THE L^p-INTEGRABILITY OF GREEN'S FUNCTIONS AND FUNDAMENTAL SOLUTIONS FOR ELLIPTIC AND PARABOLIC EQUATIONS

by

E. B. Fabes
University of Minnesota
Minneapolis, Minnesota 55455

and

D. W. Stroock
University of Colorado
Boulder, Colorado 80907

Institute for Mathematics and its Applications
University of Minnesota
Minneapolis, Minnesota 55455

1. Introduction

Given $d \geqslant 1$ and $\lambda \in (0,1)$ denote by $\mathcal{Q}_d(\lambda)$ the class of smooth, symmetric, $d \times d$ matrix-valued functions $a \equiv (a^{ij}(x))$ on R^d which satisfy

$$\lambda I \leqslant a(x) \leqslant \frac{1}{\lambda} I \quad , \qquad x \in R^d$$

in the sense of nonnegative definiteness. Set

$$L_a u = \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$$

and let

$$L_a^* v = \sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j} \left(a^{ij}(y) v(y)\right)$$

denote the adjoint of L.

In the first part of this paper we study the interior behavior of nonnegative solutions, v, of the adjoint equation, $L_a^* v = 0$, in a domain Ω of R^d. Our main result is the establishment of an interior "backward Hölder inequality" for such solutions. Specifically we show the existence of a constant, c, <u>depending only on</u> λ and not on the smoothness of the coefficients such that

$$\left[\frac{1}{|B|} \int_B v(y)^{d/d-1} dy\right]^{d-1/d} \leqslant c \, \frac{1}{|B|} \int_B v(y) dy \tag{1.1}$$

for all balls B whose concentric double is contained in Ω. ($|E|$ denotes the Lebesgue measure of the set E.) The same estimate will also be shown to be valid for the Green's function, $G_a(x,y)$, of Ω as a function of y. (Recall $L_a^*(G_a(x,\cdot))(y) = 0$, $y \in \Omega \setminus \{x\}$.) The constant, c, in this case will also be independent of the variable x.

As a consequence of the inequality (1.1) we can find $q_\lambda > d/d-1$ such that

$$\sup_{x \in \Omega} \sup_{a \in \mathcal{Q}_d(\lambda)} \int_\Omega G_a(x,y)^{q_\lambda} dy < \infty \quad \text{(Corollary 2.3)} \tag{1.2}$$

This estimate for $q_\lambda = d/d-1$ was first proved by Alexandrov [1] and Pucci [10]. Several other interesting properties of nonnegative solutions of the adjoint equation and of the Green's function follow from (1.1). Since we will not systematically use these consequences we will not dwell upon them but will instead refer the interested reader to [3] for properties of functions satisfying a backward Hölder inequality and to [2] where these properties are applied to Green's functions associated with operators, L_a, with uniformly continuous a.

In the second part of this paper we will use the estimate (1.2) to study the integrability properties of the fundamental solution, $\Gamma_a(t,x,y)$, $(t,x,y) \in (0,\infty) \times R^d \times R^d$, to the parabolic initial-value problem:

$$\frac{\partial u}{\partial t}(t,x) = L_a u(t,x) \quad , \quad u(0,x) = f(x) \quad \left(L_a = \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \right) \quad .$$

We will show that for the same exponent $q_\lambda > d/d-1$ mentioned in (1.2),

$$\sup_{x \in R^d} \sup_{a \in \mathcal{Q}_d(\lambda)} \left\| \Gamma_a(t,x,\cdot) \right\|_{L^{q_\lambda}(R^d)} < \infty \ . \tag{1.3}$$

The technique for establishing (1.3) for Γ_a, once the estimate (1.2) for G_a is known, is due to P.L. Lions [8], who proved (1.3) with q_λ replaced by $d/d-1$ and Γ_a replaced by the Green's function, $g_a(t,x,y)$, corresponding to a spatially bounded cylindrical domain. This observation of Lions will be discussed in detail in Section 3, along with a discussion of the best possible nature of the estimate (1.3).

2. A Backward Hölder Inequality for the Green's Function

In this section Ω will denote a bounded domain of R^d, B_r will denote a ball of radius r while B_{kr} will denote the ball of radius kr, concentric with B_r. Throughout this paper the letter c will denote a constant depending at most on λ and d. It is likely different at each occurrence.

We begin our proof of (1.1) by first establishing the so-called "doubling condition" for the measure whose density with respect to Lebesgue measure on R^d is a nonnegative adjoint super-solution. This property was observed in [2] and our proof here is a slight modification of the one presented there. Since the proof is relatively short and simple we present it for the sake of completeness.

Lemma 2.0. There exists a constant c, depending only on λ and d, such that for all nonnegative weak solutions of $L_a^* v \leqslant 0$ (i.e., $v \in L^1_{loc}(\Omega)$, $v \geqslant 0$, and $\int v L_a u \leqslant 0$ for all $u \geqslant 0$, $u \in c_0^{\infty}(\Omega)$) and for all balls B_r with $B_{2r} \subset \Omega$, we have

$$\int_{B_r} v(y)dy \leqslant c \int_{B_{r/2}} v(y)dy \quad .$$

Proof. We may assume that the center of B_r is the origin. For $\delta > 0$ set $h(x) = \left[(1+\delta)^2 r^2 - |x|^2\right]^2$. It is easily verified for δ sufficiently small, depending only on λ and d, that

$$L_a h \geqslant 0 \quad \text{for} \quad (1+\delta) > |x| \geqslant (1-\delta)r \quad ,$$
$$L_a h \geqslant cr^2, \quad c > 0 \ , \quad \text{for} \quad r \geqslant |x| \geqslant (1-\delta)r \ , \quad \text{and}$$
$$|L_a h| \leqslant cr^2 \quad .$$

Hence,

$$\int_{B_r \setminus B_{(1-\delta)r}} v(y)dy \leqslant c \int_{B_r \setminus B_{(1-\delta)r}} v(y) L_a(h/r^2)(y)dy \leqslant c \int_{B_{(1+\delta)r} \setminus B_{(1-\delta)r}} v(y) L_a(h/r^2)(y)dy$$

Since $L_a^* v \leqslant 0$ in the weak sense it is easy to see that $\int_{B_{(1+\delta)r}} v(y) L_a(h/r^2)(y)dy \leqslant 0$.

Hence

$$\int_{B_r} v(y)dy \leqslant c \int_{B_{(1-\delta)r}} v(y)dy \quad .$$

By a simple iteration argument it follows that

$$\int_{B_r} v(y)dy \leqslant c \int_{B_{r/2}} v(y)dy \quad .$$

We are now ready to prove (1.1), the backward or reversed Hölder inequality for nonnegative adjoint solutions. The principal idea, explicitly observed by Alexandrov and implicitly by Pucci, is that estimates for a function u in terms of Lu when $Lu \geqslant 0$ follow from corresponding estimates for the solution, z, of the Monge-Ampere equation

$$\det(\text{Hessian}\, z) = (Lu)^d \quad .$$

Theorem 2.1 There exists a constant c_λ, depending only on λ and d, such that for all $v \geqslant 0$ in Ω satisfying $L_a^* v = 0$ there and for all balls, B_r with $B_{2r} \subset \Omega$,

$$\left[\frac{1}{r^d}\int_{B_r} v(y)^{d/d-1} dy\right]^{d-1/d} \leqslant c_\lambda \frac{1}{r^d}\int_{B_r} v(y)dy \quad .$$

Proof. We can bound the $L^{d/d-1}$-norm of v over B_r by estimating the

$$\sup\left\{\int_{B_r} v(y)f(y)dy : f \in C^\infty(R^d),\ f > 0,\ \|f\|_{L^d(R^d)} \leqslant 1\right\} \quad .$$

Given such an f we consider the smooth convex function, $z_r(y)$, satisfying:

$$\det\left(\frac{\partial^2 z_r}{\partial y_i \partial y_j}(y)\right) = f^d \text{ in } B_{2r}, \quad z_r\Big|_{\partial B_{2r}} = 0 \,. \qquad [4, 9]$$

Note that $f(y) \leqslant \lambda^{-1} L_a z_r(y)$ in B_{2r}. (Write $Lz_r = \sum_{i=1}^d \alpha_i \varphi_i$ where $\alpha_i \geqslant \lambda$, $i = 1, \ldots, d$, and $\varphi_1, \ldots, \varphi_d$ are the eigenvalues of the Hessian of z_r. Then $L_a z_r \geqslant \lambda \max_i \varphi_i \geqslant \lambda(\prod_{i=1}^d \varphi_i)^{1/d} = \lambda f$.)

Now pick $\phi_r \in C_0^\infty(B_{3r/2})$ satisfying: $\phi_r \equiv 1$ on B_r, $|\nabla\phi_r| \leqslant c/r$ and $\left|\frac{\partial^2 \phi_r}{\partial y_i \partial y_j}\right| \leqslant c/r^2$ for all i and j. Clearly

$$\int_{B_r} v(y)f(y)dy \leqslant c \int v(y)\phi_r(y)L_a z_r(y)dy \leqslant c\Bigg[\int v(y)L_a(\phi_r z_r)(y)dy$$

$$+ \frac{1}{r}\int_{B_{3r/2}\setminus B_r} v(y)\,|\nabla z_r(y)|\,dy + \frac{1}{r^2}\int_{B_{3r/2}} v(y)|z_r(y)|dy\Bigg] .$$

Since $L_a^* v = 0$, the first integral on the right side of the final inequality is zero.

It is easily seen that if y_0 denotes the center of B_r, we can write

$$z_r(y) = 4r^2 w\left(\frac{y-y_0}{2r}\right)$$

where $w(\tilde{y})$ is a smooth convex function on $B_1(0)$ satisfying:

$$\det\left(\frac{\partial^2 w(\tilde{y})}{\partial\tilde{y}_i \partial\tilde{y}_j}\right) = f(y_0 + 2r\tilde{y})^d, \qquad w\big|_{\partial B_1(0)} = 0 .$$

The arguments in Pucci [10, pp. 17-19] show that

$$|w| \leqslant c\left[\int_{B_1(0)} f^d(y_0 + 2\tilde{y})d\tilde{y}\right]^{1/d} \leqslant \frac{c}{r}\,\|f\|_{L^d} \leqslant \frac{c}{r} .$$

Since w is convex and zero on the boundary of $B_1(0)$, $|\nabla w(\tilde{y})| \leqslant c(1-|\tilde{y}|)^{-1}|w(\tilde{y})|$

$$\frac{1}{r}\int_{B_{3r/2}\setminus B_r} v(y)\,|\nabla z_r(y)|\,dy \leqslant \frac{c}{r}\int_{B_{3r/2}} v(y)dy$$

and

$$\frac{1}{r^2}\int_{B_{3r/2}} v(y)\,|z_r(y)|\,dy \leqslant \frac{c}{r}\int_{B_{3r/2}} v(y)dy .$$

We conclude that

$$\int_{B_r} v(y)f(y)dy \leqslant \frac{c}{r}\int_{B_{3r/2}} v(y)dy$$

and, hence,

$$\left[\int_{B_r} v(y)^{d/d-1} dy\right]^{d-1/d} \leqslant \frac{c}{r}\int_{B_{3r/2}} v(y)dy$$

An application of Lemma 2.0 concludes the proof of Theorem 2.1.

Theorem 2.2. Let $G(x,y)$ denote the Green's function corresponding to Ω and the operator L_a. There exists c_λ, depending only on λ and d, such that for all balls B_r with $B_{4r} \subset \Omega$, we have

$$\left[\frac{1}{r^d}\int_{B_r} G(x,y)^{d/d-1} dy\right]^{d-1/d} \leqslant c_\lambda \frac{1}{r^d}\int_{B_r} G(x,y)dy \quad .$$

Proof. If $x \notin B_{2r}$ then the conclusion of Theorem 2.2 follows from Theorem 2.1. Therefore assume $x \in B_{2r}$ and let $G_r(x,y)$ denote the Green's function corresponding to L_a and B_{3r}.

Since $G(x,y) \geqslant G_r(x,y)$,

$$\frac{1}{r^d}\int_{B_r} G(x,y)^{d/d-1} dy \leqslant c\left[\frac{1}{r^d}\int_{B_r} \left[G(x,y) - G_r(x,y)\right]^{d/d-1} dy\right.$$

$$\left. + \frac{1}{r^d}\int_{B_r} G_r(x,y)^{d/d-1} dy\right] .$$

As a function of y, $G(x,y) - G_r(x,y)$ is a nonnegative solution of $L_a^* v = 0$ in B_{3r}. By Theorem 2.1

$$\frac{1}{r^d}\int_{B_r}\left[G(x,y)-G_r(x,y)\right]^{d/d-1}dy \leqslant c\left[\frac{1}{r^d}\int_{B_r}\left[G(x,y)-G_r(x,y)\right]dy\right]^{d/d-1} .$$

For simplicity we assume that the center of B_r is the origin and we write $G_r(x,y) = G_r(r\tilde{x}, r\tilde{y})$ where $|\tilde{x}| \leqslant 2$ and $|\tilde{y}| \leqslant 3$. The function $G_r(r\tilde{x}, r\tilde{y})r^{d-2}$ is the Green's function, $\tilde{G}(\tilde{x},\tilde{y})$, corresponding to B_3 and the operator L_{a_r} where $a_r(\tilde{x}) = a(r\tilde{x})$. Using this observation together with the result of Pucci and Alexandrov [10, 1] we have

$$\frac{1}{r^d}\int_{B_{3r}} G_r(x,y)^{d/d-1}dy = \int_{B_3}\tilde{G}(\tilde{x},\tilde{y})^{d/d-1}d\tilde{y} \leqslant c(r^{2-d})^{d/d-1} .$$

Also

$$\inf_{x\in B_{2r}}\frac{1}{r^d}\int_{B_{3r}} G_r(x,y)dy = \inf_{\tilde{x}\in B_2} r^{2-d}\int_{B_3}\tilde{G}(\tilde{x},\tilde{y})d\tilde{y} \geqslant cr^{2-d} .$$

These inequalities imply

$$\frac{1}{r^d}\int_{B_r} G_r(x,y)^{d/d-1}dy \leqslant c\left[\frac{1}{r^d}\int_{B_{3r}} G_r(x,y)dy\right]^{d/d-1}$$

and from Lemma 2.0

$$\int_{B_{3r}} G_r(x,y)dy \leqslant c\int_{B_r} G_r(x,y)dy .$$

The conclusion of Theorem 2.2 is now immediate.

Corollary 2.3 Let $G(x,y)$ again denote the Green's function corresponding to Ω and L_a. There exist positive numbers A and q_λ, with $q_\lambda > d/d-1$ and depending only on λ and d while A depends only on λ, d, and the diameter of Ω, such that

$$\sup_{x\in\Omega} \int_\Omega G(x,y)^{q_\lambda} dy \leqslant A .$$

Proof. Take any cube, Q, containing $\bar{\Omega}$ and also with the property that the $\mathrm{dist}(\partial Q, \bar{\Omega}) \geqslant 1$. Let $\tilde{G}$ denote the Green's function corresponding to $2Q$, the symmetric double of Q, and the operator L_a. As a function of y, $\tilde{G}(x,y)$ satisfies the "backward Hölder inequality" of Theorem 2.1 over any subcube of Q. From the theory of A_∞ weights [3] there exist q_λ and A satisfying the conclusion of Corollary (2.3) such that

$$\sup_{x\in Q} \int_Q \tilde{G}(x,y)^{q_\lambda} dy \leqslant A .$$

Since $\tilde{G} \geqslant G$ on $\Omega \times \Omega$ the same inequality for G over Ω is immediate.

3. L^p-integrability of a Fundamental Solution in the Parabolic Case

Given $a(x) = (a^{ii}(x)) \in \mathcal{Q}_d(\lambda)$ we recall that $\Gamma_a(t,x,y)$, $(t,x,y) \in (0,\infty) \times R^d \times R^d$, denotes the fundamental solution to the initial-value Cauchy problem:

$$\frac{\partial u}{\partial t}(t,x) = L_a u(t,x) , \quad (t,x) \in (0,\infty) \times R^d \quad \text{and} \quad u(0,x) = f(x) , \quad x \in R^d . \tag{3.1}$$

(As before, $L_a = \sum_{i,j} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$.) Once and for all in this section we fix the exponent $q_\lambda > d/d-1$ given in Theorem 2.2 and Corollary 2.3.

The main purpose of this part of the paper is to prove that for each $q \in [1, q_\lambda]$ there is a finite constant, $c_d(\lambda, q)$ such that

$$\sup_{x\in R^d} \sup_{a\in \mathcal{Q}_d(\lambda)} \left\| \Gamma_a(t,x,\cdot) \right\|_{L^q(R^d)} = c_d(\lambda,q) t^{-d/2q'} , \quad t > 0 , \tag{3.2}$$

where q' denotes the Hölder conjugate of q. Some preliminary remarks may be helpful in understanding the above equality.

Remark (3.3): The form of (3.2) is imposed by the underlying structure of (3.1). To be precise, let $q \in (0,\infty)$ be given and set

$$C_d(\lambda,q) = \sup_{a \in \mathcal{a}_d(\lambda)} \|\Gamma_a(1,0,\cdot)\|_{L^q(R^d)} .$$

Then (3.2) holds, with this choice of $C_d(\lambda,q)$ for all $t > 0$. Indeed, since the class $\mathcal{a}_d(\lambda)$ is invariant under translation of the independent variable, it is clear that

$$\sup_{a \in \mathcal{a}_d(\lambda)} \|\Gamma_a(1,x,\cdot)\|_{L^q(R^d)}$$

is independent of $x \in R^d$ and is therefore equal to $C_d(\lambda,q)$ for all $x \in R^d$. In addition, given $T > 0$ and $a \in \mathcal{a}_d(\lambda)$, define $a_T(x) = a(T^{1/2}x)$, $x \in R^d$. Clearly $a_T \in \mathcal{a}_d(\lambda)$. Moreover, given $f \in C_0(R^d)$, it is easily seen that the function $u(t,x)$ defined by

$$u(t,x) = T^{d/2}\int \Gamma_a(Tt, T^{1/2}x, T^{1/2}y)f(y)dy \quad , \qquad t > 0 ,$$

satisfies (3.1) where a is replaced by a_T on the right hand side of (3.1). Thus $T^{d/2}\,\Gamma_a(Tt,T^{1/2}x,T^{1/2}y) = \Gamma_{a_T}(t,x,y)$ for all $(t,x,y) \in (0,\infty) \times R^d \times R^d$; and so

$$\sup_{a \in \mathcal{a}_d(\lambda)} \|\Gamma_a(T,0,\cdot)\|_{L^q(R^d)} = T^{-d/2} \sup_{a \in \mathcal{a}_d(\lambda)} \|\Gamma_a(1,0,T^{-1/2}\cdot)\|_{L^q(R^d)}$$

$$= C_d(\lambda,q)/T^{d/2q'} .$$

In other words, in order to prove (3.2) it suffices to prove that for $q \in [1,q_\lambda]$:

$$\sup_{a \in \mathcal{a}_d(\lambda)} \|\Gamma_a(1,0,\cdot)\|_{L^q(R^d)} < \infty . \tag{3.4}$$

Finally, since $\|\Gamma_a(1,0,\cdot)\|_{L^1(R^d)} = 1$ for all $a \in \mathcal{a}_d(\lambda)$, Hölder's inequality tells us that it suffices to prove (3.4) when $q = q_\lambda$.

Remark (3.5): It is interesting to examine in what sense (3.2) is optimal. We first point out that there is no analogue of (3.2) when the coefficient matrix a is allowed to be time-dependent. In fact, even when $d = 1$ and $a : [0,\infty) \times R^1 \to [\lambda, 1/\lambda]$ is uniformly continuous, the example constructed in [5] shows that the fundamental solution to (3.1) may not be absolutely constinuous with respect to Lebesgue measure for fixed $t > 0$. On the other hand, if $\rho : (0,\infty) \to (0,\infty)$ is a nondecreasing function satisfying $\lim_{\delta \downarrow 0} \rho(\delta) = 0$ and $\mathcal{A}_d(\lambda,\rho)$ denotes the class of $a \in \mathcal{A}_d(\lambda)$ such that

$$\|a(x) - a(y)\| \leqslant \rho(|x-y|) \;, \qquad x, y \in R^d \;,$$

then it is known ([11, Chap. 9], for example) that for each $t > 0$ and $q \in [1,\infty)$

$$\sup_{x \in R^d} \; \sup_{a \in \mathcal{A}_d(\lambda,\rho)} \|\Gamma_a(t,x,\cdot)\|_{L^q(R^d)} < \infty \;.$$

Moreover, the technique used to prove the preceding is perturbation and can be modified to show that for each $q \in (0,\infty)$ there is a $\lambda \in (0,1)$ such that

$$\sup_{x \in R^d} \; \sup_{a \in \mathcal{A}_d(\lambda)} \|\Gamma_a(t,x,\cdot)\|_{L^q(R^d)} < \infty \;, \qquad t > 0 \;.$$

Finally, we will show below ((3.10)) that when $d \geqslant 2$: for each $\lambda \in (0,1)$ there is a $q \in (d/d\text{-}1, \infty)$ and for each $q \in (d/d\text{-}1, \infty)$ there is a $\lambda \in (0,1)$ such that

$$\sup_{a \in \mathcal{A}_d(\lambda)} \|\Gamma_a(1,0,\cdot)\|_{L^q(R^d)} = \infty \;.$$

We now turn to the proof of (3.4) when $q = q_\lambda$. As will be apparent, the contribution of the present authors in minor. Indeed, our starting point is the following clever observation communicated to us by P.L. Lions [8]. Let $a \in \mathcal{A}_d(\lambda)$ and $r > 0$ be given. Denote by $g_{a,r}(t,x,y)$, $(t,x,y) \in (0,\infty) \times B_r \times B_r$, where $B_r = \{x \in R^d : |x| < r\}$, the Green's function for the initial-value Cauchy problem:

$$\frac{\partial u}{\partial t}(t,x) = L_a u(t,x) , \quad (t,x)\in(0,\infty)\times B_r ,$$
$$u(0,x) = f(x) , \quad x\in B_r , \tag{3.6}$$
$$u(t,x) = 0 , \quad (t,x)\in(0,\infty)\times\partial B_r .$$

What Lions showed is that there is a $C_d(\lambda,r) < \infty$ such that

$$\sup_{x\in B_r}\ \sup_{a\in\mathcal{A}_d(\lambda)} \|g_{a,r}(t,x,\cdot)\|_{L^{d/d-1}(B_r)} \leqslant C_d(\lambda,r)/t^{d+1} , \qquad t>0 . \tag{3.7}$$

As we are about to see, (3.4) with $q = q_\lambda$ is an easy step away from the proof of (3.7) with $d/d\text{-}1$ replaced by q_λ. However, before making this step, it will be necessary to repeat Lions's argument for the exponent q_λ. This argument rests on the estimate in Corollary (2.3) and on another estimate due to Krylov [6]. What the latter author had shown is the existence of $K_d(\lambda,r) < \infty$ such that:

$$\sup_{x\in B_r}\ \sup_{a\in\mathcal{A}_d(\lambda)} \left|\int_0^\infty\int_{B_r} q_{a,r}(t,x,y)f(t,y)dydt\right| \leqslant K_d(\lambda,r)\|f\|_{L^{d+1}([0,\infty)\times B_r)} \tag{3.8}$$

for all $f\in C_0([0,\infty)\times B_r)$. Given Corollary (2.3) and (3.8), Lions's argument runs as follows. Let $f\in C_0^\infty((0,\infty)\times B_r)$ and set $u(t,x) = \int_{B_r} g_{a,r}(t,x,y)f(y)dy$. Then

$$t^{d+1}u(t,x) = -\int_0^t\int_{B_r}\left(\frac{\partial}{\partial s} - L_a\right)\left[s^{d+1}u(s,y)\right]g_{a,r}(s,x,y)dsdy$$
$$= -(d+1)\int_0^t\int_{B_r} s^d u(s,y)g_{a,r}(s,x,y)dsdy .$$

Thus, by (3.8):

$$t^{d+1}u(t,x) \leqslant (d+1)K_d(\lambda,r)\left[\int_0^t\int_{B_r}\left[s^d u(s,y)\right]^{d+1} dyds\right]^{1/d+1}$$

$$\leqslant (d+1)K_d(\lambda,r) \sup_{[0,t]\times B_r}\left[s^{d+1}u(s,y)\right]^{d/d+1}\left[\int_0^\infty\int_{B_r} u(s,y)dyds\right]^{1/d+1} .$$

But, by Corollary 2.3,

$$\sup_{y\in B_r}\int_0^\infty u(s,y)ds \leqslant c_{\lambda,r}\|f\|_{L^{q'_\lambda}(B_r)}^{c_\lambda} .$$

Hence, we conclude that there is a $C_d(\lambda, r) < \infty$ such that:

$$\sup_{[0,t]\times B_r}\left[s^{d+1}u(s,x)\right] \leqslant C_d(\lambda,r)^{1/d+1}\sup_{[0,t]\times B_r}\left[s^{d+1}u(s,x)\right]^{d/d+1}\|f\|_{L^{q'_\lambda}(R^d)}^{1/d+1} .$$

Clearly (3.7) follows immediately from this.

The first step in the passage from (3.7) to (3.4) with $q = q_\lambda$ is to prove that

$$A_d(\lambda) \equiv \sup_{a\in \mathcal{Q}_d(\lambda)}\left\|\Gamma_a(1,0,\cdot)\right\|_{L^{q_\lambda}(B_1)} < \infty . \tag{3.9}$$

The argument is basically an application of the maximum principle, but is most easily seen probabilistically. Set $\Omega = C([0,\infty);R^d)$ and for $(t,\omega)\in[0,\infty)\times\Omega$ use $x(t,\omega)$ to denote the position of ω at time t. Given $a\in\mathcal{Q}_d(\lambda)$, there is a unique probability measure P_x on Ω such that

$$P_x\left(x(t+s)\in E \,\middle|\, x(u),\ 0\leqslant u\leqslant s\right) = \int_E \Gamma_a(t,x(s),y)dy$$

for all $s,t\geqslant 0$ and $E\in\mathcal{B}_{R^d}$. Moreover, if

$$\tau_0(\omega) = \inf\{t \geqslant 0 : |x(t,\omega)| \geqslant 3\} \quad ,$$

then

$$P_x\left(x(t) \in E,\ \tau_0 > t\right) = \int_{E \wedge B_3} g_{a,3}(t,x,y)\,dy \quad .$$

Now set

$$\sigma_1(\omega) = \inf\{t \geqslant \tau_0(\omega) : |x(t,\omega)| \leqslant 2\} \quad ,$$

$$\tau_m(\omega) = \inf\{t \geqslant \sigma_m(\omega) : |x(t,\omega)| \geqslant 3\} \quad , \qquad m \geqslant 1 \ ,$$

$$\sigma_{m+1}(\omega) = \inf\{t \geqslant \tau_m(\omega) : |x(t,\omega)| \leqslant 2\} \quad , \qquad m \geqslant 1 \quad .$$

Then, by the strong Markov property:

$$P_0\left(x(1) \in E,\ \sigma_m < 1 < \tau_m\right)$$

$$= \int_{\{\omega : \sigma_m(\omega) < 1\}} P_{x(\sigma_m(\omega),\omega)}\left(x[1 - \sigma_m(\omega)] \in E,\ \tau_0 > 1 - \sigma_m(\omega)\right) P_0(d\omega)$$

$$= \int_{\{\omega : \sigma_m(\omega) < 1\}} \left[\int_E g_{a,3}\left(1 - \sigma_m(\omega),\ x[\sigma_m(\omega),\omega],\ y\right) dy\right] P_0(d\omega) \quad .$$

Hence, for $\phi \in C_0(B_1)$:

$$\int \Gamma_a(1,0,y)\phi(y)\,dy = E^{P_0}\left[\phi\left(x(1)\right)\right]$$

$$= \sum_{m=0}^{\infty} E^{P_0}\left[\phi\left(x(1)\right),\ \sigma_m < t < \tau_m\right]$$

$$= \sum_{m=0}^{\infty} \int_{\{\omega : \sigma_m(\omega) < 1\}} \left[\int g_{a,3}\left(1 - \sigma_m(\omega),\ x[\sigma_m(\omega),\omega],\ y\right)\phi(y)\,dy\right] P_0(d\omega)$$

where $\sigma_0 \equiv 0$. In particular:

$$\left\| \Gamma_a(1,0,\cdot) \right\|_{L^{q_\lambda}(B_1)}$$

$$\leqslant E^{P_0}\left[1 + \sum_1^\infty \chi_{[0,1]}(\sigma_m)\right] \sup_{0<t\leqslant 1} \sup_{|x|=2} \left\| g_{a,3}(t,x,\cdot) \right\|_{L^{q_\lambda}(B_1)} .$$

By Lemma (9.1.6) of [11], $E^{P_0}\left[\sum_1^\infty \chi_{[0,1]}(\sigma_m)\right]$ can be dominated by a finite constant which depends only on d and λ. At the same time, the Krylov-Safanov Harnack principle [7] says that there is a constant $M < \infty$, depending only on d and λ, such that

$$g_{a,3}(t,x,y) \leqslant M g_{a,3}(2,x,y)$$

for all $t \in (0,1]$, $x \in \partial B_2$, and $y \in B_1$. Thus, by (3.7) with $t = 2$, $r = 3$, and $d/d-1$ replaced by q_λ, we see that

$$\sup_{0<t\leqslant 1} \sup_{|x|=2} \left\| g_{a,3}(t,x,\cdot) \right\|_{L^{q_\lambda}(B_1)}$$

is dominated by a finite constant which depends only on d and λ. Thus (3.9) has been proved.

The step from (3.9) to (3.4) with $q = q_\lambda$ is another application of the scaling arguments used in Remark (3.3). Namely, by the argument used there, for any $n \geqslant 0$:

$$\Gamma_a(1,0,y) = 2^{-(n+1)d} \Gamma_{a_n}(4^{-(n+1)}, 0, 2^{-(n+1)} y) ,$$

where $a_n(x) = a(2^{(n+1)} x)$, $x \in R^d$. Thus, by the Harnack principle [7], there is a finite M, depending only on d and λ, such that

$\Gamma_a(1,0,y) \leqslant 2^{-(n+1)d} M \Gamma_{a_n}(1,0,2^{-(n+1)} y)$ for all $y \in R^d$ satisfying $2^n \leqslant |y| \leqslant 2^{n+1}$.

Now suppose that $d = 1$. Then $q_\lambda = \infty$ and by (3.9) and the preceding:

$$\sup_{n\geqslant 0}\ \sup_{a\in \mathcal{Q}_1(\lambda)}\ \sup_{2^n\leqslant |y|\leqslant 2^{n+1}} \Gamma_a(1,0,y) \leqslant 2^{-(n+1)} MA_1(\lambda) \quad .$$

Combined with (3.9), this certainly proves (3.4) when d = 1. Next, suppose that $d \geqslant 2$. By the preceding:

$$\int_{R^d\setminus B_1} \left[\Gamma_a(1,0,y)\right]^{q_\lambda} dy$$

$$\leqslant M^{q_\lambda} \sum_{n=0}^{\infty} \int_{2^n\leqslant |y|<2^{n+1}} 2^{-(n+1)dq_\lambda} \left[\Gamma_{a_n}(1,0,2^{-(n+1)}y)\right]^{q_\lambda} dy$$

$$\leqslant M^{q_\lambda} \sum_{n=0}^{\infty} 2^{-(n+1)d(q_\lambda-1)} \int_{B_1} \left[\Gamma_{a_n}(1,0,y)\right]^{q_\lambda} dy$$

$$\leqslant \left[M A_d(\lambda)\right]^{q_\lambda} \sum_{n=0}^{\infty} 2^{-(n-1)d(q_\lambda-1)} \quad .$$

Combined with (3.9), this proves (3.4) when $d \geqslant 2$. In view of the comments in Remark (3.3), the derivation of (3.2) is now complete.

Our next project is to show the "best possible" nature of (3.2) in the sense explained in Remark (3.5). The result which we have in this direction is that for $d \geqslant 2$, $\lambda \in (0,1)$, and $\delta > 0$:

$$\sup_{a\in \mathcal{Q}_d(\lambda)} \left\| \Gamma_a(1,0,\cdot)\right\|_{L^{q_d(\lambda)}(B_\delta)} = \infty \quad , \tag{3.10}$$

where $q_d(\lambda) = d/(d-1)(1-\lambda^2)$.

Our proof of (3.10) turns on the following observation. Let $\eta \in C^\infty(R^d)$ be a rotation invariant function satisfying $0 \leqslant \eta \leqslant 1$, $\eta = 0$ on B_1, and $\eta = 1$ off B_2. For $\lambda \in (0,1)$ and $n \geqslant 1$, define

$$a^{ij}_{\lambda,n}(x) = \lambda\left[\delta_{ij} + (\lambda^{-2} - 1)\eta(nx)\frac{x_i x_j}{|x|^2}\right], \quad 1 \leqslant i,\ j \leqslant d .$$

Set $\Gamma_{\lambda,n} = \Gamma_{a_{\lambda,n}}$, $D_\lambda = (d-1)\lambda^2 + 1$, and $K_\lambda = 1\Big/\int_0^\infty e^{-r^2/4\lambda} r^{D_\lambda - 1} dr$. We claim that for each $f \in C_b([0,\infty))$:

$$\lim_{n\to\infty} \int f(|y|)\Gamma_{\lambda,n}(1,0,y)dy = K_\lambda \int_0^\infty f(r)\, e^{-r^2/4\lambda} r^{D_\lambda - 1} dr . \tag{3.11}$$

Supposing for the moment that (3.11) is true, we show how to derive (3.10). To this end, let $\delta > 0$ and $q \in (1,\infty)$ be given and let $f \in C([0,\delta])$ satisfy

$$\left[\omega_d \int_0^\delta |f(r)|^{q'} r^{d-1} dr\right]^{1/q'} \leqslant 1 , \tag{3.12}$$

where ω_d denotes the area of the unit sphere, S^{d-1}, and $1/q + 1/q' = 1$. Noting that $\{a_{\lambda,n}\}_1^\infty \subseteq \mathcal{Q}_d(\lambda)$, we have:

$$\begin{aligned}\sup_{a\in\mathcal{Q}_d(\lambda)} \left\|\Gamma_a(1,0,\cdot)\right\|_{L^q(B_\delta)} &\geqslant \varliminf_{n\to\infty} \left\|\Gamma_{\lambda,n}(1,0,\cdot)\right\|_{L^q(B_\delta)} \\ &\geqslant \lim_{n\to\infty} \int_{B_\delta} f(|y|)\,\Gamma_{\lambda,n}(1,0,y)dy \\ &= K_\lambda \int_0^\delta f(r)\, e^{-r^2/4\lambda} r^{D_\lambda - 1} dr .\end{aligned}$$

Maximizing the last expression over $f \in C([0,\delta])$ satisfying (3.12), we obtain:

$$\sup_{a\in\mathcal{Q}_d(\lambda)} \left\|\Gamma_a(1,0,\cdot)\right\|_{L^q(B_\delta)} \geqslant K_\lambda \left[\omega_d \int_0^\delta (e^{-r^2/4\lambda} r^{D_\lambda - d})^q r^{d-1} dr\right]^{1/q} .$$

In particular, if $q = q_d(\lambda)$, then $(D_\lambda - d)q = -d$ and so (3.10) follows.

Although our proof of (3.11) is based on probabilistic thinking, we have removed most of the probability theory from our presentation. Let $\lambda \in (0,1)$ be fixed. For each $n \geqslant 1$ and $t > 0$, let $\mu_{n,t}$ be the probability measure on $[0,\infty)$ given by

$$\mu_{n,t}(dr) = \left[\int_{S^{d-1}} \Gamma_{\lambda,n}(t,0,r\omega)\,d\omega\right] r^{d-1}\,dr \quad .$$

Note that for any $f \in C^2([0,\infty))$:

$$\int_{[0,\infty)} f(r^2)\mu_{n,t}(dr) - f(0) = \frac{1}{\lambda}\int_0^t \left[\int_{[0,\infty)} \left[4r^2 f''(r) + 2D_\lambda \bar{\eta}(nr)f'(r)\right]\mu_{n,s}(dr)\right] ds \ , \tag{3.13}$$

where $\bar{\eta}(r) = \eta(r\omega)$, $r \geqslant 0$ and $\omega \in S^{d-1}$. From (3.13) we see that

$$\int_{[0,\infty)} r^2 \mu_{n,t}(dr) \leqslant \frac{2D_\lambda}{\lambda} t$$

and that there is a $C < \infty$ such that

$$\sup_{n\geqslant 1}\left|\int f(r^2)\mu_{n,t}(dr) - \int f(r^2)\mu_{n,s}(ds)\right| \leqslant C(t-s)\|f\|_{C_b^2([0,\infty))} \quad .$$

From these it is easy to show that if $\{n_k : k \geqslant 1\}$ is any sequence of positive integers tending to ∞, then there is a subsequence $\{n_{k'}\}$ tending to infinity such that $\mu_{n_{k'},t}$ converges weakly to a probability measure μ_t for all $t > 0$. Furthermore, since (3.2) tells us that

$$\lim_{\delta \downarrow 0}\ \sup_n \int_0^t \mu_{n,s}(B_\delta)ds = 0$$

for each $t > 0$, we can pass to the limit in (3.13) and thereby obtain:

$$\int_{[0,\infty)} f(r^2)\mu_t(dr) - f(0) = \frac{1}{\lambda}\int_0^t \left[\int_{[0,\infty)} \left[4r^2 f''(r^2) + 2D_\lambda f'(r)\right]\mu_s(dr)\right] ds \tag{3.14}$$

for all $f \in C_0^2([0,\infty))$. Now define

$$g(t,\rho) = \int_{[0,\infty)} e^{-\rho r^2} \mu_t(dr)$$

for $(t,\rho) \in [0,\infty)^2$. Then, from the fact that

$$\int_{[0,\infty)} r^2 \mu_t(dr) < \infty$$

combined with (3.14), we see that g is the unique solution $h \in C_b^1([0,\infty)^2)$ to the boundary value problem

$$h(t,\rho) = 1 \quad \text{if} \quad t = 0 \quad \text{or} \quad \rho = 0$$

$$\frac{\partial h}{\partial t}(t,\rho) = -\frac{4\rho^2}{\lambda}\frac{\partial h}{\partial \rho}(t,\rho) + \frac{2D_\lambda \rho}{\lambda} h(t,\rho) \quad , \quad t,\rho \geqslant 0 \ .$$

Hence,

$$g(t,\rho) = (4\lambda\rho t + 1)^{D_\lambda/2} \ .$$

At the same time,

$$\frac{K_\lambda}{t^{D_\lambda/2}} \int_0^\infty e^{-\rho r^2} e^{-r^2/4\lambda t} r^{D_\lambda - 1} dr = (4\lambda\rho t + 1)^{D_\lambda/2} , \qquad t,\rho \geqslant 0 \ .$$

We have therefore proved that

$$\mu_1(dr) = K_\lambda e^{-r^2/4\lambda} r^{D_\lambda - 1} dr \ .$$

Since μ_1 was the weak limit of an arbitrary weakly convergent subsequence of the relatively compact set $\{\mu_{n,1} : n \geq 1\}$, we can now assert that $\mu_{n,1}$ tends weakly to μ_1 as $n \to \infty$. This is precisely the content of (3.11).

We conclude this note with a couple of additional comments. In the first place, it may be confusing that although (3.2) is clearly an improvement over Lions's (3.7) for small $t > 0$, it seems less good at infinity. This circumstance is a result of our having dealt with Γ_a instead of $g_{a,r}$. Indeed, the following simple argument shows that, at infinity, the $L^{q_\lambda}(B_r)$ norm of $g_{a,r}(t,x,\cdot)$ dies exponentially fast. Note that

$$g_{a,r}(t+1, x, y) = \int_{B_r} g_{a,r}(t,x,\xi) g_{a,r}(1,\xi,y) d\xi \quad .$$

Thus, for any $q \in [1,\infty]$:

$$\left\| g_{a,r}(t+1, x, \cdot) \right\|_{L^q(B_r)} \leq u_{a,r}(t,x) \sup_{\xi \in R^d} \left\| \Gamma_a(1,\xi,\cdot) \right\|_{L^q(R^d)}$$

where

$$u_{a,r}(t,x) = \int_{B_r} g_{a,r}(t,x,\xi) d\xi$$

is the solution to

$$\frac{\partial u}{\partial t}(t,x) = L_a u(t,x) \quad , \qquad (t,x) \in (0,\infty) \times B_r$$

$$u(0,x) = 1 \quad , \qquad x \in B_r$$

$$u(t,x) = 0 \quad , \qquad (t,x) \in (0,\infty) \times \partial B_r \quad .$$

Now set $\rho = \rho_d(\lambda, r) = d\lambda^2/2r$ and $\sigma = \sigma_d(\lambda, r) = d^2\lambda^2/4r^2$, and define

$$v(t,x) = \exp(-\rho|x|^2/2 - \sigma t)/\exp(-\rho r^2/2) \quad .$$

Then for all $a \in \mathcal{Q}_d(\lambda)$:

$$\frac{\partial v}{\partial t}(t,x) \geqslant L_a v(t,x) \ , \qquad (t,x) \in (0,\infty) \times B_r \ ,$$

$$v(0,x) \geqslant 1 \ , \qquad x \in B_r \ ,$$

$$v(t,x) \geqslant 0 \ , \qquad (t,x) \in (0,\infty) \times \partial B_r \ .$$

Thus, by the maximum principle, $u_{a,r}(t,x) \leqslant v(t,x)$ in $[0,\infty) \times B_r$. In other words,

$$\sup_{x \in B_r} \ \sup_{a \in \mathcal{A}_d(\lambda)} u_{a,r}(t,x) \leqslant \exp\left[\frac{d\lambda^2 r}{4} - \frac{d^2\lambda^2}{4r^2} t\right] \ .$$

Combining this with the preceding, we arrive at

$$\sup_{x \in B_r} \ \sup_{a \in \mathcal{A}_d(\lambda)} \left\| g_{a,r}(t,x,\cdot)\right\|_{L^{q_\lambda}(B_r)} \leqslant C_d(\lambda) \exp\left(\frac{d\lambda^2 r}{4} + 1\right) \exp\left(-\frac{d^2\lambda^2}{4r^2} t\right) \tag{3.15}$$

for $t \geqslant 1$.

A second obvious question is what estimate replaces (3.2) when one allows there to be first order terms in the operator L. Obviously, the argument we have given relies heavily on homogeneity and does not go over directly to this situation. Nonetheless, an easy application of the Cameron-Martin formula allows one to show that if $\mathcal{A}_d(\lambda,\beta)$ stands for the class of pairs $\{a,b\}$ where $a \in \mathcal{A}_d(\lambda)$ and $b : R^d \to R^d$ is a smooth function satisfying $\sup_{x \in R} |b(x)| \leqslant \beta$, then for each $q \in [1, q_\lambda)$ there exist $\mu_d(q) \in (0,\infty)$ and $C_d(q,\lambda,\beta) < \infty$ such that

$$\sup_{x \in R^d} \ \sup_{\{a,b\} \in \mathcal{A}_d(\lambda,\beta)} \left\| \Gamma_{a,b}(t,x,\cdot) \right\|_{L^q(R^d)}$$

$$\leqslant \left[C_d(q,\lambda,\beta)/t^{d/2q'}\right] \exp\left[\mu_d(q)\beta^2 t/\lambda\right] \ . \tag{3.16}$$

REFERENCES

[1] A.D. Alexandrov, Uniqueness conditions and estimates of the solution of Dirichlet's problem, Vestn. Leningr. Un.-ta., 13, No. 3 (1963), 5-29.

[2] P. Bauman, Properties of nonnegative solutions of second-order elliptic equations and their adjoints, Ph.D. Thesis, University of Minnesota, Minneapolis, Minnesota (1982).

[3] R.R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math., 51 (1974), 241-250.

[4] L.C. Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, Comm. Pure Appl. Math., 35 (1982), 333-363.

[5] E.B. Fabes, C.E. Kenig, Examples of singular parabolic measures and singular transition probability densities, Duke Math. J., 48, No. 4 (1981), 845-856.

[6] N.V. Krylov, Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation, Sibirskii Mat. Ž., 17 (1976), 290-303; English translation in Siberian Math. J., 17 (1976), 226-236.

[7] N.V. Krylov and M.V. Safanov, A certain property of solutions of parabolic equations with measurable coefficients, Math. USSR Izvestija, 16 (1981), 151-164; English translation in Izv. Akad. Nauk SSSR, 44 (1980), 81-98.

[8] P.L. Lions, Some recent results in the optimal control of diffusion processes, Cahiers de Mathematiques de la Decision, No. 8302, Ceremade (1983), preprint.

[9] L. Nirenberg, The Dirichlet problem for Monge-Ampere equations, Abstracts of the workshop on nonlinear elliptic partial differential equations, Mathematical Sciences Research Institute, Berkeley, California (1983), 21-22.

[10] C. Pucci, Limitazione per soluzioni de equazione ellittichi, Ann. Math. Pura Appl., 74 (1966), 15-30.

[11] D.W. Stroock and S.R.S. Varadhan, Multidimensional Diffusion Processes, Springer-Verlag, New York Inc. (1979).