UNDECIDABILITY IN A FREE *-ALGEBRA

By

**Mihai Putinar**

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**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS**

UNIVERSITY OF MINNESOTA

400 Lind Hall

207 Church Street S.E.

Minneapolis, Minnesota 55455–0436

Phone: 612-624-6066 Fax: 612-626-7370

URL: http://www.ima.umn.edu

# UNDECIDABILITY IN A FREE *-ALGEBRA

MIHAI PUTINAR

ABSTRACT. The integral structure of the maximal weights of the finite dimensional representations of $sl_2(k)$ imply that a Tarski type decision principle for free $*$-algebras with two or more generators does not hold.

## 1. Introduction

We witness these days a proliferation of non-commutative Positiv- and Null-stellensätze in non-commutative algebras. Besides the older results of Amitsur [1] and Procesi-Schacher [16] there is a variety of such theorems, valid for representations of free $*$-algebras [7, 8, 9, 10, 11, 12], enveloping algebras of Lie algebras [19, 20] or more specialized non-commutative algebras [4, 5, 6, 14, 15, 17, 18]. This turn towards the non-commutative setting is partially motivated by the recently found striking applications of the commutative Positivstellensätze in optimization theory, and by the expectation that a similar success will result from the interaction between such non-commutative Sätze and control theory, see [13, 18] for details. Whether this is the case or not it might be premature to evaluate; but at least on the theoretical side we are today well equipped with a full spectrum of surprisingly new, rich non-commutative phenomena.

Most of these novel results were obtained via duality techniques, which in their turn inherently lead to Hilbert space methods (specifically the so-called Gelfand-Naimark-Segal construction), see for details the survey [13]. On the other hand, the classical, commutative Sätze were discovered in connection with, and are in general proved via, Tarski's decision principle for real closed fields, see for instance [2].

It is the aim of this note to refute the belief that, in spite of the abundance of non-commutative Null- and Positiv-stellensätze, a decision principle for the free $*$-algebra with two generators is possible. This is simply due to the observation that the weights of all finite dimensional representations of the Lie algebra $sl_2(\mathbb{C})$ exhaust all integers. Thus, one can define the integers within the free $*$-algebra with two generators and its finite dimensional representations. Then Gödel's Theorem will assure the existence of undecidable propositions built inside this system.

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## 2. Main result

Let $k = \mathbb{R}$ or $k = \mathbb{C}$ be the ground field. Let $A$ be a $k$-algebra with unit 1 and (anti-linear in the complex case) involution $f \mapsto f^*$. That is, $(fg)^* = g^* f^*,\ 1^* = 1$. The (maximal spectrum) point evaluations in the commutative case are customarily replaced in the non-commutative one by (irreducible) finite dimensional Hilbert space representations. We assimilate the latter with finite dimensional (real, respectively complex) Hilbert $A$-modules to the left. To be more specific, if $M$ is an Hilbert A-module with inner product $\langle \cdot, \cdot \rangle$, then

$$\langle fx, y \rangle = \langle x, f^* y \rangle, \quad x, y \in M,\ f \in A.$$

That is the involution from $A$ corresponds to taking the adjoint operator in the representation.

We denote by $k\langle t_1, ..., t_n \rangle$ the free $*$-algebra with involution, that is the $k$-algebra freely generated by $t_1, ..., t_n, t_1^*, ..., t_n^*$, with the obvious involution.

**Lemma 1.** *Let $A = k\langle t_1, t_2 \rangle$ be the free $k$-algebra with involution, with two generators. Let $X = t_1, Y = t_1^*, H = t_2 + t_2^*$. The set of those scalars $\lambda \in k$ with the property that there exists a finite dimensional Hilbert $A$-module $M$, satisfying*

$$[H, X]h = 2Xh,\ [H, Y]h = -2Yh,\ [X, Y]h = Hh, \quad h \in M,$$

*and there exists $v \in M \setminus \{0\}$ such that*

$$Hv = \lambda v,$$

*coincides with the set of all integers $\mathbb{Z}$.*

The proof uses the standard description of all irreducible representations of $sl_2(k)$, see for instance Proposition VIII.1.1 in [3].

A couple of remarks are in order. Knowing the structure of all representations of $sl_2(k)$ one can replace $M$ in the statement by an irreducible Hilbert module, and furthermore, the last requirement in the Lemma can be replaced by: "there exists $V \in A$, such that

$$V \cdot M \neq 0, \quad HVh = \lambda Vh, \quad h \in M."$$

**Theorem 2.** *There exists an undecidable statement constructed with quantifiers and the calculus of propositions from the $k$-free-$*$ algebra with two generators and its finite dimensional (irreducible) Hilbert modules to the left.*

On the other hand, there is nothing to worry from the applied side. Specifically, when working with matrix representations of the free $*$-algebra, of a fixed dimension, Tarski's principle remains valid. Indeed, let $F = k\langle t_1, ..., t_d \rangle$ be the free $*$-algebra with $d$ generators $t = (t_1, ..., t_d)$, and fix a positive integer $N$. Let $X = (X_1, ..., X_d)$ denote a $d$-tuple of complex $N \times N$-matrices, and let $f_i, g_j, i \in I, j \in J$, be finite systems of hermitian elements of $F$. The set of all tuples $X$ satisfying

$$f_i(X, X^*) \geq 0, \quad i \in I,$$

where $\geq 0$ means positive semi-definiteness as a matrix, and

$$(\exists)\ v_j \in \mathbb{C}^N,\ \langle g_j(X, X^*)v_j, v_j\rangle > 0,\ j \in J,$$

can be described, when written on the matrix elements $X = (X_{k\ell})$, by a system of polynomial inequalities.

Thus, our main result can be rephrased as follows: given a proposition $P$ constructed with elements of the free $*$-algebra and its representations, there is no dimension bound function $N(P)$ to guarantee that the proposition is true if and only if it is true for all representations of dimension less than or equal to $N(P)$. There are of course exceptions to this rule, as for instance the Striktpositivstellensätze discussed in [9, 12].

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(M.P.) Mathematics Department, University of California, Santa Barbara, CA 93106

*E-mail address*: `mputinar@math.ucsb.edu`

*URL*: `http://math.ucsb.edu/~mputinar`