SOME REMARKS ON DEFORMATIONS OF MINIMAL SURFACES

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Introduction

We consider complete minimal surfaces (c. m. s.'s) in R^3 and their deformations. M_1 is an ϵ deformation of M_0 if M_1 is a graph over M_0 in an ϵ tubular neighborhood of M_1 and M_1 is $\epsilon - C^1$ close to M_0. A c. m. s. M_0 is isolated if all minimal surfaces M_1, which are sufficiently small deformations of M_0, are congruent to M_0. Many of the classical minimal surfaces in R^3 are known to be isolated [2], however, no example was known of a nonisolated minimal surface. In this paper we construct such an example; it is modelled on a four punctured sphere and of finite total curvature. On the other hand, we prove a c. m. s. discovered by Meeks and Gorge, modelled on the sphere punctured at the four roots of unity, $\{1, -1, i, -i\}$, is isolated. The analogous surface modelled on the sphere punctured at the cube roots of unity was shown to be isolated in [2]. The question is raised there, whether deformations of the four puncture case can be realized by deforming the conformal structure; i.e., changing the cross ratio of the four points. Thus for the Meeks-Gorge example in question, the answer is no. As we shall see, the conformal structure of our example that admits deformations, does not change either. Perhaps the conformal structure never changes by small deformations? We wish to thank W. Meeks and B. Morin for helpful conversations and greatly simplifying suggestions.

I. A Deformable Surface

Let M be a c. m. s. of finite total curvature, so that M is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at a finite number of points. An end E of M is said to be bounded if E is a bounded distance from a plane. If E corresponds to the puncture p and P is the plane orthogonal to g(p), g the Gauss map, then E bounded means E is a bounded distance from P. If E is embedded then it can be expressed as the graph of $a \log R + \mathcal{O}(|R|)$ where R is the distance from the origin in P and this holds for R large. Then E is bounded if and only if $a = 0$.

Let (g, w) be a Weirstrass representation of M and ϕ_1, ϕ_2, ϕ_3 the associated analytic differentials on M. We know the real periods of each ϕ_k are zero on M since the coordinate functions, $x_k = \mathrm{Re}\int \phi_k$, are well defined on M. Suppose in addition, that each ϕ_k has no imaginary periods on M. Let $\phi_k(t) = e^{it}\phi_k$, for each $t \in R$. Then $\phi_k(t)$ are analytic differentials on M which have no periods on M. Thus $x_k(t) = \mathrm{Re}\int \phi_k(t)$ are single-valued coordinate functions on M, giving a c.m.s. $M(t)$ modelled on M. We claim that if each end of M is flat, then $M(t)$ is an ϵ-deformation of M, for t sufficiently small. We will prove this by showing each end of $M(t)$ is a deformation of the corresponding end of $M(0) = M$; this suffices since the compact part of $M(t)$, the complement of the ends, converges uniformly to the compact part of M. Let the end E of M be parameterized by the disc unity D punctured at 0. Since deforming (i.e., $(g, e^{it}w)$) commutes with rotation in R^3, we can assume $g(0) = 0$. Then g has a zero of order n at 0 and w has a pole of order $k \geqslant 2$. Clearly E is a flat end if and only if $n \geqslant k$ (look at $x_3 = \mathrm{Re}\int gw$). Now $(g, e^{it}w)$ is a Weirstrass representation of $M(t)$ and 0 is still a pole of order k of $e^{it}w$ so $E(t)$ is flat as well. Thus $x_3(t)$ is bounded in the punctured disc, hence extends to the origin. Since

$$x_3(t) = \cos(t)\, x_3(0) - \sin(t)\, \mathrm{Im}\int \phi_3$$

we conclude $\mathrm{Im}\int \phi_3$ is bounded in the punctured disc. Thus $x_3(t)$ converges to $x_3(0)$ uniformly and it follows that $E(t)$ is a deformation of E.

Thus to exhibit a nonisolated c.m.s. we will find a c.m.s. M of finite total curvature, with flat ends, all periods of the ϕ_k are zero, and $M(t)$ is not congruent to M for $t \neq 0$. Such an example follows.

Let M be the sphere punctured at the four points a_1, a_2, a_3, ∞, where the a_k are the cube roots of $-1/2$. Let

$$g(z) = -\frac{1}{2}\frac{z}{(z^3-1)} \quad , \qquad w = \frac{1}{4}\frac{(z^3-1)^2}{(z^3+\frac{1}{2})^2}\,dz$$

We claim (g, w) is a Weirstrass representation of M satisfying the conditions desired. First observe

$$dg = \frac{z^3+\frac{1}{2}}{(z^3-1)^2}\,dz$$

vanishes at each pole of w so each end is flat (each pole of w is of order two).

Next we see the forms ϕ_k have no periods: $\phi_1 = \frac{1}{2}(1-g^2)w$, $\phi_2 = \frac{i}{2}(1+g^2)w$, $\phi_3 = gw$. It suffices that w, gw and g^2w be exact. We have

$$w = \frac{1}{4}\,d\left(\frac{z^4+2z}{z^3+\frac{1}{2}}\right)$$

$$gw = +\frac{1}{8}\,d\left(\frac{z^2}{z^3+\frac{1}{2}}\right)$$

$$g^2w = -\frac{1}{48}\,d\left(\frac{1}{z^3+\frac{1}{2}}\right) \quad .$$

We leave it to the reader to verify M is a c.m.s. and the associated surface $M(t) = (g, e^{it}w)$ is not congruent to M.

This example is part of the following family, all of which admit deformations:

$$g(z) = -\frac{1}{2}\frac{z}{(z-b_1)(z-b_2)(z-b_3)} \quad , \qquad w = \frac{\pi(z-b_i)^2}{\pi(z-a_i)^2}\,dz$$

where the a's and b's satisfy:

$$a_i \neq a_j \;, \quad b_i \neq b_j \;, \quad i \neq j \quad \text{and} \quad a_i \neq b_j \;,$$

$$b_1 + b_2 + b_3 = 0 \quad , \qquad 2a_1^3 = -b_1 b_2 b_3$$

$$a_2 = e^{2\pi i/3} a_1 \quad , \qquad a_3 = e^{4\pi i/3} a_1 \quad .$$

II. An Isolated Surface

Let M be the sphere punctured at the n roots of unit. Meeks and Gorge have discovered examples of a c.m.s. modelled on M, of finite total curvature Their Weirstrass representation is:

$$g(z) = z^{n-1} \quad , \qquad w = \frac{dz}{(z^n - 1)^2} \quad .$$

When $n = 2$ this gives the catenoid; in general this surface is rotationally symmetric and has n catenoid type ends. In [2] we proved the surfaces $n = 2$ and $n = 3$ are isolated and the question was raised whether deformations for $n = 4$ could be obtained by changing the conformal structure; i.e., moving the punctures to change the cross ratio. We shall now prove this is not possible. The technique is the same as that we used for $n = 3$ and offers little hope of understanding $n > 4$.

Theorem 2.1. The c.m.s. M_0, modelled on the sphere punctured at the 4 roots of unity, defined by

$$g_0(z) = z^3 \quad , \qquad w_0 = \frac{dz}{(z^4 - 1)^2} \quad ,$$

is isolated.

Proof. We proved in [2] that if M_0 is a c.m.s. conformally equivalent to a compact Riemann surface $\overline{M}$, punctured at m points, and if M_1 is a deformation of M_0 then M_1 is conformally equivalent to $\overline{M}$ punctured at m points (not necessarily the same points) and the total curvature of M_1 equals the total curvature of M_0.

Now suppose M is a deformation of our example. Then M is conformally equivalent to the sphere punctured at four points, and we can assume the points are $\{z_0, 1, -1, i\}$. Assume $z_0 \neq \infty$; this case will be discussed later.

Let (g, w) be a Weirstrass representation of M; g is a rational map of degree three since the total curvature of M is that of M_0. Write

$$g(z) = \frac{az^3 + bz^2 + cz + d}{\alpha z^3 + \beta z^2 + \gamma z + \delta} \quad .$$

We assume $\delta \neq 0$; this case will be treated later. So normalize by $\delta = 1$. We know w has a double pole at the points z_0, 1, -1, i, the point ∞ is regular for w, and w has a zero of order $2k$ at each pole of g of order k. We write

$$w = \frac{e(\alpha z^3 + \beta z^2 + \gamma z + 1)^2}{(z - z_0)^2 (z-1)^2 (z+1)^2 (z-i)^2} \, dz \quad .$$

We know the limiting normals of an end of M_0 and M are the same, so

$$g(1) = 1 \;, \quad g(-1) = -1 \;, \quad g(i) = -i \;, \quad g(z_0) = i \quad .$$

Write

$$\frac{1}{(z-z_0)^2 (z-1)^2 (z+1)^2 (z-i)^2} = \frac{Az+B}{(z-z_0)^2} + \frac{Cz+D}{(z-1)^2} + \frac{Ez+F}{(z+1)^2} + \frac{Gz+H}{(z-i)^2} \quad .$$

Let x_k, $k = 1,2,3$, be the coordinate functions of M and y_k those of M_0; $x_k = \mathrm{Re}\int \phi_k$ and $y_k = \mathrm{Re}\int \psi_k$, ϕ_k and ψ_k the differentials obtained via the Weirstrass representation. We know the real periods of the ϕ_k and ψ_k are zero, since x_k and y_k are well defined functions on M and M_0 respectively. From this we obtain three equations:

1. $-G(-ai - b + ci + d) + (Gi + H)\left[i(-3\alpha + 2i\beta + \gamma) - (-3a + 2bi + c)\right] = 0$

2. $-E(-a + b - c + d) + (F - E)\left[3\alpha - 2\beta + \gamma - (3a - 2b + c)\right] = 0$

3. $c(a + b + c + d) + (C + D)(3\alpha + 2\beta + \gamma + 3a + 2b + c) = 0$.

Equation 1 results from $\mathrm{Re}(i\,\mathrm{Res}(\phi_1, i)) = 0$ and $\mathrm{Re}(i\,\mathrm{Res}(\phi_3, i)) = 0$. Equation 2 follows from $\mathrm{Re}(i\,\mathrm{Res}(\phi_2,-1)) = 0$ and $\mathrm{Re}(i\,\mathrm{Res}(\phi_3,-1)) = 0$; equation 3, from $\mathrm{Re}(i\,\mathrm{Res}(\phi_2,1)) = 0 = \mathrm{Re}(i\,\mathrm{Res}(\phi_3,1))$.

Now $g(1) = 1$ so near the puncture $z = 1$, the end of M is a graph over the (x_2, x_3) plane and a calculation yields:

$$x_1(z) = K \log|x_3 + ix_2| + \mathcal{O}(|x_3 + ix_2|) \quad \text{as} \quad z \to 1,$$

where $K = -e(C+D)(a+b+c+d)[3\alpha + 2\beta + \gamma - (3a+2b+c)]$. The same calculation for M_0 at this end yields:

$$y_1(z) = \frac{6}{32} \log|x_3 + ix_2| + \mathcal{O}(|x_3 + ix_2|) \quad \text{as} \quad z \to 1 .$$

Since M and M_0 are ϵ close, the coefficients of $\log|x_3 + ix_2|$ must be the same. This yields equation 4:

4. $\quad -e(C+D)(a+b+c+d)[3\alpha + 2\beta + \gamma - (3a+2b+c)] = 6/32$.

The same calculations at the punctures -1 and i yield two more equations:

5. $\quad -e(F-E)(-a+b-c+d)(3\alpha - 2\beta + \gamma + 3a - 2b + c) = 6/32$,

6. $\quad -e(Gi+H)(-ai-b+ci+d)[i(-3\alpha + 2\beta i + \gamma) + (-3a + 2bi + c)] = 6i/32$.

A calculation shows:

$$C = \frac{(1-2i)(1-z_0) - 2i}{8(1-z_0)^3}, \qquad D = \frac{(1-z_0)(-1+3i) + 2i}{8(1-z_0)^3}$$

$$E = \frac{-(1+2i)(1+z_0) - 2i}{8(1+z_0)^3}, \qquad F = (1+z_0)(-1-3i) - 2i$$

$$G = \frac{i(i-z_0) - 1}{2(i-z_0)^3}, \qquad H = \frac{3(i-z_0) + 2i}{4(i-z_0)^3} .$$

To the above 6 equations one must add the four equations $g(1) = 1$, $g(-1) = -1$, $g(i) = -i$, $g(z_0) = i$. Now we have ten equations in nine unknowns. These equations have no solution, as the courageous reader can verify.

Bibliography

[1] Meeks, W., A survey of the geometric results in the classical theory of minimal surfaces, Bol. Soc. Bras. Mat., 12 (1981), 29-86.

[2] Rosenberg, H., Deformations of complete minimal surfaces, to appear.