ON RADIALLY SYMMETRIC SIMPLE WAVES IN ELASTICITY

BY

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On Radially Symmetric Simple Waves in Elasticity

K. A. Pericak-Spector

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Introduction

In this paper I look for simple wave solution equations of nonlinear elastodynamics. The main r that, under mild constitutive restrictions, every nontrivial radially symmetric simple wave must be of the form

$$\mathbf{f}(\mathbf{x}, t) = s(|\mathbf{x}|/(t+c))\mathbf{x}.$$

An explicit solution is then obtained in the special case of a linear material.

1. Simple Waves

Let $\Omega \subset \mathbb{R} \times \mathbb{R}$ be an open set and let's consider the following problem, which, as we shall see in §2, arises in nonlinear elastodynamics: Find $u\colon \Omega \to \mathbb{R}$ that satisfies

$$u_{tt} = F_1 u_{RR} + F_2 u_{Rt} + F_3 , \qquad (1.1)$$

where $F_i = F_i(R, t, u, u_R, u_t)$, $i = 1,2,3$, are known functions. In particular, we are interested in finding simple wave solutions of (1.1), i.e., solutions that are of the form

$$u(R,t) = g(\xi(R,t)), \qquad (1.2)$$

where g satisfies the nontrivial second order ODE that is obtained by substituting (1.2) into (1.1). More precisely, let $\xi \in C^2(\Omega, \mathbb{R})$. We say that ξ generates a **simple wave** for (1.1) if there are functions $J_k \in C^\infty(\text{Range}(\xi)\times\mathbb{R}^2,\mathbb{R})$, $k = 1,2$, such that

$$\left.\begin{aligned} p(R,t) &:= \frac{F_1\xi_{RR} + F_2\xi_{Rt} - \xi_{tt}}{\xi_t^2 - F_2\xi_R\xi_t - F_1\xi_R^2} = J_1(\xi, g(\xi), g'(\xi)) \\ q(R,t) &:= \frac{F_3}{\xi_t^2 - F_2\xi_R\xi_t - F_1\xi_R^2} = J_2(\xi, g(\xi), g'(\xi)) \end{aligned}\right\} \qquad (1.3)$$

for every $g \in C^2(\mathbb{R},\mathbb{R})$ and $(R,t) \in \Omega$. Here $F_i = F_i(R,t,g(\xi),\xi_R g'(\xi),\xi_t g'(\xi))$, $i = 1,2,3$.

Remark 1.1. Let $f \in C^2(\mathbb{R},\mathbb{R})$ and $f' \neq 0$. If ξ generates a simple wave then so does $\hat{\xi} = f(\xi)$. To see this notice that since ξ generates a simple wave for all $g \in C^2(\mathbb{R},\mathbb{R})$ all that is necessary is to take $g = f\circ G$ where $G \in C^2(\mathbb{R},\mathbb{R})$.

Remark 1.2. If one substitutes (1.2) into (1.1) and divides by the coefficient of g'', one obtains

$$g''(\xi(R,t)) = p(R,t)g'(\xi(R,t)) + q(R,t). \qquad (1.4)$$

Thus if ξ generates a simple wave according to the above definition, one can view (1.4) as a second order ODE for g as a function of the variable ξ.

Remark 1.3. Notice that geometrically, in the R-t plane, along curves $\xi(R,t)$ = constant we have that $u(R,t) = g(\xi)$ must be constant also.

Remark 1.4. Although simple wave is a widely used term in gas and fluid dynamics (cf., e.g., Courant & Friedrichs [2] and Whitham [4], respectively) it appears that Courant & Friedrichs

[2] were the first to apply this method to one dimensional nonlinear elasticity. Later Varley [3] used this method in the context of the three dimensional theory.

2. Elastodynamics

The governing dynamic equation for a homogeneous elastic material is

$$\rho \underset{\sim}{f}_{tt}(\underset{\sim}{x}, t) = \text{div}(\underset{\sim}{S}(\nabla \underset{\sim}{f}(\underset{\sim}{x}, t))), \qquad (2.1)$$

where $\rho \in \mathbb{R}^+$ is the (constant) reference <u>density</u> and $\underset{\sim}{S}$ is the (Piola-Kirchhoff) <u>stress</u>. Here $\nabla \underset{\sim}{f}(\underset{\sim}{x}, t)$ is the tensor field with components $(\nabla \underset{\sim}{f})_{ij} = \partial f_i/\partial x_j$ and div is the divergence operator on $\mathbb{R}^n$; i.e., div $\underset{\sim}{S}$ is the vector field with components $\Sigma_j\ \partial S_{ij}/\partial x_j$.

We assume that the material is hyperelastic and let $W(\nabla \underset{\sim}{f}(\underset{\sim}{x}, t))$ denote the stored energy at the point $\underset{\sim}{x}$ and time t in the motion $\underset{\sim}{f}$. The stress $\underset{\sim}{S}$ then satisfies the relation

$$\underset{\sim}{S}(\underset{\sim}{F}) = \frac{dW}{d\underset{\sim}{F}}(\underset{\sim}{F})$$

for any tensor $\underset{\sim}{F}$ with det $\underset{\sim}{F} > 0$. Further we assume that the material is <u>isotropic</u>; i.e., there is a symmetric C^2 function Φ: $(\mathbb{R}^+)^n \to \mathbb{R}$ such that

$$W(\underset{\sim}{F}) = \Phi(\lambda_1, \ldots, \lambda_n),$$

where λ_i are the eigenvalues of $(FF^T)^{1/2}$.

We look for radially symmetric solutions of (2.1) so that $\underset{\sim}{f}$ takes the form

$$\underset{\sim}{f}(\underset{\sim}{x}, t) = s(R, t)\underset{\sim}{x},$$

where $R = |\underset{\sim}{x}|$. Then $(\nabla \underset{\sim}{f})_{ij} = s\delta_{ij} + s_R x_i x_j / R$ and hence (cf., e.g., Ball [1]) $\lambda_1 = s + Rs_R$, $\lambda_2 = \lambda_3 = \ldots = \lambda_n = s$. Let $\Phi,_i$ be the derivative of Φ with respect to λ_i, then (2.1) transforms into

$$\rho s_{tt} = R^{-n}[\frac{\partial}{\partial R}(R^{n-1}\Phi,_1) - (n-1)\Phi,_2 R^{n-2}] \tag{2.2}$$

or equivalently

$$\begin{aligned} \rho s_{tt} = \Phi,_{11} s_{RR} &+ R^{-1}(2\Phi,_{11} + (n-1)\Phi,_{12})s_R \\ &+ R^{-2}(n-1)(\Phi,_1 - \Phi,_2). \end{aligned} \tag{2.3}$$

Definition. We say ξ: $\Omega \subset [0,\infty) \times \mathbb{R} \to \mathbb{R}$ generates a radial simple wave for (2.1) if ξ generates a simple wave for (2.2).

Lemma. $\xi = R/(t+c)$ generates a radial simple wave for (2.1).

Proof. In equation (2.3) we set $s(R,t) = g(R/(t+c))$. Then $s_R = g'/(t+c)$, $s_{RR} = g''/(t+c)^2$, $s_t = -Rg'/(t+c)^2$, and $s_{tt} = (2R(t+c)g' + R^2 g'')/(t+c)^4$. Also $\Phi,_{11}(s+Rs_R, s, s, \ldots, s) = \Phi,_{11}(g+\xi g', g, g, \ldots, g)$. Putting this into equation (2.3) we find that

$$\begin{aligned} \frac{\rho}{(t+c)^2}[2\xi g' + \xi^2 g''] = \Phi,_{11}(\frac{1}{t+c})^2 g'' & \\ + R^{-1}(2\Phi,_{11} + (n-1)\Phi,_{12})(\frac{1}{t+c})g' & \\ + R^{-2}(n-1)(\Phi,_1 - \Phi,_2). & \end{aligned}$$

Finally, we multiply by $(t+c)^2$ to arrive at

$$\rho[2\xi g' + \xi^2 g''] = \Phi,_{11} g'' + \frac{1}{\xi}(2\Phi,_{11} + (n-1)\Phi,_{12})g' + \frac{1}{\xi^2}(n-1)(\Phi,_1 - \Phi,_2),$$

which is a <u>second order nonlinear ODE</u> for g as a function of ξ. ■

<u>Remark 2.1</u>. Note that $\xi = R/(t+c)$ generates a radial simple wave <u>for all stored energy functions</u> Φ.

We will now state and prove the main

Theorem. *Suppose there is a* $\lambda_0 > 0$ *such that*

$$\Phi,_{11}(\lambda_0, \lambda_0, \ldots, \lambda_0) \neq \Phi,_{12}(\lambda_0, \lambda_0, \ldots, \lambda_0),$$

and that

$$\left.\frac{d}{d\lambda}\Phi,_{11}(\lambda, \lambda, \ldots, \lambda)\right|_{\lambda=\lambda_0} \neq 0.$$

Then $\xi = R/(t+c)$, $\xi = \xi(t)$ *and* $\xi = \xi(R)$ *are the only functions that generate radial simple waves for* (2.1).

Proof. If we set $s(R,t) = g(\xi(t))$ or $s(R,t) = g(\xi(R))$ then ξ (trivially) generates a simple wave solution to (2.3), the latter case being the radial equation of elastostatics.

If we now set $s(R,t) = g(\xi(R,t))$ in equation (2.3) then

$$s_t = g'\xi_t, \quad s_{tt} = g''(\xi_t)^2 + g'\xi_{tt},$$
$$s_R = g'\xi_R, \quad s_{RR} = g''(\xi_R)^2 + g'\xi_{RR},$$

and equation (2.3) becomes

$$\rho((\xi_t)^2 g'' + \xi_{tt} g') = \phi,_{11}((\xi_R)^2 g'' + \xi_{RR} g')$$
$$+ R^{-1}(2\phi,_{11} + (n-1)\phi,_{12})\xi_R g'$$
$$+ R^{-2}(n-1)(\phi,_1 - \phi,_2),$$

where $\phi = \phi(g + R\xi_R g', g, \ldots, g)$ or equivalently

$$g'' = \frac{\phi,_{11}\xi_{RR} - \rho\xi_{tt} + R^{-1}\xi_R(2\phi,_{11} + (n-1)\phi,_{12})}{\rho(\xi_t)^2 - (\xi_R)^2\phi,_{11}} g' + \frac{R^{-2}(n-1)(\phi,_1 - \phi,_2)}{\rho(\xi_t)^2 - (\xi_R)^2\phi,_{11}}. \tag{2.4}$$

From the definition of a simple wave we see that

$$J_1(\xi, g(\xi), g'(\xi)) = \frac{\phi,_{11}\xi_{RR} - \rho\xi_{tt} + R^{-1}\xi_R(2\phi,_{11} + (n-1)\phi,_{12})}{\rho(\xi_t)^2 - (\xi_R)^2\phi,_{11}},$$

and

$$J_2(\xi, g(\xi), g'(\xi)) = \frac{R^{-2}(n-1)(\phi,_1 - \phi,_2)}{\rho(\xi_t)^2 - (\xi_R)^2\phi,_{11}},$$

which hold for every g. If we let $g(\xi) = \lambda_0 + \epsilon\xi$, we find that

$$J_2(\xi, g(\xi), g'(\xi)) = J_2(\xi, \lambda_0 + \epsilon\xi, \epsilon).$$

At $\epsilon = 0$, $J_2(\xi, \lambda_0, 0) = 0$[1]. If we differentiate $J_2(\xi, \lambda_0 + \epsilon\xi, \epsilon)$ with respect to ϵ and set ϵ equal to zero we conclude that

---

[1] Recall that ϕ is symmetric, i.e., $\phi(a,b,c,\ldots,c) = \phi(b,a,c,\ldots,c)$ and hence $\phi,_1(a,b,c,\ldots,c) = \phi,_2(b,a,c,\ldots,c)$.

$$P(\xi,\lambda_0) := \frac{d}{d\epsilon} J_2(\xi,\lambda_0+\epsilon\xi,\epsilon)\Big|_{\epsilon=0}$$

$$= \frac{R^{-1}(n-1)\xi_R(\Phi,_{11}(\lambda_0,\ldots,\lambda_0) - \Phi,_{21}(\lambda_0,\ldots,\lambda_0))}{\rho(\xi_t)^2 - \Phi,_{11}(\lambda_0,\ldots,\lambda_0)(\xi_R)^2} . \qquad (2.5)$$

But $\Phi,_{11}(\lambda_0,\ldots,\lambda_0) \neq \Phi,_{21}(\lambda_0,\ldots,\lambda_0)$ so that $P(\xi,\lambda_0) \not\equiv 0$ unless $\xi_R \equiv 0$ and hence $\xi = \xi(t)$.

Therefore, without loss of generality, we shall assume that $P(\xi,\lambda_0)$ is not identically zero. Note that the numerator can then be written as $h^{-2}(R,t)Q(\lambda_0)$, where $h(R,t) = |R/(n-1)\xi_R|^{1/2} > 0$. Rearranging (2.5) we obtain

$$h^2[\rho(\xi_t)^2 - \Phi,_{11}(\lambda_0,\ldots,\lambda_0)(\xi_R)^2] = \frac{Q(\lambda_0)}{P(\xi,\lambda_0)} .$$

If we write $\Phi^0,_{11} = \Phi,_{11}(\lambda_0,\ldots,\lambda_0)$ and let $\alpha(\xi) := Q(\lambda_0)/P(\xi,\lambda_0)$ then

$$\alpha(\xi) = h^2(R,t)(\rho(\xi_t)^2 - \Phi^0,_{11}(\xi_R)^2).$$

We now consider another function $g = \lambda_1+\epsilon\xi$ (for some λ_1 sufficiently close to λ_0). Then

$$\beta(\xi) = h^2(R,t)(\rho(\xi_t)^2 - \Phi^1,_{11}(\xi_R)^2),$$

where $\Phi^1,_{11} = \Phi,_{11}(\lambda_1,\ldots,\lambda_1)$ and $\beta(\xi) := Q(\lambda_1)/P(\xi,\lambda_1)$. Since $\frac{d}{d\lambda}\Phi,_{11}(\lambda,\ldots,\lambda)\Big|_{\lambda=\lambda_0} \neq 0$ we can choose λ_1 such that $\Phi^1,_{11} \neq \Phi^0,_{11}$.

We now have two equations and two unknowns. Solving these equations for $h\xi_R$ and $h\xi_t$ we see that

$$h\xi_R = \left[\frac{\alpha - \beta}{\phi^1_{,11} - \phi^0_{,11}}\right]^{1/2},$$

and

$$h\xi_t = \left[\frac{\alpha\phi^1_{,11} - \beta\phi^0_{,11}}{\rho(\phi^1_{,11} - \phi^0_{,11})}\right]^{1/2}.$$

If $\alpha \equiv \beta$ then $\xi_R \equiv 0$ and $\xi = \xi(t)$. If $\alpha\phi^1_{,11} \equiv \beta\phi^0_{,11}$ then $\xi_t \equiv 0$ and $\xi = \xi(R)$. Thus we shall assume that $\alpha-\beta \not\equiv 0$ and $\alpha\phi^1_{,11} - \beta\phi^0_{,11} \not\equiv 0$. If we now divide $h\xi_t$ by $h\xi_R$ we arrive at

$$\xi_t + f(\xi)\xi_R = 0,$$

where $f(\xi) = [(\alpha\phi^1_{,11} - \beta\phi^0_{,11})/\rho(\alpha-\beta)]^{1/2}$ (or equivalently

$$f(\xi) = [(Q(\lambda_0)\phi^1_{,11}P(\xi,\lambda_1)-Q(\lambda_1)\phi^0_{,11}P(\xi,\lambda_0))/(Q(\lambda_0)-Q(\lambda_1))]^{1/2}).$$

We can now solve this first order hyperbolic partial differential equation. Along $\frac{dR}{dt} = f(\xi(R(t),t))$ we have $\frac{d\xi}{dt}(R(t),t) = 0$ with $R(t_0) = R_0$ and $\xi(R(t_0),t_0) = \xi_0(R_0)$. Thus the solution is $\xi(R,t) = \xi_0(R_0)$ along the lines $R = (t-t_0)f(\xi_0(R_0)) + R_0$.

<u>Case 1.</u> Suppose that $f \equiv c$ is constant. Then $R_0 = R - c(t-t_0)$ and $\xi(R,t) = \xi_0(R - c(t-t_0))$. Putting this value of ξ into (2.5) we arrive at

$$P(\xi,\lambda_0) = \frac{n-1}{R\xi_0'}\,\frac{Q(\lambda_0)}{\rho-\phi^0_{,11}}.$$

The right hand side will not be a function of $\xi = \xi_0(R-c(t-t_0))$ unless $c \equiv 0$ and hence $\xi = \xi(R)$.

<u>Case 2.</u> Assume that f is not constant. Since $P(\xi,\lambda) \in C^\infty$ we know that $f \in C^\infty$. Therefore, in some neighborhood where

$\alpha\phi^1_{,11} - \beta\phi^0_{,11}$ is not zero f is invertible, i.e., $\xi_0(R_0) = f^{-1}((R-R_0)/(t-t_0))$. Hence $\xi(R,t) = f^{-1}((R-R_0)/(t-t_0)) = F((R-R_0)/(t-t_0))$ (notice that F is C^∞). Putting this into (2.5) we have that

$$P(\xi,\lambda_0) = \frac{n-1}{F'} \frac{Q(\lambda_0)}{\rho\left[\frac{R-R_0}{t-t_0}\right]^2 - \phi^0_{,11}} \frac{t-t_0}{R}.$$

Thus $\xi = F((R-R_0)/(t-t_0))$ holds if and only if $R_0 = 0$. Therefore $\xi = R/(t-t_0)$ (see Remark 1.1). ■

<u>Remark 2.2.</u> The conditions of the above theorem can be weakened. Suppose that $\phi \in C^r$ and that there exists $\lambda_0 > 0$ and integer $m \in (0,r]$ such that

$$\frac{d^m}{d\lambda_1^m}(\phi_{,1}(\lambda_1,\lambda_2,\dots,\lambda_n) - \phi_{,2}(\lambda_1,\lambda_2,\dots,\lambda_n))\Big|_{\lambda_1=\lambda_0} \neq 0$$

and

$$\frac{d}{d\lambda}\phi_{,11}(\lambda,\lambda,\dots,\lambda)\Big|_{\lambda=\lambda_0} \neq 0$$

then the conclusions of the theorem remain valid. The proof is modified by setting $g = \lambda_0 + \sum_{j=1}^{m} \frac{\epsilon^j}{j!}\xi$. If you then differentiate $J_2(\xi,\lambda_0 + \sum_{j=1}^{m}\frac{\epsilon^j}{j!}\xi, \sum_{j=1}^{m}\frac{\epsilon^j}{j!})$ with respect to ϵ m-times the result follows as above.

3. A Simple Example.

Consider the particular case $\phi = \alpha \sum_{i=1}^{n} \lambda_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j$ and $\rho \equiv 1$. Recall that $\lambda_1 = s + Rs_R$ and $\lambda_2 = \cdots = \lambda_n = s$. Then $\phi,_1 = \phi,_2 = \alpha + \sum_{j=1}^{n} \lambda_j$, and $\phi,_{11} = \phi,_{12} = 1$. Substituting this into (2.3) we arrive at

$$s_{tt} = s_{RR} + R^{-1}(n+1)s_R. \tag{3.1}$$

Set $s(R,t) = g(\xi(R,t)) = g(R/(t+c))$. Then (3.1) becomes

$$(\xi^2 - 1)g'' = (\frac{n+1}{\xi} - 2\xi)g'. \tag{3.2}$$

This is a <u>second order linear ordinary differential equation</u> which has singularities at $\xi = 0, 1, -1$ but can still be solved explicitly.

<u>Remark 3.1</u>. This is equivalent to studying the equations of linear elasticity where $\underset{\sim}{f}$, in (2.1), would be the displacement instead of the deformation.

<u>Remark 3.2</u>. Notice that $(\xi^2 - 1)$ comes from $(\xi^2 - \phi,_{11})$ and that for general ϕ, $(\xi^2 - \phi,_{11})$ will be a point of singularity.

We can now solve (3.2) for g' to arrive at

$$g'(\xi) = A|\xi^2 - 1|^{\frac{n-1}{2}} |\xi|^{-(n+1)}.$$

Recall that n is the dimension of the space. Consider first the Cauchy problem in $\mathbb{R}^3$ with all points of a homogeneously deformed body subjected to a radial initial velocity of magnitude $v_0 > 0$ (outward). The initial data is given by

$$\underset{\sim}{f}(\underset{\sim}{x},0) = \lambda\underset{\sim}{x},$$

$$\underset{\sim}{f}_t(\underset{\sim}{x},0) = \frac{v_0}{R}\,\underset{\sim}{x},$$

or in terms of the function s we have that

$$s(R,0) = \lambda,$$

$$s_t(R,0) = v_0/R.$$

If we now solve the above ODE for g we arrive at

$$g = [\operatorname{sgn}(\xi^2 - 1)]A\left[-\xi^{-1} + \frac{1}{3}\xi^{-3}\right] + B.$$

Putting the last equation back in terms of s(R, t) and applying the initial conditions, we find the solution

$$s(R,t) = \begin{cases} v_0\left[\frac{t}{R} - \frac{1}{3}\left(\frac{t}{R}\right)^3\right] + \lambda & \text{for } R \geq t \geq 0 \\ \lambda + \frac{2}{3}v_0 & \text{for } t > R \geq 0. \end{cases}$$

Remark 3.3. In this first example we set c = 0 in $\xi = R/(t+c)$ since (t+c) just represents a change in time scale.

Let us now turn our attention to Figure 1. At time $t = t_0$

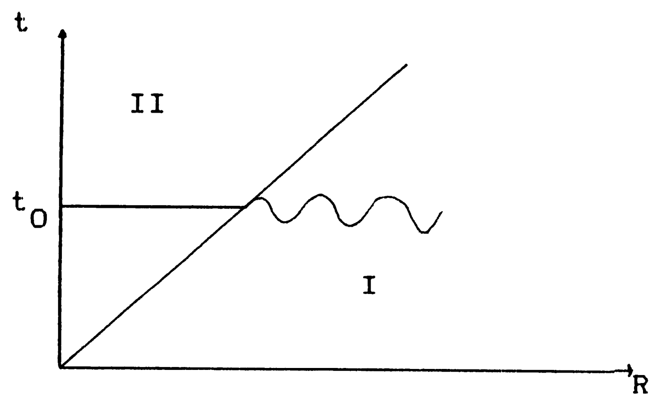

Figure 1

we have the following situation: For $R \in [0, t_0]$ (region II) the body is in a homogeneously compressed or stretched state (depending on whether $\lambda + \frac{2}{3}v_0 > 1$ or $0 < \lambda + \frac{2}{3}v_0 < 1$). In region I ($R > t_0$) a wave is propagating to infinity (recall that $v_0 > 0$). As t increases the wave moves farther away from the center of the body and the part of the body left in the final homogeneous deformation, $\underset{\sim}{f}(\underset{\sim}{x}, t) = (\lambda + \frac{2}{3}v_0)\underset{\sim}{x}$, becomes larger (region II).

Remark 3.4. In the case of a finite body of radius R_0 we consider the same initial values with a further restriction that the boundary is loaded with the solution, i.e., we specify the displacement boundary condition

$$s(R_0, t) = \begin{cases} v_0\left(\frac{t}{R_0} - \frac{1}{3}\left(\frac{t}{R_0}\right)^3\right) + \lambda & \text{for} \quad 0 \le t < R_0 \\ \lambda + \frac{2}{3}v_0 & \text{for} \quad R_0 \le t. \end{cases}$$

Physically the wave is, in some sense, being pulled out of the boundary. Notice that the final deformation is given by $\underset{\sim}{f}(\underset{\sim}{x}, t) = (\lambda + \frac{2}{3} v_0)\underset{\sim}{x}$ for all $t \ge R_0$ and $|\underset{\sim}{x}| \le R_0$.

Finally, let us consider a slightly different problem. Suppose that an acceleration wave of strength a is propagated toward the center of a finite (homogeneous) ball of radius R_0 by applying an impulse to the boundary at $t = 0$. The initial-boundary value problem we will consider (in terms of the function s) consists of the differential equation with the following initial and boundary conditions

$$s(R,0) = \lambda,$$

$$s_t(R,0) = 0,$$

$$s(R_0,t) = \begin{cases} \frac{a}{2}\left[-\frac{t-R_0}{R_0} + \frac{1}{3}\left[\frac{t-R_0}{R_0}\right]^3\right] + \lambda - \frac{a}{3} & \text{for } 0 \le t < 2R_0 \\ \lambda - \frac{2}{3}a & \text{for } 2R_0 \le t, \end{cases}$$

where $a > 0$. It is important to notice that <u>these conditions have been chosen to be compatible with a simple wave solution</u>. Applying these initial and boundary conditions the solution (for $R \in [0,R_0]$ and $t \ge 0$) is

$$s(R,t) = \begin{cases} \lambda & \text{for } R_0-t > R \ge 0 \\ \frac{a}{2}\left[-\frac{(t-R_0)}{R} + \frac{1}{3}\left[\frac{t-R_0}{R}\right]^3\right] + \lambda - \frac{a}{3} & \text{for } R_0 \ge R \ge |t-R_0| \\ \lambda - \frac{2}{3}a & \text{for } t-R_0 > R \ge 0. \end{cases}$$

Consider Figure 2. In the region labelled I we have that

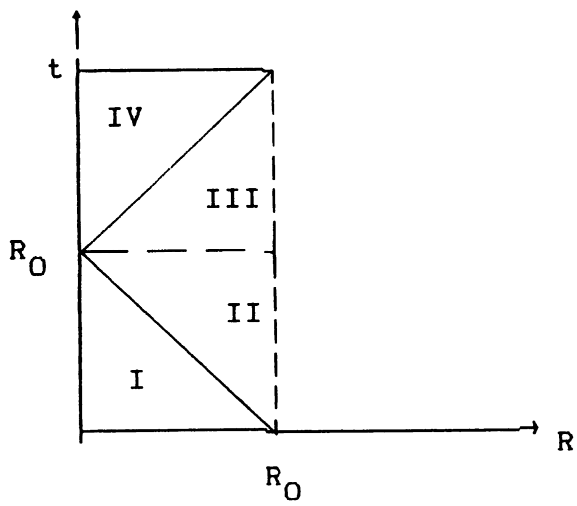

Figure 2

$R_0-t > R \ge 0$ and in the region labelled II, $R/(R_0-t) > 1$. At $t = 0$ the part of the body in region I is in a homogeneous stretched or compressed state (depending on whether $\lambda > 1$ or $0 < \lambda < 1$). At the boundary of the body ($R = R_0$) an impulse, of strength a, is applied initially and then a smooth displacement is prescribed. Notice that the boundary conditions are compatible with the initial conditions so that the deformation $\underset{\sim}{f}$ and the first derivatives of $\underset{\sim}{f}$ are continuous. As $t \to R_0^-$ the wave approaches the center of the body. Notice that the homogeneous portion of the body gets smaller as $t \to R_0^-$ (as region I shrinks so does the homogeneous part of the body shrink). At $t = R_0$ the wave focuses.

Let us now consider $t > R_0$. When $R_0 \geq R > t-R_0$ (region III) the wave propagates back to (and is pulled out of) the boundary. Finally for $R < t-R_0$ (region IV) the body is left in some new homogeneous state. As $t \to 2R_0^-$ the part of the body denoted by region IV becomes larger and the acceleration wave will be farther from the center of the body. Finally, at $t = 2R_0$, the body is left in the final deformation $\underset{\sim}{f}(\underset{\sim}{x},t) = (\lambda - \frac{2}{3}a)\underset{\sim}{x}$. Notice that the deformation and velocity are continuous functions throughout the entire body for all time. The lines $\left|\frac{R}{t-R_0}\right| = 1$ are the front of an acceleration wave of constant strength a.

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