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IMA Preprint Series # 421

July, 1988

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Abstract. Two almost explicit constructions are given satisfying the title.

1. Preliminaries. Let $[n]$ denote the set of the first n positive integers, $2^{[n]}$ its power set. Sometimes $2^{[n]}$ will be called the *Boolean lattice* and denoted by $\mathbf{B}_n$. The collection of all k-subsets of a set S is denoted by $\binom{S}{k}$. A family $\mathcal{L} = \{L_0, L_1, \dots, L_t\} \subset 2^{[n]}$ is called a *chain* if its members contain each other, $L_0 \subset L_1 \subset \cdots \subset L_t$. Such a chain is *maximal* if $t = n$, in which case $|L_i| = i$ for all i. The family $\mathcal{C} \subset 2^{[n]}$ is a *cutset* of the Boolean lattice if $\mathcal{C} \cap \mathcal{L} \neq \emptyset$ for all maximal chains $\mathcal{L}$. A *minimal cutset* $\mathcal{C}$ is a cutset with the property that for every $C \in \mathcal{C}$ some maximal chain avoids $\mathcal{C} \setminus \{C\}$. For example the whole k-th level of the Boolean lattice $\binom{[n]}{k}$ is a minimal cutset. But there are minimal cutsets of much larger size, e.g. the following family

$$\{C \subset [n] : |C \cap \{1,2\}| = 1\} \tag{1.1}$$

has size 2^{n-1}. Denote the maximum size of a minimal cutset of $\mathbf{B}_n$ by $c(n)$. Ko-Wei Lih asked whether $c(n) = 2^{n-1}$ in general. It is easy to see that

$$c(n+1) \geq 2c(n). \tag{1.2}$$

(Indeed, if $\mathcal{C}$ is a minimal cutset of $\mathbf{B}_n$ then $\mathcal{C} \cup \{C \cup \{n+1\} : C \in \mathcal{C}\}$ is a minimal cutset of $\mathbf{B}_{n+1}$.) The inequality (1.2) implies that there is a limit of the sequence $c(n)/2^n$ whenever n tends to infinity. This limit is at least 1/2 by (1.1). In [L] Ko-Wei Lih gives a construction for $n = 6$ due to Jin-Fa Chern in which $|\mathcal{C}| = 33 > 2^{n-1}$. (Unfortunately, his example contains a misprint. To fix it, the set $\{1,2,4,5,6\}$ should be replaced by $\{1,2,3,5,6\}$.) It is natural to ask whether the answer is asymptotic to 2^n. In this note we give an almost explicit construction proving that

$$\lim_{n\to\infty} c(n)/2^n = 1. \tag{1.3}$$

"Almost explicit" means that we will define a large cutset (of size $(1-o(1))2^n$) and prove that by deleting only $o(2^n)$ members of it one can obtain a minimal cutset.

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[1] This research was done while the authors visited the Institute for Mathematics and its Applications at the University of Minnesota, Minneapolis, MN 55455.

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Research supported in part by NSF grant DMS 86-06225 and Airforce Grant OSR-86-0076.

2. An Almost Deterministic Construction. Let $k \geq 3$ be an integer, and suppose that n is divisible by k. Let $S_1 \cup \ldots S_{n/k}$ be a partition of $[n]$ into k-element parts. Define the family $\mathcal{C}$ as follows.

$$\mathcal{C} = \{C \subset [n] : 0 < |S_i \cap C| < k \text{ for all } S_i\}$$
$$\cup \{C \subset [n] : \exists S_i \text{ and } S_j \text{ with } |S_i \cap C| = 0, |S_j \cap C| = k\}.$$

We claim that $\mathcal{C}$ is a cutset. Indeed, if $\emptyset = L_0 \subset L_1 \subset \ldots L_n$ is a maximal chain then define t as the largest integer such that L_t is still disjoint from some S_i. Then L_{t+1} intersects all S_i. If L_{t+1} does not contain any S_j, then then it belongs to the first part of $\mathcal{C}$. If L_{t+1} contains some S_j, then L_t belongs to the second part of $\mathcal{C}$.

A member C of a cutset $\mathcal{C}$ is *essential* if $\mathcal{C} \setminus \{C\}$ is not cutset. Define

$$\mathcal{C}_0 = \{C \subset [n] : 0 < |S_i \cap C| < k \text{ for all } S_i \text{ and } \exists S_i, S_j \text{ with } |S_i \cap C| = 1, |S_j \cap C| = k-1\}.$$

We claim that every member of $\mathcal{C}_0$ is essential in $\mathcal{C}$. Indeed, if $C \in \mathcal{C}_0$ with $|S_i \cap C| = 1$ and $|S_j \cap C| = k-1$, then every maximal chain containing $C \setminus S_i, C$ and $C \cup S_j$ avoids $\mathcal{C} \setminus \{C\}$. Starting with an arbitrary cutset one can always obtain a minimal cutset by deleting the unnecessary members one by one. But we can never delete an essential set. So all minimal cutsets contained in $\mathcal{C}$ contain $\mathcal{C}_0$. We have

$$(2.1) \quad |\mathcal{C}_0| = (2^k - 2)^{n/k} - 2(2^k - k - 2)^{n/k} + (2^k - 2k - 2)^{n/k}$$
$$> 2^n((1 - \frac{2}{2^k})^{n/k} - 2(1 - \frac{k+2}{2^k})^{n/k}) > 2^n(1 - \frac{2n}{k2^k} - 2\exp[-\frac{n}{2^k}]).$$

Here we used the inequalities $(1-x)^y \leq \exp[-xy]$, which holds for $-\infty \leq x \leq 1$ and $y \geq 0$, and $1 - xy \leq (1-x)^y$, which holds for $0 \leq x \leq 1$ and $y \geq 1$. If $n \sim 2^k \log k$, (i.e., $k \sim \log n - \log\log\log n$) then the (2.1) gives the following.

COROLLARY 2.1. *For sufficiently large n*

$$c(n) > 2^n(1 - \frac{4\log\log n}{\log n}).$$

We shall improve this result in Theorem 4.1.

3. Filters and Ideals. A subfamily $\mathcal{F}$ of $2^{[n]}$ is called a *filter* if $F \in \mathcal{F}$ and $F \subset F' \subset [n]$ imply $F' \in \mathcal{F}$. Starting with any subfamily $\mathcal{S} \subset 2^{[n]}$ one can obtain a filter $\mathcal{F}(\mathcal{S})$ as follows. $\mathcal{F}(\mathcal{S}) = \{F \subset [n] : \exists S \in \mathcal{S} \text{ such that } S \subset F\}$. $\mathcal{F}(\mathcal{S})$ is the filter *induced* by $\mathcal{S}$. A family $\mathcal{I}$ is called an *ideal* if $I \in \mathcal{I}$ and $I' \subset I$ imply $I' \in \mathcal{I}$ as well. For an arbitrary family $\mathcal{S} \subset 2^{[n]}$ we associate an ideal $\mathcal{I}(\mathcal{S})$ in the following way. $\mathcal{I}(\mathcal{S}) = \{I \subset [n] : \exists S \in \mathcal{S} \text{ such that } I \cap S = \emptyset\}$. $\mathcal{I}(\mathcal{S})$ is the ideal *induced* by $\mathcal{S}$. (Warning! This definition differs from the usual one.) In this way $\mathcal{F}(\mathcal{S})$ and $\mathcal{I}(\mathcal{S})$ consist of complementary pairs, i.e. $A \in \mathcal{F}(\mathcal{S})$ if and only if $([n] \setminus A) \in \mathcal{I}(\mathcal{S})$.

The *neighborhood* $N(\mathcal{G})$ of a family $\mathcal{G}$ is defined as the family of those subsets in $[n]$ whose Hamming distance from $\mathcal{G}$ is exactly 1, i.e. $N(\mathcal{G}) = \{N \subset [n] : N \notin \mathcal{G} \text{ and } \exists G \in \mathcal{G} \text{ such that } |N \triangle G| = 1\}$. Note that $\mathcal{G} \cap N(\mathcal{G}) = \emptyset$. The *complement* $\overline{\mathcal{G}}$ of the family $\mathcal{G}$ is defined as $\overline{\mathcal{G}} = 2^{[n]} \setminus \mathcal{G}$. The following idea underlies the construction in Section 2.

OBSERVATION 3.1. *Suppose that $\mathcal{I}$ is an ideal and $\mathcal{F}$ is a filter such that there are no two sets $I \in \mathcal{I} \setminus \mathcal{F}$ and $F \in \mathcal{F} \setminus \mathcal{I}$ such that*

$$I \subset F \text{ and } |F \setminus I| = 1. \tag{3.1}$$

Then $\mathcal{C} = (\overline{\mathcal{I}} \cap \overline{\mathcal{F}}) \cup (\mathcal{I} \cap \mathcal{F})$ is a cutset. Moreover, all members of $N(\mathcal{I}) \cap N(\mathcal{F})$ are essential. □

If we use an arbitrary family $\mathcal{S}$ to induce an ideal and a filter , then we obtain

LEMMA 3.2. *If for every S and $S' \in \mathcal{S}$ one has $|S \cap S'| \neq 1$, then the ideal $\mathcal{I}(\mathcal{S})$ and the filter $\mathcal{F}(\mathcal{S})$ fulfill Observation 3.1.*

Proof. Indeed, if $F \in \mathcal{F}(\mathcal{S}) \setminus \mathcal{I}(\mathcal{S})$ then there exists an $S_1 \in \mathcal{S}$ such that $S_1 \subset F$ and F intersects all members of $\mathcal{S}$. Moreover if $I \in \mathcal{I}(\mathcal{S}) \setminus \mathcal{F}(\mathcal{S})$ then there exists an $S_2 \in \mathcal{S}$ such that $S_2 \cap I = \emptyset$ and I does not contain any member of $\mathcal{S}$. So in this case $|F \setminus I| = 1$ would imply $S_1 \cap S_2 = F \setminus I$, a contradiction. □

4. A Random Construction. In view of Lemma 3.2, all that we need in order to construct a large minimal cutset is to find a suitable family $\mathcal{S}$ that has a filter $\mathcal{F}(\mathcal{S})$ with a big neighborhood. In this section we describe a *random* family $\mathcal{S}$ satisfying

$$|S \cap S'| \neq 1, \tag{4.1}$$

such that for some positive constant c

$$|N(\mathcal{F}(\mathcal{S}))| > 2^n(1 - c\frac{(\log n)^{3/2}}{\sqrt{n}}). \tag{4.2}$$

Of course, the same lower bound holds for $|N(\mathcal{I}(\mathcal{S}))|$ as well, thus

$$|N(\mathcal{F}(\mathcal{S})) \cap |N(\mathcal{I}(\mathcal{S}))| > 2^n(1 - 2c\frac{(\log n)^{3/2}}{\sqrt{n}}).$$

So Lemma 3.2 yields that $\mathcal{C} = (\overline{\mathcal{I}(\mathcal{S})} \cap \overline{\mathcal{F}(\mathcal{S})}) \cup (\mathcal{I}(\mathcal{S}) \cap \mathcal{F}(\mathcal{S}))$ is a cutset with a large number of essential sets.

THEOREM 4.1. *There exists a $c > 0$ such that $c(n) > 2^n(1 - c\frac{(\log n)^{3/2}}{\sqrt{n}})$.*

Proof. To find such a family $\mathcal{S}$ our method is a modified version of what was used in **[FKK]** and in **[K]** to construct a small filter with large neighborhoods. Suppose that n is divisible by 8, and let $B_1 \cup \ldots B_{n/2}$ be a partition of the underlying set into pairs. Let k be an integer $k \sim \sqrt{n/\log n}$. For every $K \in \binom{[n/2]}{k}$ let ξ_K be a random variable with

$$\mathrm{Prob}(\xi_K = 1) = \frac{(1000\log n)^{3/2}}{\sqrt{n}} \binom{n/8}{k}^{-1} = p$$
$$\mathrm{Prob}(\xi_K = 0) = 1 - p.$$

These random variables are to be chosen totally independently. Let $\mathcal{S}$ be the random family defined by

$$\mathcal{S} = \{\cup_{i \in K} B_i : \xi_K = 1\}.$$

Of course, $\mathcal{S}$ satisfies (4.1). We next show that the expected size of $N(\mathcal{F}(\mathcal{S}))$ is as large as it was given in (4.2). This implies the existence of a family $\mathcal{S}$ wich fulfils both (4.1) and (4.2), proving Theorem 4.1.

Let N be an arbitrary but fixed member of $2^{[n]}$. Denote the number of blocks B_i which are contained in N by n_2, and let $N_2 = \{i : B_i \subset N\}$. Similarly, let $N_1 = \{i : |B_i \cap N| = 1\}$, and $|N_1| = n_1$. We give an exact formula for the probability that N belongs to $N(\mathcal{F}(\mathcal{S}))$. N belongs to $N(\mathcal{F}(\mathcal{S}))$ if and only if $\xi_K = 0$ for all $K \in \binom{N_2}{k}$ and $\xi_K = 1$ for some k-set K with $|K \setminus N_2| = 1$ and $(K \setminus N_2) \subset N_1$. Since the variables ξ_K are independent, we obtain that

$$\begin{aligned} \mathrm{Prob}(N \in N(\mathcal{F}(\mathcal{S}))) &= (1-p)^{\binom{n_2}{k}}(1-(1-p)^{n_1\binom{n_2}{k-1}}) \\ &\geq (1 - p\binom{n_2}{k})(1 - \exp[-pn_1\binom{n_2}{k-1}]) \end{aligned} \tag{4.3}$$

Now suppose that N is a *typical* member of $\mathbf{B}_n$. More exactly, define the collection $\mathcal{N}$ of typical sets N by

$$\mathcal{N} = \{N \in 2^{[n]} : |n_2(N) - \frac{n}{8}| < \sqrt{n\log n} \text{ and } |n_1(N) - \frac{n}{4}| < 0.1n\}.$$

Then the well-known de Moivre-Laplace formula (see, e.g. in [**R**, p. 151]) gives that

$$|\mathcal{N}| > 2^n(1 - \frac{1}{n}). \tag{4.4}$$

There exists some positive constant c such that for every typical set N,

$$p\binom{n_2}{k} = \frac{(1000\log n)^{3/2}}{\sqrt{n}} \frac{\binom{n_2}{k}}{\binom{n/8}{k}} < c\frac{(\log n)^{3/2}}{\sqrt{n}} \tag{4.5}$$

and

$$
(4.6) \qquad pn_1\binom{n_2}{k-1} = \frac{(1000\log n)^{3/2}}{\sqrt{n}}\frac{kn_1}{n_2-k+1}\frac{\binom{n_2}{k}}{\binom{n/8}{k}} > 2\log n.
$$

(Here we used the inequalities for $(1-x)^y$ from Section 2.) Then (4.5) and (4.6) imply the following lower bound in (4.3). If $N \in \mathcal{N}$ then

$$
(4.7) \qquad \mathrm{Prob}(N \in N(\mathcal{F}(\mathcal{S}))) > 1 - c\frac{(\log n)^{3/2}}{\sqrt{n}}.
$$

Then (4.4) and (4.7) give that the expected size $E(N(\mathcal{F}(\mathcal{S})))$ fulfils (4.2). Hence there exists a family $\mathcal{S}$ satisfying (4.2). □

5. Problems, Remarks. It is a natural question how close $c(n)$ can be to 2^n. Obviously, $2^n - c(n) \geq 2^n/n$. Kostochka **[K]** proved that for every filter $\mathcal{F}$ one has $2^n - |N(\mathcal{F})| > 0.011 \cdot 2^n(\log n)^{3/2}/\sqrt{n}$. So the method presented in this note cannot give a better bound than Theorem 4.1.

Another possible direction for the further research is to extend the investigation to other (popular) posets. (Cf. **[GRS]**, [N], **[SW]**).

6. Acknowledgements. We wish to thank Ron Graham for directing our attention to the problem of Ko-Wei Lih. We are grateful to the I.M.A. for its hospitality during our stay. We also thank Martin Aigner and Rudolf Wille for organizing the meeting "Kombinatorik geordneter Mengen", April 24-30, 1988, at the Mathematisches Forschungsinstitut Oberwolfach, West Germany, where this paper was presented.

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