}u_\epsilon|^2 \leq C \tag{7.7}$$

where C is a constant independent of T, ϵ. By integrating (7.3) over Q_T we also have

$$\int_\Omega u_\epsilon(x,T)\,dx = \text{const.} = \int_\Omega u_{0\epsilon}(x)\,dx. \tag{7.8}$$

Using (7.6), (7.2) and (7.8) we deduce, by the Poincaré inequality, that

$$\int_\Omega |D^j u_\epsilon(x,T)|^2 \leq C \qquad \forall \;\; 0 \leq j \leq m-1 \tag{7.9}$$

by the boundary conditions (7.2), $D^2u, D^4u, \dots, D^mu$ have at least one zero for each T and this is enough to apply the Poincaré inequality. By the Sobolev inequality,

$$|u_\epsilon| \le A\,, \tag{7.10}$$

$$|u_\epsilon(x_1,t) - u_\epsilon(x_2,t)| \le K|x_1 - x_2| \qquad \forall\ x_1, x_2 \in \Omega, t \in (0,T) \tag{7.11}$$

where A, K are constants independent of ϵ; in fact

$$\begin{aligned} &D^j u_\epsilon(\cdot,t) \quad \text{is Lipschitz in } x,\ 0 \le j \le m-2 \\ &D^{m-1}u_\epsilon(\cdot,t) \quad \text{is Hölder continuous in } x \text{ (exponent } \frac{1}{2}) \end{aligned}$$

uniformly in t, ϵ.

Consider the function

$$h_\epsilon = (-1)^{m-1}(f(u_\epsilon) + \epsilon)D^{2m+1}u_\epsilon. \tag{7.12}$$

By (7.7) and (7.10) it follows that

$$\iint\limits_{Q_{T_0}} |h_\epsilon|^2 \le C, \qquad C \quad \text{independent of } \epsilon. \tag{7.13}$$

Using this fact and the relation

$$\frac{\partial}{\partial t}u_\epsilon + Dh_\epsilon = 0 \tag{7.14}$$

we can now repeat the argument of Lemma 2.1. Using however (7.11) (instead of (2.13)) we get

$$|u_\epsilon(x,t_1) - u(x,t_2)| \le M|t_1 - t_2|^{1/5}. \tag{7.15}$$

We can now proceed as in §3 and establish:

THEOREM 7.1. *There exists a function u, a uniform limit of the u_ϵ (for a sequence $\epsilon \to 0$) in Q_{T_0}, such that u is Lipschitz continuous in x and Hölder continuous (exponent 1/5) in t,*

$$\int\limits_\Omega |D^ju(x,t)|\, dx \le C \qquad \forall \qquad 1 \le j \le m,\ 0 < t < T_0, \tag{7.16}$$

$$f(u)\ D^{2m+1}u \in L^2(P) \qquad \text{where } P = \overline{Q}_{T_0} \setminus (\{u=0\} \cup \{t=0\}); \tag{7.17}$$

u is a classical solution of (1.2) in P, and

$$\iint_{Q_{T_0}} u\phi_t + \iint_P f(u)D^{2m+1}u \cdot \phi_x = 0 \tag{7.18}$$

for any $\phi \in Lip(\overline{Q}_{T_0})$, $\phi = 0$ *near* $t = 0$ *and near* $t = T_0$*; further,*

$$u(x,0) = u_0(x) \ , \quad x \in \Omega$$

and

u satisfies the boundary conditions (7.2)
at all points of the lateral boundary where $u \neq 0$.

REMARK 7.1. The argument of Lemma 2.1 also gives that

$$D^j u \quad \text{is Hölder continuous in } \overline{Q}_{T_0}, \text{ for all } 0 \leq j \leq m-1.$$

In fact, if $m \geq 1$,

$$D^{m-1}u \quad \text{belongs to } C_{x,t}^{\frac{1}{2},\frac{1}{4(m+1)}}$$

and if $m \geq 2$ then

$$D^j u \quad \text{belongs to } C_{x,t}^{1,\frac{1}{5+2j}} \text{ for } 0 \leq j \leq m-2 \ .$$

We next consider the case where $u_0 \geq 0$ and

$$\begin{aligned} &\int_\Omega |\log u_0| < \infty && \text{if} \quad n = 2 \ , \\ &\int_\Omega u_0^{2-n} dx < \infty && \text{if} \quad 2 < n < n^*, \\ &u_0(x) > 0 \quad \text{in} \quad \overline{\Omega} && \text{if} \quad n \geq n^* \end{aligned} \tag{7.19}$$

where $n^* = 8/3$ if $m = 2$ and $n^* = 5/2$ if $m \geq 3$.

REMARK 7.2. If $u_0 \geq 0$ and $\int_\Omega u_0^{2-n} dx < \infty, n \geq n^*$ then $u_0(x)$ must be strictly positive (cf. Remark 4.1). Indeed if $m = 2$ then $u_{0,xx} \in L^2(\Omega)$ and we argue as in the proof of Corollary 4.5; if $m \geq 3$ then $u_{0,xx}$ is continuous and thus if $u_0(x_0) = 0$ then $u_{0,x}(x_0) = 0$ and $u_0(x) \leq C|x - x_0|^2$, which implies

$$\int_\Omega u_0(x)^{2-n} dx \geq c \int_\Omega |x - x_0|^{2(2-n)} dx = \infty \quad \text{if} \quad n \geq \frac{5}{2} \quad (c > 0),$$

a contradiction.

THEOREM 7.2. *If $u_0 \geq 0$ and (6.19) holds then $u \geq 0$ and, in fact, all the assertions of Theorem 4.1 hold; further*

$$u(x,t) > 0 \quad in \quad \overline{Q}_{T_0} \qquad if \qquad n \geq n^* . \tag{7.20}$$

Proof. Multiplying (7.3) by $g_\epsilon(u_\epsilon)$ (as in §4) and integrating over Q_T we easily get after some integrations by parts,

$$\int_\Omega G_\epsilon(u_\epsilon(x,T)) + \iint_{Q_T} |D^{m+1}u_\epsilon|^2 = \int_\Omega G_\epsilon(u_{0\epsilon}(x)) . \tag{7.21}$$

This relation allows us to deduce all the assertions of Theorem 4.1. Finally, if $n \geq n^*$, $u(x,t)$ must be strictly positive in $\overline{Q}_{T_0}$; for, if $u(x_0,t_0) = 0$, then we get (cf. Remark 7.2)

$$\int_\Omega u^{2-n}(x,t_0)\, dx = \infty$$

which is a contradiction

Recall that for $n \geq n^*$ the solution u is positive and classical.

For any $u_0 \geq 0$ we can construct (by Theorem 7.2) solutions $u = \tilde{u}_\delta$ corresponding to the initial data $u_0(x) + \delta$. Taking $\delta \to 0$ we obtain a limiting function $u = \lim\ \tilde{u}_\delta$ which is ≥ 0; u is a weak solution in the sense of Theorem 7.1. Thus, for any $u_0 \geq 0$ there exists a weak solution which is ≥ 0 (cf. Theorem 4.6).

We summarize:

THEOREM 7.3. *For any $u_0 \in H^m(\Omega), u_0 \geq 0$ there exists a weak solution (in the sense of Theorem 7.1) which is ≥ 0.*

Acknowledgement. The first author wishes to thank Professor M. Bertsch for useful discussions.

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