

**EXACT SOLUTIONS OF MULTIDIMENSIONAL
NONLINEAR DIRAC'S AND SCHRÖDINGER'S EQUATIONS**

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EXACT SOLUTIONS OF MULTIDIMENSIONAL NONLINEAR DIRAC'S AND SCHRÖDINGER'S EQUATIONS

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Abstract. A class of nonlinear spinor equations invariant under the extended Poincare group and conformal group is described. New Ansätze for spinor fields are suggested. Multiparameter families of exact solutions for the multidimensional families of exact solutions for the multidimensional nonlinear Dirac and Schrödinger equations are obtained.

In this paper I present some new results, obtained in Institute of Mathematics, Academy of Sciences of the Ukrainian SSR in Kiev by R. Zhdanov, W. Shtelen, N. Serow and me on multiparameter families of exact solutions of nonlinear Dirac and Schrödinger equations

$$(1) \quad \gamma_\mu p^\mu \Psi + F_1(x, \bar{\Psi}, \Psi) \Psi = 0$$

$$(2) \quad (p_0 + \frac{1}{2m} p_a p_a) u + F_2(x, u, u^*) = 0,$$

where $\Psi \equiv \Psi(x) = (\Psi_0, \Psi_1, \Psi_2, \Psi_3)$ is 4-component spinor, $x = (x_0, x_1, x_2, x_3)$, $\bar{\Psi}$ is complex conjugated spinor, γ_μ are 4×4 Dirac Matrices, $u \equiv u(x_0, x_1, x_2, x_3)$, $x_0 \equiv t$, u^* is complex conjugated wave function,

$$p_0 = i \frac{\partial}{\partial x_0}, \quad p_j = -i \frac{\partial}{\partial x_j}, \quad \mu, \nu = \overline{0, 3}, \quad j = 1, 2, 3,$$

F_1, F_2 are arbitrary smooth function, m is the particle mass.

Fifty years ago D. Ivanenko (1938) considered the simplest equation of the type (1), the case in which

$$(3) \quad F_1 = \lambda(\bar{\psi}\psi),$$

where $\bar{\psi} \equiv \psi^\dagger \gamma_0$ is Dirac-conjugated spinor, λ is arbitrary parameter.

W. Heisenberg and his collaborators (1954-1959) have analysed the equation (1) from a different point of view with the nonlinearity

$$(4) \quad F_1 = \lambda \bar{\Psi} \gamma_\mu \gamma_4 \Psi \gamma^\mu \gamma_4, \quad \gamma_4 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$$

The main efforts of W. Heisenberg directed into the construction of unified quantum field theory based on eq. (1) with the nonlinearities (3), (4). In the works by R. Finkelstein

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and his collaborators (1951-1956) eq. (1) has been studied from the classical point of view, i.e. they studied the exact and approximate solutions of spinor systems of the type (1).

Some exact solutions of the Dirac equation were obtained by F. Kortel (1956), D. Kurgeleidze (1957), K.G. Akdeniz, A. Smailagic (1984), A.O. Barwt, B.W. Xu (1982), K. Takahashi (1979).

The classical Lie's method for finding exact solutions of multidimensional nonlinear Schrödinger equation and d'Allembert equation was applied by Fushchich (1981, 1983), Fushchich and Serov (1983, 1987) Gagnon and Winternitz (1988), Grundland, Harnad and Winternitz (1984), Tajiri (1983).

Evidently if we do not specify the functions F_1, F_2 in eqs. (1), (2) there is no hope to get any profound information about exact solutions of these equations. To specify the functions F_1 and F_2 we shall study the symmetry properties of equations (1), and (2). In what follows, I shall essentially use the classical ideas of S. Lie in application to nonlinear wave equations.

The wide symmetry of equations (1), (2) makes it possible to reduce the multidimensional partial differential equation (PDE) a set of systems of ordinary differential equations (ODE). Many of these ODEs can be solved exactly. In this way we are able to construct many parameter families of exact solutions of the multidimensional wave equations (1), (2).

§1. The symmetry of the nonlinear spinor equation. In this section we will present the theorems concerning the symmetry properties of equation (1).

THEOREM 1. *Equation (1) is invariant under the Poincare group $P(1,3)$ iff*

$$\begin{aligned} F_1(x, \psi, \psi) &= F_{11}(s) + F_{12}(s)\gamma_4 + \\ &+ F_{13}(s)\gamma^\mu(\bar{\Psi}\gamma_4\gamma_\mu\Psi) + F_{14}(s)S^{\mu\nu}(\bar{\Psi}\gamma_4 S_{\mu\nu}\Psi), \\ vs &= (\bar{\psi}\gamma_4\Psi, \bar{\Psi}\Psi), \\ S_{\mu\nu} &= [\gamma_\mu, \gamma_\nu] = \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \end{aligned}$$

where $F_{11}, F_{12}, F_{13}, F_{14}$ are arbitrary smooth scalar functions of the invariant variable s .

THEOREM 2. *Equation (1) is invariant under the extend Poincaré group $\tilde{P}(1,3)$, i.e. the $P(1,3)$ group expanded by the one-parameter group of scale transformations of the type*

$$x'_\mu = x_\mu \exp(\theta), \Psi'_{x'} = \Psi(x) \exp(\kappa\theta),$$

iff

$$(1.2) \quad F_{1i} = (\bar{\Psi}\Psi)^{-1/2k} \tilde{F}_{1i}, i = 1, 2$$

$$(1.3) \quad F_{1j} = (\bar{\Psi}\Psi)^{-\frac{(1+2k)}{2k}} \tilde{F}_{1j}, j = 3, 4,$$

where $\tilde{F}_{1i}, \tilde{F}_{1j}$ are arbitrary functions of $\frac{(\bar{\Psi}\Psi)}{(\Psi\gamma_\eta\Psi)}$

THEOREM 3. Equation (1) is invariant under the conformal group $C(1, 3) = \langle P(1, 3), \mathcal{D}, K_\mu \rangle$

$$\begin{aligned} x'_\mu &= \{x_\mu - c_\mu(x \cdot x)\}\sigma^{-1}(x), \sigma(x) = 1 - 2c_\mu x^\mu + c^2 x^2, \\ \Psi'(x) &= \sigma(x)\{1 - (\gamma \cdot c)(\gamma \cdot x)\}\Psi \end{aligned}$$

iff

$$(1.4) \quad F_{1i} = (\bar{\Psi}\Psi)^{1/3}\tilde{F}_{1i}, \quad i = 1, 2, \quad k = -3/2$$

$$(1.5) \quad F_{1j} = (\bar{\Psi}\Psi)^{-2/3}\tilde{F}_j, \quad j = 3, 4$$

Note 1. The conformally-invariant Dirac-Gürsey equation (1956)

$$(1.6) \quad \{\gamma_\mu p^\mu + D(\bar{\Psi}\Psi)^{1/3}\}\Psi =$$

belongs to the class (1), (1, 4)

Note 2. The equation of the type

$$(1.7) \quad \{\gamma_\mu p^\mu + D\gamma_\mu(\bar{\Psi}\gamma^\mu\Psi) \cdot [(\bar{\Psi}\gamma_\alpha\Psi)(\bar{\Psi}\gamma^\alpha\Psi)]^{-1/3}\}\Psi = 0$$

is invariant under the conformal group $C(1, 3)$.

§2. The Ansatzes for $\tilde{P}(1, 3)$ -invariant equation. To be specific let us consider the nonlinear spinor equation of the type

$$(2.1) \quad \{\gamma_\mu p^\mu + \lambda(\Psi\Psi)^{1/2k}\}\Psi = 0,$$

where λ, k are arbitrary constants, $k \neq 0$.

We look for solutions of (2.1) in the form Fushchich (1981)

$$(2.2) \quad \Psi = A(x)\phi(w),$$

where $A(x)$ is 4×4 matrix, $\phi(w)$ is 4-component column- function depending on three new variables $w = \{w_1, w_2, w_3\}$

For the Ansatz (2.2) to work effectively it is necessary to find $A(x), w$ in a form which after a substitution of (2.2) into (1) would yield an equation for $\phi(w)$ depending only on new variables w .

This requirement is met if the following equalities are satisfied:

$$(2.3) \quad QA(x) \equiv (\zeta^\mu \frac{\partial}{\partial x_\mu} + \eta)A(x) = 0$$

$$(2.4) \quad \zeta^\mu \frac{\partial w^\ell}{\partial x_\mu} = 0, \quad \ell = 1, 2, 3$$

where $\zeta^\mu(x)$, $\eta(x)$ are the coefficients of the infinitesimal operators $Q = \{Q_1, Q_2, \dots\}$ of the group $\tilde{P}(1, 3)$. In our case the generators of the $P(1, 3)$ have the form

$$(2.5) \quad P_\mu = p_\mu, J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu},$$

Thus the problem of describing Ansätze of the form (2.2) reduces to the construction of the general solution to the system of equations (2.3), (2.4) with the given ζ^μ, η .

As an example let us consider the case where in (2.3), (2.4) the operators Q have the simple form

$$Q = \langle Q_1 = J_{03}, Q_2 = P_1, Q_3 = P_3 \rangle$$

Then the system (2.3), (2.4) has the form

$$(2.6) \quad \begin{aligned} (x_0 p_3 - x_3 p_0 + \frac{i}{2} \gamma_0 \gamma_3) A(x) &= 0 \\ p_1 A(x) &= 0, \quad p_2 A(x) = 0 \end{aligned}$$

$$(2.7) \quad \begin{aligned} (x_0 p_3 - x_3 p_0) w &= 0 \\ p_1 w &= 0, \quad p_2 w = 0 \end{aligned}$$

It follows from (2.7) that $w = w(x_0, x_3) = x_0^2 - x_3^2$, We look for the solutions to (2.6) in the form

$$(2.8) \quad A(x) = \exp\{\gamma_0 \gamma_3 g(x)\}$$

After a substitution of (2.8) into (2.6), we obtain

$$(2.9) \quad x_0 \frac{\partial g}{\partial x_3} + x_3 \frac{\partial g}{\partial x_0} - \frac{1}{2} = 0$$

The particular solution of eq. (2.9) is given by the expression

$$g(x) = \frac{1}{2} \ln(x_0 + x_3)$$

Thus we have

$$(2.10) \quad A(x) = \exp\left\{\frac{1}{2} \gamma_0 \gamma_3 \ln(x_0 + x_3)\right\}$$

Without going into the technical details on solving the system (2.3), (2.4) we give some expressions for the matrix $A(x)$ and w .

Example 2.1.

$$(2.11) \quad \begin{aligned} A(x) &= (x_0 - x_2)^{-k} \exp\left\{\frac{1}{2a}\gamma_1(\gamma_2 - \gamma_0)\ln(x_0 - x_2)\right\} \\ w_1 &= (x_0^2 - x_1^2 - x_2^2)x_3^{-2} \\ w_2 &= (x_0 - x_2)x_3^{-2} \\ w_3 &= ax_1(x_0 - x_2)^{-1} - \ln(x_0 - x_2), \quad a \neq 0 \end{aligned}$$

If the parameter $a = 0$, then

$$A(x) = \exp\left\{\frac{x_1}{2(x_0 - x_3)}\gamma_1(\gamma_2 - \gamma_0)\right\}$$

Example 2.2.

$$(2.14) \quad \begin{aligned} A(x) &= (2x_0 + 2x_1 + \beta)^{-k/2} \exp\left\{\frac{1}{4}\gamma_0\gamma_2\ln(2x_0 + 2x_1 + \beta)\right. \\ &\quad \left. - \frac{1}{2}\gamma_2\gamma_3tg^{-1}\frac{x_2}{x_3}\right\}, \quad \beta \neq 0 \\ w_1 &= (2x_0 + 2x_1 + \beta) \exp\{2(x_1 - x_0)\beta^{-1}\} \end{aligned}$$

$$(2.15) \quad \begin{aligned} w_2 &= (2x_0 + 2x_1 + \beta)(x_2^2 + x_3^2)^{-1} \\ w_3 &= b\ln(x_2^2 + x_3^2) + 2tg^{-1}\frac{x_2}{x_3} \end{aligned}$$

§3. Reduced equations. 3.1 The Ansatz (2.2) with the matrices (2.10), (2.11), (2.14) and new variables ω gives the following reduced equations

$$\begin{aligned} &k(\gamma_2 - \gamma_0)\phi + [(\gamma_0 - \gamma_2)(w_1 + a^{-2}w_2^2w_3^2) + (\gamma_0 + \gamma_2)w_2^2 - \\ &- 2a^{-1}\gamma_1w_3w_2^2 - 2\gamma_3w_1w_2] \frac{\partial\phi}{\partial w_1} + [(\gamma_0 - \gamma_2)w_2 - \\ &- \gamma_3w_2^2] \frac{\partial\phi}{\partial w_2} + [a\gamma_1 + (\gamma_2 - \gamma_0)(w_3 + 1)] \frac{\partial\phi}{\partial w_3} = i\lambda(\bar{\phi}\phi)^{\frac{1}{2k}} \cdot \phi \end{aligned}$$

$$(3.2) \quad (\gamma_0 + \gamma_1) \frac{\partial\phi}{\partial w_1} + \gamma_2 \frac{\partial\phi}{\partial w_2} + \gamma_3 \frac{\partial\phi}{\partial w_3} = i\lambda(\bar{\phi}\phi)^{1/2k} \phi$$

$$(3.3) \quad \frac{1}{2}(1-2k)\gamma_3\phi + 2(\gamma_3 + a\gamma_2)\frac{\partial\phi}{\partial w_2} = i\lambda(\bar{\phi}\phi)^{1/2k}\phi$$

The Ansatz (2.2) with the matrix $A(x) = 1$, $w_1 = x_0 + x_3$, $w_2 = x_1$, $w_3 = x_2$ leads to the equation

$$(3.4) \quad (\gamma_0 + \gamma_3)\frac{\partial\phi}{\partial w_1} + \gamma_1\frac{\partial\phi}{\partial w_2} + \gamma_2\frac{\partial\phi}{\partial w_3} = i\lambda(\bar{\phi}\phi)^{1/2k}\phi$$

It turns out that some of the reduced equations possess substantially more symmetries than the initial equation (2.1). For example the equation (3.4) is invariant under infinite-dimensional Lie algebras. More exactly, the following statement is true:

THEOREM 4. *The system (3.4) is invariant under the infinitely dimensional algebra whose basis elements have the form*

For the case $k = 1$

$$\begin{aligned} Q_1 &= \Phi_1(w_1)\frac{\partial}{\partial w_2} + \Phi_2(w_2)\frac{\partial}{\partial w_3} + \frac{1}{2}[\Phi_1\gamma_1 + \Phi_2\gamma_2](\gamma_0 + \gamma_3), \\ Q_2 &= -w_2\frac{\partial}{\partial w_3} + w_3\frac{\partial}{\partial w_2}, \\ Q_3 &= \Phi_0(w_1)\frac{\partial}{\partial w_1} + \dot{\Phi}_0(w_1)(w_2\frac{\partial}{\partial w_3} + w_3\frac{\partial}{\partial w_2}) + \ddot{\Phi}_0 + \\ &\quad \frac{1}{2}\ddot{\Phi}_0(w_1)(\gamma_1w_2 + \gamma_2w_1)(\gamma_0 + \gamma_3) \\ Q_4 &= \Phi_3(w_1)\gamma_4(\gamma_0 + \gamma_3) \end{aligned}$$

For the case $k \neq 1$

$$\begin{aligned} Q_1 &= \frac{\partial}{\partial w_1}, Q_2 = -w_2\frac{\partial}{\partial w_3} + w_3\frac{\partial}{\partial w_2} + \frac{1}{2}\gamma_2\gamma_3, \\ Q_3 &= \Phi_1(w_1)\frac{\partial}{\partial w_2} + \Phi_2(w_1)\frac{\partial}{\partial w_3} + \frac{1}{2}[\dot{\Phi}_1(w_2)\gamma_1 + \dot{\Phi}_2(w_1)\gamma_2](\gamma_2 + \gamma_3) \\ Q_4 &= w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2} + w_3\frac{\partial}{\partial w_3} + k \\ Q_5 &= \Phi_3(w_1)\gamma_4(\gamma_0 + \gamma_3) \end{aligned}$$

where $\Phi_0(w_1)$, $\Phi_1(w_1)$, $\Phi_2(w_1)$, $\Phi_3(w_1)$ are arbitrary smooth functions, a dot designates differentiation with respect to w_1 .

3.2 The reduction of the nonlinear spinor equation to a system of ODE. Here

we give the explicit form of some ODE's to which the Dirac equation is reduced:

$$(3.5) \quad i\gamma_2 \dot{\phi}(w) = \lambda(\bar{\phi}\phi)^{1/2k} \phi$$

$$(3.6) \quad \frac{i}{2}(\gamma_0 + \gamma_3)\phi + i[w(\gamma_0 - \gamma_3)]\dot{\phi} = \lambda(\bar{\phi}\phi)^{1/2k} \phi$$

$$(3.7) \quad \frac{i}{2}w^{-1/2}\gamma_2\phi + 2iw^{1/2}\gamma_2\dot{\phi} = \lambda(\bar{\phi}\phi)^{1/2k} \phi$$

$$\frac{i}{2}[w(w+1)]^{-1}(2w+1)[\gamma_0 + \gamma_3]\phi + i(\gamma_0 + \gamma_3)\dot{\phi} = \lambda(\bar{\phi}\phi)^{1/2k} \phi$$

$$(3.9) \quad i(\gamma_0 + \gamma_3)\dot{\phi} + i[(\gamma_0 + \gamma_3)w^{-1} + \frac{1}{4}(\gamma_0 - \gamma_3)\gamma_4]\phi = \lambda(\bar{\phi}\phi)^{1/2k} \phi$$

$$(3.10) \quad iw^{-1}(\gamma_0 + \gamma_3)\phi + i(\gamma_0 + \gamma_3)\dot{\phi} = \lambda(\bar{\phi}\phi)^{1/2k} \phi$$

$$\dot{\phi} \equiv \frac{d\phi}{dw}$$

The full list of the systems of ordinary differential equations is given in Fushchich and Zhdanov (1988).

§4. The explicit solutions of the Dirac Equation. Some of the ODE's can be integrated in quadratures

I. Case $k = 1/2$, with the nonlinearity

$$(4.1) \quad \begin{aligned} &\lambda(\bar{\Psi}\Psi), w = x_2^2 + x_3^2, \\ &\Psi(x) = (x_2^2 + x_3^2)^{-1/4} \exp\left\{-\frac{1}{2}\gamma_2\gamma_3tg^{-1}\frac{x_2}{x_3}\right\} \\ &\quad \times \exp\left\{-i\lambda\frac{\bar{\chi}\chi}{2(1+a^2)}[\gamma_3 + a\gamma_2]\ln(x_2^2 + x_3^2) + 2tg^{-1}\frac{x_2}{x_3}\right\}\chi, \end{aligned}$$

χ -is constant spinor, a is a real parameter.

II. Case $k \neq 1/2$, nonlinearity

$$(4.2) \quad \begin{aligned} &\lambda(\bar{\Psi}\Psi)^{1/2k}, \quad w = x_2^2 + x_3^2 \\ &\Psi(x) = (x_2^2 + x_3^2) \exp\left\{-\frac{1}{2}\gamma_2\gamma_3tg^{-1}\frac{x_2}{x_3}\right\} \times \\ &\quad \exp\left\{\frac{2i\lambda \cdot k}{1-2k}(x_2^2 + x_3^2)^{\frac{2k-1}{4k}}\gamma_3\right\}\chi \end{aligned}$$

III. Case $k \neq \frac{1}{2}$, nonlinearity $\lambda(\bar{\Psi}\Psi)^{1/2k}$, $w = x_0 + x_3$

$$(4.3) \quad \begin{aligned} &\Psi(x) = \exp\left\{\left[-\frac{1}{2}(\dot{\Phi}_1\gamma_1 + \dot{\Phi}_2\gamma_2) + \phi_3\gamma_4\right](\gamma_0 + \gamma_3)\right\} \times \\ &\quad \times \exp\{i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_1(x_1 + \phi_1)\}\chi, \end{aligned}$$

where Φ_1, Φ_2, Φ_3 are arbitrary smooth functions of w .

IV. Case $k = 1$, nonlinearity $\lambda(\bar{\Psi}\Psi)^{1/2}$

$$(4.4) \quad \Psi(x) = \frac{\gamma_0 x_0 - \gamma_1 x_1 - \gamma_2 x_2}{(x_0^2 - x_1^2 - x_2^2)^{3/2}} \cdot \exp\left\{i\lambda(\bar{\chi}\chi)^{1/2} \frac{\gamma_0 x_0}{x_0^2 - x_1^2 - x_2^2}\right\} \chi$$

Now, making use of the Poincare invariance of the Dirac equation, it is not difficult to construct new multiparameter families of exact solutions of the equation starting from those obtained above. The two following examples illustrate the procedure of generating solutions.

V. Case $k = 1/2$, nonlinearity $\lambda(\bar{\Psi}\Psi)$

$$(4.5) \quad \begin{aligned} \Psi = & [(a \cdot z)^2 + (b \cdot z)^2]^{-1/4} \times \exp\left\{-\frac{1}{2}(\gamma \cdot a)(\gamma \cdot b)tg^{-1} \frac{a \cdot z}{b \cdot z}\right\} \times \\ & \times \exp\left\{-i\lambda \frac{\bar{\chi}\chi}{2(1+\theta^2)}(\gamma \cdot b + \theta\gamma \cdot a)[\ln|(a \cdot z)^2 + (b \cdot z)^2 + \right. \\ & \left. 2\theta tg^{-1} \frac{a \cdot z}{b \cdot z}]\right\} \chi \end{aligned}$$

$Z_\mu = x_\mu + \theta_\mu, a_\mu, b_\mu, \theta, \theta_\mu$ are arbitrary parameters satisfying conditions

$$\begin{aligned} a \cdot a = a_\mu a^\mu = -1, b \cdot b = b_\mu b^\mu = -1, a \cdot b = 0 \\ \gamma \cdot a \equiv \gamma_0 a_0 - \gamma_1 a_1 - \gamma_2 a_2 - \gamma_3 a_3, \\ \theta = (\theta_0^2 - \theta_j^2 - \theta_2^2 - \theta_3^2)^{1/2} \end{aligned}$$

VI. Case $k \neq 1/2$, nonlinearity $\lambda(\bar{\Psi}\Psi)^{\frac{1}{2k}}$

$$(4.6) \quad \begin{aligned} \Psi(x) = & [(a \cdot z)^2 + (b \cdot z)^2]^{-1/4} \times \\ & \times \exp\left\{-\frac{1}{2}(\gamma \cdot a)(\gamma \cdot b)tg^{-1} \frac{a \cdot z}{b \cdot z}\right\} \times \\ & \times \exp\left\{-i \frac{2\lambda \cdot k}{1-2k}(\gamma \cdot b)(\bar{\chi}\chi)^{1/2} [(a \cdot z)^2 + (b \cdot z)^2]^{\frac{1-2k}{4k}}\right\} \chi \end{aligned}$$

Formulas (4.5), (4.6) give multiparameter families of exact solutions of the Dirac equations. These families are nongenerating with respect to the group $\tilde{P}(1, 3)$ in the sense that solutions (4.5), (4.6) have the same symmetries as the equation (2.1).

§5. Conformally invariant solutions. Conformally invariant Dirac-Gürsey equation has the form

$$(5.1) \quad \{\gamma_\mu p^\mu + 1\}(\bar{\Psi}\Psi)^{1/3} \Psi = 0$$

With the help of a conformally invariant ansatz we can construct the following solutions

$$(5.2) \quad \Psi(x) = \frac{\gamma x}{(x \cdot x)^2} \exp\{i\lambda k(\gamma \cdot \beta)w\} \chi, k = 1/3$$

$$w = \frac{\beta_\mu x^\mu}{x_\nu x^\nu}, x_\nu x^\nu \neq 0, \quad \beta_\mu \beta^\mu > 0;$$

$$(5.3) \quad \Psi(x) = \sigma^{-2}(x)[1 - (\gamma \cdot x)(\gamma \cdot \beta)] \times$$

$$\exp\left\{\frac{\theta}{2}(\gamma \cdot a)(\gamma \cdot b)[b \cdot x - (b \cdot \theta)(x \cdot x)]\sigma_{(x)}^{-1}\right\}$$

$$\exp\left\{-\frac{i}{2}\lambda(\bar{\chi}\chi)^{1/3}(\gamma \cdot a)[2(a \cdot x) - (a \cdot \theta)(x \cdot x)]\sigma(x) +\right.$$

$$\left. + \theta(b \cdot x - (b \cdot \theta)(x \cdot x))^2\right]\sigma^{-2}(x)\}\chi$$

$$a \cdot b = b \cdot b = 0, a \cdot a = a_0^2 - a_1^2 - a_2^2 - a_3^2 = -1$$

$$c^2 = c_\alpha c^\alpha, x^2 = x_\mu x^\mu$$

Formulae (5.2), (5.3) give multiparameter families of exact solutions of the equation (5.1). The family (5.3) is nongenerating with respect to the group $C(1, 3)$.

If $\Psi_1(x)$ is a solution of equation (5.1) then

$$(5.4) \quad \Psi_2(x) = \sigma^{-2}(x)[1 - (\gamma \cdot x)(\gamma \cdot c)]\Psi_1(x'),$$

$$x' = \left\{x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(x)}\right\}$$

will also be a solution. (5.4) is the formula for multiplication of solutions of Dirac equation.

§6. How to construct solutions of the nonlinear d'Alambert equation via solutions of the Dirac equation? Complex scalar field can be represented as

$$(6.1) \quad u(x) = \bar{\Psi}\Psi \exp\{i\theta(x)\}$$

where $\Psi(x)$ is a solution of the Dirac equation, $\theta(x)$ is a phase. In the simplest case, when

$$(6.2) \quad \bar{\Psi}\Psi = c = \text{const.}, \theta(x) = \tau_\mu x^\mu$$

formula (6.1) gives a plane-wave solution of the linear d'Alambert equation. In most cases solutions of the nonlinear Dirac equation generate a scalar field (6.1) which satisfies the nonlinear d'Alambert equation

$$(6.3) \quad p_\mu p^\mu u \kappa |u|^r u,$$

κ, r are constants.

Let us exhibit some exact solutions of the equation (6.3) obtained this way

$$(6.4) \quad u(x) = c(x_1^2 + x_2^2) \exp\{i\phi_0(x_0 + x_3)\}, r = 2,$$

$$(6.5) \quad u(x) = c[(x_1 + \phi_1(x_0 + x_3))^2 + (x_2 + \phi_2(x_0 + x_3))^2]^{-1/2} \exp\{i\phi_0(x_0 + x_3)\}, \quad r = 2,$$

$$(6.6) \quad u(x) = c(x_0^2 - x_1^2 - x_2^2 - x_3^2)^{-2}, \quad r = 1/2$$

$$(6.7) \quad u(x) = c(x_2^2 + x_3^2)^{-1/2} \exp\{i\phi(x_0 + x_1)\}, r = 2$$

In formulae (6.4)-(6.7) $\phi_0, \phi_1, \phi_2, \phi$ are arbitrary smooth functions.

So, solutions of the nonlinear Dirac spinor equation give a possibility to construct solutions to the nonlinear d'Alembert equation.

All these ideas and results were considered in more detail by Fushchich (1981, 1987), Fushchich and Shtelen (1983, 1987), Fushchich, Shtelen and Zhdanov (1985), Fushchich and Zhdanov (1987, 1988, 1989), Fushchich and Nikitin (1987) (see Appendix)

§7. The solutions of the multidimensional Schrödinger equation. Let us consider the following nonlinear equation

$$(7.1) \quad \left(i \frac{\partial}{\partial x_0} - \frac{1}{2m} \Delta\right) U + F(x, u, u^*) = 0$$

$$u \equiv u(x_0 \equiv t, x_1, x_2, x_3)$$

It is well known that if $F = 0$, then linear Schrödinger equation (see Sofus Lie (1881), Hagen (1972), Niederere (1972), Kalnins and Miller (1987) is invariant under the generalised Galilei group, which will be denoted by the symbols $G_2(1, 3)$. The basis elements of the Lie algebra $AG_2(1, 3) = \langle P_0, P_1, J_{ab}, G_a, D, A, I \rangle$ have the following form:

$$(7.2) \quad \begin{aligned} P_0 &= i \frac{\partial}{\partial x_0}, P_a = -i \frac{\partial}{\partial x_a}, a = 1, 2, 3 \\ J_{ab} &= x_a P_b - x_b P_a, I = u \frac{\partial}{\partial u} \\ G_a &= x_0 P_a + m x_a \\ D &= 2x_0 P_0 - x_a P_a + 3/2 \cdot i \\ A &= x_0 \mathcal{D} + \frac{m}{2} x_a^2 \end{aligned}$$

Symbols $AG_1(1,3)$, $AG(1,3)$ denote the following Lie algebras

$$AG_1(1,3) = \langle P_0, P_a, J_{ab}, G_a, \mathcal{D}, I \rangle$$

$$AG(1,3) = \langle P_0, P_a, J_{ab}, G_a, I \rangle$$

To construct families of exact solutions of (7.1) in explicit form we have to know the symmetries of (7.1) which obviously depends on the structure of the function F .

By Lie's algorithm (as given by Ovsyannikov (1978), Olver (1986)), the following statement can be proved.

THEOREM 4. *Equation (7.1) is invariant under the following algebras:*

$$(7.3) \quad AG(1,3) \text{ iff } F = \phi(|u|)a$$

where Φ is arbitrary smooth function, and

$$(7.4) \quad AG_1(1,3) \text{ iff } F = \lambda|u|^k u$$

where λ, K are arbitrary parameters, the operator of scale transformations \mathcal{D} having the form $\mathcal{D} = x_0 P_0 - x_a P_a + 2i/k, k \neq 0$, and

$$(7.5) \quad AG_2(1,3) \text{ iff } F = \lambda|u|^{4/n} \cdot u$$

Later on we shall construct the exact solutions of the equation (7.1) with nonlinearity (7.5), i.e.

$$(7.6) \quad (p_0 - \frac{pa^2}{2m})u + \lambda|u|^{4/3} \cdot u$$

Following Fushchich (1981) we seek solutions of (7.6) with the help of the ansatz

$$(7.7) \quad U = f(x)\phi(w_1, w_2, w_3),$$

where ϕ is the function to be calculated. To construct solutions of (7.6) using ansatz (7.7) it is necessary to have the explicit form of the function $f(x)$ and the new invariant variables w_1, w_2 and w_3 . Next I shall present two Ansätze of the type (7.7).

$$1. \quad f(x) = (1 - x_0^2)^{-\frac{3}{4}} \exp\left\{\frac{im}{2} \frac{x_0 \vec{x}^2}{1 - x_0^2}\right\}$$

$$w_1 = \vec{\alpha} \vec{x} (1 - x_0^2)^{-1/2}, w_2 = \vec{x}^2 (1 - x_0^2)^{-1},$$

$$w_3 = \arctan x_0 + \arctan \frac{\vec{\beta} \vec{x}}{\vec{\gamma} x},$$

where $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are constant vectors satisfying the conditions

$$\vec{\alpha}^2 = \vec{\beta}^2 = \vec{\gamma}^2 = 1, \quad \vec{\alpha}\vec{\beta} = \vec{\beta}\vec{\gamma} = \vec{\gamma}\vec{\alpha} = 0$$

The ansätze (7.6), (7.7) give the following reduced equation

$$(7.8) \quad L\phi + 6\frac{\partial\phi}{\partial w_2} - 2im\frac{\partial\phi}{\partial w_3} + m^2 w_2 \phi - 2\lambda m |\phi|^{4/3} \phi = 0$$

$$L\phi \equiv \frac{\partial^2\phi}{\partial w_1^2} + 4w_2 \frac{\partial^2\phi}{\partial w_2^2} + (w_2 - w_1^2)^{-1} \frac{\partial^2\phi}{\partial w_3^2} + 4w_1 \frac{\partial^2\phi}{\partial w_1 \partial w_2}$$

2. The Second Ansätze has the form

$$(7.9) \quad f(x) = x_0^{-3/2} \exp\left\{-\frac{im}{2} x^2 x_0^{-1}\right\},$$

$$(7.10) \quad w_1 = (\vec{\alpha} \vec{x}) x_0^{-1}, w_2 = \vec{x}^2 x_0^{-2}$$

$$w_3 = x_0^{-1} + \arctan \frac{\vec{\beta} \vec{x}}{\vec{\gamma} \vec{x}}$$

The Ansätze (7.9), (7.10) reduce the equation (7.6) to

$$L\phi + 6\frac{\partial\phi}{\partial w_2} + 2im\frac{\partial\phi}{\partial w_3} - 2m|\phi|^{4/3}\phi = 0$$

§8. **Solutions of the equation (7.6).** In this paragraph I present some explicit solutions of the equation (7.6)

$$(8.1) \quad u = (1 - x_0^2)^{-3/4} \exp\left\{\frac{im}{2} \vec{x} (1 - x_0)^{-1}\right\}, \lambda = \frac{3}{2}i;$$

$$(8.2) \quad u = (c_0 x_0 - \vec{c} \vec{x})^{-3/2} \exp\left\{-\frac{im}{2} \vec{x}^2 x_0^{-1}\right\}, \text{ where}$$

c_0, \vec{c} are arbitrary parameters satisfying the following condition

$$\vec{c}^2 = \frac{8}{15} \lambda m;$$

$$(8.3) \quad u = x_0^{-3/2} \exp\left\{-\frac{im}{2} (\vec{x}^2 - \vec{r} \vec{x}) x_0^{-1}\right\}, \vec{r}^2 = -\frac{8\lambda}{m};$$

$$(8.4) \quad u = \left(\frac{8}{3} \lambda m \vec{x}^2\right)^{-3/4} \exp\left\{-\frac{im}{2} \vec{x}^2 x_0^{-1}\right\};$$

$$(8.5) \quad u = x_0^{-3/2} \phi(w_1) \exp\left\{-\frac{im}{2} \vec{x}^2 \cdot x_0^{-1}\right\}, \quad w_1 = \frac{\vec{\alpha} \vec{x}}{x_0},$$

function $\phi(w_1)$ is defined by the elliptic integral

$$\int_0^\phi \frac{d\tau}{(k_1 + \tau^{10/3})^{1/2}} = \left(\frac{6}{5}\lambda m\right)^{1/2}(w_1 + k_2), \text{ where}$$

k_1, k_2 are arbitrary parameters.

$$(8.6) \quad u = x_0^{-3/2} \exp\left\{-\frac{im}{2}\vec{x}^2 x_0^{-1}\right\} \phi(w_2), \quad w_2 = \vec{x}^2 x_0^{-1},$$

where function $\phi(w_2)$ is a solution of Emden-Fauler equation

$$2w_2 \frac{d^2\phi}{dw_2^2} + 3 \frac{d\phi}{dw_2} - \lambda m \phi^{7/3} = 0$$

The formulae (8.1) - (8.6) give multiparameter families of exact solutions of the equation (7.6). Some of them are of non-perturbative type due to a singularity with respect to the coupling constant λ .

In conclusion we give formulae for multiplication of solutions. If u_1 is a solution of the equation with the nonlinearity 4/3 then the functions U_2, U_3 defined by

$$U_2 = U_1(x_0, \vec{x} + \vec{v}x_0) \exp\left\{im\left(\frac{\vec{v}x_0}{2} + \vec{v}\vec{x}\right)\right\},$$

$$U_3 = U_2\left(\frac{x_0}{dx_0 - 1}, \frac{\vec{x}}{1 - dx_0}\right) (1 - dx_0)^{-3/2}$$

$$\times \exp\left\{\frac{im}{2} \frac{d\vec{2}}{1 - dx_0}\right\},$$

also satisfy equation (7.6). Here d, \vec{v} are arbitrary parameters.

The Ansätze that have been presented here may also be applied to the equation

$$i \frac{\partial u}{\partial t} + \frac{1}{2m} \Delta U + \lambda \left\{ \frac{\partial |U|^2}{\partial x_a} \frac{\partial |U|^{-1}}{\partial x_a} \right\} U = 0,$$

which is also invariant under the group $G_2(1, 3)$.

More full consideration solutions of equation (7.6) was given Fushchich and Serov (1987), Fushchich and Cherniha (1986).

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