SPHERE COVERINGS OF THE HYPERCUBE WITH INCOMPARABLE CENTERS

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Sphere coverings of the hypercube with incomparable centers

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Abstract. It is shown that the shadow of a Sperner family can cover 10 percent of the Boolean algebra. Whether this can be improved to $(100 - o(1))\%$ remains open.

1. SHADOWS OF SPERNER FAMILIES

Let $[n]$ denote the set of the first n integers, $2^{[n]}$ its power set. The collection of all k-subsets of a set S is denoted by $\binom{S}{k}$. Let $\mathcal{F}$ be a subfamily of $2^{[n]}$. The *neighborhood* of $\mathcal{F}$, $N(\mathcal{F})$, is defined as the family of sets in $[n]$ whose Hamming distance is exactly 1 from $\mathcal{F}$, i.e., $N(\mathcal{F}) = \{N \subset [n] : N \notin \mathcal{F}$ and there exists an $F \in \mathcal{F}$ such that $|\ N \ \triangle \ F\ | = 1\}$. (If we identify the subsets of $[n]$ with the vertices of the n-dimensional unit-cube, then $N(\mathcal{F})$ is the usual neighborhood in the graph Q^n.) The *shadow* of $\mathcal{F}$, $\partial\mathcal{F}$, consists of those members of $N(\mathcal{F})$ which are covered by a member of $\mathcal{F}$, i.e., $\partial\mathcal{F} = \{S : S \notin \mathcal{F}$ and there exists an $F \in \mathcal{F}$ such that $S \subset F$, $|F \setminus S| = 1\}$.

The family $\mathcal{F}$ is a *Sperner family* if no two of its members contain each other. One of the oldest results in the theory of finite sets states that the size of the largest Sperner family is $\binom{n}{\lfloor n/2 \rfloor}$ and the extremal family consists of all members of $2^{[n]}$ of size either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ (Sperner [Sp]). The size of the shadow of such a family is again a binomial coefficient, so it is not more than $\binom{n}{\lfloor n/2 \rfloor}$. Engel [E] and independently Zuev [Z] conjectured that there exists a positive real C such that

$$|\partial\mathcal{F}| < C\binom{n}{n/2} < C'\frac{2^n}{\sqrt{n}} \tag{1.1}$$

holds for every Sperner family $\mathcal{F}$. This was disproved by Kospanov [Ko] who showed that

$$\max |\partial\mathcal{F}| > cn^{-1/6}2^n.$$

Griggs [G] also also constructed a family whose shadow was larger than $\log n\binom{n}{n/2}$. The aim of this note is to prove

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THEOREM. *There exists a Sperner family $\mathcal{S}$ over n elements such that $|\partial\mathcal{S}| > 0.1 \cdot 2^n$ (for all $n > n_0$).*

CONJECTURE. *There exists a $c < 1$ such that $|\partial\mathcal{S}| < c2^n$ holds for every Sperner family $\mathcal{S}$.*

A theorem of Kostochka [Kos] implies that

$$|\partial\mathcal{S}| < (1 - \frac{(\log n)^{3/2}}{100\sqrt{n}})2^n,$$

which is the best upper bound we know.

2. The Random Construction

We use a random construction. The problem of finding an explicit construction giving a similar bound remains open. Let t be an integer, $t = (1 + o(1))\sqrt{n/2}$, and denote $\lfloor (n-t)/2 \rfloor$ by s. Then the size of the middle t levels of the Boolean lattice is

$$(2.1) \qquad \sum_{a=s+1}^{s+t} \binom{n}{a} = (1 + o(1))2^n(\Phi(\frac{1}{\sqrt{2}}) - \Phi(-\frac{1}{\sqrt{2}})) = (1 + o(1))0.520\ldots \cdot 2^n.$$

Here $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$, as usual. Let k be an integer, $k = (1+o(1))\sqrt{n/2}$. We are going to define disjoint random families $\mathcal{K}(1), \ldots, \mathcal{K}(t)$ of k-sets. define p by the equation

$$tp\binom{s+t}{k} = c.$$

For every $K \in \binom{[n]}{k}$ let ξ_K be a random variable with

$$\text{Prob}(\xi_K = 0) = 1 - tp$$
$$\text{Prob}(\xi_K = i) = p$$

for $i = 1, \ldots, t$. These random variables are to be chosen totally independently. Let $\mathcal{K}(i)$ be the random family defined by $\mathcal{K}(i) = \{K \in \binom{[n]}{k} : \xi_K = i\}$. Finally, we define the family $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_t$ where $\mathcal{S}_i$ is the family of those $s+i+1$-element sets which contain a member of $\mathcal{K}(i)$ but do not contain any members of $\mathcal{K}(j)$ with $1 \le j < i$. Obviously, $\mathcal{S}$ is a Sperner family.

We next show that the expected size of the shadow of $\mathcal{S}$ is greater than $0.1 \cdot 2^n$ (if $n > n_0$). This implies the existence of a Sperner family with such a large shadow. To prove this we show that every a-element set A belongs to $\partial\mathcal{S}$ with a probability at least 0.2 if $s+1 \le a \le s+t$ and $A \subset [n]$, and then we use (2.1). For a family $\mathcal{F}$ and a set A

we use the notation $\mathcal{F}_A$ for the induced subfamily, i.e., $\mathcal{F}_A = \{F \in \mathcal{F} : F \subset A\}$. Let $\mathcal{K}([i])$ denote $\mathcal{K}(1) \cup \cdots \cup \mathcal{K}(i)$.

(2.2)

$$\begin{aligned}
&\operatorname{Prob}(A \in \partial\mathcal{S}_i)\\
&\geqq \operatorname{Prob}(\mathcal{K}([i])_A = \emptyset) \operatorname{Prob}(\exists x : A \cup \{x\} \in \mathcal{S}_i | \mathcal{K}([i])_A = \emptyset)\\
&= \operatorname{Prob}(\mathcal{K}([i])_A = \emptyset)(1 - \operatorname{Prob}(\forall x \in [n] \setminus A : A \cup \{x\} \notin \mathcal{S}_i | \mathcal{K}([i])_A = \emptyset))\\
&= \operatorname{Prob}(\mathcal{K}([i])_A = \emptyset)(1 - (\operatorname{Prob}(A \cup \{x\} \notin \mathcal{S}_i | \mathcal{K}([i])_A = \emptyset))^{n-a})\\
&= \operatorname{Prob}(\mathcal{K}([i])_A = \emptyset)(1 - (1 - \operatorname{Prob}(A \cup \{x\} \in \mathcal{S}_i | \mathcal{K}([i])_A = \emptyset))^{n-a})\\
&\geqq \operatorname{Prob}(\mathcal{K}([i])_A = \emptyset)(1 - \exp[-\operatorname{Prob}(A \cup \{x\} \in \mathcal{S}_i | \mathcal{K}([i])_A = \emptyset)(n-a)])
\end{aligned}$$

Here we used the inequality $(1-x)^y \leqq \exp[-xy]$ which holds for all reals $x \leqq 1$ and $y \geqq 0$. We estimate separately the two probabilities in the last line of (2.2).

$$\begin{aligned}
\operatorname{Prob}(\mathcal{K}([i])_A = \emptyset) = (1 - ip)^{\binom{a}{k}} \geq (1 - tp)^{\binom{s+t}{k}} &= (1 + o(1)) \exp[-tp\binom{s+t}{k}]\\
&= (1 + o(1)) \exp[-c]. \qquad (2.3)
\end{aligned}$$

Moreover

$$\begin{aligned}
\operatorname{Prob}(A \cup \{x\} \in \mathcal{S}_i | \mathcal{K}([i])_A = \emptyset) &= (1 - (i-1)p)^{\binom{a}{k-1}} - (1 - ip)^{\binom{a}{k-1}}\\
&\geqq p\binom{a}{k-1}(1 - ip)^{\binom{a}{k-1}} \qquad (2.4)
\end{aligned}$$

Here the last factor is $1 - o(1)$, because

$$(1 - ip)^{\binom{a}{k-1}} \geq 1 - ip\binom{a}{k-1} = 1 - ip\binom{a}{k}\frac{k}{a-k+1} \geq 1 - \frac{ck}{a-k+1}.$$

Moreover we have (see, e.g., in [**R**, p.151]) that

$$\binom{a}{k} \geq \binom{s+t}{k} \exp[-tk/s](1 - o(1)). \qquad (2.5)$$

Applying this to (2.4), we obtain

$$\operatorname{Prob}(A \cup \{x\} \in \mathcal{S}_i | \mathcal{K}([i])_A = \emptyset) \geq p\binom{a}{k-1}(1 - o(1)) =$$

$$\frac{1 - o(1)}{a-k+1} kp\binom{a}{k} \geq \frac{1 - o(1)}{a-k+1} kp\binom{s+t}{k} \exp[-1 + o(1)] = (1 + o(1))\frac{c}{es}.$$

Using this result in (2.2) we obtain

$$\operatorname{Prob}(A \in \partial\mathcal{S}_i) \geq (1 - o(1)) \exp[-c](1 - \exp[-(n-a)\frac{c}{es}]) = (1 - o(1)) \exp[-c](1 - \exp[-\frac{2c}{e}]) > 0.2003\ldots$$

Remark. See also [**Kos**] for a similar, though simpler, construction.

3. The Complexity of The Boolean Functions

The minimum number of conjuctions. Let $f(\mathbf{x})$ be a Boolean function of n variables, $f(x_1,\ldots,x_n) : \{0,1\}^n \to \{0,1\}$. Let $d(f)$ be the smallest integer d such that one can write f in a disjunctive normal form of d conjuctions, i.e., $d(f) =: \min\{d : \exists K_1 \ldots K_d$ such that $f(\mathbf{x}) = K_1 \vee \cdots \vee K_d\}$, where every term K has the form

$$K = x_{i_1}^{\varepsilon_1} \ldots x_{i_r}^{\varepsilon_r} \qquad \text{where} \qquad x^{\varepsilon} = \begin{cases} x \text{ if } \varepsilon = 1, \\ \bar{x} \text{ if } \varepsilon = -1. \end{cases}$$

Korshunov [K1] proved that there are positive reals c_1 and c_2 such that

$$c_1 \frac{2^n}{\log n \log\log n} < d(f) < c_2 \frac{2^n}{\log n \log\log n} \tag{3.1}$$

holds for almost all Boolean function f. Sapozhenko [S2] gave a simple algorithm wich provides a disjunctive normal form of length $c2^n/\log n$ for almost all Boolean function.

They also investigated the length of the longest irreducible normal form of f. A disjunctive normal form of the Boolean function f is called *irreducible* if by removal of a conjunction or of a letter one obtains a disjunctive normal form which does not generate f. Let $d_{\max}(f)$ denote the maximum number of conjunctions among all irreducible disjunctive normal forms which generate f. Sapozhenko [S1] proved that $d_{\max}(f) \sim 2^{n-1}$ for almost all f. For a short proof see Korshunov [K2].

Representations by systems of linear inequalities. In [BJ] and [J] Balas and Jeroslow introduced the following notion. Let Z be a subset of $\{0,1\}^n$, i.e., a finite point set in $\mathbb{R}^n$. Then let $\ell(Z)$ denote the minimum number ℓ of linear inequalities

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad \text{where} \qquad i = 1,\ldots,\ell \tag{3.2}$$

such that the set of all $0-1$ solutions of (3.2) is exactly Z. If we identify the Boolean function f by its zero set, then this definition can be extended, i.e., let $Z(f) =: \{\mathbf{x} : f(\mathbf{x}) = 0\}$ and set $\ell(f) = \ell(Z(f))$. Denote by Q^n the graph of the *n-dimensional cube* , i.e., the vertex set of Q^n consists of all the (0,1)-vectors of length n, and two vectors $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ are adjacent if they differ from each other in exactly one component. For a graph $\mathcal{G}$ we denote the number of connected components by $c(\mathcal{G})$. Let $\bar{Z}$ denote the complement of Z in $\{0,1\}^n$. Then it is easy to see ([J],[HIP]) that

$$c(Q^n_{\bar{Z}}) \le \ell(Z) \le 2^{n-1},$$

and that ([Z])

$$\ell(f) \le d(f).$$

An asymptotic formula, analogous to (3.1), is not known for $\ell(f)$. It is possible, for example, that $\ell(f)=1$ while $d(f)=\binom{n}{\lfloor n/2\rfloor}$. Zuev [Z] proved that for almost all Boolean function f

$$\ell(f) \geqq \frac{2^n}{n^2}$$

holds.

Monotone Boolean functions. A subset $Z \subset \{0,1\}^n$ is called *monotone* if $\mathbf{x} \in Z$ and $\mathbf{x} \leq \mathbf{y}$ imply $\mathbf{y} \in Z$. A Boolean function φ is monotone if $Z(\varphi)$ is monotone. Hammer, Ibaraki and Peled [HIP] proved that

$$\frac{1}{n}\binom{n}{\lfloor n/2\rfloor} \leq \max_{\varphi} \ell(\varphi) \leq \binom{n}{\lfloor n/2\rfloor}, \tag{3.3}$$

where φ runs over monotone functions. This was improved by Zuev [Z]

$$\ell(\varphi) \leq N(n)\frac{1+\log n}{n} + 1, \tag{3.4}$$

where $N(n)$ denotes the maximum size of the neighborhood of a Sperner family in $2^{[n]}$. (Actually, his proof was not completely clear for the authors of this paper.) Then (3.4) implies that $\ell(\varphi) \leq \frac{c \log n}{n} 2^n$ holds for all monotone φ. He conjectures that the true order of the magnitude of $\max_\varphi \ell(\varphi)$ is given by the lower bound in (3.3).

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