

THE EXTRAPOLATION FOR BOUNDARY FINITE ELEMENTS

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I. Introduction

It has been shown in [1]-[3] that the Richardson extrapolation can be applied to the elliptic Ritz projection with linear finite elements and increase the second-order accuracy of linear finite elements

$$U_h(z) = u(z) + O(h^2 | \ln 1/h |)$$

in mesh point z at least to order three or four

$$U_h(z) \equiv \frac{1}{3} (4 U_{h/2}(z) - U_h(z)) = u(x) + O(h^3 | \ln 1/h |) \text{ (or } O(h^4 | \ln 1/h |))$$

Where T_h is uniform triangulation and $T_{h/2}$ is generated from T_h by dividing each triangle as usual into four congruent subtriangles.

In the present paper above basic results have been extended to boundary finite element and the coupling of finite elements and boundary finite elements.

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2. The Extrapolation for Boundary Finite Elements

Let us consider the following boundary value problem

$$-\Delta u = f \quad \text{in } \Omega \quad (2.1)$$

$$u = u_0 \quad \text{on } \Gamma$$

where $\Omega \in \mathbb{R}^2$ is a bounded domain with Lipschitz boundary. The equivalent integral equation is that

$$\int_{\Gamma} q(x) E(x;y) ds_x = \int_{\Gamma} u_0(x) \frac{\partial E(x;y)}{\partial n} ds_x + \frac{1}{2} u_0(y) - \int_{\Omega} f(x) E(x;y) dx \quad (2.2)$$

$$\forall y \in \Gamma$$

$$u(y) = \int_{\Omega} f(x) E(x;y) dx + \int_{\Gamma} q(x) E(x;y) ds_x - \int_{\Gamma} u_0(x) \frac{\partial}{\partial n} E(x;y) ds_x \quad (2.3)$$

$$\forall y \in \Omega$$

Where $E(x;y)$ is a fundamental solution of Laplace equation, i.e.

$$\Delta_x E(x;y) + \delta(x-y) = 0$$

and $q = \frac{\partial u}{\partial n} \Big|_{\Gamma}$. For two dimensional problem it is known

$$E(x;y) = -\frac{1}{2\pi} \ln |x-y| \quad (2.4)$$

The Galerkin variational formulation of (2.2) is that

$$\text{find } q \in H^{-1/2}(\Gamma) \text{ such that} \quad (2.5)$$

$$b(q, q^*) = \mathfrak{L}((u_0, f), q^*) \quad \forall q^* \in H^{-1/2}(\Gamma)$$

where

$$b(q, q^*) = \int_{\Gamma} \int_{\Gamma} q(x) q^*(y) E(x; y) ds_x ds_y$$

is a symmetry and coercive bilinear form on $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$

$$\begin{aligned} \mathfrak{L}((u_0, f), q^*) &= \int_{\Gamma} \int_{\Gamma} u_0(x) q^*(y) \frac{\partial}{\partial n_x} E(x; y) ds_x ds_y + \frac{1}{2} \int_{\Gamma} u_0(x) ds_x \\ &\quad - \int_{\Gamma} \int_{\Omega} f(x) q^*(y) E(x; y) dx ds_y \end{aligned}$$

is a bounded linear functional on $H^{-1/2}(\Gamma)$.

The approximation problem corresponding to (2.5) can be expressed as the following

$$\text{find } q_h \in V_h \subset H^{-1/2}(\Gamma) \text{ such that} \quad (2.6)$$

$$b(q_h, q_h^*) = \mathfrak{L}((u_0, f), q_h^*), \quad \forall q_h^* \in V_h$$

Where V_h is a boundary finite element subspace on Γ , and h is mesh size parameter. Later on, we suppose V_h is a linear elements subspace on Γ .

Lemma 1 Assume $E(x;y)$ is a fundamental solution defined by (2.4). Then there exists $\tilde{E}(x;y) \in V_h$ such that

$$\begin{aligned} \|\tilde{E}(\cdot;y)\|_{0,2,\Gamma} &< c |\ln h|^{1/2} \\ \|\tilde{E}(\cdot;y)\|_{1/2,\Gamma} &< c |\ln h|^{1/2} \\ \|\tilde{E}(\cdot;y) - E(\cdot,y)\|_{0,1,\Gamma} &< c h |\ln h| \end{aligned} \tag{2.7}$$

where c is a positive constant independent of h and y .

Proof. Suppose

$$M_y = \{x \mid \|x-y\| < 1\}, \quad \Gamma_1 = \Gamma \cap M_y, \quad \Gamma_2 = \Gamma \setminus \Gamma_1$$

Because of $E(x;y)$ satisfying

$$-\Delta E(x;y) = \delta(x-y) \quad \text{in } M_y$$

$$E(x;y) = 0 \quad \text{on } \partial M_y$$

we have $E(x;y) \in H_0^{1-\epsilon}(M_y)$ and

$$(\nabla E(x;y), \nabla v) = v(y) \quad \forall v \in H_0^{1-\epsilon}(M_y)$$

By [5], there exists $E_h(x;y) \in W_h \subset H_0^1(M_y)$ (where W_h is piecewise linear interpolation finite element subspace of $H_0^1(M_y)$), such that

$$(\nabla E_h(\cdot; y), \nabla v) = v(y) \quad \forall v \in W_h$$

and

$$\|E_h(\cdot; y)\|_{1,2,M_y} \leq c |1nh|^{1/2}$$

$$\|E_h(\cdot; y) - E(x; y)\|_{1,1,M_y} \leq ch |1nh|$$

so

$$\|E_h(\cdot; y)\|_{1,2,M_y \cap \Omega} \leq c |1nh|^{1/2}$$

$$\|E(\cdot; y) - E_h(\cdot; y)\|_{1,1,M_y \cap \Omega} \leq ch |1nh|$$

By virtue of trace theory, we can get

$$\|E_h(\cdot; y)\|_{0,2,\partial(M_y \cap \Omega)} \leq c |1nh|^{1/2}$$

$$\|E_h(\cdot; y) - E(\cdot; y)\|_{0,1,\partial(M_y \cap \Omega)} \leq ch |1nh|$$

In view of $\Gamma_1 \subset (M_y \cap \Omega)$, we have

$$\|E_h(\cdot; y)\|_{0,2,\Gamma_1} \leq c |1nh|^{1/2}$$

$$\|E_h(\cdot; y) - E(\cdot; y)\|_{0,1,\Gamma_1} \leq ch |1nh|$$

Because of $\|x-y\| > 1, \quad \forall x \in \Gamma_2$

so $\|E(\cdot; y)\|_{2,\infty,\Gamma_2} < 1$

Suppose $E^1(\cdot; y)$ is a piecewise linear interpolation of $E(x; y)$, then

$$\| E^I(\cdot; y) - E(\cdot; y) \|_{0,1,\Gamma_2} < ch^2 \| E(\cdot; y) \|_{2,1,\Gamma_2} < \tilde{c}h^2$$

$$\| E^I(\cdot; y) \|_{0,\Gamma_2} < c$$

$$\| E^I(\cdot; y) \|_{1,\Omega} \quad (My \quad \Omega) < c$$

Setting

$$E(x; y) = \begin{cases} E_h(x; y) & x \in \Gamma_1 \\ E^I(x; y) & x \in \Gamma_2 \end{cases}$$

then

$$\| \tilde{E}(\cdot; y) \|_{0,\Gamma} < \| E_h(\cdot; y) \|_{0,\Gamma_1} + \| E^I(\cdot; y) \|_{0,\Gamma_2} < c | \ln h |^{1/2}$$

$$\| \tilde{E}(\cdot; y) \|_{1,\Omega} < c | \ln h |^{1/2}$$

$$\| \tilde{E}(\cdot; y) \|_{1/2,\Gamma} = \inf \{ \| v \|_{1,\Omega}, \gamma_0 E(t, y) = \gamma_0 v \}$$

$$< \| E(\cdot; y) \|_{1,\Omega} < c | \ln h |^{1/2}$$

where γ_0 is the trace operator of 0-order on the Γ . In addition,

$$\| \tilde{E}(\cdot; y) - E(\cdot; y) \|_{0,1,\Gamma} < \| \tilde{E}(\cdot; y) - E(\cdot; y) \|_{0,1,\Gamma_1}$$

$$+ \| \tilde{E}(\cdot; y) - E(\cdot; y) \|_{0,1,\Gamma_2} < ch \| \ln h \|$$

The proof ends.

Remark 1 In view of (7) it is clear that

$$\| \tilde{E}(\cdot; y) \|_{0,1,\Gamma} \leq c | \ln h |^{1/2}$$

Lemma 2 Suppose G_z is the Green function of boundary integral equation (2.2) ,

$$b(q, G_z) = q(z) \quad \forall q \in H^{-1/2}(\Gamma)$$

$G_{zh} \in V_h$ is the approximation of G_z

$$b(q, G_{zh}) = q(z) \quad \forall q \in V_h \subset H^{-1/2}(\Gamma)$$

Then we have

$$\| G_{zh} \|_{-3/2, \Gamma} \leq c | \ln h |^{1/2} \tag{2.8}$$

$$\| G_{zh} - G_z \|_{-3/2, \Gamma} \leq c h$$

where c is a positive constant independent of h and z .

Proof. (1). $\forall g \in H^{3/2}(\Gamma)$, there exists a $q_g \in V_h \subset H^{1/2}(\Gamma)$

such that

$$b(q_g, q') = \langle g, q' \rangle \quad \forall q' \in V_h$$

and

$$\| q_g \|_{1/2, \Gamma} \leq c \| g \|_{3/2, \Gamma}$$

By [6] ,

$$\langle g, G_{zh} \rangle = b(q_g, G_{zh}) = q_g(z) \leq \| q_g \|_{0, \infty, \Gamma} \leq \tag{[6]}$$

$$c | \ln h |^{1/2} \|g_g\|_{1/2, \Gamma} < c | \ln h |^{1/2} \|g\|_{3/2, \Gamma}$$

So

$$\|G_{Zh}\|_{-3/2, \Gamma} = \sup_{g \in H^{3/2}(\Gamma)} \frac{\langle g, G_{Zh} \rangle}{\|g\|_{3/2, \Gamma}} < c | \ln h |^{1/2}$$

(2) In similar manner, $\forall g \in H^{3/2}(\Gamma)$, there exist $q_g \in H^{1/2}(\Gamma)$, $q_h \in V_h \subset H^{1/2}(\Gamma)$ such that

$$b(q_g, q') = \langle g, q' \rangle \quad \forall q' \in H^{-1/2}(\Gamma)$$

$$b(q_h, q') = \langle g, q' \rangle \quad \forall q' \in V_h$$

By [11],

$$\|q_g - q_h\|_{0, \Gamma} < ch^{3/2} \|q_g\|_{1/2, \Gamma} < c h^{3/2} \|g\|_{3/2, \Gamma}$$

So

$$\begin{aligned} \langle g, G_Z - G_{ZR} \rangle &= b(q_g - q_h, G_Z - G_{Zh}) = b(q_g - q_h, G_Z) \\ &= q_g(Z) - q_h(Z) < \|q_g - q_h\|_{0, \infty, \Gamma} < ch^{-1/2} \|q_g - q_h\|_{0, \Gamma} \\ &< c h \|g\|_{3/2, \Gamma} \end{aligned}$$

$$\|G_Z - G_{Zh}\|_{-3/2, \Gamma} = \sup_{g \in H^{3/2}(\Gamma)} \frac{\langle g, G_Z - G_{Zh} \rangle}{\|g\|_{3/2, \Gamma}} < ch \quad \#$$

Assume $F(\Gamma_h)$ is a finite element regular family on Γ which satisfies that the distance from any point on the arc $S_i S_{i+1}$ to line $\overline{S_i S_{i+1}}$ is shorter than $\theta(h^2)$. Then we have

Lemma 3 Suppose $q \in C^3(\Gamma)$, $q^I \in V_h$ is the interpolation of q , then

$$b(q^I - q, G_{zh}) = \frac{h^2}{12} b(q'', G_{zh}) + O(h^3 | \ln h |^{3/2}) \|q\|_{3, \infty, \Gamma} \quad (2.9)$$

Proof. In fact, on the $\overline{S_i S_{i+1}}$

$$\begin{aligned} q^I(s) &= \frac{S_{i+1} - s}{h} q(S_i) + \frac{s - S_i}{h} q(S_{i+1}) \\ &= \frac{S_{i+1} - s}{h} (q(s) + (S_i - s) q'(s) + \frac{1}{2} (S_i - s)^2 q''(s) + \frac{1}{6} (S_i - s)^3 q'''(\xi)) \\ &\quad + \frac{s - S_i}{h} (q(s) + (S_{i+1} - s) q'(s) + \frac{1}{2} (S_{i+1} - s)^2 q''(s) + \frac{1}{6} (S_{i+1} - s)^3 q'''(\eta)) \\ &= q(s) + \frac{1}{2} (S_{i+1} - s)(S_i - s) q''(s) + O(h^3) \|q\|_{3, \infty, \Gamma} \end{aligned}$$

Therefore we have

$$\begin{aligned} I(y) &= \int_{\Gamma} (q^I(s) - q(s)) E(x; y) ds_x = \int_{\Gamma} (q^I - q) \tilde{E}(x; y) ds_x \\ &\quad + \int_{\Gamma} (q^I - q) (E(x; y) - \tilde{E}(x; y)) ds_x = I_1 + I_2 \end{aligned} \quad (2.10)$$

By lemma 1 and Remark 1 we have

$$I_2 \leq c h^2 \|q\|_{2, \infty, \Gamma} \int_{\Gamma} |\tilde{E}(x; y) - E(x; y)| ds_x = O(h^3 | \ln h |) \|q\|_{2, \infty, \Gamma}$$

$$I_1 = \sum_i \int_{\overline{S_i S_{i+1}}} (q^I - q) \tilde{E}(x;y) ds_x$$

$$= \sum_i \int_{\overline{S_i S_{i+1}}} (q^I - q) \tilde{E}(x;y) ds_x + \sum_i \left(\int_{\overline{S_i S_{i+1}}} - \int_{\overline{S_i S_{i+1}}} \right) (q^I - q) \tilde{E}(x;y) ds_x = L_1 + L_2$$

$$L_2 < c h^2 \int_{\Gamma} \left| \frac{\partial}{\partial n} ((q^I - q) \tilde{E}(x;y)) \right| ds_x$$

$$= ch^2 \{ \|q - q\|_{1, \infty, \Gamma} \|\tilde{E}\|_{1,1, \Gamma} + \|q^I - q\|_{1/2, \Gamma} \|\frac{\partial \tilde{E}}{\partial n}\|_{1/2, \Gamma} \}$$

$$< ch^3 |\ln h|^{1/2} \|q\|_{2, \infty, \Gamma} + ch^{3+1/2} \|q\|_{2, \Gamma} \|\tilde{E}\|_{1/2, \Gamma} \} < ch^3 |\ln h|^{1/2} \|q\|_{2, \infty, \Gamma} \quad (2.11)$$

on the $\overline{S_i S_{i+1}}$, by virtue of $\tilde{E}(x;y) \in Vh$,

$$\tilde{E}(x;y) = \frac{S_{i+1} - S}{h} \tilde{E}(x_i; y) + \frac{S - S_i}{h} \tilde{E}(x_{i+1}; y) = N_i(s) \tilde{E}(x_i; y) + N_{i+1}(S) \tilde{E}(x_{i+1}; y)$$

$$\int_{\overline{S_i S_{i+1}}} N_i(s) ds = \frac{1}{h} \int_{S_i}^{S_{i+1}} (S_{i+1} - s) ds = h/2$$

$$\int_{\overline{S_i S_{i+1}}} N_{i+1}(s) ds = h/2$$

Hence

$$L_1 = \sum_i \int_{\overline{S_i S_{i+1}}} (q^I - q) \tilde{E}(x;y) ds_x = \sum_i \int_{\overline{S_i S_{i+1}}} \left(\frac{1}{2} (S_{i+1} - S)(S - S_i) q''(s) \right)$$

$$(N_i(s) \tilde{E}(x_i; y) + N_{i+1}(s) \tilde{E}(x_{i+1}; y)) ds_x + O(h^3) \|q\|_{3, \infty, \Gamma} \int_{\Gamma} |\tilde{E}(x;y)| ds_x$$

$$= \sum_i q''(\xi) \left(\frac{h^3}{24} \tilde{E}(x_i; y) + \frac{h^3}{24} \tilde{E}(x_{i+1}; y) \right) + O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma}$$

$$= \sum_i \frac{h^2}{12} \{q''(\xi) \left(\int_{S_i S_{i+1}} (N_i(s) \tilde{E}(x_i; y) + N_{i+1}(s) \tilde{E}(x_{i+1}; y)) ds_x \right)\} + O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma}$$

$$L_1 = \frac{h^2}{12} \int_{\Gamma} q''(s) E(x; y) ds_x + \frac{h^2}{12} \sum_i \int_{S_i S_{i+1}} (q''(\xi) - q''(s)) \tilde{E}(x; y) ds_x \\ + O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma}$$

but

$$\int_{S_i S_{i+1}} (q''(\xi) - q''(s)) \tilde{E}(x; y) ds_x = O(h |\ln h|^{1/2})$$

So

$$L_1 = \frac{h^2}{12} \int_{\Gamma} q''(s) \tilde{E}(x; y) ds_x + O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma} \\ = \frac{h^2}{12} \int_{\Gamma} q''(s) E(x; y) ds_x + \frac{h^2}{12} \int_{\Gamma} q''(s) (\tilde{E}(x; y) - E(x; y)) ds_x \\ + O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma} \quad (\text{by lemma 1})$$

$$= \frac{h^2}{12} \int_{\Gamma} q''(s) E(x; y) ds_x + O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma} \quad (2.12)$$

Consequently,

$$I(y) = I_1 + I_2 = L_1 + L_2 + I_2 = \frac{h^2}{12} \int_{\Gamma} q''(s) E(x;y) ds_x + O(h^3 |\ln h|) \|q\|_{3, \infty, \Gamma} \quad (2.13)$$

Therefore,

$$\begin{aligned} b(q^I - q_1, G_{zh}) &= \int_{\Gamma} I(y) G_{zh}(y) ds_y = \frac{h^2}{12} \int_{\Gamma} \int_{\Gamma} q''(x) G_{zh}(y) E(x;y) ds_x ds_y \\ &+ O(h^3 |\ln h|) \|q\|_{3, \infty, \Gamma} \|G_{zh}\|_{-3/2, \Gamma} \\ &= \frac{h^2}{12} \int_{\Gamma} \int_{\Gamma} q''(x) G_z(y) E(x;y) ds_x ds_y + \frac{h^2}{12} \int_{\Gamma} \int_{\Gamma} q''(x) (G_{zh}(y) - G_z(y)) E(x;y) ds_x ds_y \\ &+ O(h^3 |\ln h|^{3/2}) \|q\|_{3, \infty, \Gamma} \\ &= \frac{h^2}{12} b(q'', G_z) + O(h^2 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma} \|G_{zh} - G_z\|_{0, 1, \Gamma} + O(h^3 |\ln h|^{3/2}) \end{aligned}$$

Owing to that

$$\|G_{zh} - G_z\|_{0, 1, \Gamma} \leq |\Omega| \|G_{zh} - G_h\|_{-3/2} \leq c h$$

Finally, we obtain

$$b(q^I - q, G_{zh}) = \frac{h^2}{12} b(q'', G_z) + O(h^3 |\ln h|^{3/2}) \|q\|_{3, \infty, \Gamma}$$

Theorem 1 Suppose q is the solution of (2.5), $q \in C^3(\Gamma)$, $q_h \in V_h \subset H^{-1/2}(\Gamma)$ is a solution of (2.6), then for any mesh point Z we have

$$\frac{1}{3} (4 q_{h/2}(Z) - q_h(Z)) = q(Z) + O(h^3 |\ln h|^{3/2}) \|q\|_{3, \infty, \Gamma} \quad (2.14)$$

Proof since

$$b(q - q_h, G_{2h}) = 0$$

$$b(q, G) = q(z), \quad b(q_h, G_{zh}) = q_h(z)$$

∀ mesh point z we have

$$q(z) - q_h(z) = q^I(z) - q_h(z) = b(q^I - q_h, G_{zh})$$

$$= b(q^I - q, G_{zh}) = \frac{h^2}{12} b(q'', G_{zh}) + O(h^3 | \ln h |^{3/2}) \|q\|_{3, \infty, \Gamma}$$

Hence (2.14) is obtained.

3. Extrapolation for the Collocation Method

The integral equation (2.2) can be expressed as the following

find $q \in H^{-1/2}(\Gamma)$ such that

$$\int_{\Gamma} q(x) E(x;y) dS_x = F(y) \quad (3.1)$$

Where

$$F(y) = \int_{\Gamma} u_0(x) \frac{\partial}{\partial n} E(x;y) ds_x + \frac{1}{2} u_0(y) - \int_{\Omega} f(x) E(x;y) dx$$

The collocation method for (3.1) is that

find $q_h \in V_h \subset H^{-1/2}(\Gamma)$ such that

$$\int_{\Gamma} q_h(x) E(x;y_j) ds_x = F(y_j) \quad j = 1, 2, \dots, n \quad (3.2)$$

where y_j are mesh points of Γ_h .

From (3.1) and (3.2) we have

$$\int_{\Gamma} (q(x) - q_h(x)) E(x;y_j) ds_x = 0, \quad j = 1, 2, \dots$$

Hence

$$\int_{\Gamma} (q^I(x) - q_h(x)) E(x;y_j) ds_x = \int_{\Gamma} (q^I(x) - q(x)) E(x;y_j) ds_x, \quad j=1, 2, \dots, n \quad (3.3)$$

i.e.

$$\int_{\Gamma} (q^I(x) - q_h(x)) E(x;y_j) ds_x = I(y_j) \quad j = 1, 2, 3, \dots, n \quad (3.4)$$

where $I(y)$ is defined by (2.10). By virtue of (2.13) we have

$$\int_{\Gamma} [q^I(x) - q_h(x) - \frac{h^2}{12} q''(s)] E(x; y_j) ds_x = O(h^3 \|q\|_{3, \infty, \Gamma}) \quad (3.5)$$

$$j = 1, 2, \dots, n.$$

Let N_i, N_{i+1} are shape function of linear element in one dimensional space.

$$N_i(s) = \frac{s_{i+1} - s}{h}, \quad N_{i+1}(s) = \frac{s - s_i}{h}$$

Then

$$(q^I(x) - q_h(x) - \frac{h^2}{12} q''(s)) \Big|_{s_i s_{i+1}} = \alpha_i N_i(s) - \alpha_{i+1} N_{i+1}(s)$$

The algebraic equations (3.5) can be expressed as

$$AX = F$$

where $X^T = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $F^T = \{f_1, f_2, \dots, f_n\}$

$$A = (a_{ij}),$$

$$a_{ij} = \int_{s_{i-1} s_i} N_{i+1}(s) E(x; y_j) ds_x + \int_{s_i s_{i+1}} N_i(s) E(x; y_j) ds_x$$

$$f_i = O(h^3 \|q\|_{3, \infty, \Gamma})$$

It is clear that

$$\left| \int_{s_i s_{i+1}} N_i(s) E(x; y_j) ds_x \right| = \left| \int_{s_i s_{i+1}} N_i(s) (E(x, y_j) - \tilde{E}(x_i, y_j)) ds_y \right|$$

$$+ \int_{s_i s_{i+1}} N_i(s) \tilde{E}(x; y_j) ds_x | \leq c h |\ln h| + c |\ln h|^{1/2}$$

Hence

$$a_{ij} = O(|\ln h|^{1/2})$$

$$\alpha_i = O\left(\frac{(|\ln h|^{1/2})^{n-1}}{(|\ln h|^{1/2})^n} h^3 |\ln h|\right) \|q\|_{3, \infty, \Gamma}$$

$$= O(h^3 |\ln h|^{1/2})$$

Finally, we obtain: \forall mesh point z

$$q^I(z) - q_h(z) = \frac{h^2}{12} q''(x) + O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma}$$

By virtue of $q^I(z) = q(z)$ we have

Lemma 4 Suppose $q \in C^\infty(\Omega)$ is a solution of (2.5) and q_h is a collocation solution of (3.2), then

$$q(z) - q_h(z) = \frac{h^2}{12} q''(x) + O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma} \quad (3.6)$$

for any mesh points z

From lemma 4 we deduce the extrapolation formula for collocation method

$$\frac{1}{3} (4 q_{h/2}(z) - q_h(z)) - q(z) = O(h^3 |\ln h|^{1/2}) \|q\|_{3, \infty, \Gamma} \quad (3.7)$$

4. Numerical Test

Let us consider the following problems

$$-\Delta u(x) = -2\pi\delta(x) \quad \forall x \in \Omega = \{x \mid |x| < 0.5\}.$$

$$u|_{\Gamma} = g(0.5) \quad \text{on } \Gamma = \partial\Omega$$

Exact and Approximation solutions

$\tilde{q}_h(x) = \frac{1}{3} (4q_{h/2}(x) + q_h(x))$	$q_h(x)$	$q_{h/2}(x)$	$q(x)$
1.9999950	1.9986870	1.9996680	2.0000000
1.9999770	1.9986910	1.9996560	2.0000000
1.9999810	1.9986830	1.9996570	2.0000000
1.9999740	1.9986800	1.9996520	2.0000000
1.9999790	1.9986900	1.9996570	2.0000000
1.9999790	1.9986830	1.9996550	2.0000000
1.9999700	1.9986910	1.9996500	2.0000000
1.9999800	1.9986860	1.9996570	2.0000000
1.9999640	1.9986880	1.9996450	2.0000000
1.9999880	1.9986890	1.9996630	2.0000000
1.9999650	1.9986860	1.9996450	2.0000000
1.9999800	1.9986890	1.9996580	2.0000000
1.9999720	1.9986830	1.9996500	2.0000000
1.9999800	1.9986920	1.9996580	2.0000000
1.9999550	1.9986830	1.9996370	2.0000000
1.9999860	1.9986880	1.9996620	2.0000000
1.9999720	1.9986880	1.9996510	2.0000000

$q_h(x) = \frac{1}{3} (4q_{h/2}(x) + q_h(x))$	$q_h(x)$	$q_{h/2}(x)$	$q(x)$
1.9999510	1.9986830	1.9996340	2.0000000
1.9999760	1.9986910	1.9996550	2.0000000
1.9999840	1.9986860	1.9996590	2.0000000
1.9999930	1.9986870	1.9996660	2.0000000
2.0000000	1.9986920	1.9996730	2.0000000
1.9999860	1.9986880	1.9996610	2.0000000
1.9999890	1.9986800	1.9996610	2.0000000
1.9999580	1.9986900	1.9996410	2.0000000
1.9999960	1.9986870	1.9996680	2.0000000
1.9999810	1.9986880	1.9996580	2.0000000
1.9999690	1.9986880	1.9996490	2.0000000
1.9999740	1.9986860	1.9996520	2.0000000
1.9999830	1.9986850	1.9996580	2.0000000
1.9999720	1.9986910	1.9996520	2.0000000
1.9999820	1.9986860	1.9996580	2.0000000

Error

$q(x) - \tilde{q}_h(x)$	$q(x) - q_h(x)$	$q(x) - q_{h/2}(x)$
.0000049	.0013126	.0003319
.0000225	.0013094	.0003443
.0000190	.0013167	.0003433
.0000263	.0013115	.0003477
.0000210	.0013096	.0003432

.0000215	.0013167	.0003452
.0000302	.0013092	.0003500
.0000196	.0013143	.0003432
.0000362	.0013115	.0003550
.0000119	.0013114	.0003368
.0000352	.0013142	.0003549
.0000197	.0013107	.0003425
.0000283	.0013169	.0003505
.0000197	.0013076	.0003417
.0000452	.0013165	.0003630
.0000137	.0013118	.0003382
.0000285	.0013118	.0003493
.0000491	.0013167	.0003660
.0000240	.0013086	.0003451
.0000161	.0013142	.0003406
.0000070	.0013133	.0003335
.0000001	.0013083	.0003272
.0000144	.0013117	.0003388
.0000114	.0013201	.0003387
.0000423	.0013101	.0003593
.0000044	.0013126	.0003315
.0000190	.0013118	.0003421
.0000305	.0013120	.0003510
.0000257	.0013144	.0003480
.0000170	.0013150	.0003415
.0000283	.0013087	.0003483
.0000185	.0013142	.0003424

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