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Abstract. This paper discusses the quenching behavior of a semilinear parabolic equation. It is shown that there are only finitely many quenching points. A theorem on the asymptotic behavior at the quenching point is proved by using the method of blow-up. By using an integral identity of Pohozaev's type, we prove a nonexistence theorem for a nonlinear ordinary differential equation.

Key words. quenching, blow-up, semilinear parabolic equation, asymptotic behavior, identity of Pohozaev's type, nonexistence, ordinary differential equation

AMS(MOS) subject classifications. 34B34, 35B40, 35K55

1. Introduction. Consider the system

$$\text{(1.1)} \qquad \begin{cases} u_t - u_{xx} = \frac{\epsilon}{(1-u)^\beta} & \text{in} \quad Q_T \equiv (-1,1)\times(0,T), \\ u(\pm 1, t) = 0 & \text{if} \quad t \geq 0, \\ u(x,0) = u_0(x) & \text{for} \quad |x| \leq 1, \end{cases}$$

where $\beta > 1$, $\epsilon > 0$, $0 \leq u_0 < 1$, and $u_0(\pm 1) = 0$. Hereafter the subscript x or t will represent the partial derivative with respect to that variable. We say that u quenches if u reaches 1 in finite time. The quenching problem (1.1) has been studied by some authors (see e.g. [1], [7], [8]). Recently, it has been shown in [8] that for any $\beta > 1$ there exists an $\epsilon(\beta) > 0$ such that u quenches at finite time T if $\epsilon > \epsilon(\beta)$. We say that a is a quenching point for u if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to a$ and $t_n \uparrow T$ such that $u(x_n, t_n) \to 1$ as $n \to \infty$. Throughout this paper we shall assume that u quenches at finite time T, and that u_0 is smooth and satisfies

$$\text{(1.2)} \qquad u_0'' + \frac{\epsilon}{(1-u_0)^\beta} \geq 0,$$

i.e., $u_t \geq 0$ at $t = 0$.

In §2 we shall prove that only finitely many quenching points occur. In §3 we shall study the asymptotic behavior of the solution near a quenching point by using the method of [5]. In §4 we shall prove a nonexistence theorem for a nonlinear ordinary differential equation which will be used in §3.

By introducing the function

$$v = \frac{1}{(1-u)},$$

v satisfies

$$\begin{cases} v_t - v_{xx} + 2\,\frac{v_x^2}{v} - \epsilon\, v^{2+\beta} = 0 & \text{in} \quad Q_T, \\ v(\pm 1, t) = 1 & \text{if} \quad t \geq 0, \\ v(x,0) = v_0(x) & \text{for} \quad |x| \leq 1, \end{cases}$$

where $v_0(x) = \frac{1}{1-u_0(x)} \geq 1$. We observe, by the strong maximum principle, that $v > 1$ in Q_T. Also, for $w = v_t$, we have

$$\begin{cases} w_t - w_{xx} + (4\,\frac{v_x}{v})\,w_x - [\epsilon\,(2+\beta)\,v^{1+\beta} + 2\,\frac{v_x{}^2}{v^2}]\,w = 0 \quad \text{in} \quad Q_T, \\ w(x,0) \geq 0 \quad \text{if} \quad |x| \leq 1, \quad \text{by} \quad (1.2), \\ w(\pm 1, t) = 0 \quad \text{if} \quad t > 0. \end{cases}$$

By the strong maximum principle, we conclude that

$$v_t > 0 \quad \text{in} \quad Q_T.$$

Note that u quenches at time T if and only if v blows up at time T; a is a quenching point for u if and only if a is a blow-up point for v.

2. Finitely many quenching points. In this section we shall prove that there are only finitely many quenching points for u. Set

$$\text{sgn}(b) = \begin{cases} +1, & \text{if} \quad b > 0; \\ 0, & \text{if} \quad b = 0; \\ -1, & \text{if} \quad b < 0. \end{cases}$$

LEMMA 2.1. *For any $a \in [-1,1]$, the limit*

$$\lim_{t \uparrow T} sgn(v_x(a,t))$$

exists.

Proof. We shall adopt the method used in [3]. Assume that $a \in (-1, 0)$. For any $T_0 < T$, consider the function

$$w(x,t) = v(x,t) - v(2a - x, t)$$

in the rectangle $R = (-1, a) \times (0, T_0)$. Then w satisfies

$$\begin{cases} w_t - w_{xx} + b_1\, w_x + b_2\, w = 0 \quad \text{in} \quad R, \\ w(-1,t) = 1 - v(2a+1, t) < 0 \quad \text{for} \quad t > 0, \\ w(a,t) = 0 \quad \text{for} \quad t > 0, \end{cases}$$

where b_1 and b_2 are bounded functions in R. Introduce the function

$$W(y,t) = \exp\,(-\frac{1}{2}\int_{-1}^{y} b_1(x,t)\,dx)\,w(y,t).$$

Then W satisfies

$$\begin{cases} W_t - W_{yy} + q\,W = 0 & \text{in} \quad R, \\ W(-1,t) < 0, W(a,t) = 0 & \text{for} \quad t > 0, \end{cases}$$

for a suitably bounded function $q(y,t)$. Define

$$U(y,t) = \begin{cases} W(y,t), & -1 \le y \le a, \\ -W(2a-y,t), & a \le y \le 2a+1, \end{cases}$$

and let

$$Q(y,t) = \begin{cases} q(y,t), & -1 \le y \le a, \\ q(2a-y,t), & a \le y \le 2a+1. \end{cases}$$

Then U satisfies

$$\begin{cases} U_t - U_{yy} + Q\,U = 0 & \text{in} \quad (-1, 2a+1) \times (0, T_0), \\ U(-1,t) < 0, U(2a+1,t) > 0 & \text{for} \quad t > 0, \end{cases}$$

where Q is bounded. Denote by $z(t)$ the number of zeros of $U(\cdot,t)$ in the interval $[-1, 2a+1]$, i.e., $z(t) = \#\{y \in [-1, 2a+1]; U(y,t) = 0\}$. Applying Theorem D in [2] (see also [10]) we conclude that $z(t)$ is nonincreasing in t and is finite for all $0 < t < T_0$. Moreover, if there exists a (y_0, t_0) such that $U(y_0,t_0) = U_y(y_0,t_0) = 0$, then $z(t_1) < z(t_2)$ for all $t_1 > t_0 > t_2$. Since $z(t) = 2m(t) - 1$, where $m(t) = \#\{x \in [-1, a]; w(x,t) = 0\}$, we see that $m(t)$ is also nonincreasing in t and is finite for all $0 < t < T$, and if there exists a (x_0, t_0) such that $w(x_0,t_0) = w_x(x_0,t_0) = 0$, then $m(t_1) < m(t_2)$ for all $t_1 > t_0 > t_2$.

Suppose that the lemma is not true. Then $v_x(a,t)$ changes sign infinitely many times as $t \uparrow T$. Since $w_x(a,t) = 2\,v_x(a,t)$, we see that $w_x(a,t)$ changes sign infintely many times as $t \uparrow T$. Then there exists a sequence $t_n \to T$ as $n \to \infty$ such that $w_x(a, t_n) = 0$, for all n. Since $w(a,t) \equiv 0$, we have $m(t_n) > m(t_{n+1})$ for all n, and therefore

$$\begin{aligned} m(t_1) &= \sum_{i=1}^{N} [m(t_i) - m(t_{i+1})] + m(t_{N+1}) \\ &\ge N + m(t_{N+1}) \end{aligned}$$

for any $N \ge 1$. This implies that $m(t_1) = \infty$, which is a contradiction. Hence the lemma is proved for the case $a \in (-1, 0)$. The proof for $a \in (0,1)$ is similar.

For the case $a = 0$, we have

$$\begin{cases} w_t - w_{xx} + b_1\,w_x + b_2\,w = 0 & \text{if} \quad (x,t) \in R \\ w(-1,t) = w(0,t) = 0, & t > 0. \end{cases}$$

Then we can apply Theorem C in [2] directly and conclude that $m(t)$ is nonincreasing in t and is finite for all $0 < t < T$. Moreover, if there exists a point $(x_0, t_0) \in \bar{R}$ such that $w(x_0,t_0) = w_x(x_0,t_0) = 0$, then $m(t_1) < m(t_2)$ for all $t_1 > t_0 > t_2$. Proceeding as before, we conclude that the lemma is true for the case $a = 0$. In the case $a = \pm 1$, the lemma is trivial, since $v_x(-1,t) > 0, v_x(1,t) < 0$. Hence the lemma is proved. □

LEMMA 2.2. *There exists* $t^* \in (0,T)$ *such that*

$$n(t) \equiv \#\{a \in [-1,1]; v_x(a,t) = 0\}$$

is a constant ≥ 1 *for all* $t \geq t^*$. *Moreover, there are* C^1 *functions* $s_1(t), \cdots, s_m(t)$ *from* $[t^*, T)$ *to* $[-1,1]$ *such that*

$$\begin{aligned} &s_1(t) < \cdots < s_m(t), \quad t \geq t^* \\ &\{a \in [-1,1]; v_x(a,t) = 0\} = \{s_1(t), \cdots, s_m(t)\} \end{aligned}$$

and the limit $s_i = \lim_{t \uparrow T} s_i(t)$ *exists for all* $1 \leq i \leq m$.

Proof. Let $w = v_x$. Then w satisfies the equation

$$w_t - w_{xx} + b_1 w_x + b_2 w = 0$$

in Q_T, where $b_1 = 4\frac{v_x}{v}$ and $b_2 = -[2(\frac{v_x}{v})^2 + \epsilon(2+\beta)v^{1+\beta}]$. Also, $w(-1,t) > 0$ and $w(1,t) < 0$ for all $t > 0$. Applying Theorem D in [2] and arguing as in the proof of previous lemma, we conclude that $n(t)$ is finite and nonincreasing in t for $0 < t < T$.

We argue by contradiction. Suppose that there is no such t^*. Then we can find a sequence $t_n \uparrow T$ as $n \to \infty$ such that $n(t_i) > n(t_{i+1})$ for $i = 1, 2, \cdots$. Hence we obtain

$$\begin{aligned} n(t_1) &= \sum_{i=1}^{N}[n(t_i) - n(t_{i+1})] + n(t_{N+1}) \\ &\geq N + n(t_{N+1}) \end{aligned}$$

for any $N \geq 1$. This implies that $n(t_1) = \infty$, which is a contradiction. Hence there are $t^* \in (0,T)$ and a constant m such that $n(t) = m$ for all $t \geq t^*$. Consequently, for a given $t \geq t^*$ there exist $a_1, \cdots, a_m \in [-1,1]$ such that

$$v_x(a_i,t) = 0, \quad v_{xx}(a_i,t) \neq 0, \quad 1 \leq i \leq m.$$

Applying the implicit function theorem, we obtain C^1 functions $s_1(t), \cdots, s_m(t)$ from $[t^*, T)$ to $[-1,1]$ such that

$$\begin{aligned} &s_1(t) < \cdots < s_m(t), \quad t \geq t^* \\ &\{a \in [-1,1]; v_x(a,t) = 0\} = \{s_1(t), \cdots, s_m(t)\}. \end{aligned}$$

Futhermore, by Lemma 2.1, the limit, $\lim_{t \uparrow T} s_i(t)$, exists for all i. The proof is completed. □

LEMMA 2.3. *Let $[a,b]$ be an interval $\subset [-1,1] \setminus \{s_1, \cdots, s_m\}$. If there is a blow-up point c for v in (a,b), then*

$$\lim_{t\uparrow T} v(x,t) = \infty$$

either for all $x \in [c,b]$ or for all $x \in [a,c]$.

Proof. Choose T_0 so large so that v_x does not change sign in $[a,b] \times [T_0, T)$. Suppose that $v_x > 0$ in $[a,b] \times [T_0,T)$. Then $\lim_{t\uparrow T} v(x,t) = \infty$ for all $x \in [c,b]$ follows from the fact $v(x,t) \geq v(c,t)$ and $\lim_{t\uparrow T} v(c,t) = \infty$.

In the case $v_x < 0$, we can get $\lim_{t\uparrow T} v(x,t) = \infty$ for all $x \in [a,c]$. □

THEOREM 2.4. *Under the assumption (1.2), there are only finitely many quenching points for u.*

This theorem follows from the following lemma.

LEMMA 2.5. *There is no blow-up for v in $[-1,1] \setminus \{s_1, \cdots, s_m\}$.*

Proof. First of all, since $v(\pm 1, t) = 1$ for $t < T$, we see that v cannot blow up at $x = \pm 1$, and that $s_1 > -1, s_m < 1$.

Next, set $s_0 = -1$ and $s_{m+1} = 1$. Assume that there is a blow-up point c with $s_i < c < s_{i+1}$ for some $i \in \{0, 1, \cdots, m, m+1\}$. Set $a = \frac{s_i + c}{2}$ and $b = \frac{s_{i+1}+c}{2}$. Then there is a $T_0 \in (0,T)$ such that v_x has a fixed sign in the rectangle $R_0 = [a,b] \times [T_0, T)$. Without loss of generality, we may assume that $v_x > 0$ in R_0. Following the method of [4], consider the function

$$J(x,t) = v_x(x,t) - \eta\, \epsilon\, h(x)\, v^2(x,t)$$

in the rectangle $R = [c,b] \times [T_0, T)$, where $h(x) = \sin \frac{(x-c)}{(b-c)} \pi$ and $\eta > 0$ is a constant to be determined.

By a simple computation,

$$\begin{aligned} J_t - J_{xx} &= (\epsilon\, v^{2+\beta} - 2\, \frac{v_x^2}{v})_x - 2\, \eta\, \epsilon\, hv(\epsilon\, v^{2+\beta} - 2\, \frac{v_x^2}{v}) \\ &\quad + 2\eta \epsilon\, h v_x^2 + 4\eta\epsilon\, h' v v_x + \eta \epsilon\, h''\, v^2 \qquad (2.1) \\ &= b_1\, J_x + b_2\, J + \eta\, \epsilon\, h\, v^2 \left[\beta\, \epsilon\, v^{\beta+1} - \left(\frac{\pi}{(b-c)} \right)^2 \right] \quad \text{in} \quad R, \end{aligned}$$

for some functions b_1 and b_2, here we used the identities

$$\begin{aligned} v_x &= J + \eta \epsilon\, h v^2 \\ v_{xx} &= J_x + 2\eta\epsilon\, h v v_x + \eta \epsilon\, h'\, v^2. \end{aligned}$$

By Lemma 2.3, we can find a $T_1 \geq T_0$ such that if $t \geq T_1$, then

$$\beta \epsilon v^{\beta+1} \geq \left(\frac{\pi}{(b-c)}\right)^2$$

for all $x \in [c, b]$. For this T_1 we can choose $\eta > 0$ so small that $J \geq 0$ at $t = T_1$. Since $J > 0$ on $x = c$ and $x = b$ for $t \geq T_1$, the maximum principle implies that $J \geq 0$ in the rectangle $[c, b] \times [T_1, T)$. This means that

$$v_x \geq \eta \epsilon h v^2 \quad \text{or} \quad \frac{v_x}{v^2} \geq \eta \epsilon h$$

in $[c, b] \times [T_1, T)$. Integrating the last inequality from c to b at time $t \geq T_1$ we obtain

$$\frac{1}{v(c,t)} - \frac{1}{v(b,t)} \geq \eta \epsilon \int_c^b \sin\left(\frac{x-c}{b-c}\pi\right) dx. \tag{2.2}$$

Letting $t \uparrow T$, the left hand side of (2.2) tends to 0, but the right hand side of (2.2) remains positive. This is a contradiction. Hence the lemma is proved. □

3. Asymptotic behavior at the quenching point. In this section we shall study the asymptotic behavior of the solution at the quenching point. First, we prove an apriori estimate.

LEMMA 3.1. *Assume that the initial data u_0 satisfies the condition (1.2). Then we have the estimate*

$$v(x,t) \leq B(T-t)^{-\gamma}, \gamma = \frac{1}{\beta+1}, \tag{3.1}$$

for $|x| \leq 1, 0 \leq t < T$, for some positive constant B.

Proof. Recall that $v_t > 0$ in Q_T. Choose $\delta > 0$ so that $\delta < \frac{1}{2}\min(s_1 + 1, 1 - s_m)$. Then there is a positive constant C_0 such that $v_t \geq C_0$ on the parabolic boundary of $Q_\delta \equiv \{(x,t); |x| < 1-\delta, \delta < t < T\}$. Following [4], consider the function

$$J = v_t - \eta v^{2+\beta} \quad \text{in} \quad Q_\delta$$

where $\eta > 0$ is a constant so small that $J \geq 0$ on the parabolic boundary of Q_δ. By a simple computation,

$$J_t - J_{xx} + b_1 J_x + b_2 J = \eta \beta (\beta+1) v^\beta v_x^2$$

for some functions b_1 and b_2. Therefore, the maximum principle implies that $J > 0$ in Q_δ. Hence we have

$$\frac{v_t}{v^{2+\beta}} \geq \eta \quad \text{in} \quad Q_\delta.$$

It follows that

$$\gamma v^{-(\beta+1)}(x,t) \geq \int_t^T \frac{v_t}{v^{2+\beta}}\, dt \geq \eta\,(T-t),$$

and then

$$v(x,t) \leq (\frac{\gamma}{\eta})^\gamma (T-t)^{-\gamma}$$

in Q_δ. Since v is bounded in the set $Q_T \setminus Q_\delta$, we conclude that (3.1) holds in Q_T for some constant B. □

The proof of the following lemma is essentially as same as Proposition 1 in [5]. For reader's convenience we present its proof here.

LEMMA 3.2. *Under the assumption (1.2), for any $\delta \in (0, \delta_0)$, there exists a constant $M = M(\epsilon, \beta, B, \delta) > 0$ such that*

$$|v_x(x,t)| \leq M\,(T-t)^{-(\gamma+1/2)}$$
$$|v_{xx}(x,t)| \leq M\,(T-t)^{-(\gamma+1)}$$
$$|v_t(x,t)| \leq M\,(T-t)^{-(\gamma+1)}$$

for all $(x,t) \in R_\delta \equiv \{|x| \leq 1-\delta, 0 < t < T\}$, where $\delta_0 = \frac{1}{2}\min(s_1+1, 1-s_m)$.

Proof. Fix a $\delta \in (0, \delta_0)$. Without loss of generality we may assume that $T > 1/2$. For any η with $0 < \eta \leq \frac{\delta^2}{2}$, v is bounded in the set $S_\eta \equiv (-1,1) \times (0, T-\eta)$. Because u satisfies the equation

$$u_t - u_{xx} = \epsilon\, v^\beta$$

in Q_T, by L^p estimates

$$\iint_{S_\eta} (|u_t|^p + |u_x|^p + |u_{xx}|^p)\, dx\, dt \leq C$$

where $C = C(p, \epsilon, \beta, B, \eta, T)$ is a constant and $1 < p < \infty$. The Sobolev inequality and the Schauder estimate imply that $u \in C^{2+\alpha}(\bar{S}_\eta)$ for some $\alpha \in (0,1)$ and

$$|u_t|, |u_x|, |u_{xx}| \leq K \quad \text{in} \quad S_\eta \tag{3.2}$$

where $K = K(\epsilon, \beta, B, \eta, T)$ is a constant. But since $v_t = \frac{u_t}{(1-u)^2}$, $v_x = \frac{u_x}{(1-u)^2}$, and $v_{xx} = \frac{u_{xx}}{(1-u)^2} - 2\,\frac{u_x{}^2}{(1-u)^3}$, it follows from the (3.2) and the boundedness of v that

$$|v_t|, |v_x|, |v_{xx}| \leq L \quad \text{in} \quad S_\eta \tag{3.3}$$

for some constant $L = L(\epsilon, \beta, B, \eta, T)$.

For any point $(\bar{x}, \bar{t}) \in R_\delta \setminus S_\eta$, consider the function

$$z(y,s) = \lambda^{2\gamma} v(\bar{x} + \lambda y, T - \lambda^2 (T - s))$$

where $\lambda = \sqrt{2(T - \bar{t})}$. Then z satisfies the equation

$$z_s - z_{yy} + 2\frac{{z_y}^2}{z} = \epsilon z^{2+\beta}$$

in a domain D containing Q_T. Also, by Lemma 3.1, z satisfies the estimate

$$0 < z(y,s) \leq B(T-s)^{-\gamma}.$$

Since z satisfies the same equation and the same estimate as v we conclude that

$$|z_s|, |z_y|, |z_{yy}| \leq L \quad \text{at the point} \quad (0, T - 1/2) \tag{3.4}$$

which are equivalent to

$$|v_x(\bar{x}, \bar{t})| \leq M(T-t)^{-(\gamma+1/2)}$$
$$|v_{xx}(\bar{x}, \bar{t})| \leq M(T-t)^{-(\gamma+1)}$$
$$|v_t(\bar{x}, \bar{t})| \leq M(T-t)^{-(\gamma+1)}$$

where $M = L\,2^{-(\gamma+1/2)}$. Hence the proof is completed. □

Given any $a \in (-1,1)$, set $y = \frac{(x-a)}{\sqrt{T-t}}$, $s = -\ln(T-t)$, and $z^a(y,s) = v(x,t)(T-t)^\gamma$. Then z^a satisfies the equation

$$z_s - z_{yy} + \frac{1}{2} y z_y + 2\frac{{z_y}^2}{z} + \gamma z - \epsilon z^{2+\beta} = 0 \tag{3.5}$$

in the set $W = \{(y,s); |y \exp(-\frac{s}{2}) + a| \leq 1, s \geq -\ln T\}$. In the sequel we will omit the superscript a if there is no danger of ambiguity.

COROLLARY 3.3. *Under the assumption (1.2), we have*

$$\exp(-\gamma s) \leq z \leq B \quad \text{in} \quad W,$$
$$|z_y| \leq M, |z_{yy}| \leq M, |z_s| \leq M(1 + \frac{1}{2}|y|) + \gamma B \quad \text{in} \quad W_\delta,$$

where $W_\delta = \{(y,s); |y \exp(-\frac{s}{2}) + a| \leq (1-\delta), s \geq -\ln T\}$, *$B$ is the constant in Lemma 3.1, and δ and M are the constants in Lemma 3.2.*

The following lemma is similar to a result in [4].

LEMMA 3.4. *Given $t_* \in (0,T)$ sufficiently close to T and let $u_* = \max_{|x|\le 1} u(x,t_*)$. Then*

$$\frac{1}{2}\,{u_x}^2 \le H(u) \quad \text{in} \quad Q_{t_*},$$

where $Q_{t_*} \equiv \{(x,t); |x|<1, t<t_*\}$ and

$$H(u) = \frac{1}{\beta-1}\,[\frac{1}{(1-u_*)^{\beta-1}} - \frac{1}{(1-u)^{\beta-1}}].$$

Proof. Consider the function

$$J(x,t) = \frac{1}{2}\,{u_x}^2 - H(u) \quad \text{in} \quad Q_{t_*}.$$

By a simple computation,

$$J_t - J_{xx} = \frac{1}{(1-u)^{2\,\beta}} - {u_{xx}}^2. \tag{3.6}$$

Since

$$J_x = u_x\,u_{xx} + \frac{u_x}{(1-u)^\beta}, \tag{3.7}$$

we have

$${u_{xx}}^2 = \frac{1}{{u_x}^2}\left(J_x - \frac{u_x}{(1-u)^\beta}\right)^2. \tag{3.8}$$

By (3.6) and (3.8), we obtain

$$J_t - J_{xx} + b\,J_x = 0$$

for some function b. At $t=0$, we have

$$J(x,0) = \frac{1}{2}\,({u_0}')^2 - \frac{1}{\beta-1}\,[\frac{1}{(1-u_*)^{\beta-1}} - \frac{1}{(1-u_0)^{\beta-1}}],$$

which is nonpositive if t_* is sufficiently close to T. On $|x|=1$, we have $u_t=0$ and then

$$J_x = u_x\,[u_{xx} + \frac{1}{(1-u)^\beta}] = 0.$$

By the maximum principle, we then conclude that $J \le 0$ in Q_{t_*}, and the lemma is proved. □

COROLLARY 3.5. *There is a constant C depending only on the constants B and β, such that*

$$\frac{|z_y|}{z^2} \le C \quad \text{in} \quad W. \tag{3.9}$$

Proof. By Lemmas 3.1 and 3.4,

$$\frac{1}{2} {u_x}^2(x,t) \le C\,(T-t)^{-(\beta-1)\,\gamma}.$$

The lemma then follows from the identities

$$\frac{|z_y|}{z^2} = |(1/z)_y| = |u_x|\,(T-t)^{1/2-\gamma}.$$

□

Remark 3.6. Note that the constant C in Corollary 3.5 is independent of $a \in (-1,1)$. Let $w(y,s) = 1/z(y,s)$. Then $|w_y| \le C$ in W, and hence

$$B^{-1} \le w(y_2,s) \le w(y_1,s) + C\,|y_2 - y_1| \tag{3.10}$$

for any $(y_i,s) \in W, i = 1,2$.

In order to prove the asymptotic behavior at any quenching point we need the following nondegeneracy lemma which is due to Giga and Kohn [6]. For completeness, we present its proof here.

LEMMA 3.7. *Given $\delta > 0$ arbitrary, there exists a positive constant η such that if*

$$[v(x,t)\,(T-t)^{\gamma}] \le \eta$$

for all $(x,t) \in \{(x,t); |x-a| < \delta, T-\delta < t < T\}$, then a is not a blow-up point for v.

Proof. Without loss of generality we may assume that $a = 0, T = 0$, and $\delta = 1$. Let ϕ be a cutoff function in B_1 such that $0 \le \phi \le 1$ in B_1 and $\phi = 1$ in $B_{1/2}$. Consider the function

$$w(x,t) = v(x,t)\,\phi(x).$$

Then we have

$$w_t - w_{xx} = \epsilon\,\phi\,v^{2+\beta} - 2\,\phi\,\frac{{v_x}^2}{v} - 2\,(\phi_x\,v)_x + \phi_{xx}\,v.$$

Recall the properties of the fundamental solution of heat equation,

$$\Gamma(x,t;\xi,\tau) = \frac{1}{\sqrt{4\,\pi\,(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4\,(t-\tau)}\right),$$

$$\int_{-\infty}^{\infty} \Gamma(x,t;\xi,\tau)\,d\xi = 1,$$

$$\int_{-\infty}^{\infty} |\Gamma_\xi(x,t;\xi,\tau)|\,d\xi \le \frac{1}{\sqrt{\pi}}\,(t-\tau)^{-1/2}.$$

Since

$$w(x,t) = \int_{-\infty}^{\infty} w(\xi,-1)\,\Gamma(x,t;\xi,-1)\,d\xi + \int_{-1}^{t}\int_{-\infty}^{\infty} [w_t - w_{xx}]\,\Gamma(x,t;\xi,\tau)\,d\xi\,d\tau,$$

we have, by setting $M(t) = \max_{x\in\mathbf{R}} w(x,t)$,

$$\begin{aligned} M(t) \le M(-1) &+ \epsilon \int_{-1}^{t} M(\tau)\,[\eta\,(-\tau)^{-\gamma}]^{1+\beta}\,d\tau \\ &+ \|\phi''\|_{L^\infty} \int_{-1}^{t} \eta\,(-\tau)^{-\gamma}\,d\tau \\ &+ 2\,\|\phi'\|_{L^\infty} \int_{-1}^{t} \eta\,(-\tau)^{-\gamma}\,\frac{1}{\sqrt{\pi}}\,(t-\tau)^{-1/2}\,d\tau. \end{aligned}$$

Therefore,

$$M(t) \le C_1 + C_2 \int_{-1}^{t} (-\tau)^{-1}\,M(\tau)\,d\tau$$

for some constants C_1 and C_2, where $0 < C_1 < \infty$ and $C_2 = \epsilon\,\eta^{1+\beta}$, and then

$$M(t) \le C_1\,(-t)^{-C_2}.$$

Choose $\eta > 0$ so that $C_2 < \gamma$. By a bootstrap argument, we conclude that v is bounded in $B_{1/2} \times (-1,0)$. Hence a is not a blow-up point for v. The lemma is proved. □

From Lemma 3.7, we can deduce the following lemma by the same argument as in [9].

LEMMA 3.8. *For $a \in (-1,1)$, if $z^a(y,s) \to 0$ as $s \to \infty$ uniformly in $|y| \le C < \infty$ for any constant C, then a is not a blow-up point for v.*

We now state the main theorem of this section.

THEOREM 3.9. *Under the assumption (1.2), for any quenching point a,*

$$\lim_{t\uparrow T} [1 - u(x,t)]\,(T-t)^{-\gamma} = \left(\frac{\epsilon}{\gamma}\right)^{\gamma}$$

uniformly for $|x-a| \le C\sqrt{T-t}$ for any constant C provided that $\beta \ge 3$.

Proof. The proof is similar to the proof of Proposition 7 in [5]. Let $\{s_j\}$ be a sequence with $s_j \to \infty$. Without loss of generality we may assume that $a = 0$ and $s_{j+1} - s_j \to \infty$ as $j \to \infty$. Define

$$z_j(y,s) = z(y, s + s_j)$$

for all $j \geq 1$ and $(y,s) \in W_\delta$. Here $\delta \in (0,\delta_0)$ is a fixed small constant such that the set $\{(y,s); |y| \leq s, s \geq \bar{s}\}$ is contained in W_δ for some $\bar{s} \geq s_0$. By Corollary 3.3 and the Arzela-Ascoli theorem, there is a subsequence $\{j_m\}$ such that

$$z_{j_m}(y,s) \to z_\infty(y,s), \quad \text{as} \quad m \to \infty \tag{3.11}$$

uniformly in compact subsets of W_δ. Also, by a diagonal process, we have a subsequence $\{j_l\}$ of $\{j_m\}$ such that

$$z_{j_l,y}(y,m) \to z_{\infty,y}(y,m), \quad \text{as} \quad l \to \infty \tag{3.12}$$

for almost all y and for all integers m. Without loss of generality we may assume that (3.11) and (3.12) are true for the full sequence. Multiplying the equation (3.5) by

$$\rho(y)\, z^{-4}(y,s)\, z_s(y,s), \quad \text{where} \quad \rho(y) = \exp\left(-\frac{y^2}{4}\right),$$

we obtain

$$\rho \frac{{z_s}^2}{z^4} - (\rho\, z_y\, z^{-2})_y\, z^{-2}\, z_s + \gamma\, \rho\, z^{-3}\, z_s - \epsilon\, \rho\, z^{\beta-2}\, z_s = 0. \tag{3.13}$$

Integrating the equation (3.13) from $-R$ to R at time s, after an integration by parts, we get

$$\begin{aligned} - \int_{-R}^{R} \rho \frac{{z_s}^2}{z^4}\, dy &= \int_{-R}^{R} (\rho\, z_y\, z^{-2})\,(z^{-2}\, z_s)_y\, dy - (\rho\, z_y\, z^{-4}\, z_s)(R,s) \\ &+ (\rho\, z_y\, z^{-4}\, z_s)(-R,s) + \int_{-R}^{R} \gamma\, \rho\, z^{-3}\, z_s\, dy - \int_{-R}^{R} \epsilon\, \rho\, z^{\beta-2}\, z_s\, dy. \end{aligned} \tag{3.14}$$

Define the energy of z over $(-R,R)$ by

$$E_R[z](s) = \frac{1}{2} \int_{-R}^{R} \rho(y)\, \frac{{z_y}^2(y,s)}{z^4(y,s)}\, dy - \frac{\gamma}{2} \int_{-R}^{R} \rho(y)\, \frac{dy}{z^2(y,s)} - \frac{\epsilon}{\beta - 1} \int_{-R}^{R} \rho(y)\, z^{\beta-1}(y,s)\, dy.$$

Using (3.14) and setting $R(s) = s$, we obtain

$$\frac{d}{ds} E_s[z](s) = - \int_{-s}^{s} \frac{{z_s}^2}{z^4}\, \rho\, dy + J(s) \tag{3.15}$$

where

$$\begin{aligned} J(s) &= \frac{1}{2}\rho(s)\,\left[\frac{{z_y}^2(s,s)}{z^4(s,s)} + \frac{{z_y}^2(-s,s)}{z^4(-s,s)}\right] - \frac{\gamma}{2}\rho(s)\,\left[\frac{1}{z^2(s,s)} + \frac{1}{z^2(-s,s)}\right] \\ &- \frac{\epsilon}{\beta-1}\rho(s)\,[z^{\beta-1}(s,s) + z^{\beta-1}(-s,s)] \\ &+ \rho(s)\,\left[\frac{z_y(s,s)\, z_s(s,s)}{z^4(s,s)} - \frac{z_y(-s,s)\, z_s(-s,s)}{z^4(-s,s)}\right]. \end{aligned} \tag{3.16}$$

Set $w_\infty = 1/z_\infty$. From (3.10), we conclude that either $w_\infty \equiv \infty$ or $w_\infty < \infty$ in $\mathbf{R}^2$. Hence we obtain that either $z_\infty \equiv 0$ or $z_\infty > 0$ in $\mathbf{R}^2$. The case $z_\infty \equiv 0$ is ruled out by Lemma 3.8, and we conclude that $z_\infty > 0$. Therefore, by (3.10), we obtain

$$B^{-1} \le w_\infty \le (c_1 + c_2\,|y|) \quad \text{or} \quad B \ge z_\infty \ge (c_1 + c_2\,|y|)^{-1} \tag{3.17}$$

for some positive constants c_1 and c_2. By Corollary 3.3, (3.9), and (3.17), we see that $J(s)$ is integrable over $(\bar{s}, \infty)$ and $E_\infty[z_\infty](m)$ is well-defined for any intger m. Proceeding as in the proof of Proposition 7 in [5], and using (3.15) and (3.16), we obtain that z_∞ is independent of s and satisfies the equation

$$z'' - \frac{1}{2}\,y\,z' - 2\,\frac{z'^2}{z} - \gamma\,z + \epsilon\,z^{2+\beta} = 0, \quad \text{in} \quad \mathbf{R}.$$

Then w_∞ satisfies the equation

$$w'' - \frac{1}{2}\,y\,w' + \gamma\,w - \epsilon\,w^{-\beta} = 0, y \in \mathbf{R}.$$

It follows from (3.17) and Theorem 4.1 section 4 that $w \equiv$ const. $= (\frac{\epsilon}{\gamma})^\gamma$. Since this assertion is independent of the choice of $\{s_j\}$, the theorem follows. □

4. Nonexistence of nonconstant solution. In this section we prove the following theorem which is used in section 3.

THEOREM 4.1. *Let $f(w) = \gamma\,w - \epsilon\,w^{-\beta}$, $k = (\frac{\epsilon}{\gamma})^\gamma$, $\gamma = \frac{1}{\beta+1}$, and $\beta \ge 3$. Then the only global solution, w, of the equation*

$$w'' - \frac{1}{2}\,y\,w' + f(w) = 0, y \in \mathbf{R}, \tag{4.1}$$

which is greater than or equal to some positive constant c and which grows at most polynomially as $|y| \to \infty$, is $w \equiv k$.

First, we prove

LEMMA 4.2. *Suppose that $w = w(y)$ is a global solution of (4.1) with $w \ge c$ for some constant $c > 0$, and w grows at most polynomially as $|y| \to \infty$. Then*

$$\rho_\delta(y)\,w'(y) \to 0, \quad \text{as} \quad y \to \pm\infty$$

for any $\delta > 0$, where $\rho_\delta(y) = \exp(-\delta\,y^2)$.

Proof. For a given $\delta > 0$,

$$\begin{aligned}(\rho_\delta\,w' - a\,\delta\,y\,\rho_\delta\,w)' &= -\rho_\delta\,f(w) - a\,\delta\,\rho_\delta\,w + 2a\,\delta^2\,y^2\,\rho_\delta\,w \\ &\equiv I, \quad \text{where} \quad a = \frac{1}{2\,\delta} - 2.\end{aligned}$$

Since I is absolutely integrable to ∞ and $(a\,\delta\,y\,\rho_\delta\,w)(y) \to 0$ as $y \to \infty$, we conclude that

$$\lim_{y\to\infty} \rho_\delta(y)\,w'(y) = l$$

exists and $l < \infty$.

We claim that $l = 0$. Suppose that $l \neq 0$. Then w' is unbounded as $y \to \infty$ since $\rho_\delta(y) \to 0$ as $y \to \infty$. Therefore, w' must changes sign infinitely many times as $y \to \infty$. Otherwise, if w' has a fixed sign ultimately then w is unbounded which is a contradiction. Hence we can find a sequence $y_n \to \infty$ such that $w'(y_n) = 0$ for all n. Hence $\lim_{n\to\infty} \rho_\delta(y_n)\,w'(y_n) = 0$. This gives that $l = 0$.

The case for $y \to -\infty$ is similar. □

LEMMA 4.3. *Under the hypotheses of Lemma 4.2, we have*

$$\int_{-\infty}^{\infty} [(\beta - 3) + \frac{\beta+1}{2}\,y^2]\,\rho(y)\,w'(y)^2\,dy = 0, \tag{4.2}$$

which is an identity of Pohozaev's type (cf. [5]).

Proof. We rewrite equation (4.1) as

$$(\rho\,w')' + \rho\,f(w) = 0, \quad -\infty < y < \infty.$$

Set $G = \frac{1}{2}\,\rho\,(w')^2 - \rho\,F(w)$, where $F(w) = \frac{\gamma}{2}\,w^2 + \frac{\epsilon}{\beta-1}\,w^{1-\beta}$. For any C^1 functions a and h we have the identity

$$[h\,G - (aw + hw')\,G_p]' = h'\,G + h\,G_y - (a'w + h'w')G_p - a(pG_p + wG_w) \tag{4.3}$$

where $G_p = \frac{\partial G}{\partial w'}$, $G_w = \frac{\partial G}{\partial w}$, and $G_y = \frac{\partial G}{\partial y}$. Introduce the functions $a(y) = y^2 - 2$ and $h(y) = 2\,(\beta - 1)\,y$ and integrate the identity (4.3) from $-R$ to R to get

$$\begin{aligned} &\int_{-R}^{R} \{(\frac{1}{2}y^2 - 1)\,\rho\,w^2 - [(\beta - 3) + \frac{\beta+1}{2}\,y^2]\,\rho\,(w')^2 - \rho\,a'\,w\,w'\}\,dy \\ &= [h\,G - (aw + hw')\,G_p](R) - [h\,G - (aw + hw')\,G_p](-R). \end{aligned} \tag{4.4}$$

Integrating the last integral in the left hand side of (4.4) by parts gives

$$-\int_{-R}^{R} \rho\,a'\,w\,w'\,dy = \int_{-R}^{R} (1 - \frac{1}{2}\,y^2)\,\rho\,w^2\,dy - [\frac{1}{2}\,\rho\,a'\,w^2](R) + [\frac{1}{2}\,\rho\,a'\,w^2](-R). \tag{4.5}$$

If we let $R \to \infty$ and apply Lemma 4.2, then the lemma follows by combining the equations (4.4) and (4.5). □

Proof of Theorem 4.1. Since the integrand in (4.2) is nonnegative, $w' \equiv 0$ and the theorem follows. □

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