SIMPLICIAL POLYTOPES WITHOUT THE ISOTOPY PROPERTY

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Abstract. Using the recent counterexample by Jaggi and Mani-Levitska to the isotopy conjecture for uniform oriented matroids, we construct a class of simplicial polytopes whose realization spaces are disconnected.

1. Introduction. A celebrated classical theorem due to E. Steinitz states that any two combinatorially equivalent 3-polytopes P and Q can be connected modulo reflections by a path of polytopes within the same class [11],[8]. In this note we prove that Steinitz' isotopy result fails for simplicial polytopes in dimension 61 and higher.

We recall some basic definitions concerning the topology of polytopes and oriented matroids. See [2],[9],[12] for details and further references. A rank r oriented matroid χ on a set $E = \{1, \dots, n\}$ of *points* is an alternating map $\chi : E^r \to \{-, 0, +\}$ which satisfies a certain signed bases exchange axiom. For our purposes we may assume that χ is *realizable*, that is, there exist $x_1, \dots, x_n \in \mathbf{R}^r$ such that $\chi(e_1, \dots, e_r) = \det(x_{e_1}, \dots, x_{e_r})$ for all $(e_1, \dots, e_r) \in E^r$. The space of all such vector realizations of χ modulo the action of $GL(r, \mathbf{R})$ is denoted $\mathcal{R}(\chi)$ and called the *realization space* of χ. Using Plücker coordinates for the Grassmann manifold of r-flats in $\mathbf{R}^n$, $\mathcal{R}(\chi)$ embeds into a real projective space of dimension $\binom{n}{r} - 1$:

$$(1) \qquad \mathcal{R}(\chi) \quad = \quad \{ \, \xi \in (\wedge_r \mathbf{R}^n)/\mathbf{R} \;\; | \;\; \xi \text{ decomposable} \quad \text{and} \quad \operatorname{sign}(\xi) = \chi \, \}.$$

Suppose that χ is *uniform* (i.e. $\operatorname{Im}(\chi) \subset \{-, +\}$) and *totally acyclic*, that is, the positive hull $\mathcal{C}$ of $x_1, \dots, x_n \in \mathbf{R}^r$ is a pointed cone. Then the boundary complex of $\mathcal{C}$ is isomorphic to a simplicial $(r-2)$-sphere $\mathcal{S}$ on E which depends only on the oriented matroid χ [7]. Conversely, given any simplicial $(r-2)$-sphere $\mathcal{S}$ on E, we define its *convex realization space* by

$$(2) \qquad \mathcal{R}(\mathcal{S}) \quad := \quad \bigcup \{ \, \mathcal{R}(\chi) \;\; | \;\; \mathcal{S} \text{ is the boundary complex of } \chi \, \}.$$

The space $\mathcal{R}(\mathcal{S})$ is disconnected if and only if Steinitz' isotopy theorem fails for the corresponding $(r-1)$-polytope $\mathcal{C} \cap H$ where $H \subset \mathbf{R}^r$ is a suitable affine hyperplane.

Very recently, B. Jaggi and P. Mani-Levitska disproved the isotopy conjecture for simple line arrangements [6]. Under the well-known correspondence between realizable rank 3 oriented matroids and projective line arrangements [4],[10] their result reads

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THEOREM 1. [Jaggi & Mani-Levitska] *There exists a uniform rank 3 oriented matroid* χ^{JM} *on* $n = 17$ *points whose realization space* $\mathcal{R}(\chi^{JM})$ *is disconnected.*

The first counterexample to the isotopy conjecture for general (non-uniform) oriented matroids had been given by N. White [13], and from this we constructed a non-simplicial 44-polytope without the isotopy property [1]. It is the objective of the present note to settle the isotopy problem for the remaining open case of simplicial polytopes.

THEOREM 2. *Let* χ *be a rank 3 oriented matroid on* $E = \{1, \ldots, n\}$. *Then there exists a simplicial sphere* $\mathcal{S}_\chi$ *of dimension* $n + \binom{n}{3} - 5$ *with* $n + \binom{n}{3}$ *vertices such that if* $\mathcal{R}(\mathcal{S}_\chi)$ *is connected then* $\mathcal{R}(\chi)$ *is connected.*

COROLLARY 3. *There exists a simplicial 692-sphere* $\mathcal{S}_{\chi^{JM}}$ *with 697 vertices whose convex realization space* $\mathcal{R}(\mathcal{S}_{\chi^{JM}})$ *is disconnected.*

After completing the proof of Theorem 2, we will show that the number $\binom{n}{3} = O(n^3)$ can be replaced by the number $p_3(\chi)$ of triangular cells in the projective line arrangement associated with χ. By results of Shannon and Roudneff (see [10]), we have the sharp bounds $n \leq p_3 \leq n(n-1)/3$. Since the line arrangement χ^{JM} of Jaggi and Mani-Levitska has 48 triangular cells, we obtain a simplicial polytope without the isotopy property in dimension 61.

COROLLARY 4. *There exists a simplicial 60-sphere* $\mathcal{S}'$ *with 65 vertices whose convex realization space* $\mathcal{R}(\mathcal{S}')$ *is disconnected.*

2. Proof of the main result.

Let $\mathcal{S}$ be a simplicial $(r-2)$-sphere with vertex set E. We view $\mathcal{S}$ as an oriented manifold with *intrinsic orientation* $O_\mathcal{S}$. $O_\mathcal{S}$ is a map from the set of ordered facets of $\mathcal{S}$ into $\{-1, +1\}$. In the following we identify oriented matroids χ and oriented spheres $O_\mathcal{S}$ with their negatives $-\chi$ and $-O_\mathcal{S}$ respectively.

LEMMA 1. *Let* χ *be a rank* r *uniform oriented matroid on* E *with boundary complex* $\mathcal{S}$. *Then we have* $O_\mathcal{S}(F) = \chi(F, e)$ *for all facets* $F \subset E$ *and all* $e \in E \setminus F$.

See [12, Section 2.2] for a proof of Lemma 1. If χ is realizable then Lemma 1 translates into the following well known fact.

LEMMA 1'. *Let* P *be a simplicial* $(r-1)$*-polytope with vertices* $v_1, \ldots, v_n$. *For every facet* $F := \{v_{i_1}, \ldots, v_{i_{r-1}}\}$ *of* P *and every vertex* $v_j \notin F$ *the orientation of the* $(r-1)$*-simplex* $[v_{i_1}, \ldots, v_{i_{r-1}}, v_j]$ *equals the intrinsic* $(r-2)$*-dimensional orientation* $O_\mathcal{S}(F)$ *of* F *in the boundary complex* $\mathcal{S}$ *of* P.

Next we describe the construction of the sphere $\mathcal{S}_\chi$ promised by Theorem 2. Proceeding by lexicographical order on the triples (i, j, k) where $1 \leq i < j < k \leq n$, we define *principal extensions* $e_{ijk} := [i^-, j^-, k^-]$. Let χ_{ext} denote the resulting uniform rank 3 oriented matroid on $E' := E \cup \{e_{123}, e_{124}, \ldots, e_{n-2\,n-1\,n}\}$. See [2, Section 2] and [3, Section 5] for details on principal extensions of oriented matroids. This construction can be carried out geometrically as follows. Let $(x_1, \ldots, x_n) \in (\mathbf{R}^3)^n$ be any vector realization of χ. Define successively

$$(3) \qquad x_{ijk} \quad := \quad -x_i \; - \; \epsilon \cdot x_j \; - \; \delta \cdot x_k$$

where $\delta \ll \epsilon \ll 1$ are sufficiently small positive numbers. Then $(x_1, \ldots, x_n, x_{123}, x_{124}, \ldots, x_{n-2\,n-1\,n}) \in (\mathbf{R}^3)^{n+\binom{n}{3}}$ is a vector realization of χ_{ext}.

Let $\mathcal{S}_\chi$ be the boundary complex of the uniform oriented matroid χ^*_{ext}, the dual of χ_{ext}. In the language of Grünbaum [5]: $(x_1, \ldots, x_n, x_{123}, x_{124}, \ldots, x_{n-2\,n-1\,n})$ is a *Gale transform* of the simplicial polytope with boundary complex $\mathcal{S}_\chi$.

LEMMA 2. *Let* Ψ *be any rank* 3 *oriented matroid on* E' *such that its dual* Ψ^* *has boundary complex* $\mathcal{S}_\chi$. *Then* χ *equals the restriction* $\Psi|_E$ *of* Ψ *to* E.

Proof. Let $1 \le i < j < k \le n$. By the definition of e_{ijk}, $\{i, j, k, e_{ijk}\}$ is a positive circuit of χ_{ext} and hence $F_{ijk} := E \setminus \{i, j, k, e_{ijk}\}$ is a facet of $\mathcal{S}_\chi$. Using Lemma 1 we can write

$$\Psi(i,j,k) \;=\; \Psi^*(E \setminus \{i,j,k\}) \;=\; \pm\Psi^*(F_{ijk}, e_{ijk}) \;=\; \pm O_{\mathcal{S}_\chi}(F_{ijk})$$

where the sign factor "$\pm$" does not depend on the specific oriented matroid Ψ but only on (i, j, k). This implies $\Psi(i,j,k) = \chi_{ext}(i,j,k)$ and therefore $\Psi|_E(i,j,k) = \chi(i,j,k)$. □

Proof of Theorem 2. Via the *Hodge star* $* : \wedge_r \mathbf{R}^n \to \wedge_{n-r} \mathbf{R}^n$ applied to formula (1), every oriented matroid has the same realization space as its dual, and we can write

$$(4) \qquad \mathcal{R}(\mathcal{S}_\chi) \quad = \quad \bigcup \{\, \mathcal{R}(\Psi) \mid \mathcal{S}_\chi \text{ is the boundary complex of } \Psi^* \,\}.$$

(The simplicial polytopes realizing $\mathcal{S}_\chi$ are parameterized by their Gale transforms). Suppose that $\mathcal{R}(\mathcal{S}_\chi)$ is connected.

Given any two vector realizations $(x^0_1 \ldots, x^0_n)$ and $(x^1_1, \ldots, x^1_n)$ of χ, we will show that they can be connected by a path in $\mathcal{R}(\chi)$. Using (3), we find vectors $x^0_{ijk}, x^1_{ijk} \in \mathbf{R}^3$ such that $X^0 := (x^0_1, \ldots, x^0_n, x^0_{123}, x^0_{124}, \ldots, x^0_{n-2\,n-1\,n})$ and $X^1 := (x^1_1, \ldots, x^1_n, x^1_{123}, x^1_{124}, \ldots, x^1_{n-2\,n-1\,n})$ are realizations of χ_{ext}. X^0 and X^1 are in $\mathcal{R}(\mathcal{S}_\chi)$ by (4). Since $\mathcal{R}(\mathcal{S}_\chi)$ is connected, there exists a path

$$X : [0,1] \to \mathcal{R}(\mathcal{S}_\chi), \quad t \mapsto X^t = (x^t_1, \ldots, x^t_n, x^t_{123}, x^t_{124}, \ldots, x^t_{n-2\,n-1\,n})$$

connecting X^0 and X^1. By Lemma 2 we have $(x^t_1, \ldots, x^t_n) \in \mathcal{R}(\chi)$ for all $t \in [0,1]$. Hence $\mathcal{R}(\chi)$ is connected, and the proofs of Theorem 2 and Corollary 3 are complete. □

Proof of Corollary 4. We modify our above argument by defining χ_{ext} through successive principal extensions $e_{ijk} := [i^-, j^-, k^-]$ only for those triples which correspond to triangles in the line arrangement of χ. Let $\mathcal{S}'$ be the boundary complex of χ^*_{ext}. By the results of [10], the triangular cells of χ are in one-to-one correspondence with the local changes or *mutations* of χ. Given any realization $X_0 \in \mathcal{R}(\mathcal{S}')$ of χ_{ext}, let X_1 be in the same connected component of $\mathcal{R}(\mathcal{S}')$. Under deletion of the new points e_{ijk} both X_0 and X_1 are mapped into $\mathcal{R}(\chi)$, and, moreover, they are mapped into the same connected component of $\mathcal{R}(\chi)$. This proves that in Theorem 2 the number $\binom{n}{3}$ may be replaced by the number $p_3(\chi)$ of mutations of χ, and Corollary 4 follows. □

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