A MODEL OF DISLOCATIONS AND THE ASSOCIATED FREE BOUNDARY PROBLEM

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IMA Preprint Series # 391

January 1988

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§1 The dislocation problem.

Dislocations occur during plastic deformation of metals and they move through the material causing plastic strain. This phenomenon has been studied by various authors; we refer to [5] [7] [8] [9] and the references given there. Recently, Head, Howison, Ockendon, Titchener and Wilmott [6] have proposed, in the two-dimensional case, a theory describing the motion of dislocations along parallel lines (to the x-axis). Their model is based on linear elasticity but with an inhomogeneous term which depends on the dislocation density $\omega(x, y); \omega > 0$.

The stress at any point (x, y) due to the dislocations is $\nabla\phi$ where

$$\phi(x,y) = \frac{1}{2\pi} \iint_{\Omega_0} \omega(\xi,\eta) \log\{(x-\xi)^2 + (y-\eta)^2\}^{1/2} d\xi\, d\eta \tag{1.1}$$

and where Ω_0 is the region occupied by the dislocations. Let us denote by (σ, σ_1) the externally imposed stress field. Then the velocity of the dislocations at any point (x, y) is given by the vector $(\sigma + \phi_x, 0)$. In the stationary case we then have

$$\sigma + \phi_x = 0 \quad \text{in} \quad \Omega_0 \,. \tag{1.2}$$

From (1.1) we also get

$$\Delta\phi = \omega \quad \text{in} \quad \Omega_0 \tag{1.3}$$

and

$$\Delta\phi = 0 \quad \text{in} \quad \Omega_1 \setminus \Omega_0 \tag{1.4}$$

where Ω_1 is the region occupied by the metal. It is further assumed in [6] that

$$\phi \quad \text{and } \nabla\phi \quad \text{are continuous across } \partial\Omega_0 \tag{1.5}$$

*This work was partially supported by National Science Foundation Grants MCS 79–15171 and DMS-86-12880

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and that the total number of dislocations along any line of motion $\Omega_{0,y} = \{x; (x,y) \in \Omega_0\}$ is a given function:

(1.6)
$$\int_{\Omega_{0,y}} \omega(x,y)\, dx = 2G(y)\ .$$

In case of screw dislocations σ is a harmonic function, i.e.,

(1.7)
$$\Delta\sigma = 0 \quad \text{in } \Omega_1\ .$$

The model (1.1) – (1.7) was introduced in [6]. In the special case where $\omega =$ const, $\sigma = -Ax$ ($A > 0$) and $\Omega_1 = \mathbb{R}^2$, explicit solution was constructed in [6] with

$$\phi \sim \frac{N}{2\pi} \log r \quad \text{as} \quad r^2 = x^2 + y^2 \to \infty$$

(N positive constant); $\partial\Omega_0$ is then an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ .$$

Note that (1.1)–(1.6) represent a free boundary problem with the free boundary $\partial\Omega_0 \cap \Omega_1$.

In the present paper we shall assume that Ω_1 is a rectangle

$$\Omega_1 = \{(x,y);\ -a < x < a,\ 0 < y < b\}.$$

Let us prescribe, for definiteness, boundary conditions

(1.8)
$$\begin{aligned} &\phi(x,0),\ \phi(x,b) \quad \text{for} \quad -a \le x \le a, \\ &\text{and} \quad \phi_x(-a,y),\ \phi_x(a,y) \quad \text{for} \quad -b < y < b \end{aligned}$$

and seek a solution ϕ which is symmetric in x and for which the interval $\{(0,y), 0 < y < b\}$ lies in Ω_0; the results of this paper can be extended to other boundary conditions. We can then restrict our attention to the region $\Omega_1 \cap \{x \ge 0\}$.

We shall assume from now on that

(1.9)
$$\begin{aligned} &G \in C^\alpha[0,b] \cap C^{m+\alpha}(0,b) \qquad \text{some integer } m \ge 0 \text{ and } 0 < \alpha < 1, \\ &G(y) > 0 \quad \text{for} \quad 0 \le y \le b. \end{aligned}$$

In §2 we shall prove that the free boundary problem can be reduced to a quasi– variational inequality (*q.v.i.*)

In §3 we consider truncated versions of the *q.v.i.* and establish the existence of a solution with continuous free boundary. In §4 we take the limit of the truncations and obtain a solution of the original *q.v.i.*, thereby solving the dislocation problem; the free boundary is given by $x = f(y)$ with $f(y)$ continuous for $0 \le y \le b$ and positive for $0 < y < b$. In §5 we prove that $f \in C^{m+1+\alpha}(0,b)$.

We finally remark that C. Baiocchi (oral communication) has a method of transforming the dislocation problem into a variational inequality of the fourth order provided $\phi_x \ge \sigma$; it is however not known under what conditions this inequality is valid. (The solution whose existence is established in this paper satisfies $\phi_x + \sigma \le 0$.)

§2. Reduction to q.v.i.

Set

$$E = \{(x,y);\ 0 < x < a,\ 0 < y < b\}.$$

We anticipate the dislocation region to be given by

$$\Omega = \{(x,y);\ 0 \le x \le f(y),\ 0 \le y \le b\}$$

where

$$\Gamma\ :\ \{x = f(y)\}$$

is the free boundary. Setting

$$\Sigma\,(x,y) = \int_0^x \sigma(\xi,y)\,d\xi \tag{2.1}$$

and recalling that $\phi_x + \sigma = 0$ in Ω_0, we then have

$$\phi(x,y) - \phi(0,y) = -\Sigma\,(x,y) \quad \text{in} \quad \Omega\,. \tag{2.2}$$

From the relations

$$(\Delta\phi)_x = -\Delta\sigma = 0 \quad \text{in} \quad \Omega$$

we deduce that $\Delta\phi$ is independent of x. This fact together with the relation

$$\int_{\Omega_y} (\Delta\phi)(x,y)\,dx = G(y)$$

where

$$\Omega_y = \{x;\ (x,y) \in \Omega\},$$

which follows from (1.3), (1.6), give

$$\Delta\phi(x,y) = \frac{G(y)}{f(y)} \quad \text{in} \quad \Omega\,. \tag{2.3}$$

Introduce a function $w_0(y)$ by

$$\begin{aligned} w_0''(y) &= \frac{G(y)}{f(y)}\ , \\ w_0(0) = \sigma(0,0), &\quad w_0(b) = \sigma(0,b)\ , \end{aligned} \tag{2.4}$$

and set

$$v = w_0 - \phi \; . \tag{2.5}$$

Then, by (2.3), (2.4),

$$\begin{aligned} -v_{yy}(0,y) = \phi_{yy}(0,y) - w_0''(y) &= \Delta\phi(0,y) - \phi_{xx}(0,y) - w_0(y) \\ &= -\phi_{xx}(0,y) = \sigma_x(0,y). \end{aligned}$$

Hence

$$v(0,y) = v_0(y) \tag{2.6}$$

where v_0 is defined by

$$v_0(y) = -\int_0^y \int_0^{y'} \sigma_x(0,\eta)\,d\eta\,dy' + c_1 y, \quad v_0(b) = 0 \tag{2.7}$$

where c_1 is an appropriate constant.

With v_0 thus defined in terms of σ, we introduce the function

$$\psi(x,y) = v_0(y) + \Sigma\,(x,y) \; . \tag{2.8}$$

One can easily check that, since $\Delta\sigma = 0$,

$$\Delta\psi = 0 \quad \text{in} \quad E \; . \tag{2.9}$$

From (2.3), (2.4), (2.5) we get

$$\Delta v = 0 \quad \text{in} \quad \Omega \; ; \tag{2.10}$$

also, by (1.4), (2.4), (2.5),

$$\Delta v = w_0'' = \frac{G}{f} \quad \text{in} \quad E \backslash \Omega \; . \tag{2.11}$$

Next, by (2.2), (2.5), (2.8)

$$v(x,y) = \psi(x,y) \quad \text{in} \quad \Omega \; . \tag{2.12}$$

Finally we hope that

$$v(x,y) > \psi(x,y) \quad \text{in} \quad E \backslash \Omega \; , \tag{2.13}$$

for, in that case, v will become a solution of the variational inequality

$$(2.14)\qquad \begin{aligned} &-\Delta v \geq -\frac{G}{f}\,, \quad v \geq \psi\,, \\ &(\Delta v - \frac{G}{f})\,(v-\psi) = 0 \quad \text{a.e. in} \quad E \end{aligned}$$

with the appropriate boundary conditions. Setting

$$(2.15)\qquad u = v - \psi$$

and recalling (2.8), (2.9), we obtain for u the variational inequality

$$(2.16)\qquad \begin{aligned} &-\Delta u \geq \frac{G}{f}\,,\ u \geq 0, \\ &(\Delta u - \frac{G}{f})\,u = 0 \quad \text{a.e. in} \quad E \end{aligned}$$

with boundary conditions

$$(2.17)\qquad \begin{aligned} &u(x,0) = k_0(x),\ u(x,b) = k_1(x) \qquad (0 < x < a)\,, \\ &u(0,y) = 0,\ u_x(a,y) = \ell(y) \qquad (0 < y < b) \end{aligned}$$

and

$$(2.18)\qquad \{u > 0\} \cap E = \{(x,y) \in E\,; \quad f(y) < x < a,\ 0 < y < b\}\,,$$

where

$$(2.19)\qquad \begin{aligned} &k_0(x) = \sigma(0,0) - \int_0^x \sigma(\xi,0)\,d\xi - \phi(x,0), \\ &k_1(x) = \sigma(0,b) - \int_0^x \sigma(\xi,b)\,d\xi - \phi(x,b), \\ &\ell(y) = -\sigma(a,y) - \phi_x(a,y)\,. \end{aligned}$$

In the sequel we assume that $k_i \in C^{2+\beta}[0,a], \ell \in C^{2+\beta}[0,b]$ for some $0 < \beta < 1$ and that

$$(2.20)\qquad \begin{aligned} &k_i(0) = k_i'(0) = 0, k_i(a) > 0, \\ &k_i(x) \geq 0, k_i'(x) \geq 0 \quad \text{if } 0 < x < a, \\ &\ell(y) > 0 \quad \text{if} \quad 0 \leq y \leq b, \quad \text{and} \\ &k_0'(a) = \ell(0), \quad k_1'(a) = \ell(b). \end{aligned}$$

Notice that the main restrictions on k_i, ℓ are

$$k_i'(x) \geq 0, \quad \ell(y) > 0,$$

which amount to the assumptions

$$\begin{aligned} \phi_x + \sigma \leq 0 \qquad & \text{on} \qquad \{y = 0\} \cup \{y = b\} \ ; \\ \phi_x + \sigma < 0 \qquad & \text{on} \qquad \{x = a\} \ . \end{aligned}$$

These inequalities hold, for instance, for the special solution derived in [6].

We summarize:

LEMMA 2.1. *If (u, f) is a solution of (2.16)–(2.18) then the corresponding ϕ defined by (2.5), (2.4) is a solution of the dislocation problem in E, and the dislocations occupy the region $\{(x,y);\ 0 < x < f(y),\ 0 < y < b\}$.*

For technical reasons it will be convenient to work first with the truncated problem in which (2.16) is replaced by

$$\begin{aligned} & -\Delta u \geq -\frac{G}{\epsilon + f}\,, \quad u \geq 0, \\ & (\Delta u - \frac{G}{\epsilon + f}\,)\, u = 0 \quad \text{a.e. in} \quad E. \end{aligned} \tag{2.21}$$

Set

$$\mathcal{A} = \{f \in L^\infty(0, b), \quad 0 \leq f(y) \leq a\}$$

For any $f \in \mathcal{A}$ we consider the variational inequality (2.21), (2.17) and denote its solution by u.

LEMMA 2.2. *The solution u satisfies*

$$u_x \geq 0 \tag{2.22}$$

and, consequently,

$$\{u > 0\} = \{(x,y) \in E, \quad g(y) < x < a, \quad 0 \leq y \leq b\} \tag{2.23}$$

for some function $g(y)$ in $\mathcal{A}$.

Proof. Since Δu is bounded in E and the boundary condition are consistent at the corners of $\partial E, u$ belongs to $W^{2,p}(E)$ for any $1 < p < \infty$ and, in particular, u_x is continuous in $\overline{E}$. Observe also that $u_x \geq 0$ on $\partial E \backslash \{x = 0\}$ as well as on $\{x = 0\}$ (since $u(0, y) = 0,\ u \geq 0$). Since u_x is harmonic in $\{u > 0\}$ we can apply the maximum principle to deduce that $u_x > 0$ in $\{u > 0\}$. Thus (2.22) holds, and then also (2.23) is valid.

REMARK 2.1. Since $u_x(a,y) > 0$ and $u_x \in C(\overline{E}), u_x(g(y)-0,y) = 0$, we must have

$$g(y) < b \quad \text{for all} \quad 0 \le y \le b \,. \tag{2.24}$$

Define a mapping T by

$$g = Tf \,.$$

If T has a fixed point f and if f is continuous, then it provides a solution of (2.21), (2.17), (2.18).

Since T is defined by means of a variational inequality, the problem (2.21), (2.17), (2.18) is referred to as a *quasi variational inequality* (*q.v.i*); the same applies to (2.16)–(2.18).

§3. Existence for the *q.v.i.* (2.21), (2.17), (2.18).

It is not clear whether the mapping T is continuous. T is also not a monotone increasing mapping; in fact, if $f_1 \ge f_2$ then $Tf_1 \le Tf_2$. Thus there is no fixed-point theorem available for establishing the existence of a solution of $Tf = f$. We shall therefore proceed somewhat indirectly, by considering a family of approximating problems whereby the free boundary is approximated by a polygonal curve. This method is due to Friedman and Jensen [4] who used it for another type of *q.v.i.* Our goal is to prove:

THEOREM 3.1. *The q.v.i. (2.21), (2.17), (2.18) has a solution (u,f) with $f(y)$ continuous for $0 \le y \le b$.*

We shall need several lemmas.

LEMMA 3.2. *Let U be a solution of the variational inequality*

$$-\Delta U \ge -F, \; U \ge 0, \; (\Delta U - F)U = 0 \quad \text{a.e. in} \quad E \,,$$

with $\{U > 0\} = \{g(y) < x \le a, \; 0 \le y \le b\}$ and let $F \in L^\infty, F \ge \lambda > 0, \quad |U_y| \le N$. Then for any points $0 < y' < \overline{y} < y'' < b$ there holds:

$$g(\overline{y}) \ge \min \,\{g(y'), g(y'')\} - M \mid y'' - y' \mid^{1/2} \tag{3.1}$$

where M is a positive constant depending only on λ, N and b.

This is a well known result which is proved by comparing U in $\{0 < x < \min\{g(y'), g(y'')\}, y' < y < y''\}$ with a function of the form $U_0 = [(C - C_1x)^+]^2$ with suitable constants C, C_1; cf. also the proof of Lemma 4.4.

LEMMA 3.3. *The function $g(y)$ defined in Lemma 3.2 is continuous for $0 \le y \le b$.*

Proof. From Lemma 3.2 we know that $g(y)$ cannot oscillate near any point $y = y_0$, i.e., $g(y_0+0)$ and $g(y_0-0)$ exist. Thus it remains to show that $g(y_0+0) = g(y_0-0)$ if $0 < y_0 < b$. Suppose this is not true and take, for definiteness,

$$g(y_0+0) > g(y_0-0) \,.$$

Let

$$\sigma = \{(x, y_0)\ ;\ g\ (y_0 - 0) < x < g\ (y_0 + 0)\}\ .$$

The function u_x is positive in $\{u > 0\}$ (by Lemma 2.2) and $u_x = 0$ on σ. Since u_x is harmonic in $\{u > 0\}$, the strong maximum principle gives

$$u_{xy} > 0 \quad \text{on} \quad \sigma\ ,$$

i.e., $u_y(x, y_0)$ is strictly increasing along σ. This is impossible since $\nabla u = 0$ on $\partial\{u > 0\} \cap E$.

For any positive integer n introduce points $y_j = jb/n \quad (j = 0, 1, \dots, n)$ and the admissible subclass $\mathcal{A}_n$ of $\mathcal{A}$:

$$\mathcal{A}_n = \{f_n(y) \quad \text{in} \quad \mathcal{A}\ ; \quad (f_n(y), y) \quad \text{is a polygonal curve with vertices } (f_n(y_j), y_j)\}.$$

For any $f_n \in \mathcal{A}_n$ we consider the variational inequality

$$-\Delta u \geq -\frac{G}{\epsilon + f_n}\ , u \geq 0,\ (\Delta u - \frac{G}{\epsilon + f_n})\ u \geq 0 \quad \text{a.e. in} \quad E$$

with the boundary conditions (2.17), and denote the solution by u_n and its free boundary by $\{x = g_n(y)\}$. We introduce the polygonal curve $(h_n(y), y)$ with vertices $(g_n(y_j), y_j)$ and set

$$h_n = T_n\ f_n\ ;$$

thus T_n maps $\mathcal{A}_n$ into itself.

Notice that, for any j,

(3.2) $$u_n(x, y_j) > 0 \quad \text{if and only if} \quad h_n(y_j) < x \leq a\ .$$

LEMMA 3.4. *T_n is a continuous mapping in the $L^\infty(0, b)$-norm.*

This means that

$$\text{if } \|f_{n,i} - f_n\|_{L^\infty(0,b)} \to 0 \quad \text{then} \quad \|h_{n,i} - h_n\|_{L^\infty(0,b)} \to 0$$

where $h_{n,i} = T_n\ f_{n,i}\ , \quad h_n = T_n\ f_n$.

Proof. The proof is similar to the proof of Lemma 2.3 in [4]. Dropping the index n and noting that the corresponding solution $u_i = u_{n,i}$ is uniformly convergent in E to $u = u_n$, one easily verifies that

(3.3) $$g(y) \geq \varliminf_{i \to \infty}\ g_i(y)\ ;$$

therefore also

$$h(y) \geq \varliminf_{i\to\infty} h_i(y) \ . \tag{3.4}$$

Next,

$$0 = \int_{\{u=0\}} \Delta u \leftarrow \int_{\{u=0\}} \Delta u_i \geq \frac{\min G}{\epsilon + a} \int_{\{u=0\}\setminus\{u_i=0\}} 1$$

that is

$$\text{meas } \{g_i < g\} \to 0 \quad \text{ if } \quad i \to \infty \ ,$$

which together with (3.3) imply that

$$g_i \to g \quad \text{in measure} \ . \tag{3.5}$$

Suppose now that the lemma is not true. Then in view of (3.4) there must exist a point y_0 of the form $j\ b/n$ such that

$$h(y_0) > \varlimsup_{i\to\infty} \ h_i(y_0) + 2c$$

for some $c > 0$; hence also

$$g(y_0) > \varlimsup_{i\to\infty} \ g_i(y_0) + 2c \ .$$

Recalling (3.5) and using also the continuity of g we deduce that for any $\delta > 0$ there exist points $y_1 < y_0 < y_2$ with $\mid y_2 - y_1 \mid < \delta$ such that

$$\min \{g_i(y_1) - g_i(y_0),\ g_i(y_2) - g_i(y_0)\} > c$$

for some i large enough. But this is a contradiction to the inequality (3.1) for g_i.

Having proved Lemma 3.4 we can now use the Brouwer fixed point theorem to deduce that there exists an $f_n \in \mathcal{A}_n$ such that $T_n\ f_n = f_n$. A slight modification of the proof of Lemma 3.2 (cf. [4: p.60]) yields:

$$f_n(\overline{y}) \geq \min\{f_n(y'), f_n(y''\} - M \mid y'' - y' \mid^{1/2} \ . \tag{3.6}$$

We shall denote the solution of the variational inequality (2.22), (2.17) corresponding to $f = f_n$ by u_n, and its free boundary by $\{x = g_n(y)\}$. Then

$$g_n(y_i) = f_n(y_i) \qquad \text{for all } y_i = \frac{i\ b}{n} \tag{3.7}$$

where $i = 0, 1, \ldots, n$.

By Lemma 3.3 of [4], a uniformly bounded and weakly convergent sequence (in $L^p(0,b)$) of functions f_n which satisfy (3.6) for all $0 \le y' < \overline{y} < y'' \le b$ has a subsequence which is convergent in measure. Thus we may assume that the sequence f_n satisfies:

$$f_n \to f \qquad \text{in measure .}$$

It follows that for a subsequence, denoted again by f_n,

$$(3.8) \qquad \begin{aligned} &f_n(x) \to f(x) \qquad \text{for all } x \in S = [0,b]\backslash N, \\ &\text{where meas } (N) = 0 \end{aligned}$$

and S contains all the points of the form $ib/m \quad (i = 0, 1, \ldots, m;\ m = 1, 2, \ldots)$. Further, by (3.6),

$$f(\overline{y}) \ge \min\ \{f(y'), f(y'')\} - M \mid y'' - y' \mid \qquad \text{if} \quad y', y'', \overline{y} \in S\ , \quad y' < \overline{y} < y''\ ,$$

so that f restricted to S is right continuous and left continuous.

We may also assume that

$$u_n \to u \qquad \text{uniformly in } E\ .$$

Then u is the solution of the variational inequality (2.22), (2.17) with the same f as in (3.7). Denote by $\{x = g(y)\}$ the free boundary of u; g is continuous (by Lemma 3.3). Since $u_n(f_n(y_i), y) = 0$, for all $y_i = ib/n \quad (i = 0, 1, \ldots, n)$, we easily get (cf. [4; p. 64])

$$(3.9) \qquad f(y) \le g(y)\ .$$

If we prove that

$$(3.10) \qquad f(y) \ge g(y)$$

then (u, f) forms a solution of the $q.v.i.$ as asserted in Theorem 3.1. Suppose (3.10) is not true. Then there exists a point $y_0 \in (0, b)$ and $\eta > 0$ such that for any small $\delta_0 > 0$

$$\begin{gathered} f(y) < g(y) - 5\eta \quad \text{if} \quad y \in K, \quad K \subset \{y_0 < y < y_0 + \delta_0\}\ , \\ \text{meas } (K) > 0\ . \end{gathered}$$

Since g is continuous and f restricted to S has right limits, we deduce that

$$(3.11) \qquad f(y) < g(y) - 4\eta \quad \text{if} \quad y_1 < y < y_1 + \delta\ , \quad y \in S$$

for some $y_0 < y_1 < y_0 + \delta$ and $0 < \delta < \delta_0$.

Suppose

$$f_n(y_{in}) > g(y_{in}) - 2\eta$$

for some vertices y_{1n}, y_{2n} in $\{y_1 < y < y_1 + \delta\}$ (i.e., y_{in} has the form $k_i b/n$, k_i integer). Then by continuity of g, and by Lemma 3.2 applied to u_n and (3.7),

$$f_n(y) > g(y) - 3\eta \quad \text{if} \quad y_{1n} < y < y_{2n}$$

provided δ is chosen sufficiently small. Taking $n \to \infty$ we get a contradiction to (3.11) unless $y_{2n} - y_{1n} \to 0$.

The above remark implies that there exists an interval (β, γ) in $(y_1, y_1 + \delta)$ such that

$$f_n(y_i) \leq g(y_i) - 2\eta \qquad \text{if} \qquad y_i \in (\beta, \gamma)$$

for all vertex points y_i, i.e., $y_i = k\ b/n$ where k is an integer. Recalling that $\{x = f_n\}$ is a polygonal curve and using the continuity of g, we conclude that for any $\beta < \beta_1 < \gamma_1 < \gamma$,

$$f_n(y) < g(y) - \eta \quad \text{if} \quad \beta_1 < y < \gamma_1 \tag{3.12}$$

provided n is sufficiently large.

Next from the relations

$$0 = \int\limits_{\{x<g(y)\}} \Delta u \leftarrow \int\limits_{\{x<g(y)\}} \Delta u_n = \int\limits_{\{g_n<x<g\}} \Delta u_n \geq \frac{\min G}{\epsilon + a} \text{ meas } \{g_n < g\}$$

we deduce that

$$\text{meas } \{g_n < g\} \to 0 \qquad \text{if} \qquad n \to \infty\,.$$

Consequently we can find sequences $\tilde{y}_{1n}, \tilde{y}_{2n}$ in (β_1, γ_1) such that

$$g_n(\tilde{y}_{in}) > g(\tilde{y}_{in}) - \frac{\eta}{4}\,, \quad \tilde{y}_{2n} > \tilde{y}_{1n} + c$$

where $c = (\gamma_1 - \beta_1)/2$. Set $\mu = \gamma_1 - \beta_1$. Applying Lemma 3.2 to u_n with $y' = \tilde{y}_{1n}$, $y'' = \tilde{y}_{2n}$ we deduce that

$$g_n(y) > \min_{\beta_1 \leq y \leq \gamma_1} g - \frac{\eta}{2} - M\mu^{1/2}\,,$$

Hence, if μ is chosen sufficiently small then

$$g_n(y) > g(y) - \eta \quad \text{for} \quad \beta_1 \leq y \leq \delta_1$$

provided n is sufficiently large. But this is a contradiction to (3.12) (if we choose y as a vertex point) and the proof of (3.10) is thereby completed.

§4. Existence of a solution of the *q.v.i.* (2.16) - (2.18).

In this section we shall establish a priori estimates on the solutions (u_ϵ, f_ϵ) of the *q.v.i.* (2.22), (2.17), (2.18) as $\epsilon \to 0$ and then prove that a subsequence of (u_ϵ, f_ϵ) is convergent to a solution of the *q.v.i.* (2.16) - (2.18).

LEMMA 4.1. *There exists a positive constant C independent of ϵ such that*

$$u_{\epsilon,xx} \geq -C \quad in \quad E\,. \tag{4.1}$$

Proof. Consider the penalized problem

$$-\Delta u + \beta_\delta(u) = -\frac{G}{\epsilon + f_\epsilon} \quad \text{in} \quad E \quad (\delta > 0), \tag{4.2}$$

with the boundary conditions (2.17), where $\beta_\delta(t)$ is a C^2 function satisfying: $\beta'_\delta(t) \geq 0$, $\beta''_\delta(t) \leq 0, \beta_\delta(0) = 0$, $\beta_\delta(t) \to -\infty$ if $t < 0, \delta \to 0$ and $\beta_\delta(t) \to 0$ if $t > 0, \delta \to 0$. Then, as $\delta \to 0$, the solution $u = u_{\epsilon,\delta}$ converges to the solution u_ϵ of (2.22), (2.17), uniformly in E, and for any $1 < p < \infty$,

$$| u |_{W^{2,p}(E)} \leq C_{\epsilon,\delta} \qquad (u = u_{\epsilon,\delta})\,. \tag{4.3}$$

Differentiating (4.2) twice in x and setting $w = u_{xx}$, we get

$$-\Delta w + \beta'_\delta(u) w \geq 0 \quad \text{in} \quad E\,. \tag{4.4}$$

Further,

$$w(x,0) = k''_0(x) \geq -c, \quad w(x,b) = k''_1(x) \geq -c\,,$$
$$w(0,y) = -u_{yy}(0,y) + \beta_\delta(u) + \frac{G}{\epsilon + f_\epsilon} = \frac{G}{\epsilon + f_\epsilon} > 0$$

and, on $\{x = a\}$,

$$w_x = -u_{yyx} + \beta'_\delta(u) u_x = -\ell''(y) + \beta'_\delta(u)\ell \geq -\ell''(y) \geq -c$$

since $\beta'_\delta \geq 0, \ell > 0$. Thus, by formally applying the maximum principle to $w + c(x+1)$ we deduce that $w + c(x+1) \geq 0$ in E, and (4.1) follows. In order to justify the use of the maximum principle, we must investigate the behavior of w near the vertices of ∂E where w may not be continuous, or even bounded. Since $w \in L^p(E)$ or any $1 < p < \infty$, we can apply a version of the Phragmen-Lindelof theorem [2; Lemma 5.1] in order to deduce that

$$\begin{aligned} \liminf(w(x,y) + cx + c) \geq 0 \quad \text{as} \quad & (x,y) \to (0,0) \quad \text{or} \\ & (x,y) \to (0,b)\,. \end{aligned} \tag{4.5}$$

Next, since $u_{\epsilon,\delta} \to u_\epsilon$ and $u_\epsilon \geq c_0 > 0$ in a neighborhood N of $(a,0)$, $\beta_\delta(u) = 0$ in N provided δ is small enough. From the consistency condition $k'_0(a) = \ell(0)$ it then follows that $u \in C^2$ in a neighborhood of $(a,0)$ and thus w is continuous at $(a,0)$. Similarly w is continuous at (a,b). Consequently (4.5) holds also as $(x,y) \to (a,0)$ or $(x,y) \to (a,b)$, and the proof that $w + cx + c \geq 0$ in E follows.

LEMMA 4.2. *There exists a positive constant C independent of ϵ such that*

$$u_{\epsilon,x} \leq C \ , \tag{4.6}$$

$$0 \leq u_\epsilon \leq C \ . \tag{4.7}$$

Proof. Note first that $u_{\epsilon,x} = 0$ at $(f_\epsilon(y), y)$ if $f_\epsilon(y) > 0$ and $u_{\epsilon,x} \geq 0$ at $(f_\epsilon(y), y)$ if $f_\epsilon(y) = 0$. Also $u_{\epsilon,x}(a, y) = \ell(y) \leq C$. Using (4.1) the assertion (4.6) then immediately follows. Finally, (4.7) is a consequence of (4.6).

LEMMA 4.3. *For any $\eta > 0$ there exists a positive constant C_η independent of ϵ such that*

$$\int_\eta^{b-\eta} \frac{dy}{\epsilon + f_\epsilon(y)} \leq C_\eta \ . \tag{4.8}$$

Proof. Represent u_ϵ by means of Green's function K of Δ in E :

$$\begin{aligned} u_\epsilon(x, y) = &\int_{\partial E} u_\epsilon(\xi, \zeta) \frac{\partial K(x, y; \xi, \zeta)}{\partial \nu} \, ds \\ &- \iint_E \frac{G(\zeta)}{\epsilon + f_\epsilon(\zeta)} K(x, y; \xi, \zeta) \, d\xi d\zeta \ . \end{aligned} \tag{4.9}$$

Since $0 \leq u_\epsilon \leq C$ we get

$$\iint_E \frac{G(\zeta)}{\epsilon + f_\epsilon(\zeta)} K(x, y; \xi, \zeta) \, d\xi d\zeta \leq C \tag{4.10}$$

and the assertion (4.8) then follows by using the positivity of K.

LEMMA 4.4. *Let $\eta < y_0 - \delta < y_0 + \delta < b - \eta$, $\eta > 0$ and set*

$$\gamma = f(y_0), \qquad \sup_{y_0 - \delta \leq y \leq y_0} f(y) = f(y_1), \qquad \sup_{y_0 \leq y \leq y_0 + \delta} f(y) = f(y_2)$$

where $y_1 \in [y_0 - \delta, y_0]$, $y_2 \in [y_0, y_0 + \delta]$. Let also $\sigma = \min\{f(y_1), f(y_2)\}$. Then

$$\frac{(\sigma - \gamma)^2}{\sigma + \epsilon} \leq C \, \delta \ , \tag{4.11}$$

and

$$\sigma \le 2\epsilon + 4C\delta + 2\gamma \tag{4.12}$$

where $C = C_\eta$ is a positive constant independent of y_0, δ, ϵ.

Proof. We may suppose that $\sigma > \gamma$. Denote by D the subset of $\{u_\epsilon > 0\}$ which is contained in $\{x < \sigma, \quad y_1 < y < y_2\}$. Then, for any $\gamma_0 > \gamma$, the function

$$w = u_\epsilon - \frac{c}{\sigma + \epsilon}(x - \gamma_0)^2 \qquad (c > 0)$$

satisfies

$$\Delta w = \frac{G}{\epsilon + f_\epsilon} - \frac{2c}{\sigma + \epsilon} > 0 \quad \text{in } D \text{ if } \quad c < \frac{G}{2}\,,$$

and $w < 0$ on $\partial D \cap \{x < \sigma\}, \quad w(\gamma_0, y_0) > 0$. By the maximum principle it follows that w attains its positive maximum in $\overline{D}$ on $\{x = \sigma\}$, say at a point $(\sigma, \overline{y})$; consequently

$$u_\epsilon(\sigma, \overline{y}) > \frac{c(\sigma - \gamma_0)^2}{\sigma + \epsilon} \tag{4.13}$$

On the other hand, since by Lemma 4.1

$$u_{\epsilon,yy} = \frac{G}{\epsilon + f_\epsilon} - u_{\epsilon,xx} \le \frac{G}{\epsilon + f_\epsilon} + C\,, \tag{4.14}$$

we have, using Lemma 4.3,

$$\int_{y_*}^{y} u_{\epsilon,yy}(\sigma, y)\, dy \le C_\eta$$

where y_* is the smallest y-coordinate in the interval $\partial D \cap \{x = \sigma\}$. Since $u_y(\sigma, y_*) = u(\sigma, y_*) = 0$, we conclude that

$$u(\sigma, y) \le 2C_\eta\, \delta\,.$$

Substituting this with $y = \overline{y}$ into (4.13) and taking $\gamma_0 \downarrow \gamma$, the assertion (4.11) follows. To prove (4.12) suppose it is not true. Then

$$\frac{(\sigma - \gamma)^2}{\sigma + \epsilon} > \frac{\sigma/2}{2\sigma}(2\epsilon + 4C\delta + 2\gamma) > C\delta$$

which contradicts (4.11).

LEMMA 4.5. *For any $\eta > 0$ there exists a positive constant c_η independent of ϵ such that*

$$f_\epsilon(y) \geq c_\eta \quad \text{if} \quad \eta \leq y \leq b-\eta \tag{4.15}$$

for all ϵ small enough.

Proof. Let $0 < \eta < y_0 < b - \eta$ and suppose that $\gamma \equiv f(y_0)$ is small. Introduce the intervals

$$I_k^+ = \{y_0 + 2^{-k-1} < y < y_0 + 2^{-k}\} ,$$
$$I_k^- = \{y_0 - 2^{-k-1} < y < y_0 - 2^{-k}\}$$

for $k = k_0 + 1,\ k_0 + 2, \ldots$, where k_0 is the smallest positive integer such that

$$\frac{\eta}{2} < y_0 - 2^{-k_0} < y_0 + 2^{-k_0} < b - \frac{\eta}{2} .$$

By (4.11), (4.12) with $\delta = 2^{-k}$,

$$\Bigg(\int\limits_{I_k^+} + \int\limits_{I_k^-}\Bigg) \frac{dy}{\epsilon + f_\epsilon(y)} \geq \frac{2^{-k-1}}{\epsilon+\sigma} \leq \frac{1}{2}\,\frac{2^{-k}}{3\epsilon + 4C2^{-k} + 2f_\epsilon(y_0)} .$$

Hence, by Lemma 4.3,

$$\begin{aligned} C_\eta \geq \int\limits_{\eta/2}^{b-\eta} \frac{dy}{\epsilon + f} &\geq \sum_{k=k_0}^{\infty} \Bigg(\int\limits_{I_k^+} + \int\limits_{I_k^-}\Bigg) \frac{dy}{\epsilon + f_\epsilon(y)} \\ &\geq \frac{1}{2} \sum_{k=k_0}^{\infty} \frac{2^{-k}}{3\epsilon + 4C2^{-k} + 2f_\epsilon(y_0)} . \end{aligned} \tag{4.16}$$

If $f_\epsilon(y_0) < \epsilon$ then the right-hand side is larger than

$$c \sum_{2^{-k} \geq \epsilon} \frac{2^{-k}}{\epsilon + 2^{-k}} \geq \frac{1}{2}\, c \sum_{2^{-k} \geq \epsilon} 1 \geq \frac{1}{2}\, c\, \log \frac{1}{\epsilon} \qquad (c > 0)$$

and this contradicts (4.16) if ϵ is small enough. Thus we must have

$$f_\epsilon(y_0) \geq \epsilon$$

and then, by (4.16),

$$C_\eta \geq c \sum_{k=k_0}^{\infty} \frac{2^{-k}}{2^{-k}+f_\epsilon(y_0)} \geq \frac{1}{2}\, c \sum_{k<\log 1/f_\epsilon(y_0)} 1$$
$$\geq \frac{1}{2}\, c\, \log \frac{1}{f_\epsilon(y_0)} \qquad (c>0)$$

and (4.15) follows.

Having proved Lemma 4.4 we can now take a sequence $\epsilon \to 0$ for which

$$u_\epsilon \to u \quad \text{weakly in } W^{2,p}[E \cap (\eta < y < b-\eta)]$$
$$\text{and uniformly in } \overline{E} \cap (\eta < y < b-\eta)$$

for any $\eta > 0$, and

$$f_\epsilon \to f \quad \text{in measure}$$

(by Lemma 2.3 of [4]), and establish (as in §3) that the free boundary of u in a continuous curve $\{x = g(y),\quad 0 < y < b\}$ with $g(y) = f(y)$.

We now state the main result of this section:

THEOREM 4.6. *The pair (u, f) forms a solution of the q.v.i. (2.16)–(2.18) with $u \epsilon C^0(\overline{E})$, $f(y)$ in $C^0[0,b]$, and $f(y) > 0$ if $0 < y < b$.*

Proof. In view of what we have already proved it only remains to show that $f(y)$ is continuous at $y = 0$ and $y = b$, and u is continuous up to $\{y = 0\}$ and $\{y = b\}$. It will suffice to establish the continuity at $y = 0$ only.

Suppose f is not continuous at $y = 0$, i.e.,

$$\liminf_{y\to 0}\ f(y) < \limsup_{y\to 0}\ f(y)\ .$$

Then there is a sequence of arcs

$$\{x = f(y), \gamma_n < y < \delta_n\} \quad \text{with} \quad \delta_n \to 0 \quad \text{such that}$$

$$\lambda_n = \max_{\gamma_n \leq y \leq \delta_n} f(y) = f(\gamma_n) = f(\delta_n) \to \lambda, \quad \lambda < b\ ,$$
$$\mu_n = \min_{\gamma_n \leq y \leq \delta_n} f(y) \to \mu, \quad 0 \leq \mu < \lambda\ .$$

Applying (4.13) (which holds for (u, f) with $\epsilon = 0$) with $y_1 = \gamma_n, y_2 = \delta_n,\ \sigma = \lambda_n,\ \gamma = \mu_n$, we get

$$\frac{(\lambda_n - \mu_n)^2}{\lambda_n} \leq C\ u(\lambda_n, \overline{y}) \tag{4.17}$$

where $\overline{y} = \overline{y}_n$ is a point of maximum of $u(\lambda_n, y), \gamma_n \le y \le \delta_n$; since $\overline{y} \in (\gamma_n, \delta_n)$ we have

$$u_y(\lambda_n, \overline{y}) = 0 \; . \tag{4.18}$$

From (4.10) which holds for $\epsilon = 0$ and the fact that

$$\int_0^a K(x, y; \xi, \zeta)\, d\xi \ge c\zeta \qquad \text{near} \quad \zeta = 0 \quad (c > 0)$$

provided (x, y) is restricted to a compact subset of E, we deduce that

$$\int_0^{b/2} \frac{\eta\, G(\eta)}{f(\eta)} \le c \, . \tag{4.19}$$

Since

$$y u_{yy}(\lambda_n, y) = -y\; u_{xx}(\lambda_n, y) + \frac{y\; G(y)}{f(y)} \, ,$$

it follows that

$$\int_{\overline{y}}^{\delta_n} y u_{yy}(\lambda_n, y) \le C\; \delta_n + \int_{\overline{y}}^{\delta_n} \frac{y\; G(y)}{f(y)}\, dy \to 0 \quad \text{as } \delta_n \to 0 \tag{4.20}$$

where (4.19) was used. We now compute:

$$\begin{aligned} u(\lambda_n, \overline{y}) &= -\int_{\overline{y}}^{\delta_n} u_y(\lambda_n, y)\; f dy \qquad (\text{since} \quad u(\lambda_n, \delta_n) = 0) \\ &= -[y u_y(\lambda_n, y)]_{\bar{y}}^{\delta_n} + \int_{\overline{y}}^{\delta_n} y u_{yy}(\lambda_n, y)\; dy \\ &= \int_{\overline{y}}^{\delta_n} y\; u_{yy}(\lambda_n, y)\; dy \qquad (\text{by (4.18) and} \quad u_y(\lambda_n, \gamma_n) = 0)). \end{aligned}$$

Using (4.20) we conclude that $u(\lambda_n, \overline{y}) \to 0$ if $n \to \infty$, which is a contradiction to (4.17), since $\lambda - \mu > 0$.

Having proved that $f(0) = \lim_{y \to 0} f(y)$ exists, we proceed to prove that $u(x, y)$ is continuous up to $y = 0$ with $u(x, 0) = k_0(x)$. Consider first the case

$$k_0(x) = 0 \quad \text{if} \quad 0 \le x \le \rho_0 \; , \; \rho_0 > 0 \; , \tag{4.21}$$

and begin with the subcase

$$f(0) = 0 \; . \tag{4.22}$$

From (4.9) we deduce that

$$u_\epsilon(x,y) \le \int\limits_{\partial E} u_\epsilon(\xi,\eta) \, \frac{\partial K(x,y;\xi,\eta)}{\partial \nu} \, ds \tag{4.23}$$

and hence also

$$u(x,y) \le \int\limits_{\partial E} u(\xi,\eta) \, \frac{\partial K(x,y;\xi,\eta)}{\partial \nu} \, ds \; . \tag{4.24}$$

Since $u(\xi,0) = 0$ if $0 < \xi < \rho_0$, it follows that

$$\overline{\lim_{y\to 0}} \; u(x,y) \le 0 \quad \text{if} \quad 0 < x < \rho_0 \; .$$

Recalling that $u \ge 0$, we deduce that

$$\lim_{y\to 0} u(x,y) = k_0(x) = 0 \quad \text{uniformly if} \quad 0 < x < \rho_0 \, . \tag{4.25}$$

For any $0 < \rho^* < \rho_0$ there exists a $\delta = \delta(\rho^*) > 0$ such that for all ϵ sufficiently small

$$f_\epsilon(y) \le \rho^* \quad \text{if} \quad 0 < y < \delta \; . \tag{4.26}$$

Indeed, since $f(0+) = 0$ and $f_\epsilon(y) \to f(y)$ a.e., if (4.26) is not true then we can apply the proof of (4.13) to deduce that

$$c\rho^* \le C \; u_\epsilon(\rho^*, \overline{y}_\epsilon) \quad \text{with} \quad \overline{y}_\epsilon \to 0 \; (C > c > 0). \tag{4.27}$$

But since $u_\epsilon(\rho^*, \overline{y}_\epsilon) \to 0$ by (4.23), this is a contradiction.

From (4.26) we deduce that

$$u_\epsilon > 0 \quad \text{in} \quad R = \{\rho_* < x < a, \; 0 < y < \delta\} \tag{4.28}$$

for some $\delta > 0$. Since $u_{\epsilon,x}$ is harmonic in R and takes smooth boundary values on $\{x = a\}$ and $\{y = 0\}$, we have

$$u_{\epsilon,x} \to v \quad \text{uniformly in } R_0 = \{\rho_* + \eta < x < a, 0 < y < \frac{\delta}{2}\}$$

for any $\eta > 0$; take $\eta < \rho_0 - \rho_*$. Writing

$$u_\epsilon(x,y) - u_\epsilon(\rho_* + \eta, y) = \int_{\rho_*+\eta}^{x} u_{\epsilon,x}(\xi,y)\, d\xi$$

and taking $\epsilon \to 0$ we get

$$u(x,y) - u(\rho_* + \eta, y) = \int_{\rho_*+\eta}^{x} v(\xi,y)\, d\xi. \tag{4.29}$$

Recalling (4.25) we deduce that, if $\rho_0 \le x \le a$,

$$\lim_{y\to 0} u(x,y) = \int_{\rho_*+\eta}^{x} k_0'(\xi)\, d\xi = k(x)\ .$$

Similar conclusion holds if $0 < f(0) < \rho_0$. Consider next the case $f(0) \ge \rho_0$. Then

$$f_\epsilon(y) \ge \frac{1}{2}\rho_0 \quad \text{if} \quad 0 < y < \delta \tag{4.30}$$

for any $\delta < b$; otherwise f_ϵ again has "fingers" and the proof of (4.13) yields

$$c_0 \le u_\epsilon(\tilde{x}, y_\epsilon) \qquad \text{for some} \quad \tilde{x} < \rho_0$$

with $y_\epsilon \to 0$. This is a contradiction since $u_\epsilon(\tilde{x}, y_\epsilon) \to 0$, by (4.23). From (4.30) it follows that $\mid \Delta u_\epsilon \mid \le C$ and we can then easily deduce that $u_\epsilon \to u$ uniformly in $\{0 \le y \le \delta\}$. Since, finally, $u(x,0) = k_0(x)$, it follows that $f(0) = \rho_0$.

So far we have assumed that (4.21) holds. If this is not the case then $k_0(x) > 0$ for all $0 < x < b$. Since however $k'(0) = 0$, (4.25) can be replaced by

$$\overline{\lim_{y\to 0}}\, u(x,y) \le C_0 x^2\ , \quad C_0 > 0; \tag{4.31}$$

Similarly,

$$\overline{\lim_{y\to 0}}\, u_\epsilon(x,y) \le C_0\ x^2. \tag{4.32}$$

Next, using $f_\epsilon(0) = 0$, the proof of (4.27) extends to any small ρ_* (c is independent of ρ_*) and, combined with (4.32) it shows that (4.28) is still valid. We can therefore proceed to derive (4.29) as before and, recalling (4.32), deduce that

$$\overline{\lim_{y\to 0}}\ \mid u(x,y) - k(x) \mid \le 2C_0(\rho_* + \eta)^2\ .$$

Taking $\rho_* + \eta \to 0$, the continuity of u up to $\{y = 0\}$ follows. This implies, of course, that $f(0) = 0$.

REMARK 4.1. The solution established in Theorem 4.6 satisfies: $u_{xx} \geq -C$ and $u_x \geq 0$. The last inequality means, in terms of the physical quantities σ and ϕ, that $\phi_x + \sigma \leq 0$.

§5. Regularity of the free boundary.

Let $y_0 \in (0, b)$, $x_0 = f(x_0)$. Denote by $d = d(x, y)$ the distance from (x, y) to the free boundary.

LEMMA 5.1. *For (x, y) in a small neighborhood N of (x_0, y_0) there holds:*

$$u(x,y) \leq C\, d^2 \ , \tag{5.1}$$

$$| \nabla u(x,y) | \leq C\, d \ . \tag{5.2}$$

Proof. For any $\rho > 0$ set $D_\rho = \{x^2 + y^2 < \rho^2\}$. For $(x_1, y_1) \in \{u > 0\}$ and near (x_0, y_0), let $\epsilon = d(x_1, y_1) > 0$, and set

$$w_\epsilon(x,y) = \frac{u(x_1 + \epsilon x,\ y_1 + \epsilon y)}{\epsilon^2} \qquad \text{in} \qquad D_1$$

Denote by z the solution of

$$\Delta z(x,y) = -\frac{G(y_1 + \epsilon y)}{f(y_1 + \epsilon y)} \qquad \text{in} \qquad D_1 \ ,$$
$$z = 0 \qquad \text{on} \qquad \partial D_1 \ .$$

Then $\Delta(w_\epsilon + z) = 0$ in D_1 and $w_\epsilon + z = w_\epsilon \geq 0$ on ∂D_1. Consequently $w_\epsilon + z > 0$ in D_1. Suppose

$$w_\epsilon(0,0) > M \ , \quad M \quad \text{large.} \tag{5.3}$$

Then $(w_\epsilon + z)(0,0) > \dfrac{M}{2}$ and, by Harnack's inequality,

$$w_\epsilon + z > 2cM \quad \text{in} \quad D_{1/2} \qquad (c > 0) \ .$$

Comparing $w_\epsilon + z$ in $D_1 \backslash D_{1/2}$ with $(2c\, M\, \log \frac{1}{r})/ \log 2$ we deduce that

$$w_\epsilon + z > (2cM\ \log \frac{1}{r})/ \log 2 \quad \text{in } D_1 \backslash D_{1/2}$$

and equality holds at the point $(\overline{x}, \overline{y})$ on ∂D_1 for which $(x_1 + \epsilon \overline{x}, y_1 + \epsilon \overline{y})$ is the nearest free boundary point to $x_1, y_1)$. It follows that

$$\frac{\partial}{\partial \nu} (w_\epsilon + z) > \frac{\partial}{\partial \nu} (2cM\ \log \frac{1}{r})/ \log 2 \qquad (\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial r})$$

at $(\overline{x}, \overline{y})$, and thus

$$\frac{\partial}{\partial \nu} w_\epsilon (\overline{x}, \overline{y}) > 0$$

if M is large enough, a contradiction to $\nabla u(x_1 + \epsilon \overline{x}, y_1 + \epsilon \overline{y}) = 0$.

We have thus proved that (5.3) cannot hold with large enough M (independent of ϵ); hence $w_\epsilon(0,0) \leq C$ and, by the same proof, $w_\epsilon \leq C$ in $D_{1/4}$, say, with C independent of ϵ. It follows that also $| \nabla w_\epsilon | \leq C$ in $D_{1/8}$ and (5.1), (5.2) thus hold.

Lemma 5.1 allows us to work with blow-up sequences.

$$v_\epsilon(x,y) = \frac{1}{\epsilon^2} u(x_0 + \epsilon x, y_0 + \epsilon y) \qquad (x_0 = f(y_0));$$

any such sequence has a subsequence which is convergent in C^1 (and weakly in $W^{2,p}$) to a solution of the variational inequality

$$\begin{aligned} & -\Delta v \geq -\gamma , \ v \geq 0, (\Delta v - \gamma) v = 0 \quad \text{a.e. in } \mathbf{R}^2 , \\ & v(x,y) \leq C \ (x^2 + y^2), \\ & | \nabla v(x,y) | \leq C \ (x^2 + y^2)^{1/2} \end{aligned} \tag{5.4}$$

where

$$\gamma = \frac{G(y_0)}{f(y_0)} ;$$

further, since $u_x \geq 0$ we also have

$$v_x \geq 0 . \tag{5.5}$$

LEMMA 5.2. *The subsequence v_ϵ can be chosen so that the free boundary of v is a straight line.*

Proof. The function v is convex. Indeed, $v \in C^{1,1}$ and any direction ξ

$$v_{\xi\xi} \geq -c \ (\log \frac{1}{d})^{-1/2} \qquad \text{in} \qquad \{v > 0\} \qquad (c > 0)$$

(by [1] or [3; p. 156]) where d is the distance to the free boundary and c is an absolute constant. Applying this inequality to

$$v_\delta(x,y) = \frac{1}{\delta^2} v(x_1 + \delta x, y_1 + \delta y)$$

for any $(x_1, y_1) \in \{v > 0\}$ and $\delta \to 0$, we deduce that

$$v_{\xi\xi}(x_1, y_1) \geq -c \ (\log \frac{1}{\delta})^{-1/2} \to 0 ;$$

thus $v_{\xi\xi}(x_1, y_1) \geq 0$.

From the convexity of v and (5.5) it follows that either

$$\text{the free boundary of } v \text{ lies in } \{y = 0\} \tag{5.6}$$

or $\{v = 0\}$ is bounded on the right by a curve $x = \phi(y), -\infty < y < \infty$ and, further, by nondegeneracy and [2; p. 132] $\phi(y)$ is smooth (in fact, analytic) in y. If $\{x = \phi(y)\}$ is not a straight line then the scaled solutions

$$w_\epsilon(x, y) = \epsilon^2\, v(\frac{x}{\epsilon}, \frac{y}{\epsilon})$$

have a subsequence which converges to a solution of (5.4), (5.5) whose free boundary consists of two rays forming a curve $x = \psi(y)$ which is not differentiable at $(0, 0)$; this is a contradiction, since ψ must be smooth (by [2; p. 132]).

Consider now the case where (5.6) holds for any blow-up limit v. Then, for any $\delta > 0$, the free boundary of u cannot have points (x, y) near (x_0, y_0) with

$$\left|\frac{y - y_0}{x - x_0}\right| > \delta .$$

Indeed otherwise we can construct a blow-up limit v whose free boundary has points outside $\{y = 0\}$.

From the above remark it follows that the free boundary of u belong near (x_0, y_0) must either belong to the region.

$$\left|\frac{y - y_0}{x - x_0}\right| < \delta, \quad x < x_0 , \tag{5.7}$$

or to

$$\left|\frac{y - y_0}{x - x_0}\right| < \delta, \quad x > x_0 , \tag{5.8}$$

or it must cross from (5.7) to (5.8) as y crosses $y = y_0$. In the last case the free boundary of v is the whole line $\{y = 0\}$. Thus it suffices to show that cases (5.7), (5.8) cannot hold. It will be enough to prove it for (5.7). If (5.7) holds then by comparison

$$u_x > c\, r^\alpha \cos\alpha\theta \quad \text{in} \quad \{\frac{\pi}{2\alpha} < \theta < \frac{\pi}{2\alpha};, 0 < r < r_0\}$$

for any $\alpha \in (\frac{1}{2}, 1)$ where θ is the angle between $(x - x_0, y - y_0)$ and the ray $\{y = y_0,\ x > x_0\}$. Since, on the other hand,

$$u_x \leq C\, r \qquad \text{(by Lemma 5.1)},$$

we get $cr^\alpha \leq Cr$, which is a contradiction.

LEMMA 5.3. *For any compact subset E_0 of E there exists a positive constant c such that*

$$u(x,y) \geq cd(x,y) \qquad \textit{in} \qquad E_0 \cap \{u > 0\} \ .$$

Proof. If the assertion is not true then there exists a sequence (x_n, y_n) in $E_0 \cap \{u > 0\}$ with $d_n \equiv d(x_n, y_n) \to 0$ such that $(x_n, y_n) \to (x_0, y_0) = (f(y_0), y_0)$ and

$$\frac{1}{d_n^2}\, u(x_n, y_n) \to 0 \ . \tag{5.9}$$

Consider the functions

$$v_n(x,y) = \frac{u(x_0 + d_n x\ y_0 + d_n y)}{d_n^2} \ . \tag{5.10}$$

The point

$$e_n = \left(\frac{x_n - x_0}{d_n} \ , \quad \frac{y_n - y_0}{d_n} \right)$$

has distance 1 to the coincidence set of v_n. For a subsequence,

$$e_n \to e, \quad v_n \to v$$

and, by Lemma 5.2, the free boundary of v is a straight line ℓ through the origin. But then, by nondegeneracy, the free boundary of v_n must lie in a small neighborhood of the line ℓ (see [2; p. 253]). It follows that dist $(e_n, \ell) \geq \frac{1}{2}$ for n large, and therefore also dist $(e, \ell) \geq \frac{1}{2}$. This is a contradiction since $v(e) = 0$ by (5.9), (5.10).

LEMMA 5.4. $\in C^{1,1}_{\text{loc}}(E)$.

Proof. The function u_x is harmonic in $E \cap \{u > 0\}$ and $|u_x(x,y)| \leq C\ d(x,y)$. Hence, by interior estimates for harmonic functions, $|\nabla u_x| \leq C$ in $E_0 \cap \{u > 0\}$ for any compact subset E_0 of E, i.e., u_{xx} and u_{xy} are in bounded $E_0 \cap \{u > 0\}$. Finally, from the equation $\Delta u = G/f$ we deduce that also u_{yy} is bounded in $E_0 \cap \{u > 0\}$.

LEMMA 5.5. *The function $|\nabla u|^2/u$ is bounded in $E_0 \cap \{u > 0\}$, for any compact subset E_0 of E.*

Proof. If the assertion is not true then there exists a sequence of points (x_n, y_n) in $E_0 \cap \{u > 0\}$ such that $(x_n, y_n) \to (x_0, y_0) = (f(y_0), y_0)$ for some $0 < y_0 < b$ and

$$\frac{|\nabla u(x_n, y_n)|^2}{u(x_n, y_n)} \longrightarrow \infty \ . \tag{5.11}$$

Set $d_n^2 = u(x_n, y_n)$. By Lemmas 5.1, 5.3

$$cd_n \le d(x_n, y_n) \le C\ d_n \tag{5.12}$$

for some positive constants c, C. Consider the function

$$v_n(x,y) = \frac{u(x_n + d_n x,\ y_n + d_n y)}{d_n^2}$$

for (x, y) in the closed unit disc $\overline{D}_1$. By Lemma 5.1 and (5.12), $v_n \le C$ in $\overline{D}_1$, and, by Lemma 5.4, $|D^2 v_n| \le C$ in $\overline{D}_1$. From the Taylor expansion

$$v_n(e) = v_n(0) + e \cdot \nabla v_n(0) + \quad O(D^2 v_n) \quad (e \epsilon \partial D_1)$$

we then deduce that $|\nabla v_n(0)| \le C$, i.e.,

$$\frac{|\nabla u(x_n, y_n)|}{d_n} \le C\ ,$$

which is a contradiction to (5.11).

LEMMA 5.6. *For any compact subset E_0 of E there exist $\alpha \epsilon (0,1), \alpha$ near 1 and a positive constant c such that*

$$u_x \ge cu^\alpha \qquad in \quad E_0 \cap \{u > 0\}\ . \tag{5.13}$$

Proof. For any compact subset E_1 of E we have, in $E_1 \cap \{u > 0\}$,

$$\begin{aligned} \Delta u^\alpha &= \alpha u^{\alpha-1} \Delta u + \alpha(\alpha-1) u^{\alpha-2} |\nabla u|^2 \\ &= \alpha u^{\alpha-1} \left[\frac{G}{f} + (\alpha - 1)\ \frac{|\nabla u|^2}{u} \right] > 0 \end{aligned}$$

if $1 - \alpha$ is small, since $|\nabla u|^2/u$ is bounded (by Lemma 5.5). We conclude that, for any $c > 0$,

$$\Delta(u_x - cu^\alpha) < 0 \qquad \text{in} \qquad E_1 \cap \{u > 0\}\ .$$

We can find small discs B_1 and B_2 such that $B_i \subset E_0 \cap \{u > 0\}$ (where E_0 is a compact subset of E) and ∂B_i is tangent to the free boundary at some point $(x_i, y_i) \equiv (f(y_i), x_i)$. Construct a curve γ in $E \cap \{u > 0\}$ which begins along a normal segment ν_1 to ∂B_1 at (x_1, y_1) and ends along a normal segment ν_2 to ∂B_2 at (x_2, y_2).

The function u_x is positive and harmonic in $E \cap \{u > 0\}$, and takes its minimum zero on the free boundary. Since the inner ball condition holds at (x_i, y_i),

$$u_x \quad \text{increases linearly along } \nu_1 \text{ and } \nu_2.$$

On the other hand, u is bounded by $Cd^2(x,y)$ along ν_1 and ν_2. Hence

$$u_x \geq cu^\alpha \quad \text{along } \nu_1 \text{ and } \nu_2$$

for any $\frac{1}{2} \leq \alpha < 1$ and some $c > 0$. Since also $u_x - cu^\alpha \geq 0$ along the remaining portion of γ (if c is small enough) and $u_x - cu^\alpha = 0$ on the free boundary, we can apply the maximum principle to the superharmonic function $u_x - cu^\alpha$, and deduce that (5.13) holds in the region bounded by γ and the free boundary. Since we can exhaust $E \cap \{u > 0\}$ by such regions, the proof of the lemma is complete.

LEMMA 5.7. *The function $f(y)$ is Hölder continuous for $0 < y < b$.*

Proof. Take any free boundary point $(x_0, y_0) = (f(y_0), y_0)$. Since $u_x(x, y_0) \geq cu^\alpha(x, y_0)$, it follows by integration that

$$u(x, y_0) \geq c_1(x - x_0)^{\frac{1}{1-\alpha}} \qquad (c_1 > 0)$$

for $x > x_0$. Since $u \in C^1$, we deduce that

$$u(x,y) \geq c_1(x - x_0)^{\frac{1}{1-\alpha}} - |\nabla u||y - y_0| > 0$$

if $|y - y_0| \leq c|x - x_0|^{\frac{1}{1-\alpha}}$ for some $c > 0$. It follows that

$$f(y) \leq f(y_0) + |x - x_0| \qquad \text{if} \qquad |y - y_0| \leq c|x - x_0|^{\frac{1}{1-\alpha}},$$

and thus

$$f(y) \leq f(y_0) + C|y - y_0|^{1-\alpha} .$$

Similarly

$$f(y) \geq f(y_0) - C|y - y_0|^{1-\alpha} .$$

LEMMA 5.8. *The free boundary of u is a continuously differentiable curve and for any $0 < y_0 < b, u \in C^2\ (\overline{\{u > 0\}} \cap N)$ where N is a neighborhood of $(f(y_0), y_0)$.*

Indeed, this follows by Lemma 5.2 and the Hölder continuity of the right-hand side of the variational inequality; see [1] or [3; Chapters 4,5].

We can now improve Lemma 5.2:

LEMMA 5.9. *The subsequence v_ϵ can be chosen so that the free boundary of v is a straight line which is not parallel to the x-axis.*

Proof. By Lemma 5.8, the set $\{y; f'(y)\}$ exists is open. Thus, if the assertion is not true there is an interval $\{y_0 < y < y_1\}$ with $0 < y_0 < b$ such that $f \in C^1$ in this interval, but any blow-up v about $(f(y_0), y_0)$ has the free boundary $\{y = y_0\}$. The free boundary near $(f(y_0), y_0)$ is then a continuously differentiable curve $x = k(y)$ with $k'(y_0) = 0$. Thus, since the free boundary is a graph with respect to the y-axis as well as the x-axis (by $u_x \geq 0$), it must be a monotone curve; let us assume for definiteness that $f'(y) \leq 0$ for $| y - y_0 | < \delta$; f' may take the values $-\infty$ for some $y_0 - \delta < y \leq y_0$.

We can now proceed as in the proof of Th. 6.1 p. 177 of [3] to deduce that for some $c > 0, K > 0$,

(5.14) $$cu_x - u_y - Ku > 0 \quad \text{if} \quad f(y) < x < x_2, \ | y - y_0 | < \delta \ ;$$

here we use the fact that

$$\Delta(cu_x - u_y - Ku) = \frac{f'G}{f^2} + O(1) < 0 \ .$$

It follows that

$$u_y \leq cu_x \quad \text{if} \quad f(y) < x < x_1, \quad y_0 < y < y_0 + \delta \ .$$

Since $u_y = cu_x$ on the free boundary, we deduce that

(5.15) $$u_{yy} < cu_{xy} \quad \text{on} \quad x = f(y), \quad y_0 < y < y_0 + \delta \ ;$$

recall (by Lemma 5.8) that $u \in C^2$ if $f(y) \leq x \leq x_1, y_0 < y < y_0 + \delta$.

Differentiating the relation $u_y(f(y), y) = 0$ we get

(5.16) $$u_{xy} \ f' + u_{yy} = 0 \quad \text{on} \quad x = f(y), \quad y_0 < y < y_0 + \delta \ .$$

Similarly, from $u_x(f(y), y) = 0$ we get

$$u_{xx} f' + u_{xy} = 0 \quad \text{on} \quad x = f(y), y_0 < y < y_0 + \delta \ .$$

Recalling also that $u_{xx} + u_{yy} = G/f$, we get

(5.17) $$u_{xy} - u_{yy} f' = \frac{G}{f} \, f'$$

From (5.16), (5.17) we find that

$$u_{yy} = \frac{G}{f} \cdot \frac{f'^2}{1 + f'^2}, \quad u_{xy} = -\frac{G}{f} \cdot \frac{f'}{1 + f'^2} \quad .$$

Substituting this into (5.15), it follows that $| f'(y) | \leq c$ if $y_0 < y < y_0 + \delta$, which is a contradiction.

Having proved Lemma 5.9 we now deduce that $f'(y)$ is continuous for $0 < y < b$. Since $G \in C^{m+\alpha}(0, b)$, by applying [2; p.132] step–by–step we deduce:

THEOREM 5.5. *The function* $f(y)$ *belongs to* $C^{m+1+\alpha}(0, b)$.

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