

# POLYNOMIAL OPTIMIZATION ON ODD-DIMENSIONAL SPHERES

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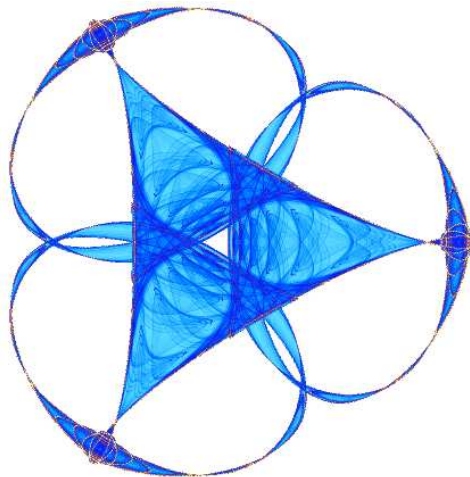
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# Polynomial optimization on odd-dimensional spheres

John P. D'Angelo and Mihai Putinar

**Abstract.** The sphere  $S^{2d-1}$  naturally embeds into the complex affine space  $\mathbb{C}^d$ . We show how the complex variables in  $\mathbb{C}^d$  simplify the known Striktpositivstellensätze, when the supports are restricted to semi-algebraic subsets of odd dimensional spheres.

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## 1. Preliminaries

Let  $\mathbb{C}^d$  denote complex Euclidean space with Euclidean norm given by  $|z|^2 = \sum_{j=1}^d |z_j|^2$ . The unit, odd dimensional sphere

$$S^{2d-1} = \{z \in \mathbb{C}^d; |z| = 1\}$$

is a particularly important example of a Cauchy-Riemann (usually abbreviated CR) manifold. This note will show how one can study problems of polynomial optimization over semi-algebraic subsets of  $S^{2d-1}$  by using the induced Cauchy-Riemann structure. Our results can be regarded as multivariate analogues of classical phenomena about positive trigonometric polynomials, known for a long time in dimension one ( $d = 1$ ). They are also related to results concerning proper holomorphic mappings between balls in different dimensional complex Euclidean spaces and the geometry of holomorphic vector bundles.

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A polynomial map  $p : \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}$  is called *Hermitian symmetric* if

$$p(z, \bar{w}) = \overline{p(w, \bar{z})}$$

for all  $z$  and  $w$ . By polarization one can recover a Hermitian symmetric polynomial from its real values  $p(z, \bar{z})$ . We therefore work on the diagonal (where  $w = z$ ) and let  $\mathcal{H} \subset \mathbb{C}[z, \bar{z}]$  denote the space of Hermitian symmetric polynomials on  $\mathbb{C}^d$ . Note that  $\mathcal{H}$  is a real algebra, naturally isomorphic to the polynomial algebra  $\mathbb{R}[x, y]$ , where  $z = x + iy \in \mathbb{R}^d + i\mathbb{R}^d$ . Henceforth we will freely identify a Hermitian symmetric polynomial  $P(z, \bar{z})$  with its real form  $p(x, y) = P(x + iy, x - iy)$ .

We denote by  $\Sigma^2\mathcal{H}$  the convex cone consisting of sums of squares of Hermitian polynomials. We denote by  $\Sigma_h^2\mathcal{H}$  the convex cone consisting of polynomials which are squared norms of (holomorphic) polynomial mappings. Thus  $R \in \Sigma_h^2\mathcal{H}$  if there exist polynomials  $p_j \in \mathbb{C}[z]$  such that

$$R(z, \bar{z}) = \sum_{j=1}^m |p_j(z)|^2.$$

See [11] and [1] for various characterizations of  $\Sigma_h^2\mathcal{H}$ . We have the containment

$$\Sigma_h^2\mathcal{H} \subset \Sigma^2\mathcal{H},$$

simply because

$$|p|^2 = \left(\frac{p + \bar{p}}{2}\right)^2 + \left(\frac{p - \bar{p}}{2i}\right)^2 = u^2 + v^2,$$

where  $u$  and  $v$  are the real and imaginary parts of  $p$ . The containment is strict as illustrated by the following two examples.

**Example a).** In one variable we define a polynomial  $R$  by

$$R(z, \bar{z}) = (z + \bar{z})^2 = 4x^2.$$

It is evidently a square but not in  $\Sigma_h^2\mathcal{H}$ . Note that the zero set of an element in  $\Sigma_h^2\mathcal{H}$  must be a complex variety and thus cannot be the imaginary axis.

**Example b).** In two variables we define  $R(z, \bar{z}) = (|z_1|^2 - |z_2|^2)^2$ . Again  $R$  lies in  $\Sigma^2\mathcal{H}$  but not in  $\Sigma_h^2\mathcal{H}$ . Here one can observe that elements of  $\Sigma_h^2\mathcal{H}$  must be plurisubharmonic but that  $R$  is not. In 3.3 we will show additionally that  $R$  cannot be written as a squared norm on the unit sphere.

In this paper we are primarily concerned with optimization on the sphere. We therefore first let  $I = I(S^{2d-1})$  be the ideal of  $\mathcal{H}$  consisting of all polynomials vanishing on  $S^{2d-1}$ . We then define

$$\mathcal{H}(S^{2d-1}) = \mathcal{H}/I,$$

and regard it as a space of polynomial functions defined on the sphere. As a matter of fact, each real-valued polynomial  $p$  has a representative in  $\mathcal{H}(S^{2d-1})$ , when  $p$  is regarded as a function on the sphere.

In analogy with the above notations we denote by  $\Sigma^2\mathcal{H}(S^{2d-1})$  the convex cone consisting of sums of squares of Hermitian polynomials on the sphere. We denote by  $\Sigma_h^2\mathcal{H}(S^{2d-1})$  the convex hull of Hermitian squares:

$$\Sigma_h^2\mathcal{H}(S^{2d-1}) = \text{co}\{|p(z)|^2; p \in \mathbb{C}[z]\} \pmod{I}.$$

Let us pause for a moment and recall a classical one-dimensional result which is guiding our investigation. We include its elementary proof for convenience.

**Lemma 1.1 (Riesz-Fejér).** *A non-negative trigonometric polynomial is the squared modulus of a trigonometric polynomial.*

*Proof.* Let  $p(e^{i\theta}) = \sum_{-d}^d c_j e^{ij\theta}$  and assume that  $p(e^{i\theta}) \geq 0$ ,  $\theta \in [0, 2\pi]$ . Since  $p$  is real-valued  $c_{-j} = \overline{c_j}$  for all  $j$ . We set  $z = |z|e^{i\theta}$  and extend  $p$  to the rational function defined by  $p(z) = \sum_{-d}^d c_j z^j$ . It follows that  $p(z) = \overline{p(1/\overline{z})}$ ; furthermore its zeros and poles are symmetrical (in the sense of Schwarz) with respect to the unit circle.

Write  $z^d p(z) = q(z)$ . Then  $q$  is a polynomial of degree  $2d$  whose modulus  $|q|$  satisfies  $|q| = |p| = p$  on the unit circle. In view of the mentioned symmetry one finds

$$q(z) = cz^\nu \prod_j (z - \lambda_j)^2 \prod_k (z - \mu_k)(z - 1/\overline{\mu_k}),$$

where  $c$  is a constant,  $|\lambda_j| = 1$  and  $0 < |\mu_k| < 1$ .

Evaluating on the circle and using  $|\zeta^2| = |\zeta|^2$  we obtain

$$\begin{aligned} p(e^{i\theta}) &= |p(e^{i\theta})| = |q(e^{i\theta})| = \\ &= |c| \prod_j |e^{i\theta} - \lambda_j|^2 \prod_k \frac{|e^{i\theta} - \mu_k|^2}{|\mu_k|^2}, \end{aligned}$$

and hence  $p$  is the squared modulus of a trigonometric polynomial.  $\square$

This fundamental lemma has deeply influenced twentieth century functional analysis. For instance the Riesz-Fejér Lemma is equivalent to the spectral theorem for unitary operators; see [28].

When invoking duality, the above is not less interesting. It was in this form that Riesz-Fejér Lemma was first generalized to an arbitrary dimension.

**Lemma 1.2.** *Let  $L \in \mathcal{H}(S^{2d-1})'$  be a linear functional which is non-negative on  $\Sigma_h^2 \mathcal{H}(S^{2d-1})$ . Then  $L$  is represented by a positive Borel measure supported on the sphere.*

The proof has implicitly appeared in the works of Ito [16], Yoshino [31], Lubin [21] and Athavale [3], all dealing with subnormality criteria for commuting tuples of bounded linear operators. Without aiming at completeness, here is the main idea.

*Proof.* (Sketch) Let  $L$  be a non-negative functional on  $\Sigma_h^2 \mathcal{H}(S^{2d-1})$ . Fix a polynomial  $p \in \mathbb{C}[z]$  and consider the functional

$$f(r_1^2, \dots, r_d^2) \mapsto L(f|p(z)|^2), \quad f \in \mathbb{R}[r_1^2, \dots, r_d^2],$$

where  $r_j^2 = |z_j|^2$ . Since

$$1 - |z_j|^2 = \sum_{k \neq j} |z_k|^2,$$

$$L\left(\prod_j [(1 - r_j^2)^{n_j} r_j^{2m_j}] |p|^2\right) \geq 0, \quad n_j, m_j \geq 0.$$

By a classical result of Haviland, see for instance [2], there exists a positive Borel measure  $\mu_{|p|^2}$  on the simplex  $\Delta$  defined by

$$\Delta = \{(r_1^2, \dots, r_d^2); r_1^2 + \dots + r_d^2 = 1\},$$

with the property

$$L(f|p(z)|^2) = \int_{\Delta} f d\mu_{|p|^2}.$$

The total mass of  $\mu_{|p|^2}$  is  $L(|p|^2)$ .

By polarization, one can define complex valued measures by

$$L(fp\bar{q}) = \int_{\Delta} f d\mu_{p\bar{q}}, \quad f \in \mathbb{R}[r_1^2, \dots, r_d^2], \quad p, q \in \mathbb{C}[z],$$

so that the sesqui-linear kernel  $(p, q) \mapsto \mu_{p\bar{q}}$  is positive semi-definite.

In short, the functional  $L$  can be extended to the linear space of functions (on the sphere) of the form

$$F(r, z) = \sum_{|\alpha| \leq n} c_\alpha(r) z^\alpha,$$

where  $c_\alpha(r)$  are bounded, Borel measurable functions on the simplex  $\Delta$ . The extended functional  $\tilde{L}$  still satisfies

$$\tilde{L}(|F(r, z)|^2) \geq 0.$$

Next we pass to polar coordinates  $z_j = r_j \omega_j$ ,  $|\omega_j| = 1$  and remark that multiplication by  $\omega_j$  satisfies the isometric condition

$$\tilde{L}(|\omega_j F(r, z)|^2) = \tilde{L}(|F(r, z)|^2).$$

Thus, we can further extend the functional  $\tilde{L}$  to all polynomials in  $r$  and  $\omega, \bar{\omega}$ , so that

$$\tilde{L}(|\omega_j^{-1} F(r, z)|^2) = \tilde{L}(|F(r, z)|^2)$$

and

$$\tilde{L}(|p(r, \omega, \bar{\omega})|^2) \geq 0.$$

We refer to [31] or [29] for the details how this extension is constructed. By rewriting the latter positivity condition we have in particular

$$\tilde{L}(|h(z, \bar{z})|^2) \geq 0, \quad h \in \mathbb{C}[z, \bar{z}],$$

whence, by the Stone-Weierstrass Theorem and the Riesz Representation Theorem, the functional  $\tilde{L}$  is represented by a positive Borel measure, supported on the sphere.

The representing measure is unique by the Stone-Weierstrass Theorem.  $\square$

## 2. A Striktpositivstellensatz

We now turn to the basic question considered in this paper. We are given a finite set of real polynomials in  $2d$  variables  $p, q_1, \dots, q_r$ , or equivalently, Hermitian symmetric polynomials in  $d$  complex variables. We suppose that  $p(z, \bar{z})$  is strictly positive on the subset of  $S^{2d-1}$  where each  $q_j$  is nonnegative. Can we write  $p$  as a weighted sum of squared norms with  $q_i$  as weights, as the real affine Striktpositivstellensatz (see for instance [22]) suggests? The answer is yes, and we can offer at least two different reasons why it is so.

**Theorem 2.1.** *Let  $p, q_1, \dots, q_r \in \mathbb{R}[x, y]$ , where  $x + iy = z \in \mathbb{C}^d$ . If*

$$(|z| = 1, q_i(z, \bar{z}) \geq 0, 1 \leq i \leq r) \Rightarrow (p(z, \bar{z}) > 0),$$

*then*

$$p \in \Sigma_h^2 + q_1 \Sigma_h^2 + \dots + q_r \Sigma_h^2 + I(S^{2d-1}).$$

First we discuss the history of such Hermitian squares decompositions, in the case where there are no constraints. A Hermitian symmetric polynomial  $p$  is called bihomogeneous of degree  $(m, m)$  if

$$p(\lambda z, \bar{\lambda} z) = |\lambda|^{2m} p(z, \bar{z})$$

for all complex numbers  $\lambda$  and all  $z \in \mathbb{C}^d$ . The values of a bihomogeneous polynomial are determined by its values on the sphere. When  $p$  is bihomogeneous and strictly positive on the sphere, Quillen [27] proved that there is an integer  $k$  and a homogeneous polynomial vector-valued mapping  $h(z)$  such that

$$|z|^{2k} p(z, \bar{z}) = |h(z)|^2.$$

This result was discovered independently by the first author and Catlin [6] in conjunction with the first author's work on proper mappings between balls in different dimensions. The proof in [6] uses the Bergman projection and some facts about compact operators, and it generalizes to provide an isometric imbedding theorem for certain holomorphic vector bundles [7].

It is worth noting that the integer  $k$  and the number of components of  $h$  can be arbitrarily large, even for polynomials  $p$  of total degree four in two variables. The result naturally fits into the phenomena encoded into the old or recent Positivstellensätze, see for instance [22]. For the specific case of Hermitian polynomials on spheres see [8] for considerable discussion and generalizations.

Using a process of bihomogenization, Catlin and the first author (see [6], [8] and [9]) proved that if  $p$  is arbitrary (not necessarily bihomogeneous) and strictly positive on the sphere, then  $p$  agrees with a squared norm on the sphere; in other words,  $p \in \Sigma_h^2 + I(S^{2d-1})$ . Thus Theorem 1 holds when there are no constraints. Our proof of Theorem 1 first considers the case of no constraints, but we approach this case in a completely different manner.

Strict positivity is required for these results. The polynomial  $(|z_1|^2 - |z_2|^2)^2$  is bihomogeneous and nonnegative everywhere, but there is no element in  $\Sigma_h^2$  agreeing with it on the sphere. See Example 3.3 below.

*Proof.* (of Theorem 1) Suppose first that no  $q_i$ 's are present and assume by contradiction that  $p \notin \Sigma_h^2$ , all regarded as elements of  $\mathcal{H}(S^{2d-1})$ . Since the constant function 1 belongs to the algebraic interior of the convex cone  $\mathcal{H}(S^{2d-1})$ , the separation lemma due to Eidelheit-Kakutani [12, 17] provides a linear functional  $L \in \mathcal{H}(S^{2d-1})'$ , satisfying both  $L(1) > 0$  and

$$L(p) \leq 0 \leq L|_{\Sigma_h^2}.$$

According to Lemma 2, there exists a positive Borel measure  $\mu$ , supported on the unit sphere, which represents  $L$ . Therefore

$$0 \geq L(p) = \int pd\mu > 0,$$

a contradiction.

The proof of the general case is similar, with the difference that we have to prove that the support of the measure  $\mu$  is contained in the non-negativity set defined by the functions  $q_i$ . To this aim, fix an index  $i$ , and remark that

$$\int q_i |p|^2 d\mu \geq 0$$

for all  $p \in \mathbb{C}[z]$ . Now, by the first case, every positive polynomial  $P(z, \bar{z})$  is in the convex hull of the Hermitian squares, whence

$$\int q_i P(z, \bar{z}) d\mu \geq 0$$

whenever  $P(z, \bar{z}) > 0$  on the sphere, that is whenever  $P(z, \bar{z}) \geq 0$  on the sphere. In view of Stone-Weierstrass Theorem, every continuous functions  $f$  on the sphere can be uniformly approximated by real polynomials. In particular, we infer

$$\int q_i f^2 d\mu \geq 0, \quad f \in C(S^{2d-1}).$$

But this inequality holds only if the support of  $\mu$  is contained in the non-negativity set  $q_i(z, \bar{z}) \geq 0$ . □

### 3. Examples

#### 3.1. Optimization on the closed disk

The following simple example shows that Hermitian sums of squares do not suffice as positivity certificates on more general semi-algebraic sets. Specifically, let

$$P(z, \bar{z}) = 1 - \frac{4}{3}|z|^2 + a|z|^4,$$



with  $\frac{1}{3} < a$ . Note that

$$P(z, \bar{z}) = \left(1 - \frac{2}{3}|z|^2\right)^2 + \left(a - \frac{4}{9}\right)|z|^4,$$

and hence  $P \in \Sigma^2\mathcal{H}$  when  $a \geq \frac{4}{9}$ . Hence we assume  $\frac{1}{3} < a < \frac{4}{9}$ . The polynomial  $1 - \frac{4}{3}t + at^2$  is decreasing for  $0 < t < 1$  when  $a < \frac{2}{3}$ ; therefore  $|z| \leq 1$  implies  $P(z, \bar{z}) \geq 1 + a - \frac{4}{3} > 0$ .

On the other hand,

$$P \notin \Sigma_h^2 + (1 - |z|^2)\Sigma_h^2.$$

To see that  $P$  is not in this set, we apply the hereditary calculus. See [1] for details. We replace  $z$  with a contractive operator  $T$  and replace  $\bar{z}$  with  $T^*$ . We follow the usual convention of putting all  $T^*$ 's to the left of the powers of  $T$ . If  $P$  were in this set, we would obtain

$$\|T\| \leq 1 \Rightarrow p(T, \bar{T}) \geq 0.$$

In particular let  $T$  be the  $2 \times 2$  Jordan block with 1 above the diagonal. We obtain a contradiction by computing that  $P(T, T^*)$  is the diagonal matrix with eigenvalues 1 and  $-\frac{1}{3}$ .

On the other hand, the larger convex cone  $\Sigma^2 + (1 - |z|^2)\Sigma_h^2$  is appropriate in this case, see [24, 26].

### 3.2. Squared norms

Recall that  $\Sigma_h^2\mathcal{H}$  denotes the convex cone consisting of polynomials which are squared norms of (holomorphic) polynomial mappings. In all dimensions the zero set of an element in  $\Sigma_h^2\mathcal{H}$  must be a complex variety.

Suppose  $R(z, \bar{z}) \geq 0$  for all  $z$ . Even in one dimension we cannot conclude that  $R \in \Sigma_h^2\mathcal{H}$ . We noted earlier, where  $x = \operatorname{Re}(z)$ , the example

$$R(z, \bar{z}) = (z + \bar{z})^2 = 4x^2.$$

The zero set of  $R$  is the imaginary axis, which has no complex structure. In one dimension of course, the zero set of an element in  $\Sigma_h^2\mathcal{H}$  must be either all of  $\mathbb{C}$  or a finite set.

Things are more complicated and interesting in higher dimensions. Consider the following example from [8]. Define a Hermitian bihomogeneous polynomial in three variables by

$$p(z, \bar{z}) = (|z_1 z_2|^2 - |z_3|^4)^2 + |z_1|^8.$$

This polynomial  $p$  is nonnegative for all  $z$ , and its zero set is the complex plane given by  $z_1 = z_3 = 0$  with  $z_2$  arbitrary. Yet  $p$  is not a sum of squared moduli; even more striking is that  $p$  cannot be written as the quotient  $\frac{|a(z)|^2}{|b(z)|^2}$  where  $a$  and  $b$  are sums of squared moduli. See [10] for additional information on this example and several tests for deciding whether a non-negative polynomial  $R$  can be written as a quotient of squared norms. See [30] for a necessary and sufficient condition involving the zeroes of  $R$ .

We give an additional example in one dimension. Define  $p$  by

$$p(z, \bar{z}) = 1 + bz^2 + \bar{b}\bar{z}^2 + c|z|^2 + |z|^4.$$

The condition for being a quotient of squared norms is that one of the following three statements holds:

$$c > 2|b|^2 - 2,$$

$$b = 0, c > -2,$$

$$|b| = 1, c = 0.$$

The condition for being nonnegative is simpler:  $c \geq 2|b| - 2$ .

### 3.3. Proof of Example b).

We claimed earlier that the polynomial  $(|z_1|^2 - |z_2|^2)^2$  is bihomogeneous and non-negative everywhere, but there is no element in  $\Sigma_h^2$  agreeing with it on the sphere.

*Proof.* Put  $R(z, \bar{z}) = (|z_1|^2 - |z_2|^2)^2$ , and let  $V(R)$  denote its zero set. We note that  $V(R) \cap S^{2n-1}$  is the torus  $T$  defined by  $|z_1|^2 = |z_2|^2 = \frac{1}{2}$ . Suppose for some polynomial mapping  $z \rightarrow P(z)$  we have  $R = |P|^2$  on the unit sphere. Note first that the zero set of  $|P|^2$  is a complex variety. We have  $|P(z)|^2 = 0$  for  $z \in T$ . We claim that  $P$  is identically zero. For each fixed  $z_2$  with  $|z_2| = 1$ , the vector-valued polynomial mapping  $z_1 \rightarrow P(z_1, z_2)$  vanishes on the circle  $|z_1|^2 = \frac{1}{2}$  and hence vanishes identically. Since  $z_2$  was an arbitrary point with  $|z_2|^2 = \frac{1}{2}$  we conclude that the mapping  $(z_1, z_2) \rightarrow P(z_1, z_2)$  vanishes whenever  $z_1 \in \mathbf{C}$  and  $z_2$  lies on a circle. By symmetry it also vanishes with the roles of the variables switched. It follows that the zero set of  $P$  (which is a complex variety) is at least three real dimensions, and hence  $P$  vanishes identically. Since  $R$  does not vanish identically on the sphere we obtain a contradiction.  $\square$

### 3.4. Example

There exist non-negative polynomials  $R$  such that  $R$  is not in  $\Sigma_h^2\mathcal{H}$ , yet there is a positive integer  $N$  for which  $R^N \in \Sigma_h^2\mathcal{H}$ . The bihomogeneous polynomial  $R_\lambda$  given by

$$R_\lambda(z, \bar{z}) = (|z_1|^2 + |z_2|^2)^4 - \lambda|z_1z_2|^4$$

satisfies this property whenever  $\lambda < 8$ . See [30]. For  $\lambda < 16$ ,  $R_\lambda > 0$  on the sphere. By Theorem 1 it agrees with a squared norm *on the sphere*.

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