LARGE DEVIATIONS ANALYSIS OF REFLECTED DIFFUSIONS AND CONSTRAINED STOCHASTIC APPROXIMATION ALGORITHMS IN CONVEX SETS

by

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Abstract

Let G be a bounded convex set. Then the solution to the Skorokhod problem for a given path ϕ may be considered as a reflected version of ϕ with respect to G. By use of estimates on the behavior of this deterministic transformation, large deviations type asymptotic estimates for reflected diffusions are shown to hold. Let Π_G be the projection onto G and $\{\xi_n\}$ be an i.i.d. Gaussian sequence. Then from the diffusion results estimates on the asymptotic behavior of recursive algorithms of the type
$X^\varepsilon_{n+1} = \Pi_G(X^\varepsilon_n + \varepsilon(b(X^\varepsilon_n) + \sigma(X^\varepsilon_n)\xi_n))$ (or $X_{n+1} = \Pi_G(x_n + a_n(b(X_n) + \sigma(X_n)\xi_n))$, $0 < a_n \to 0$, $\Sigma a_n = \infty$) are shown to follow. Let θ be a stable point of the algorithm, let D be a neighborhood of θ with respect to G, let $A = \{\phi \in C_\theta[0,T] : \phi(t) \notin D$ for some $t < T\}$, and define $x^\varepsilon(\cdot)$ to be the piecewise linear interpolation of X^ε_n starting at θ and having interpolation interval ε. Then estimates on $P\{x^\varepsilon(\cdot) \in A\}$, the probability of escape from D before time T, are obtained. This analysis yields an alternative to convergence results on the 'asymptotic normality' of errors about θ, which are in any case not applicable if $\theta \in \partial G$. These and other estimates provided by the large deviations methods are often more useful in applications. Extensions are outlined for correlated noise, unbounded domains, and domains that are smooth with 'convex corners'.

1. Introduction

We shall be concerned in this paper with proving large deviations type theorems for dynamical systems driven by small noises and which are constrained in some way or other to lie in a given bounded, convex subset of R^d. We shall consider first the case of a reflected diffusion in which the domain G is convex and the reflection is along the normal (or normals). From this we derive results for a class of stochastic approximation (SA) algorithms which are driven by white Gaussian noise and which are constrained to lie in G via projection. We then indicate extensions for both types of processes such as the case in which ∂G is an intersection of smooth manifolds with 'convex corners', the case of correlated driving noises in the SA algorithms, and unbounded domains.

In the case where ∂G is C^2 the reflected diffusion problem has been treated previously by Anderson and Orey [1], and Doss and Priouret [3]. In both cases localization arguments were used which cannot be applied when the set is not assumed to have a smooth boundary. These references, as well as [6], consider the application of large deviations theory to the analysis of singularly perturbed partial differential equations.

Tanaka [11] considers the problems of construction and uniqueness of solutions for a normally reflected diffusion in a convex set, and we shall make use of several facts derived there. See also Snitzman and Lions [8], who consider these problems for more general domains and types of reflection.

Typical results are the following. To be specific, we concentrate first on the reflected diffusion. Let $x^{\varepsilon}(\cdot)$ denote the solution to a stochastic differential equation with reflection with drift b and diffusion matrix $\varepsilon\sigma$ (a precise definition will be given later). Let x be the initial value of $x^{\varepsilon}(\cdot)$, let P_x denote probability given $x^{\varepsilon}(0) = x$, and let $C_x[0,T]$ be the set of continuous paths on $[0,T]$ with initial point x and values in R^d. Then it is shown that there is a functional (known as an action functional, or rate functional)

$$S_x(T,\cdot) : C_x[0,T] \to [0,\infty]$$

such that for any set of paths $A \subset C_x[0,T]$

(1.1) $$- \inf_{\phi \in A^\circ} S_x(T,\phi) \leq \underline{\lim}_{\varepsilon \to 0} \varepsilon^2 \log P_x\{x^\varepsilon(\cdot) \in A\}$$

$$\leq \overline{\lim}_{\varepsilon \to 0} \varepsilon^2 \log P_x\{x^\varepsilon(\cdot) \in A\}$$

$$\leq - \inf_{\phi \in \overline{A}} S_x(T,\phi)$$

where A° and $\overline{A}$ are the interior and closure of A, respectively.

Let $\Pi_G(x)$ be the projection of x onto the point in G nearest it, and consider the projected recursive (or stochastic approximation) algorithm

(1.2) $$X_{n+1}^\varepsilon = \Pi_G \left(X_n^\varepsilon + \varepsilon \left(b(X_n^\varepsilon) + \sigma(X_n^\varepsilon) \xi_n\right)\right), \; X_0^\varepsilon = x.$$

where ξ_n is a sequence of i.i.d. Gaussian random vectors.

If we define $x^\varepsilon(\cdot)$ as the piecewise linear interpolated version of X_n^ε with interpolation interval ε then estimates analogous to (1.1) are obtained.

Let θ be a stable point of the "mean" algorithm given by (1.2) (described below). Let D be a neighborhood of θ with respect to G. By defining $A = \{\phi \in C_\theta[0,T]: \phi(t) \notin D \text{ some } t \leq T\}$, (1.1) gives an estimate of important quantities such as $P_\theta\{\tau_D^\varepsilon \leq T\}$, where τ_D^ε is the time of first escape from D. Under additional hypotheses, quantities such as $E_\theta \tau_D^\varepsilon$ and most likely escape routes may be estimated [5]. As an example of how such estimates are useful one may consider the information they provide when comparing the relative stability of different algorithms. Alternative methods such as those based on linearization of errors about θ use only local information, to prove (for example) that if $\theta \in G$ then the normalized errors $(X^\varepsilon(\cdot) - \theta)/\varepsilon$ converge weakly to the solution of a stable Itô equation. Such a result does not provide a good estimate for $P_\theta\{\tau_D \leq T\}$, and unlike (1.1) it reflects only the local structure of the dynamics. If $\theta \in \partial G$ the such a linearization cannot in any case be done.

The paper is organized as follows. Section 2 introduces the basic assumptions and definitions, defines a deterministic transformation on paths which we shall consider to be the effect of reflection, and obtains estimates on this transformation. Section 3 states and proves the large deviations theorem for reflected diffusions, except for several important properties of the action functional that are proved in section 4. In section 5 the results for SA algorithms are shown to follow from those for reflected diffusions, and in section 6 various extensions are outlined.

2. Assumptions, definitions and the Skorokhod problem.

Throughout this paper we shall assume

(i) We are given a d-dimensional drift $b(\cdot)$, and a $d \times d_1$-dimensional diffusion matrix $\sigma(\cdot)$, satisfying

$$|b(\cdot)| \leq C, \quad |\sigma(\cdot)| \leq C$$

$$|b(x) - b(y)| \leq K|x-y|, \quad |\sigma(x) - \sigma(y)| \leq K|x-y|.$$

(ii) The matrix $a(x) = \sigma(x)\sigma'(x)$ is either uniformly nondegenerate or else it has the following particular degenerate form

$$a(x) = \begin{bmatrix} a_{11}(x) & 0 \\ & \\ 0 & 0 \end{bmatrix}$$

where $a_{11}(\cdot)$ is $d_2 \times d_2$, $1 \leq d_2 < d$, and there is $\alpha > 0$ such that $a_{11}(x) \geq \alpha|x|^2$ (i.e. $a_{11}(\cdot)$ is uniformly nondegenerate).

In sections 2 through 5 we shall assume

(iii) G is convex and compact.

The Skorokhod Problem.

Let $n(x)$ be the set valued function giving the outward normals at $x \in \partial G$, i.e.

$$\gamma \in n(x) \Leftrightarrow \forall x' \in G, \langle \gamma, x-x' \rangle \geq 0, \text{ and } |\gamma| = 1.$$

Denote the total variation of a function $\eta(\cdot)$ on the interval $[0,T]$ by $|\eta|(T)$. We will say that a triple of functions (ϕ,ψ,η) are associated, or that they satisfy the Skorokhod problem, (with respect to G and on the interval $[0,T]$ being understood) if $|\eta|(T)$ is finite, and if

$$\phi(\cdot) = \psi(\cdot) + \eta(\cdot), \ \phi(0) = \psi(0)$$

$$\phi(t) \in \bar{G} \text{ for } t \in [0,T]$$

$$|\eta|(t) = \int_0^t I_{\{\phi(s) \in \partial G\}} \, d|\eta|(s)$$

$$\eta(t) = - \int_0^t \gamma(s) d|\eta|(s)$$

where $\gamma(s) \in n(\phi(s))$. (In the remainder of the paper we substitute $n(\phi(s))$ for $\gamma(s)$, the above interpretation being understood.) Hence $\phi(\cdot)$ never leaves G, $\eta(t)$ changes only when $\phi(t) \in \partial G$, in which case the change points in the direction $- n(\phi(t))$. We interpret $\phi(\cdot)$ as the reflected version of $\psi(\cdot)$. We next present several lemmas concerning the dependence of ϕ on ψ.

Lemma 2.1. *Let the triples* (ϕ_i, ψ_i, η_i) *be associated for* $i=1,2$, *with* $\psi_1(0) = \psi_2(0) = x$. *Then*

$$\sup_{0\leq t\leq T} |\phi_1(t) - \phi_2(t)|^2 \leq \sup_{0\leq t\leq T} |\psi_1(t) - \psi_2(t)|^2$$

$$+ 4 \sup_{0\leq t\leq T} |\psi_1(t) - \psi_2(t)|(|\eta_1|(T) + |\eta_2|(T))$$

Proof.

$$d|\phi_1(t) - \phi_2(t)|^2$$

$$= 2\langle \phi_1(t) - \phi_2(t), d(\psi_1(t) - \psi_2(t) + \eta_1(t) - \eta_2(t))\rangle$$

$$\leq 2\langle \phi_1(t) - \phi_2(t), d(\psi_1(t) - \psi_2(t))\rangle$$

$$= d|\psi_1(t) - \psi_2(t)|^2 + 2\langle\eta_1(t) - \eta_2(t), d(\psi_1(t) - \psi_2(t))\rangle,$$

where the second inequality follows from $\langle\phi_i(t) - \phi_{3-i}(t), d\eta_i(t)\rangle \le 0$. Hence

$$|\phi_1(t) - \phi_2(t)|^2 \le |\psi_1(t) - \psi_2(t)|^2$$

$$+ 2 \int_0^t \langle\eta_1(s) - \eta_2(s), d(\psi_1(s) - \psi_2(s))\rangle ds$$

$$= |\psi_1(t) - \psi_2(t)|^2 + 2\langle\eta_1(t) - \eta_2(t), \psi_1(t) - \psi_2(t)\rangle$$
$$- 2 \int_0^t \langle\psi_1(s) - \psi_2(s), d(\eta_1(s) - \eta_2(s))\rangle ds .$$

Q.E.D.

Lemma 2.2 _Let the triple_ (ϕ,ψ,η) _be associated_. _Then for_ $s \ge 0$, $h>0$,

$$\sup_{s\le t\le s+h} |\phi(t) - \phi(s)|^2 \le \sup_{s\le t\le s+h} |\psi(t) - \psi(s)|^2$$

$$+ 4 \sup_{s\le t\le s+h} | \psi(t) - \psi(s)|(|\eta|(s+h) - |\eta|(s)).$$

Proof. In Lemma 2.1, substitute

$$\psi_1(t) = \psi(t+s), \ \psi_2(t) = \psi(s),$$

and note that $(\phi(\cdot+s) - \phi(s) + \psi(s), \psi(\cdot+s), \ \phi(\cdot+s) - \phi(s) - \psi(\cdot+s) + \psi(s))$

and $(\psi(s), \psi(s),0)$ are associated triples. Q.E.D.

Remark. Note that no assumption is made on the continuity of the triples in Lemmas 2.1 and 2.2.

Lemma 2.3. _Let the triple_ (ϕ,ψ,η) _be associated_. _Let_ B _be larger than the diameter of_ G, _and pick_ $x_0 \in G$ _and_ $\beta>0$ _such that_ $\{x: |x-x_0| \le \beta\} \subset G$. _Then there exists_ $\gamma>0$ _and_ $K' < \infty$ _depending only on_ β _and_ B _(and not on_ $\psi(0)$) _such that if for_ $h>0$ _we have_

(2.1) $$\sup \{|\psi(t) - \psi(s)| \mid 0 \le s \le t \le T, t-s \le h\} \le \gamma$$

then for $0 \le s \le t \le T$,

$$|\eta|(t) - |\eta|(s) \le (T/h + 1)K' \sup_{0\le s\le t\le T} |\psi(t) - \psi(s)|$$

Remark. The proof is contained in the proof of [11; Theorem 2.1]. Although Tanaka assumes that ψ is continuous, the proof is valid in general. The basis for the proof is the following. If there were a unit vector e and $c>0$ satisfying $\langle e,n(x)\rangle \ge c$ for all $x \in \partial G$, then we would have

$$\langle e,\phi(t) - \phi(s)\rangle = \langle e,\psi(t) - \psi(s)\rangle + \langle e,\eta(t) - \eta(s)\rangle,$$

so that

$$c(|\eta|(t) - |\eta|(s)) \le |\phi(t) - \phi(s)| + |\psi(t) - \psi(s)|. \tag{2.2}$$

From Lemma 2.2 we obtain that for some $K_1 < \infty$

$$|\phi(t) - \phi(s)| \le K_1 \sup_{0\le s\le t\le T} |\psi(t) - \psi(s)| + \frac{c}{2}(|\eta|(t) - |\eta|(s)),$$

which together with (2.2) gives the correct bound. The boundedness and convexity of G is then used to show that such an e can be found for each of a finite number of patches of ∂G, and then the condition (2.1) is used to ensure that ϕ will only visit T/h of these patches in $[0,T]$. Similar assumptions were used in [8] in the case of a more complicated domain.

Using these estimates Tanaka shows by approximating a given $\psi \in C_x[0,T]$ by piecewise continuous functions that there exist (ϕ,η) such that (ϕ,ψ,η) are associated. Uniqueness follows from Lemmas 2.1 and 2.3. If $\psi_n \in C_x[0,T]$ goes to ψ uniformly, then there are $h>0$ and $n'<\infty$ such that (2.1) holds for $n \ge n'$. Hence by Lemma 2.1 the restriction of the mapping $\psi \to \phi$ to $C_x[0,T]$ is continuous. We shall denote this mapping both by $\phi = F_x(\psi)$ and $\phi = \tilde{\psi}$. We shall consider as a solution of the stochastic differential equation with reflection (SDER) with initial point x a triple

$$\left(x(t),\ \int_0^t b(x(s))ds + \int_0^t \sigma(x(s))dw(s) + x,\ \xi(t)\right)$$

which is associated (almost surely) and for which $x(t)$ is a continuous semimartingale which is adapted to some σ-algebra $\mathcal{F}(t)$, which has the property that w_t is a Brownian motion with respect to $\mathcal{F}(t)$. We shall denote $z(t) = x(t) - \xi(t)$, and refer to $z(\cdot)$ as the unrestricted process.

Consider now a sequence $(x^\varepsilon, z^\varepsilon, \xi^\varepsilon)$ of solutions of the SDER, where ε multiplies the diffusion matrix $\sigma(\cdot)$, i.e.

$$x^\varepsilon(t) = \int_0^t b(x^\varepsilon(s))ds + \varepsilon \int_0^t \sigma(x^\varepsilon(s))dw(s) + x + \xi^\varepsilon(t) .$$

Our goal is Ventsel-Freidlin type large deviations estimates for $x^\varepsilon(\cdot)$, which describe the limiting behavior of $x^\varepsilon(\cdot)$ as $\varepsilon \to 0$.

3. Large Deviations for Reflected Diffusions

We pause now to define the action functional associated with the process x^ε. For $a(\cdot)$ uniformly nondegenerate let

$$L(x,\beta) = (\beta - b(x))' a^{-1}(x)(\beta - b(x))/2,$$

and for the special degenerate case considered above let

$$L(x,\beta) = \begin{cases} (\beta_1 - b_1(x))' a_{11}^{-1}(x)(\beta_1 - b_1(x))/2, & \beta_2 = b_2(x) \\ \infty & , \quad \beta_2 \neq b_2(x). \end{cases}$$

For $\phi \in C_x[0,T]$ the action functional is then given by

$$S_x(T,\phi) = \inf \{\int_0^T L(\phi,\dot{\psi})dt\}$$

where the infimum is over all $\psi(\cdot)$ satisfying $\phi(\cdot) = \tilde{\psi}(\cdot)$. Here and below we set $\int_0^T L(\phi,\dot{\psi})dt = \infty$ if ψ is not absolutely continuous (AC), or if there is no ψ satisfying $\phi = \tilde{\psi}$. Note that if $\phi(t) \notin G$ for some $t < T$, then $S_x(T,\phi) = \infty$, and if $\phi(t) \in G$ for $t < T$, then there is at least one ψ satisfying $\phi = \tilde{\psi}$, namely $\psi = \phi$.

Define $\Phi_x(s) = \{\phi \in C_x[0,T]: S_x(T,\phi) < s\}$. Our main result is the following.

Theorem 3.1. *The process* x^ε *defined above satisfies a large deviations principle with action functional* $S_x(T,\cdot)$ *and sequence* ε^2, *i.e.*

(i) $S_x(T,\cdot)$ *is lower semicontinuous on* $C_x[0,T]$,

(ii) $\Phi_x(s)$ *is compact*,

for $A \subset C_x[0,T]$

(iii) $\overline{\lim_{\varepsilon\to 0}} \; \varepsilon^2 \log P_x\{x^\varepsilon(\cdot) \in A\} \le - \inf_{\phi \in \bar{A}} S_x(T,\phi)$,

(iv) $\underline{\lim_{\varepsilon\to 0}} \; \varepsilon^2 \log P_x\{x^\varepsilon(\cdot) \in A\} \ge - \inf_{\phi \in A^o} S_x(T,\phi)$.

Comment. Let $N_c(\phi)$ be the c-neighborhood of the path ϕ. Then for small c and ε, (iii) and (iv) roughly say that

$$\varepsilon^2 \log P_x\{x^\varepsilon(\cdot) \in N_c(\phi)\} \approx - \inf_{\psi:\tilde{\psi}=\phi} \int_0^T L(\phi,\dot{\psi})dt. \tag{3.1}$$

If the functional $F_x(\psi)$ were Lipschitz, we could actually prove (as in [1]) a large deviations result for the unrestricted process z^ε that would imply

$$\varepsilon^2 \log P_x\{z^\varepsilon(\cdot) \in N_c(\psi)\} \approx - \int_0^T L(\tilde{\psi},\dot{\psi})\, dt.$$

Hence we may interpret (3.1) as stating that for small ε the probability that x^ε is near ϕ may be found by examining those unrestricted paths ψ which get reflected into ϕ, and choosing that which maximizes $P_x\{z^\varepsilon(\cdot) \in N_c(\psi)\}$. Contributions to $P_x\{x^\varepsilon(\cdot) \in N_c(\phi)\}$ from the neighborhoods of other paths that are reflected into ϕ and having larger action $\int_0^T L(\tilde{\psi},\dot{\psi})dt$ are asymptotically negligible. This is, of course, simply a reflection of the contraction principle, which states that if z^ε satisfies a large deviations theorem with functional $S_1(\cdot)$ (in the sense of Theorem 3.1), and $x^\varepsilon = F(z^\varepsilon)$, with F being continuous, then x^ε satisfies a large deviations theorem with functional $S_2(\phi) = \inf_{\psi:F(\psi) = \phi} S_1(\psi)$ [6; Theorem 3.3.1], [12; p. 5].

We delay the proofs of parts (i) and (ii) until section 4, where we consider these properties of the action functional and also give more useful alternative expressions for $S_x(T,\phi)$. Before proving (iii) and (iv) we present the following estimate which will be used repeatedly.

Lemma 3.1 Let $\sigma(s)$ be nonanticipative with respect to the d-dimensional Wiener process $w(s)$, and bounded by C_1. Then For $c>0$,

$$P\{\sup_{0\le t\le T_1} |\varepsilon \int_0^t \sigma(s)dw(s)| \ge c\}$$

$$\le 2d \exp - (c^2/2\varepsilon^2 C_1^2 d^2 T_1).$$

Proof. The coefficient 2d appears since we work with one component at a time, and assuming scalar σ and w estimate

$$P\{\sup_{0\le t\le T_1} \varepsilon \int_0^t \sigma(s)dw(s) \ge c/d\}.$$

By the martingale inequality, for $\alpha > 0$,

$$P\{\sup_{0\le t\le T_1} \exp \alpha\, \varepsilon \int_0^t \sigma(s)\, dw(s) \ge \exp \alpha c/d\}$$

$$\le \exp - \alpha c/d\; E \exp \alpha\, \varepsilon \int_0^{T_1} \sigma(s)dw(s)$$

$$\le \exp - \alpha c/d \cdot \exp \alpha^2 \varepsilon^2 C_1^2 T_1/2.$$

We finish by minimizing with respect to $\alpha>0$. Q.E.D.

Consider now the x^ε process:

$$x^\varepsilon(t) = \int_0^t b(x^\varepsilon(s))ds + \varepsilon \int_0^t \sigma(x^\varepsilon(s))dw(s) + x + \xi^\varepsilon(t).$$

To study this process, we adapt the technique of Azencott, and for $\Delta>0$ consider an associated triple $(x^{\xi,\Delta}, z^{\xi,\Delta}, \xi^{\xi,\Delta})$ constructed as follows. We set $x^{\varepsilon,\Delta}(0) = z^{\varepsilon,\Delta}(0) = x$. We then define the process recursively on intervals of the form $[i\Delta, i\Delta+\Delta]$ by requiring that

$$(x^{\varepsilon,\Delta}(t), \int_{i\Delta}^t b(x^{\varepsilon,\Delta}(s))ds + \varepsilon \int_{i\Delta}^t \sigma(x^{\varepsilon,\Delta}(i\Delta))dw(s) + x^{\varepsilon,\Delta}(i\Delta), \beta^{\varepsilon,\Delta}(t))$$

be associated on $[i\Delta, i\Delta + \Delta]$ and then setting

$$\xi^{\varepsilon,\Delta}(t) = \beta^{\varepsilon,\Delta}(t) + z^{\varepsilon,\Delta}(i\Delta)$$
$$z^{\varepsilon,\Delta}(t) = x^{\varepsilon,\Delta}(t) - z^{\varepsilon,\Delta}(t)$$

on $[i\Delta, i\Delta+\Delta]$. Let $\pi_\Delta(t) = [t/\Delta]\Delta$, where $[t]$ is the integer part of t. If we define the mapping $H: C_0[0,T] \to C_x[0,T]$ by $f = H(g)$ when

$$f(t) = F_x\left(\int_0^t b(f(s))ds + \int_0^t \sigma(f(\pi_\Delta(s)))dg(s) + x\right),$$

then by the continuity of F_x and since the coefficient σ is constant on Δ-intervals, the composed map H is continuous. Since $x^{\varepsilon,\Delta}(\cdot) = H(\varepsilon w(\cdot))$, by the contraction principle and the large deviations properties of $\varepsilon w(\cdot)$ [10], [12], $x^{\varepsilon,\Delta}$ satisfies a large deviations principle with action functional

$$S_x^\Delta(T,\phi) = \inf_{\psi:\ \phi=F_x(\psi)} \int_0^T (\dot\psi(t) - b(\phi(t)))' a^{-1}(\phi(\pi_\Delta(t)))(\dot\psi(t) - b(\phi(t)))dt/2,$$

in the nondegenerate case and with an analogous formula in the degenerate case.

We next give an estimate on the sup norm distance between $x^{\varepsilon,\Delta}$ and x^ε. This estimate will be strong enough to ensure that the large deviations properties of x^ε may be deduced from those of $x^{\varepsilon,\Delta}$.

<u>Lemma</u> 3.2. <u>Given</u> $c>0$, $M<\infty$ <u>there is an arbitrarily small</u> $\Delta>0$ <u>and there is</u> $\varepsilon_0>0$ <u>such that for</u> $\varepsilon<\varepsilon_0$

$$P_x\{\sup_{0\le t\le T} |x^{\varepsilon,\Delta}(t) - x^\varepsilon(t)| > c\} < \exp - M/\varepsilon^2 .$$

<u>Proof</u>. Define $\tilde{x}^{\varepsilon,\Delta} = x^\varepsilon - x^{\varepsilon,\Delta}$. Then

$$\tilde{x}^{\varepsilon,\Delta} = \int_0^t b^{\varepsilon,\Delta}(s)ds + \varepsilon\int_0^t \sigma^{\varepsilon,\Delta}(s)\, dw(s) + \tilde{\xi}^{\varepsilon,\Delta}(t)$$

where

$$b^{\varepsilon,\Delta}(s) = b(x^\varepsilon(s)) - b(x^{\varepsilon,\Delta}(s))$$

$$\sigma^{\varepsilon,\Delta}(s) = \sigma(x^\varepsilon(s)) - \sigma(x^{\varepsilon,\Delta}(\pi_\Delta(s)))$$

$$\tilde{\xi}^{\varepsilon,\Delta}(t) = \xi^\varepsilon(t) - \xi^{\varepsilon,\Delta}(t).$$

By the Lipschitz conditions on b, σ, we have

$$|b^{\varepsilon,\Delta}(s)| \leq K|\tilde{x}^{\varepsilon,\Delta}(s)|$$

$$|\sigma^{\varepsilon,\Delta}(s)| \leq K(|\tilde{x}^{\varepsilon,\Delta}(s)| + \sup_{0\leq s\leq T}|x^{\varepsilon,\Delta}(s) - x^{\varepsilon,\Delta}(\pi_\Delta(s))|).$$

Define the stopping time $\tau_1 = \tau_1(K_2)$ by $\tau_1 = \inf\{t \geq 0: |z^{\varepsilon,\Delta}(t) - x| \geq K_2\}$. Then Lemma 3.1 and the bound $|b| \leq C$ imply that if $K_2 \geq 2CT$

$$P\{\tau_1 < T\} \leq 2d \exp - (K_2^2/8\varepsilon^2C^2d^2T).$$

Pick $K_2 < \infty$ so that $K_2^2/8C^2d^2T \geq 2M$.

Now let γ be as in Lemma 2.3, and define the stopping time $\tau_2 = \tau_2(h)$ by

$$\tau_2 = \inf\{t\geq 0: \sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq h}} |z^{\varepsilon,\Delta}(t) - z^{\varepsilon,\Delta}(s)| \geq \gamma\}.$$

We obtain a bound on $P\{\tau_2 < T\}$ as follows. Partition $[0,T]$ into Δ - intervals. Then by Lemma 3.1 and the bound on $|b|$, if $h \leq \gamma/4C$

$$P\{\sup_{i\Delta\leq s\leq t\leq i\Delta+2h} |z^{\varepsilon,\Delta}(t) - z^{\varepsilon,\Delta}(s)| \geq \gamma\}$$

$$\leq 4d_1 \exp - (\gamma^2/32\varepsilon^2C^2d^2h)$$

for any $i \leq T/\Delta$. If $\Delta < h$ then it follows that

$$P\{\tau_2 < T\} \leq \left(\frac{4dT}{\Delta}\right) \exp - (\gamma^2/32\varepsilon^2C^2d^2h).$$

Pick $h>0$ so that $\gamma/32C^2d^2h \geq 2M$.

Define $\tau_3 = \tau_3(\rho)$ by

$$\tau_3 = \inf\{t\geq 0: |z^{\varepsilon,\Delta}(t) - z^{\varepsilon,\Delta}(\pi_\Delta(t))| \geq \rho\}.$$

If $\Delta \leq \rho/2C$ then

$$P\{\tau_3 < T\} \leq \left(\frac{2dT}{\Delta}\right) \exp - (\rho^2/8\varepsilon^2C^2d^2\Delta).$$

By Lemmas 2.2 and 2.3, on the set where $\tau_4 \equiv \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge T = T$,

$$\sup_{0 \le t \le T} |x^{\varepsilon,\Delta}(t) - x^{\varepsilon,\Delta}(\pi_\Delta(t))|$$

$$\le \rho^2 + 4\rho(T/h + 1)K'K_2.$$

Finally, given $c_1 > 0$, we may pick $\rho > 0$ such that

$$\rho^2 + 4\rho(T/h + 1)\, K'K_2 < c_1.$$

If $\Delta > 0$ is chosen so that $\rho^2/8C^2d^2\Delta \ge 2M$, then

$$P\{\tau_4 < T\} = P\{\sup_{0 \le t \le T} |x^{\varepsilon,\Delta}(t) - x^{\varepsilon,\Delta}(\pi_\Delta(t))| \ge c_1\}$$

$$\le 2d(1+3T/\Delta)\exp - 2M/\varepsilon^2$$

Now define τ_5 by

$$\tau_5 = \inf \{ t : |\tilde{x}^{\varepsilon,\Delta}(t)| \ge c\} .$$

Then

$$P\{\tau_5 < T\} \le P\{\tau_5 < T, \tau_4 = T\}$$

$$+ P\{\tau_4 < T\}.$$

But note that on the set $\{\tau_5 < T,\ \tau_4 = T\}$ we have

$$|b^{\varepsilon,\Delta}(s)|,\ |\sigma^{\varepsilon,\Delta}(s)| \le K(|\tilde{x}^{\varepsilon,\Delta}(s)| + c_1).$$

Define $\phi(x) = (c_1^2 + |x|^2)^{1/\varepsilon^2}$. By Ito's formula

$$d\phi(\tilde{x}^{\varepsilon,\Delta}(t)) = \frac{1}{\varepsilon^2}\,(c_1^2 + |\tilde{x}^{\varepsilon,\Delta}(t)|^2)^{((1/\varepsilon^2)\ -1)} \langle 2\tilde{x}^{\varepsilon,\Delta}(t),(b^{\varepsilon,\Delta}(t) + d\tilde{\xi}^{\varepsilon,\Delta}(t))\rangle dt$$

$$+ \varepsilon^2 \text{ trace } (\sigma^{\varepsilon,\Delta}(t)\ \sigma^{\varepsilon,\Delta'}(t)\ \Delta\phi\ (\tilde{x}^{\varepsilon,\Delta}(t)))dt$$

$$+ dw(s) \text{ terms.}$$

$$\le \frac{1}{\varepsilon^2}\,(c_1^2 + |\tilde{x}^{\varepsilon,\Delta}(t)|^2)^{((1/\varepsilon^2)\ -1)} \langle 2\tilde{x}^{\varepsilon,\Delta}(t),(b^{\varepsilon,\Delta}(t) + d\tilde{\xi}^{\varepsilon,\Delta}(t))\rangle dt$$

$$+ K'(c_1^2 + |\tilde{x}^{\varepsilon,\Delta}(t)|^2).$$

$$[4(c_1^2 + |\tilde{x}^{\varepsilon,\Delta}(t)|^2)^{((1/\varepsilon^2) - 2)} |\tilde{x}^{\varepsilon,\Delta}(t)|^2 + 2(c_1^2 + |\tilde{x}^{\varepsilon,\Delta}(t)|^2)^{((1/\varepsilon^2)-1)}]dt$$

$$+ \; dw(s) \text{ terms}.$$

Since $\langle \tilde{x}^{\varepsilon,\Delta}, d\xi^{\varepsilon,\Delta}(t)\rangle \le 0$, the coefficient of the dt term can be bounded above by

$$K_3 \; \phi(x^{\varepsilon,\Delta}(t))/\varepsilon^2 .$$

Hence $\exp - (K_3(\tau_4 \wedge \tau_5 \wedge t)/\varepsilon^2)\phi(x^{\varepsilon,\Delta}(\tau_4 \wedge \tau_5 \wedge t))$

is a supermartingale. By the supermartingale property

$$E\{(\exp - (K_3(\tau_4 \wedge \tau_5 \wedge T)/\varepsilon^2))(c_1^2 + |\tilde{x}^{\varepsilon,\Delta}(\tau_4 \wedge \tau_5 \wedge T)|^2)^{1/\varepsilon^2}\}$$
$$\le c_1^{2/\varepsilon^2}$$

Since on the set $\{\tau_4 = T, \tau_5 < T\}$ we have $|\tilde{x}^{\varepsilon,\Delta}(\tau_5)| = c$ it follows that

$$E\{(\exp - (K_3\tau_5/\varepsilon^2))(c_1^2 + c^2)^{1/\varepsilon^2} I_{\{\tau_4 = T, \tau_5 < T\}}\} \le c_1^{2/\varepsilon^2}$$

so that

$$P\{\tau_4 = T, \; \tau_5 < T\} \le \exp K_3T/\varepsilon^2 \left(\frac{c_1^2}{c_1^2+c^2}\right)^{1/\varepsilon^2} .$$

Now pick $c_1 > 0$ small enough so that

$$(c_1^2/(c_1^2 + c^2))^{1/\varepsilon^2} \le \exp - (2M + K_3T)/\varepsilon^2 .$$

Combining the above estimates gives

$$P\{\tau_5 < T\} \le (1+2d(1+3T/\Delta))\exp - \frac{2M}{\varepsilon^2} .$$

Picking ε_0 small yields the lemma. Q.E.D.

For any set A, define $N_c(A)$ to be the open neighborhood of radius c about A. In order to complete the proof of (iii) and (iv) via the estimate in Lemma 3.2, we shall use the fact that for any closed set D, (3.2) $$\lim_{c\to 0} \inf_{\phi\in\overline{N_c(D)}} S_x(T,\phi) = \inf_{\phi\in D} S_x(T,\phi)$$

(3.3) $$\lim_{\Delta\to 0} \inf_{\phi\in D} S_x^{\Delta}(T,\phi) = \inf_{\phi\in D} S_x(T,\phi) .$$

(3.2) follows simply from the l.s.c. of $S_x(T,\phi)$. (3.3) will be proved in Lemma 4.5 of section 4. If $\inf S_x(T,\phi) < \infty$, then the l.s.c. of $S_x(T,\cdot)$ and the compactness of

$$\{(\psi,\phi): \tilde{\psi} = \phi \text{ and } S_x(T,\phi) \leq s\}$$

for all $s < \infty$ (both proved in section 4) imply the existence of a pair (ψ^*,ϕ^*) such that

$$\int_0^T L(\phi^*,\dot{\psi}^*)dt = \inf_{\phi\in D} S_x(T,\phi).$$

Proof of (iii) and (iv).

Fix $h>0$. Choose $c>0$, $M < \infty$, and then $\Delta>0$ such that

$$\inf_{\phi\in\overline{A}} S_x(T,\phi) \leq \inf_{\phi\in\overline{N_c(A)}} S_x(T,\phi) + h \leq \inf_{\phi\in\overline{N_c(A)}} S_x^{\Delta}(T,\phi) + 2h$$

and

(3.4) $$P_x \{ \sup_{0\leq t\leq T} |x^{\varepsilon}(t) - x^{\varepsilon,\Delta}(t)| \geq c\} \leq \exp - M/\varepsilon^2 .$$

It follows from the large deviations properties of $x^{\varepsilon,\Delta}$ that for small $\varepsilon>0$,

$$\begin{aligned} P_x\{x^{\varepsilon}(\cdot) \in \overline{A}\} &\leq P_x\{x^{\varepsilon,\Delta}(\cdot) \in \overline{N_c(A)}\} \\ &\quad + P_x\{ \sup_{0\leq t\leq T} |x^{\varepsilon}(t) - x^{\varepsilon,\Delta}(t)| \geq c\} \\ &\leq \exp - (S_x(T,\phi) + 3h)/\varepsilon^2 \\ &\quad + \exp - M/\varepsilon^2 \end{aligned}$$

Now let $h\to 0$ and $M \to \infty$.

To obtain (iv), choose any $\phi \in A^{\circ}$. Then there is $c>0$ such that $N_{2c}(\phi) \subset A^{\circ}$. For $h>0$, $M<\infty$, choose $\Delta>0$ such that for small ε (3.4) holds and

$$S_x(T,\phi) \geq S_x^{\Delta}(T,\phi) - h.$$

Then for small ε

$$\begin{aligned} P_x\{x^{\varepsilon}(\cdot) \in A^{\circ}\} &\geq P_x\{x^{\varepsilon}(\cdot) \in N_{2c}(\phi)\} \\ &\geq P_x\{x^{\varepsilon,\Delta}(\cdot) \in N_c(\phi)\} \\ &- P_x\{\sup_{0\leq t\leq T} |x^{\varepsilon}(t) - x^{\varepsilon,\Delta}(t)| \geq c\} \\ &\geq \exp - (S_x(T,\phi) + 2h)/\varepsilon^2 - \exp - M/\varepsilon^2. \end{aligned}$$

Letting $h\to 0$ and $M\to\infty$ gives

$$\varliminf_{\varepsilon\to 0} \varepsilon^2 \log P_x\{x^{\varepsilon}(\cdot) \quad A^{\circ}\} \geq - S_x(T,\phi)$$

for any $\phi \subset A^{\circ}$. Now infimize over all such ϕ. Q.E.D.

Time dependent coefficients.

Anticipating a need in section 5 to allow $\sigma(\cdot)$ to depend on t, we point out here that if $b(x,t)$, $\sigma(x,t)$ satisfy the same boundedness, Lipschitz continuously, and nondegeneracy conditions as given in section 2 as functions of the vector (x,t), then the above proof shows that the analogous nonhomogeneous reflected diffusion satisfies a large deviations principle with rate function given by

$$L(x,\beta,t) = (\beta - b(x,t))' \, a^{-1}(x,t)(\beta - b(x,t))$$

$$S_x(T,\phi) = \inf_{\psi:\tilde{\psi}=\phi} \int_0^T L(\phi,\dot{\psi},t)\, dt$$

in the uniformly nondegenerate case and with an analogous formula in the degenerate case. The proof follows from defining t as a state variable and considering this new (degenerate) diffusion as being reflected in $G \times [0,T]$.

4. Properties of the action functional.

Lemma 4.1 $\int_0^T L(\phi,\dot{\psi})dt$ is jointly lower semicontinuous in $(\phi,\psi) \in C_x[0,T] \times C_x[0,T]$.

Proof. The proof is a simple adaptation of the standard result used in large deviations (where one considers $\int_0^T L(\phi,\dot{\phi})dt$). See for example Stroock [10;p.5].

Lemma 4.2. *For any collection* $\{(\phi_\alpha, \psi_\alpha)\}$, $(\phi_\alpha, \psi_\alpha) \in C_x[0,T] \times C_x[0,T]$, *if*

$$\sup_\alpha \int_0^T L(\phi_\alpha, \dot{\psi}_\alpha)dt < \infty,$$

then $\{\psi_\alpha\}$ *is precompact*.

Proof. The result follows from the inequality

$\dot{\psi}_\alpha^2 < K_1 L(\phi_\alpha, \dot{\psi}_\alpha) + K_2$, where the K_i are constants independent of α, and the fact that sets of the form

$$\{\psi \in C_x[0,T]: \int_0^T \dot{\psi}^2 \, dt < s\}$$

are compact [10; p. 5]. Q.E.D.

Lemma 4.3. $S_x(T,\phi)$ *is l.s.c.*.

Proof. Fix ϕ such that $S_x(T,\phi) < \infty$. Choose ψ_n such that $\phi = \tilde{\psi}_n$ and

$$\int_0^T L(\phi,\dot{\psi}_n)dt \to S_x(T,\phi).$$

By Lemma 4.2 we may choose a subsequence (which we also denote by n) such that $\psi = \lim \psi_n$ exists.

Let (ϕ,ψ_n, η_n), (ϕ^*,ψ,η) all be associated. By Lemma 2.3 we can select a bound for $|\eta_n|(T)$, $|\eta|(T)$ that is independent of n. By Lemma 2.1, $\phi^* = \phi$. By Lemma 4.1, $\int_0^T L(\phi, \dot{\psi}) \, dt = S_x(T,\phi)$.

Hence an infimizing ψ always exists, and we denote such a minimizer by ψ_ϕ. Now let $\phi_n \to \phi$. If $\underline{\lim}\, S_x(T,\phi_n) = \infty$, we are done. Otherwise choose a subsequence (denoted by n') such that

$$\lim S_x(T,\phi_{n'}) = \underline{\lim}\, S_x(T,\phi_n) < \infty.$$

By Lemma 4.2, choose a further subsequence (again denoted by n') such that $\psi_{\phi_{n'}}$ converges, and call the limit ψ. By the same argument as above, $\phi = \tilde{\psi}$. By Lemma 4.1

$$\underline{\lim}\, S_x(T,\phi_n) \geq \int_0^T L(\phi, \dot{\psi})dt \geq S_x(T,\phi).$$

Q.E.D.

Lemma 4.4. *The set of paths* $\Phi_x(s)$ *defined by*

$$\Phi_x(s) = \{\phi \in C_x[0,T]: S_x(T,\phi) \leq s\}$$

is compact.

Proof. By Lemma 4.2,

$$\{\psi_\phi : \phi \in C_x[0,T],\ S_x(T,\phi) \leq s\}$$

is precompact. Since the mapping given by the tilde is continuous, $\Phi_x(s)$ is precompact. By l.s.c. of $S_x(T,\cdot)$, $\Phi_x(s)$ is compact. Q.E.D.

Lemma 4.5. *Let* D *be a closed set of paths in* $C_x[0,T]$, *and let* $S_x^\Delta(T,\phi)$ *and* $S_x(T,\phi)$ *be as defined in section* 3. *Then*

$$\lim_{\Delta\to 0}\ \inf_{\phi\in D} S_x^\Delta(T,\phi) = \inf_{\phi\in D} S_x(T,\phi). \tag{4.1}$$

Proof. If the right hand side of (4.1) is infinite then the uniform non-degeneracy assumptions on $a(\cdot)$ (or $a_{11}(\cdot)$) imply that the left hand side is also infinite. Now assume that the right hand side is finite. By the l.s.c. of $S_x(T,\cdot)$ and the compactness of level sets, an infimizing path ϕ^* exists. As in Lemma 4.3 we may select ψ^* so that $\phi^* = \tilde{\psi}^*$ and

$$S_x(T,\phi^*) = \int_0^T L(\phi^*,\dot{\psi}^*)dt.$$

By dominated convergence

$$(4.2)\qquad \overline{\lim_{\Delta\to 0}}\ \inf_{\phi\in D}\ S_x^{\Delta}(T,\phi) \le \overline{\lim_{\Delta\to 0}}\int_0^T (\dot{\psi}^* - b(\phi^*))'a^{-1}(\phi^*(\pi_\Delta(t)))(\dot{\psi}^*-b(\phi^*))dt/2$$
$$= S_x(T,\phi^*)$$
$$= \inf_{\phi\in D} S_x(T,\phi),$$

with a corresponding inequality in the degenerate case. Now let ϕ^Δ, ψ^Δ be such that $\phi^\Delta \subseteq D$ and $\phi^\Delta = \tilde{\psi}^\Delta$ and

$$\int_0^T (\dot{\psi}^\Delta - b(\phi^\Delta))'a^{-1}(\phi^\Delta(\pi_\Delta(t)))(\dot{\psi}^\Delta-b(\phi^\Delta))dt/2$$
$$\le \inf_{\phi\in D} S_x^\Delta(T,\phi) + \Delta$$

(with an analogous statement here and below holding for the degenerate case). By (4.2) and an inequality like that used in Lemma 4.2 we have a uniform bound on $\int_0^T(\dot{\psi}^\Delta)^2dt$. By precompactness of $\{\phi^\Delta\}$ we may assume ϕ^Δ converges. This implies the equality statement below:

$$\inf_{\phi\in D} S_x(T,\phi) \le \underline{\lim_{\Delta\to 0}} \int_0^T (\dot{\psi}^\Delta-b(\phi^\Delta))'a^{-1}(\phi^\Delta)(\dot{\psi}^\Delta-b(\phi^\Delta))dt/2$$
$$= \underline{\lim_{\Delta\to 0}} \int_0^T (\dot{\psi}^\Delta - b(\phi^\Delta))'a^{-1}(\phi^\Delta(\pi_\Delta(t)))(\dot{\psi}^\Delta-b(\phi^\Delta))dt/2$$
$$\le \underline{\lim_{\Delta\to 0}}\ \inf_{\phi\in D} S_x^\Delta(T,\phi).$$

Q.E.D.

We now show that a simpler and more easily computed expression for the action functional may be found. Before presenting this formula, a few more definitions are needed.

Recall that $\Pi_G(x)$ is the projection of x onto the nearest point in G. Then $\Pi_G(x) = y$, $y \in \partial G$ if and only if $y - x = \alpha n(x)$, for some $\alpha > 0$. For $x \in G$, and any vector v, define

$$\Pi_G(x,v) = \lim_{\delta\downarrow 0} \frac{\Pi_G(x+\delta v) - x}{\delta},$$

which we will call the projection of v at x. Finally, define

$$B_G(x,v) = \{u : \Pi_G(x,u) = \Pi_G(x,v)\},$$

those vectors having the same projection at x as v. We note that for $x \in G^\circ$, $B_G(x,v) = \{v\}$, and that if absolutely continuous ϕ satisfies $\phi(t) \in G$ for $t<T$, then $\Pi_G(\phi(s), \dot{\phi}(s)) = \dot{\phi}(s)$ a.s..

Lemma 4.6. Let $\phi \in C_x[0,T]$ be absolutely continuous and satisfy $\phi(t) \in G$ for $t<T$. Then a measurable version of $\inf_{\beta \in B_G(\phi(s),\dot{\phi}(s))} L(\phi(s),\beta)$ exists and

$$S_x(T,\phi) = \int_0^T \inf_{\beta \in B_G(\phi(s),\dot{\phi}(s))} L(\phi(s), \beta)ds.$$

Proof: The existence of such a measurable selection may be found in [9]. The proof of the second part of the statement is equivalent to showing that for absolutely continuous ψ we have $\tilde{\psi} = \phi$ if and only if

(4.3) $$\dot{\psi}(s) \subseteq B_G(\phi(s), \dot{\phi}(s)) \text{ a.s..}$$

We first assume that for AC ψ we have $\tilde{\psi} = \phi$. Let $\eta = \phi-\psi$, so that (ϕ,ψ,η) are associated. Decompose η into an AC part η_a and a singular part η_s. By uniqueness of such decompositions η_s is also the singular part of ϕ. Assume that $\phi(t) \in \partial G$ for $s_1 < t < s_2$. Since $d\eta_s(t)$ points in the direction $-n(\phi(t))$, the constraint $\phi(t) \in \partial G$ implies $d\eta_s(t) = 0$ on $s_1 < t < s_2$. As $d\eta_s(t) = 0$ when $\phi(t) \in G^\circ$, $\eta_s \equiv 0$, and therefore ϕ is AC.

Consider again an interval $[s_1,s_2]$ on which $\phi(t) \in \partial G$. Then for a.e. t in $[s_1,s_2]$,

(4.4) $$\langle n(\phi(t)), \dot{\phi}(t)\rangle = 0.$$

To see this, note that it is true when $\dot{\phi}(t) = 0$, and if it were not true for a.e. t for which $\dot{\phi}(t) \neq 0$ it would contradict the a.e. differentiability of ϕ, since ϕ is assumed to lie in ∂G.

Since $\dot{\psi}(t) = \dot{\phi}(t) + \alpha(t)\, n(\phi(t))$, $\alpha(t) > 0$, (4.4) implies that

or that
$$\Pi_G(\phi,\dot\psi) = \dot\phi \qquad \text{a.s. in } [s_1, s_2],$$

$$\dot\psi \in B_G(\phi,\dot\phi) \qquad \text{a.s. in } [s_1, s_2].$$

Since $B_G(\phi, \dot\phi) = \{\dot\phi\}$ and $\dot\psi = \dot\phi$ when $\phi \in G^\circ$, $\dot\psi \in B_G(\phi, \dot\phi)$ a.s. on $[0,T]$.

Now assume ϕ is AC and $\dot\psi \in B_G(\phi,\dot\phi)$ a.s., i.e., $\Pi_G(\phi, \dot\psi) = \dot\phi$ a.s. Then (4.4) implies that $\dot\psi - \dot\phi = \alpha n(\phi)$ a.s., and hence $(\phi, \psi, \phi-\psi)$ are associated, so $\tilde\psi = \phi$.

An alternate expression for $S_x(T,\phi)$ may be obtained by replacing the noise in the system equation by a control, that is

$$S_x(T,\phi) = \int_0^T |u(t)|^2 dt$$

where $u(t)$ is chosen to minimize $|u(t)|^2$ subject to $\Pi_G(\phi(t), b(\phi(t)) + \sigma(\phi(t))u(t)) = \dot\phi(t)$. With $u(\cdot) \equiv 0$ we obtain $\dot\phi(t) = \Pi_G(\phi(t), b(\phi(t))$, which gives the dynamics in the absence of noise, and also the 'mean' dynamics of the SDER. Hence via this equivalent 'control' formulation we can replace variational problems such as

$$\inf_{\phi \in A} S_x(T,\phi)$$

by 'optimal control' problems.

5. Stochastic Approximation

We now turn our attention to the constrained stochastic approximation (SA) algorithms

(5.1) $$X_{n+1} = \Pi_G\,(X_n + a_n b(X_n) + a_n\sigma(X_n)\xi_n),\ X_0 = x$$

(5.2) $$X_{n+1} = \Pi_G(X_n + a_n b(X_n) + a_n\sigma(X_n)\xi_n/c_n),\quad X_0 = x$$

where $\Pi_G(\cdot)$ is the nearest point projection of section 4, ξ_n is a sequence of $N(0,I)$ random vectors, a_n is a positive sequence satisfying $a_n \to 0$ as $n \to \infty$, $\sum_{n=0}^{\infty} a_n = \infty$, and c_n is a positive sequence going to zero more slowly than a_n. (5.1) is of Robbins - Monro (RM) type and (5.2) is of Kiefer-Wolfowitz

(KW) type. The SA algorithms and related recursive algorithms occur frequently in applications in control and communication.

In an earlier paper [5], the form

$$X_{n+1} = \Pi_G(X_n + a_n b(X_n, \xi_n))$$

but with bounded (and possibly correllated) noise was studied.

To study the asymptotic properties of (5.1) we define for $n \geq N$

$$X_{n+1}^N = \Pi_G(X_n^N + a_n b(X_n^N) + a_n \sigma(X_n^N)\xi_n) \; , \; X_n^N = x,$$

and define the analogous process for (5.2). We define piecewise linear and constant interpolated versions with interpolation intervals a_n by defining

$$t_n = \sum_0^{n-1} a_i$$

and setting

$$\hat{x}^N(t) = \frac{X_{n+1}^N(t-t_n + t_N) - X_n^N(t-t_{n+1} + t_N)}{t_{n+1} - t_n}$$

$$\bar{x}^N(t) = X_n^N$$

for $t \in [t_n, t_{n+1})$.

Let $m(t) = \max \{n: t_n \leq t\}$. Then $m(t_n) = n$. Let $m_N(t) = m(t_N + t)$. Then $m_N(t)$ is the index corresponding to the interpolation interval a_n in which t lies, if the first interpolation interval (that which abuts t=0) is a_N. Hence $a_{m_N}(t)$ is roughly by the size of the coefficient at time t. We shall in the RM case require that

$$a_{m_N(t)} / a_N \to h(t)$$

uniformly on $[0,T]$ as $N \to \infty$, where $h(t)$ is continuous and positive on $[0,T]$. The analogous condition in the KW case is

$$(a_{m_N(t)}/c_{m_N(t)})/(a_N/c_N) \to h(t).$$

Examples. If $a_n = 1/n$, then $m_N(t) \sim N \exp t$ in the sense that $m_N(t)/N\exp t \to 1$ uniformly on $[0,T]$ and $h(t) = \exp - t$; if $a_n = 1/n^\rho$, $0 < \rho < 1$, then $m_N(t) \sim N + tN^\rho$, and $h(t) = 1$; if $a_n = 1/n$, $c_n = 1/n^\gamma$, $2\gamma < 1$, then $h(t) = \exp (2\gamma - 1)t$; if $a_n = 1/n^\rho$, $0 < \rho < 1$, $c_n = 1/n^\gamma$, $2\gamma < \rho$, then $h(t) = 1$.

We shall prove the large deviations properties of $\hat{x}^N$ by showing that it is in a certain sense sufficiently close to an appropriate reflected diffusion, and then use the results of section 3. We first consider the following alternative way of formulating the large deviations property. The equivalence of this definition and that of Theorem 3.1 is discussed in [6, sect. 3.3].

Lemma 5.1. *The conditions in Theorem 3.1 ((i) - (iv)) under which the process $x^\varepsilon(\cdot)$ is said to satisfy a large deviations principle with functional $S_x(T,\phi)$ and rate ε^2 are equivalent to (i) and (ii) together with*

(iii') *For any* $\delta > 0$, $h>0$, *and* $\phi \in C_x[0,T]$, *there is an* $\varepsilon_0>0$ *such that for* $\varepsilon < \varepsilon_0$

$$P_x\{d(x^\varepsilon, \phi) < \delta\} < \exp - \varepsilon^2 [S_x(T,\phi) + h]$$

(iv') *For any* $\delta>0$, $h>0$ *and* $s>0$ *there is an* $\varepsilon_0 > 0$ *such that for* $\varepsilon<\varepsilon_0$

$$P_x\{d(x^\varepsilon, \Phi_x(s)) > \delta\} < \exp - \varepsilon^2[s-h].$$

We now consider an alternative way of expressing $\overline{X}^N(\cdot)$ in the RM case. We may assume that the independent vectors ξ_i are imbedded in a Wiener process in the sense that for $t \in [t_n, t_{n+1})$

$$\overline{x}^N(t) = \Pi_G(\overline{x}^N(t_{n-1}) + \int_{t_{n-1}}^{t_n} b(\overline{x}^N(t_{n-1}))ds + a_n^{1/2} \int_{t_{n-1}}^{t_n} \sigma(\overline{x}^N(t_{n-1}))dw(s)).$$

Since on this interval $a_n^{1/2} \approx a_N^{1/2} h(t)^{1/2}$, it is clear that the appropriate reflected diffusion to compare $\overline{x}^N$ with is that which is driven by the same Wiener process, and which has drift $b(\cdot)$, diffusion coefficient $h(t)^{1/2}\sigma(\cdot)$, and coefficient $\varepsilon = a_N^{1/2}$. In keeping with our notation of the previous sections, we denote this process by $x^{a_N^{1/2}}(\cdot)$.

Lemma 5.2 *Under the assumptions on the sequence* $\{a_n\}$ *above, for given* $c>0$, $M<\infty$, *there is* $N_0 < \infty$ *such that for* $N>N_0$

$$P_x\{\sup_{0\le t\le T} |\overline{x}^N(t) - x^{a_N^{1/2}}(t)| > c\}$$

$$\le \exp - M/a_N.$$

Proof. The method of proof is very similar to that used to show Lemma 3.2, and will only be sketched. Define

$$\overline{z}^N(t) = \sum_{N\le n<m_N(t)} [b(\overline{x}^n(t_{n-1})a_n + a_n^{1/2}\, \sigma(\overline{x}^N(t_{n-1}))(w(t_n)-w(t_{n-1}))] + x,$$

$$\overline{\xi}^N(t) = \overline{x}^N(t) - \overline{z}^N(t).$$

Then the triple $(\overline{x}^N, \overline{z}^N, \overline{\xi}^N)$ is associated. ($\overline{z}^N(\cdot)$ is just the piecewise constant version of the algorithm without the projections,

$$\sum_{n=N}^{i} [a_n b(X_n) + a_n\sigma(X_n)\xi_n] + x,$$

with interpolation intervals a_i). Finally define

$$\overline{\overline{z}}^N(t) = \int_0^t b(\overline{x}^N(s))ds + a_N^{1/2} \int_0^t h(s)^{1/2}\, \sigma(\overline{x}^N(s))dw(s) + x,$$

and $\overline{\overline{x}}^n$, $\overline{\overline{\xi}}^N$ such that $(\overline{\overline{x}}^N, \overline{\overline{z}}^N, \overline{\overline{\xi}}^N)$ are associated. Define

$$\tau_4 = \inf \{t : |\overline{x}^N(t) - \overline{\overline{x}}^N(t)| \ge c_1\} \wedge T.$$

Using Lemmas 2.1, 2.2, 2.3, and 3.1 in a manner similar to that used in Lemma 3.2, we may show that for large N

$$P_x\{\tau_4 < T\} \le \exp - M/a_N.$$

More specifically we use Lemma 3.1 and the assumption on the a_n sequence to show $\overline{z}^N$ and $\overline{\overline{z}}^N$ are close. We then bound $|\overline{\xi}^N|$ and $|\overline{\overline{\xi}}^N|$ except on a sufficiently small set via Lemma 2.3, and then use Lemma 2.1 to show that $\overline{x}^N$ and $\overline{\overline{x}}^N$ are close. Subsequent to this we use the same argument as in Lemma 3.2 to show that $\bar{\bar{x}}^N$ and $x^{a_N^{1/2}}$ are close, by using that

$$|\bar{x}^N - x^{a_N^{1/2}}| \leq |\bar{\bar{x}}^N - x^{a_N^{1/2}}| + c_1$$

on the set where $\tau_4 = T$. Q.E.D.

Theorem 5.1 Under the assumptions on $\{a_n\}$ above, the RM process $\hat{x}^N(\cdot)$ satisfies a large deviations property in the sense of Theorem 3.1 with the functional corresponding to the reflected diffusion with drift $b(\cdot)$ and diffusion matrix $h(t)^{1/2}\sigma(\cdot)$, and sequence a_N. Under the assumptions on $\{a_n\}$ and $\{c_n\}$ given above, the KW process $\hat{x}^N(\cdot)$ satisfies a large deviations property with the same functional and sequence a_N/c_N^2.

Proof. First consider the RM case. Given the estimate of Lemma 5.2 on the $\bar{x}^N$ process it is a simple matter to show that for $c>0$, $M<\infty$ there is N_0 such that for $N>N_0$

$$P_x\{\sup_{0\leq t\leq T} |\hat{x}^N(t) - x^{a_N^{1/2}}(t)| > c\}$$

$$\leq \exp - M/a_N .$$

The theorem now follows from this estimate, Lemma 5.1, and Theorem 3.1. The proof for the KW case is entirely analogous. Q.E.D.

It is interesting to note that the L-functional for this system is given in the nondegenerate case by

$$L(x,\beta,t) = h^{-1}(t)(\beta-b(x))'a^{-1}(x)(\beta-b(x)).$$

The weighting factor $h^{-1}(t)$ becomes larger in regions where the a_n interpolation intervals 'bunch up', and where the probability that the system tracks a given path is relatively less likely than where $h^{-1}(t)$ is smaller, since a relatively greater number of the independent random variables must have the 'correct' value.

We note in passing that the constant gain recursive algorithm considered in section 1 may be treated with the same methods used here, and is in fact simpler since the convergence assumption on a_n/a_N may be dispensed with.

6. Extensions and comments

6.1 Correlated Gaussian driving noise.

By using the lemmas derived in section 2 and the method of Lemma 5.2 it is a simple matter to extend the results for i.i.d. driving noises to correlated driving noises of the form

$$\xi_{n+1} = A\xi_n + B\theta_n$$

where θ_n are i.i.d. mean zero Gaussian and the roots of A are contained in the unit circle. For simplicity, we consider only the KW case. We will obtain results for

$$X_{n+1} = \Pi_G(X_n + a_n\sigma(X_n) + a_n\sigma(X_n)\xi_n/c_n),\ X_N = x$$

by comparing it with

$$Y_{n+1} = \Pi_G(Y_n + a_n b(Y_n) + a_n\sigma(Y_n)(-(A-I)^{-1} B\theta_n / c_n)),\ Y_N = x$$

for which results have already been obtained. The intermediate process that will be needed is defined by

$$\overline{Y}_{n+1} = \Pi_G(\overline{Y}_n + a_n b(X_n) + a_n\sigma(X_n)(-(A-I)^{-1}B\theta_n / c_n)),\ Y_N = x\ .$$

Here X_n, Y_n, $\overline{Y}_n$ play the roles of $\overline{x}^N$, $\overline{x}^{a_N^{1/2}}$, $\overline{\overline{x}}^N$, respectively in the proof of Lemma 5.2.

In order to complete the argument an estimate analogous to that of Lemma 3.1 is required.

<u>Lemma</u> 6.1. <u>Let</u> σ_n <u>be nonanticipative with respect to the sequence</u> $\{\theta_n\}$, <u>and bounded by</u> C_1. <u>Then for</u> $n<\infty$, $c>0$ <u>there is</u> $N_0 < \infty$ <u>such that for</u> $N>N_0$

$$P\{\sup_{N\le n\le m_N(T)} |\sum_{i=N}^{n} a_i\sigma_i\ (\xi_i + (A-I)^{-1}\ B\theta_i)/c_i| > c\} < \exp - Mc_N^2/a_N.$$

The proof follows from using $\xi_i = (A-I)^{-1}(A-I)\xi_i = (A-I)^{-1}[\xi_{i+1} - \xi_i - B\theta_i]$ and a summation by parts to get

$$\sum_{i=N}^{n} a_i \ \sigma_i(\xi_i + (A-I)^{-1} B\theta_i)/c_i$$

$$= a_n \sigma_n \xi_{n+1}/c_n - a_N \sigma_N \ \xi_N/c_N$$

$$+ \sum_{i=N}^{n} (a_{i+1} \ \sigma_{i+1}/c_{i+1} - a_i \ \sigma_i/c_i) \ \xi_{i+1}$$

Estimates on these terms, which are lengthy but straightforward, may be found in [4].

6.2. Unbounded Convex Domains

The results for the various systems described above continue to hold when the domain G is an unbounded convex set, but the large deviations estimates may no longer be uniform with respect to the starting position, since the estimates of Lemma 2.3 need not hold uniformly in $x \in G$. See [11] for a discussion on this point.

The method of proof may be outlined as follows. First note that for given $M < \infty$ there is $R < \infty$ such that the infimum of $S_x(T,\phi)$ over

$$\{\phi: \phi(0) = x, \ \phi(t) \in G \setminus N_R(x) \ \text{ for some } \ t < T\}$$

is greater than M. This follows from the inequality

$$\inf_{\beta \in B_G(\phi,\dot{\phi})} L(\phi,\beta) > K_1 \dot{\phi}^2 - K_2$$

and the fact the path ϕ connecting x to any point y and minimizing $\int_0^T \dot{\phi}^2 dt$ is given by

$$\phi(t) = ((y-x)/T)t+x.$$

(We may in fact take $R > ((K_2T^2+MT)/K_1)^{1/2}$).

Consider now the alternative formulation of the large deviations property given in Lemma 5.1. For simplicity we make all comments with reference to the reflected diffusion. Define x_R^ε to be the strong solution of the SDER in the convex domain $G \cap N_R(x)$, which agrees with x^ε up till the first hitting time of $\partial N_R(x)$. For fixed ϕ or $\Phi_x(s)$, by choosing R suitably large we have

$$P_x\{d(x^\varepsilon,\phi) < \delta\} = P_x\{d(x_R^\varepsilon, \phi) < \delta\}$$

$$P_x\{d(x^\varepsilon,\Phi_x(s)) > \delta\} = P_x\{d(x_R^\varepsilon , \Phi_x(s)) > \delta\}.$$

Since the action functionals for the different paths under consideration agree (at least for large R) the result is immediate.

6.3. Domains with smooth boundaries except at convex corners:

The results presented above are also valid if the boundary of G is given by a collection of smooth surfaces which intersect in a convex way (by which we mean that if at a point of intersection the surfaces were to be replaced by their tangent hyperplanes, then the domain formed would be convex.) Such domains are natural, at least in the SA case, where one may be is attempting to optimize subject to constraints, which often take the form of inequalities involving smooth functions.

For the proof of these results in the case of SDER, no more is required than substituting for Lemmas 2.1, 2.2, 2.3, the analogous lemmas for the type of domain considered here. Proofs of such lemmas for more general domains may be found in [8], which in fact covers the 'convex corner' case described here as well as the convex case considered above.

Since these lemmas are proved only for continuous paths extensions are required to obtain the results in the SA case, due to the possibility of jumps outside of G. The extensions can in fact be obtained, but only for paths for which the sizes of all the jump discontinuities are bounded by a small number depending on G (essentially the distance c for which we may define a unique nearest point projection $\pi_G(\cdot)$ on $N_c(G)$). Since one can easily show that the probability of sample paths having one or more jumps larger than a fixed positive number is negligible in the large deviations exponential sense ($< \exp - M/a_N$ for any M if N is sufficiently large), the indicated results may be obtained.

6.4 Oblique Reflections

In the case of smooth boundaries, results similar to those above have been obtained [1], [3], (for the SDER), when the normal vector field on ∂G is replaced by a smoothly varying vector field satisfying a uniform non-tangency condition. Smooth mappings of the state space are used to reduce to the case of normal reflect.

For nonsmooth boundaries, such mappings do not exist in any generality and we were only able to obtain results when such a map fortuitously did happen to exist. This is not surprising, since the estimates on the Skorokhod problem are no longer available in this general setting. For more detail and some examples, see [8].

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