ON GLOBAL EXISTENCE OF SOLUTIONS TO A CROSS-DIFFUSION SYSTEM

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ON GLOBAL EXISTENCE OF SOLUTIONS TO A CROSS-DIFFUSION SYSTEM

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Abstract. We consider a strongly coupled nonlinear parabolic system which arises in population dynamics in $n-$dimensional domains ($n \geq 1$). We prove the global existence of classical solutions to the system for $n < 10$.

1. Introduction.

Modeling spatial segregation phenomena of competing species in population dynamics, Shigesada, Kawasaki and Teramoto proposed [17] in 1979 to study some nonlinear parabolic systems which include the following problem

$$
\left\{
\begin{array}{lll}
u_t & = d_1\Delta[(1+\alpha v+\gamma u)u] + u(a_1 - b_1 u - c_1 v) & \text{in } Q_T,\\
v_t & = d_2\Delta[(1+\delta v+\beta u)v] + v(a_2 - b_2 u - c_2 v) & \text{in } Q_T,\\
\frac{\partial u}{\partial \nu} & = \frac{\partial v}{\partial \nu} = 0 & \text{on } S_T,\\
u(& x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \Omega,
\end{array}
\right. \tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ (for $n \in \mathbb{N}$) with smooth boundary $\partial\Omega$, T is a positive number, $Q_T = \Omega \times [0,T)$, $S_T = \partial\Omega \times [0,T)$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$ is the Laplacian, $\partial/\partial\nu$ denotes the directional derivative along the outward normal on $\partial\Omega$, a_i, b_i, c_i, d_i $(i = 1,2)$ are given positive constants and $\alpha, \gamma, \delta, \beta$ are non-negative constants. In the system (1.1) u and v are non-negative functions which represent population densities of two competing species, d_1 and d_2 are respectively their diffusion rates. Parameters a_1 and a_2 are intrinsic growth rates, b_1 and c_2 are coefficients for intra-specific competitions, b_2 and c_1 are coefficients for inter-specific competitions. Parameters γ and δ are usually called self-diffusion rates, and α and β are called cross-diffusion rates. The homogeneous Neumann boundary condition means there is no migration crossing the boundary $\partial\Omega$. When $\alpha = \gamma = \delta = \beta = 0$, (1.1) reduces to the well-known Lotka-Volterra competition-diffusion system.

Mathematically, the problem (1.1) has received a lot of attention. Local existence (in time) of solutions to (1.1) was established by Amann in a series of important papers [1], [2], [3]. His results can be summarized as follows.

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Theorem 1.1. *Assume that u_0, v_0 are in $W_p^1(\Omega)$ for some $p > n$. The system (1.3) has a unique non-negative smooth solution $u(x,t), v(x,t)$ in $C((0,T), W_p^1(\Omega)) \cap C^\infty((0,T), C^\infty(\overline{\Omega}))$ with maximal existence time T. Moreover, if the solution (u,v) satisfies the estimate*

$$\sup_{0\le t<T} ||u(.,t)||_{W_p^1(\Omega)} < \infty \quad \textit{and} \quad \sup_{0\le t<T} ||v(.,t)||_{W_p^1(\Omega)} < \infty, \tag{1.2}$$

then $T = \infty$.

However, little is known about global existence of solutions to (1.1). In particular, when α or β is positive, (1.1) is a strongly coupled parabolic system which occurs frequently in biological and chemical models and it is very difficult to analyze. In 1984, Kim [9] proved the global existence of classical solutions to (1.1) for $n = 1, d_1 = d_2$, $\alpha > 0, \beta > 0$ and $\gamma = \delta = 0$. Recently, Shim [18] improved Kim's results and established uniform bounds of solutions by using a different method. When $n = 2$, Yagi [19] showed that (1.1) possesses a global solution if either $8\gamma > \alpha > 0$ and $8\delta > \beta > 0$ or else $\delta = \beta = 0$ and $\gamma > 0$. For any dimension n, under the smallness condition on the cross-diffusion pressures α, β or initial values u_0, v_0, Deuring [8] established the global existence of solutions to (1.1) when $\gamma = \delta = 0$. For further recent results on global existence of the full system (1.1) with any n and various conditions on coefficients, one can see [11] and [14].

In this paper, we are specially interested in global existence of solutions to (1.1) when $\beta = 0$. Precisely, we shall prove the global existence of solutions to the following system of equations

$$\begin{cases} u_t &= d_1\Delta[(1+\alpha v+\gamma u)u] + \ u(a_1 - b_1u - c_1v) \quad \text{in } Q_T, \\ v_t &= d_2\Delta[(1+\delta v)v] + \ v(a_2 - b_2u - c_2v) \quad \text{in } Q_T, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad \text{on } S_T, \\ u(x,0) &= u_0(x) \ge 0, \quad v(x,0) = v_0(x) \ge 0 \quad \text{in } \Omega. \end{cases} \tag{1.3}$$

Using the L^p-estimates, Lou, Ni and Wu established in 1998 [15] global existence of solutions to (1.3) for $n = 1, 2$ with $\gamma \ge 0$, $\alpha > 0$ and $\delta \ge 0$. In 2003, 2004 Choi, Lui and Yamada considered the problem and they obtained some results on the global existence of the solutions to (1.3) in [5], [6]. Their results, however, have the restriction $n < 6$ when $\delta > 0$. Independently from [6], D. Le, L. Nguyen and T. Nguyen used the semi-group techniques and obtained global attractor and global existence results of (1.3) [13]. These results also have the restriction $n < 6$.

In another approach, Le and Nguyen constructed a special test function for (1.3) and obtained a global existence result with any n [12]. Their result, however, has some various restrictions on the coefficients of (1.3). Recently, by a non-trivial application of maximum principle (see [16]), we improved the results of Le and Nguyen. Our result in [16] also has some restrictions on the coefficients of (1.3).

Here, we would like to remark that the equation of u can be written as

$$u_t = d_1\nabla\cdot[(1+\alpha v+2\gamma u)\nabla u + \alpha u\nabla v] + u(a_1 - b_1u - c_1v).$$

Therefore, to establish the L^p-estimates of u, it is vital to understand the regularity of ∇v. However, it is not an easy task to obtain the L^p-estimates of ∇v directly from the second equation of (1.3) since this equation also depends on u. To overcome this difficulty, these authors mentioned above used Sobolev embedding theorems. Therefore, they were only able to construct the global existence results for low n. It is then an open and interesting question to understand the problem for higher n.

In this paper, we first establish a result on L^p-estimates of gradients of solutions to a class of nonlinear parabolic equations. This result enables us to obtain the L^4-estimate of ∇v directly from the equation of v. Then, by an iteration method, we show that u is in L^r for any $r \geq 1$ when $n < 10$. The global existence of the solutions then follows. More importantly, our result on L^p-estimates of gradients of solutions to non-linear parabolic equations is a topic of independent interests and we believe that it may have other applications. The results on the global existence of the system (1.3) is summarized as the following theorem.

Theorem 1.2. *Assume that $\gamma > 0$ and $n < 10$. Assume also that $u_0 \geq 0$, $v_0 \geq 0$ satisfy zero Neumann boundary conditions and belong to $C^{2+\lambda}(\overline{\Omega})$ for some $\lambda > 0$. Then (1.3) possesses a unique non-negative solution $u, v \in C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0,\infty))$.*

The paper is organized as follows. In Section 1, we introduce our main result. The L^p-estimates of gradients of solutions to a class nonlinear parabolic equations and some useful results will be presented in Section 2. In Section 3, we establish L^r-estimates of the solution u of (1.3) and in the last Section we give a proof of Theorem 1.2.

2. Preliminary results

Let $p \geq 1$ be any arbitrary number. Throughout this paper, $g \in W_p^1(\Omega)$ means g and ∇g are in $L^p(\Omega)$ and $g \in W_p^{2,1}(Q_T)$ means $g, g_t, g_{x_i}, g_{x_i x_j}$ are in $L^p(Q_T)$ for $i, j = 1, 2...n$ with norm

$$||g||_{W_p^{2,1}(Q_T)} = ||g||_{L^p(Q_T)} + ||g_t||_{L^p(Q_T)} + ||\nabla g||_{L^p(Q_T)} + ||\nabla^2 g||_{L^p(Q_T)}.$$

In addition, for any function g on Q_T, we say g in $V_2(Q_T)$ if the following norm is finite

$$||g||_{V_2(Q_T)} = \sup_{0 \leq t \leq T} ||g(.,t)||_{L^2(\Omega)} + ||\nabla g||_{L^2(Q_T)}.$$

Let $a(x,t,\xi)$ be continuous and (x,ξ)-differentiable, where $(x,t,\xi) \in Q_T \times \mathbb{R}$. Assume also that $a(x,t,\xi)$ satisfies the following conditions

(i) There is $d > 0$ such that $a(x,t,\xi) \geq d$ and $a_\xi(x,t,\xi) \geq 0$ for all $(x,t) \in Q_T$ and ξ in $\mathbb{R}$.
(ii) There is a continuous function M on $\mathbb{R}$ such that $a(x,t,\xi) \leq M(\xi)$, for all $(x,t) \in Q_T$.
(iii) For any bounded measurable function g on Q_T, $|a_x(.,.,g(.,.))|$ is in $L^{2p}(Q_T)$, where $a_x = (a_{x_1}, a_{x_2}, .., a_{x_n})$.

We then introducing the following proposition which will be used in the next sections.

Proposition 2.1. *Assume that $w \in W_p^{2,1}(Q_T) \cap C^{2,1}(\overline{\Omega} \times [0,T))$ is a bounded function satisfying*

$$w_t \leq a(x,t,w)\Delta w + f(x,t) \text{ in } Q_T, \tag{2.1}$$

with boundary condition $\dfrac{\partial w}{\partial \nu} \leq 0$ on S_T, where $f \in L^p(Q_T)$. Then, ∇w is in $L^{2p}(Q_T)$.

Proof. By replacing $a(x,t,\inf_{Q_T} w+\xi)$ for $a(x,t,\xi)$, and $w-\inf_{Q_T} w$ for w, we can assume that $w\geq 0$. For any fixed $0<t<T$, multiplying (2.1) by $w|\nabla w|^{2(p-1)}$ and integrating the result over Ω, we have

$$\begin{aligned}
\int_\Omega w_t w|\nabla w|^{2(p-1)}dx &\leq \int_\Omega \Delta w(aw|\nabla w|^{2(p-1)})dx+\int_\Omega fw|\nabla w|^{2(p-1)}dx\\
&\leq \int_{\partial\Omega} aw|\nabla w|^{2(p-1)}\frac{\partial w}{\partial\nu}-\int_\Omega \nabla w\cdot\nabla(aw|\nabla w|^{2(p-1)})dx\\
&\qquad\qquad+\int_\Omega fw|\nabla w|^{2(p-1)}dx\\
&\leq -\int_\Omega a|\nabla w|^{2p}dx-\int_\Omega w\nabla w\cdot\nabla(a|\nabla w|^{2(p-1)})dx\\
&\qquad\qquad+\int_\Omega fw|\nabla w|^{2(p-1)}dx\\
&\leq -d\int_\Omega |\nabla w|^{2p}dx-\int_\Omega wa_\xi|\nabla w|^{2p}dx-\int_\Omega w|\nabla w|^{2(p-1)}\nabla w\cdot a_x dx\\
&\qquad -\int_\Omega wa\nabla w\cdot\nabla(|\nabla w|^{2(p-1)})dx+\int_\Omega fw|\nabla w|^{2(p-1)}dx\\
&\leq -d\int_\Omega |\nabla w|^{2p}dx+\int_\Omega w|\nabla w|^{2p-1}|a_x|dx\\
&\quad +2(p-1)\sum_{i,j=1}^n\int_\Omega wa|\nabla w|^{2(p-1)}|w_{x_ix_j}|dx+\int_\Omega w|f||\nabla w|^{2(p-1)}dx
\end{aligned}$$

Since w is bounded, we can find a constant $C_1>0$ such that

$$\begin{aligned}
d\int_\Omega |\nabla w|^{2p}dx &\leq C_1\Bigg[\int_\Omega |w_t||\nabla w|^{2(p-1)}dx+\int_\Omega |\nabla w|^{2p-1}|a_x|dx\\
&\qquad+\sum_{i,j=1}^n\int_\Omega |\nabla w|^{2(p-1)}|w_{x_ix_j}|dx+\int_\Omega |f||\nabla w|^{2(p-1)}dx\Bigg]. \qquad (2.2)
\end{aligned}$$

Using the Young's inequality, we can find a constant $C_2>0$ such that

$$\begin{aligned}
|w_t||\nabla w|^{2(p-1)} &\leq \frac{d}{8C_1}|\nabla w|^{2p}+C_2|w_t|^p,\\
|\nabla w|^{2p-1}|a_x| &\leq \frac{d}{8C_1}|\nabla w|^{2p}+C_2|a_x|^{2p},\\
\sum_{i,j=1}^n|\nabla w|^{2(p-1)}|w_{x_ix_j}| &\leq \frac{d}{8C_1}|\nabla w|^{2p}+C_2\sum_{i,j=1}^n|w_{x_ix_j}|^p,\\
|f||\nabla w|^{2(p-1)} &\leq \frac{d}{8C_1}|\nabla w|^{2p}+C_2|f|^p.
\end{aligned}$$

Therefore, (2.2) becomes

$$\begin{aligned}
\frac{d}{2}\int_\Omega |\nabla w|^{2p}dx &\leq C_1C_2\Big[||w_t(.,t)||^p_{L^p(\Omega)}+||w_{xx}(.,t)||^p_{L^p(\Omega)}\\
&\qquad +||f(.,t)||^p_{L^p(\Omega)}+||a_x(.,t,w(.,t))||^{2p}_{L^{2p}(\Omega)}\Big] \qquad (2.3)
\end{aligned}$$

Integrating (2.3) inequality with respect to t from 0 to T, we obtain

$$\int_{Q_T} |\nabla w|^{2p} dxdt \le C\left[||w||^p_{W^{2,1}_p(Q_T)} + ||f||^p_{L^p(Q_T)} + ||a_x(.,.,w)||^{2p}_{L^{2p}(Q_T)}\right], \tag{2.4}$$

where $C > 0$ is a constant depending on $\sup w$ and a. From the hypotheses of the lemma, we get the desired result. □

Following the proof of Proposition 2.1, we see that the same result also holds for parabolic equations in divergence forms. We state this claim in the following remark.

Remark 2.1. *Assume that $w \in W^{2,1}_p(Q_T) \cap C^{2,1}(\overline{\Omega} \times [0,T))$ is a bounded solution of the following equation*

$$w_t = D_i[a_{ij}(x,t)D_j w] + f(x,t) \text{ in } Q_T, \tag{2.5}$$

with boundary condition

$$[\nu_i a_{ij} D_j w]w \le 0 \text{ on } S_T,$$

where $f \in L^p(Q_T)$ and a_{ij} are measurable, bounded functions and there is a fixed constant $d > 0$ such that

$$a_{ij}(x,t)\xi_i\xi_j \ge d|\xi|^2, \ \forall\ \xi = (\xi_1,..,\xi_n) \in \mathbb{R}^n, \ \forall\ (x,t) \in Q_T.$$

Then, ∇w is in $L^{2p}(Q_T)$.

Now, we recall the following known results which are going to be used in the next sections.

Lemma 2.1 (Lemma 2.4 [6])**.** *Let $q > 1$, $\bar{q} = 2 + 4q/n(q+1)$, $\bar{\beta}$ be in $(0,1)$ and let $C_T > 0$ be any number which may depend on T. Then there is a constant M_1 depending on $q, n, \Omega, \bar{\beta}$ and C_T such that for any g in $C([0,T), W^1_2(\Omega))$ with $(\int_\Omega |g(.,t)|^{\bar{\beta}} dx)^{1/\bar{\beta}} \le C_T$ for all $t \in [0,T]$, we have the following inequality*

$$||g||_{L^{\bar{q}}(Q_T)} \le M_1 \left\{1 + (\sup_{0\le t\le T} ||g(.,t)||_{L^{2q/(q+1)}(\Omega)})^{4q/n(q+1)\bar{q}} ||\nabla g||^{2/\bar{q}}_{L^2(Q_T)}\right\}.$$

Lemma 2.2 (Theorem 9.1 [10] p.341-342 and its remark on p. 351.)**.** *Let $3 < q < \infty$ and $\bar{w}$ be a solution of the equation*

$$\begin{cases} \bar{w}_t - \bar{a}_{ij}(x,t)\bar{w}_{x_i x_j} = h(x,t), & (x,t) \in Q_T \\ \frac{\partial \bar{w}}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ \bar{w}(x,0) = \bar{w}_0(x) & x \in \Omega, \end{cases} \tag{2.6}$$

where $T < \infty$ and $\bar{a}_{ij}$ are bounded continuous functions on $\overline{Q}_T$ satisfying

$$\lambda|\xi|^2 \le \bar{a}_{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \forall \xi \in \mathbb{R}^n,$$

where λ, Λ are positive constants. Suppose $h \in L^q(Q_T)$. Then there exists a constant c_q depending on the bounds of $\{a_{ij}(x,t)\}_{i,j=1,..n}, \lambda, \Lambda, \Omega, T$ and q such that

$$||\bar{w}||_{W^{2,1}_q(Q_T)} \le c_q \left(||h||_{L^q(Q_T)} + ||\bar{w}_0||_{W^{2-2/q}_q(\Omega)}\right), \tag{2.7}$$

where the constant c_q remains bounded for finite values of T and $\bar{w}_0(x)$ satisfies the compatibility condition $\dfrac{\partial \bar{w}_0}{\partial \nu} = 0$ on $\partial\Omega$.

3. L^r-ESTIMATES FOR u.

Let $T = T_{\max}$, where $T_{\max}$ is as in Theorem 1.1. Assume by contradiction that $T < \infty$. First, we introduce the following lemma.

Lemma 3.1. *There exists a constant $C_1(T) > 0$ such that*

$$||\nabla v||_{L^4(Q_T)} \leq C_1(T).$$

Proof. Let $w_1 = (1+\delta v)v$. By using the maximum principle to the equation of v, we see that v is bounded. Therefore, w_1 is also bounded. By Lemma 2.2 [6], we have $w_1 \in W_2^{2,1}(Q_T)$. Moreover, simple calculation shows that w_1 satisfies

$$\begin{aligned} w_{1t} &\leq d_2(1+2\delta v)\Delta w_1 + (1+2\delta v)v \\ &= d_2\sqrt{1+4\delta w_1}\Delta w_1 + (1+2\delta v)v. \end{aligned}$$

Applying Proposition 2.1 with $p = 2, a(x,t,\xi) = d_2\sqrt{1+4\delta\xi},\ f(x,t) = (1 + 2\delta v(x,t))v(x,t)$, we obtain the desired result. □

Lemma 3.2. *Let $r > 2$ and $p_r = \dfrac{2r}{r-2}$ be two positive numbers. Assume that $\gamma > 0$ and assume also that there is a constant $M_{r,T} > 0$ depending only on r, T, Ω and the coefficients of (1.3) such that*

$$||\nabla v||_{L^r(Q_T)} \leq M_{r,T}.$$

Then for any $q > 1$, there exists a constant $C(r,q,T) > 0$ such that

$$\begin{aligned} ||u(.,t)||^q_{L^q(\Omega)} &+ ||\nabla(u^{q/2})||^2_{L^2(Q_t)} + ||\nabla(u^{(q+1)/2})||^2_{L^2(Q_t)} \\ &\leq C(r,q,T)\left(1 + ||u||^{q-1}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\right),\ \forall\ t\ \in\ [0,T). \end{aligned} \tag{3.1}$$

Proof. For any constant $q > 1$, multiplying the equation of u in (1.3) by qu^{q-1} and using the integration by parts, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\int_\Omega u^q dx &= q\int_\Omega u^{q-1}\{d_1\Delta[(1+\alpha v+\gamma u)u] + u(a_1 - b_1 u - c_1 v)\}dx \\ &= qd_1\int_\Omega u^{q-1}\{\nabla\cdot[(1+\alpha v+2\gamma u)\nabla u] + \alpha\nabla\cdot[u\nabla v]\}dx \\ &\quad + q\int_\Omega u^q(a_1 - b_1 u - c_1 v)dx \\ &= -q(q-1)d_1\int_\Omega u^{q-2}(1+\alpha v+2\gamma u)|\nabla u|^2 dx \\ &\quad - (q-1)d_1\alpha\int_\Omega \nabla(u^q)\cdot\nabla v + q\int_\Omega u^q(a_1 - b_1 u - c_1 v)dx \\ &\leq -q(q-1)d_1\int_\Omega u^{q-2}(1+2\gamma u)|\nabla u|^2 dx \qquad (3.2) \\ &\quad - (q-1)d_1\alpha\int_\Omega \nabla(u^q)\cdot\nabla v + q\int_\Omega u^q(a_1 - b_1 u - c_1 v)dx. \end{aligned}$$

Integrating (3.2) from 0 to t we get

$$\int_\Omega u(x,t)^q dx - \int_\Omega u_0(x)^q dx + q(q-1)d_1 \int_{Q_t} u^{q-2}(1+2\gamma u)|\nabla u|^2 dxds$$
$$\leq -(q-1)d_1\alpha \int_{Q_t} \nabla(u^q)\cdot\nabla v + q\int_{Q_t} u^q(a_1 - b_1 u - c_1 v) \tag{3.3}$$

Since,

$$\nabla(u^{\frac{q}{2}}) = \frac{q}{2}u^{\frac{q-2}{2}}\nabla u, \quad \nabla(u^{\frac{q+1}{2}}) = \frac{q+1}{2}u^{\frac{q-1}{2}}\nabla u.$$

We see that (3.3) is equivalent to

$$\int_\Omega u^q(x,t)dx + 4d_1\frac{q-1}{q}\int_{Q_t}|\nabla(u^{q/2})|^2 + \frac{8\gamma q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla(u^{(q+1)/2})|^2$$
$$\leq -(q-1)d_1\alpha\int_{Q_t}\nabla(u^q)\nabla v + q\int_{Q_t}u^q(a_1-b_1u-c_1v) + \int_\Omega u_0(x)^q.$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned} q\int_{Q_t} u^q(a_1-b_1u-c_1v) &\leq -qb_1\int_{Q_t}u^{q+1}dxds + qa_1\int_{Q_t}u^q dxds \\ &\leq -qb_1||u||^{q+1}_{L^{q+1}(Q_t)} + qa_1|Q_T|^{1/(q+1)}||u||^q_{L^{q+1}(Q_t)} \\ &\leq c, \ \forall\ t\ \in\ [0,T), \end{aligned}$$

where c is a finite constant which depends only on $T, q, |\Omega|$ and the coefficients a_1, b_1 of (1.3). Therefore, there is a constant $C_1(q,T) > 0$ depending on q, T, Ω coefficients of (1.3) and initial data u_0 such that

$$\int_\Omega u^q(x,t) + 4d_1\frac{q-1}{q}\int_{Q_t}|\nabla(u^{q/2})|^2 + 8\gamma\frac{q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla(u^{(q+1)/2})|^2$$
$$\leq -(q-1)d_1\alpha\int_{Q_t}\nabla(u^q)\nabla v + C_1(q,T),\ \forall\ t\in[0,T). \tag{3.4}$$

Since that $\dfrac{1}{r} + \dfrac{1}{2} + \dfrac{1}{p_r} = 1$ and ∇v is in $L^r(Q_T)$, using the Hölder's inequality, we have

$$\begin{aligned} |\int_{Q_t}\nabla(u^q)\nabla v| &= \frac{2q}{q+1}\left|\int_{Q_t}u^{(q-1)/2}\nabla(u^{(q+1)/2})\nabla v dxds\right| \\ &\leq \frac{2q}{q+1}||u||^{(q-1)/2}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}||\nabla(u^{(n+1)/2})||_{L^2(Q_t)}||\nabla v||_{L^r(Q_T)} \\ &\leq \frac{2q}{q+1}M_{r,T}||u||^{(q-1)/2}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}||\nabla(u^{(n+1)/2})||_{L^2(Q_t)} \\ &\leq \frac{2\epsilon q M_{r,T}}{q+1}||\nabla(u^{(n+1)/2})||^2_{L^2(Q_t)} + \frac{qM_{r,T}}{2(q+1)\epsilon}\,||u|\,|^{q-1}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}, \end{aligned} \tag{3.5}$$

for any $\epsilon > 0$. From (3.4), (3.5) and by choosing a sufficiently small ϵ, we get (3.1). This completes the proof of the lemma. □

Next, for any number a, we denote $a_+ = \max\{a, 0\}$. Then, we have the following lemma.

Lemma 3.3. *Assume that $\gamma > 0$. Then there is a constant $C_2(T) > 0$ such that $||u||_{V_2(Q_T)} \leq C_2(T)$. Moreover, for any constant $r < \dfrac{4(n+1)}{(n-2)_+}$, there is positive constant $C_{r,T}$ such that $||u||_{L^r(Q_T)} \leq C_{r,T}$.*

Proof. For any $q > 1$, we define $w := u^{(q+1)/2}$. Also, we introduce

$$\begin{aligned} E &= \sup_{0\leq t\leq T} \int_\Omega u^q(x,t)dx + \int_{Q_T} |\nabla(u^{(q+1)/2})|^2 dxdt \\ &= \sup_{0\leq t\leq T} \int_\Omega w^{2q/(q+1)} dx + \int_{Q_T} |\nabla w|^2 dxdt. \end{aligned}$$

Let $r_0 = 4$, $p_0 = \frac{2r_0}{r_0-2}$. By Lemma 3.1, we see that ∇v is in $L^{r_0}(Q_T)$. So, from Lemma 3.2, we have

$$E + ||\nabla(u^{q/2})||^2_{L^2(Q_T)} \leq C(r_0,q,T)\left(1 + ||w||^{\frac{2(q-1)}{q+1}}_{L^{\frac{p_0(q-1)}{q+1}}(Q_T)}\right), \tag{3.6}$$

for some constant $C(r_0,q,T) > 0$ depending only on T, Ω, initial data u_0, v_0 and the coefficients of (1.3) Now, setting $\bar\beta = 2/(q+1) \in (0,1)$, by Lemma 2.2 [15] we see that there is a constant $C(T) > 0$ such that

$$||w(.,t)||_{L^{\bar\beta}(\Omega)} = ||u(.,t)||^{1/\bar\beta}_{L^1(\Omega)} \leq C(T)^{1/\bar\beta},\ \forall\, t \in [0,T]. \tag{3.7}$$

So, in addition to $q > 1$, if we restrict our q so that

$$(np_0 - 2n - 4)q \leq 2n + np_0 \tag{3.8}$$

Then $\dfrac{p_0(q-1)}{q+1} \leq \bar q$, where $\bar q$ is as in Lemma 2.1. Therefore, by Hölder's inequality, (3.7), Lemma 2.1 and the definition of E, we have

$$\begin{aligned} ||w||_{L^{\frac{p_0(q-1)}{q+1}}(Q_T)} &\leq C_3(q,T)\,||w||_{L^{\bar q}(Q_T)} \\ &\leq C_3(q,T)M_1\left\{1 + E^{2/n\bar q}E^{1/\bar q}\right\}, \end{aligned} \tag{3.9}$$

for some finite constant $C_3(q,T) > 0$ depending on $|\Omega|$, T, p_0 and q. From (3.6) and (3.9) we can find a constant $C_4(q,T) > 0$ such that

$$E \leq C_4(q,T)(1 + E^\mu E^\nu), \tag{3.10}$$

with

$$\mu = \frac{4(q-1)}{n\bar q(q+1)} \text{ and } \nu = \frac{2(q-1)}{\bar q(q+1)}.$$

Since

$$\begin{aligned} \mu + \nu &= \frac{2(q-1)}{\bar q(q+1)}\left[\frac{2}{n} + 1\right] \\ &< \frac{1}{\bar q}\left[\frac{4q}{n(q+1)} + 2\right] = 1, \end{aligned}$$

we see from (3.10) that E is bounded from above by a constant depending only on q, T, Ω and the coefficients of the system (1.3). Therefore, from (3.9) we get

$w \in L^{\bar{q}}(Q_T)$ which in turn implies $u \in L^r(Q_T)$ for $r = \bar{q}(q+1)/2$ for any q satisfying (3.8). Now, looking at (3.8), we see that if $n \le 2$, we have

$$np_0 - 2n - 4 = \frac{4n + 8 - 4r_0}{r_0 - 2} = 2(n-2) \le 0, \tag{3.11}$$

then (3.8) holds for all q. So for $n \le 2$, u is in $L^r(Q_T)$ for all $r > 1$. Now, suppose that $n > 2$ we see that (3.8) is equivalent to

$$1 < q \le q_0 := \frac{2n + np_0}{np_0 - 2n - 4} = \frac{n(r_0 - 1)}{n + 2 - r_0} = \frac{3n}{n-2}. \tag{3.12}$$

Then, we have

$$\frac{\bar{q}(q+1)}{2} = q + 1 + \frac{2q}{n} \le \overline{r}_1 := q_0 + 1 + \frac{2q_0}{n} = \frac{(n+1)r_0}{n + 2 - r_0} = \frac{4(n+1)}{n-2}. \tag{3.13}$$

So, we see that u is in $L^r(Q_T)$ for all $1 < r \le \overline{r}_1$ and this completes the proof of the second assertion of the lemma. Finally, since (3.8) holds true for $q = 2$. So when we take $q = 2$, we have E is finite. Therefore, from (3.6) and (3.9), we see that $||u||_{V_2(Q_T)}$ is bounded by a constant depending only on T and the coefficients of (1.3). This completes the proof of the lemma. □

Remark 3.1. *In* [6]*, for the case $\delta > 0$, the authors were only able to obtain $u \in V_2(Q_T)$ for $n < 6$ (see Proposition 3.1 of* [6]*). Here, u is in $V_2(Q_T)$ for any n.*

4. Proof of Theorem 1.2.

We begin with the following lemma

Lemma 4.1. *Assume that there are $r_1 > \max\{\frac{n+2}{2}, 3\}$ and a positive constant $C_{r_1,T}$ such that*

$$||u||_{L^{r_1}(Q_T)} \le C_{r_1,T}.$$

Then, u is in $L^r(Q_T)$ for any $r > 1$.

Proof. First of all, the equation of v can be written in the divergence form as

$$v_t = d_2 \nabla \cdot [(1 + 2\delta v)\nabla v] + v(a_2 - b_2 u - c_2 v). \tag{4.1}$$

Since $1 + 2\delta v$ is bounded in $\overline{Q}_T$ and $v(a_2 - b_2 u - c_2 v)$ is in $L^{r_1}(Q_T)$ with $r_1 > \frac{n+2}{2}$, applying Theorem 1.3 [7, p. 43] to (4.1)(see also Theorem 10.1 [10, p. 204]), we see that there is a constant $\beta > 0$ such that

$$v \in C^{\beta,\beta/2}(\overline{Q}_T). \tag{4.2}$$

Moreover, we have

$$(w_1)_t = d_2(1 + 2\delta v)\Delta w_1 + h_1, \tag{4.3}$$

where $w_1 = (1+\delta v)v$ is as in the proof of Lemma 3.1, $h_1 = v(1+2\delta v)(a_2 - b_2 u - c_2 v)$. Since v is bounded and by the assumption of this lemma, we see that $h_1(x,t)$ is in $L^{r_1}(Q_T)$. From (4.2), we can apply Lemma 2.2 to the equation (4.3). Then we have

$$w_1 \in W^{2,1}_{r_1}(Q_T). \tag{4.4}$$

This implies $\nabla v = \dfrac{1}{1+2\delta v}\nabla w_1$ in $L^{r_1}(Q_T)$. Now, following the proof of Lemma 3.3 with r_1 instead of r_0 and $p_1 = \dfrac{2r_1}{r_1-2}$ instead of p_0, we see that either u is in $L^r(Q_T)$ for any $r>1$ or else u is in $L^{r_2}(Q_T)$ with

$$r_2 := \frac{(n+1)r_1}{n+2-r_1}.$$

The later case happens if and only if

$$n+2-r_1>0.$$

If this case happens, we see that h_1 is in $L^{r_2}(Q_T)$. Therefore, applying the regularity result again, we get ∇v in $L^{r_2}(Q_T)$. Then we go back and do the same argument again. Keep doing like this we will get a sequence of numbers

$$r_{k+1} := \frac{(n+1)r_k}{n+2-r_k}. \tag{4.5}$$

We stop and get the conclusion that u is in $L^r(Q_T)$ for any $r>1$ when

$$n+2-r_k \le 0. \tag{4.6}$$

By the formula (4.5) and since $r_1>3$, we can prove by induction that $r_k>3$ for any $k=1,2,...$ Then, we have

$$\frac{r_{k+1}}{r_k} = \frac{n+1}{n+2-r_k} \ge \frac{n+1}{n-1} > 1. \tag{4.7}$$

This implies that the sequence r_k is strictly increasing. Therefore, there must be some k such that (4.6) holds. We stop at this k and conclude that u is in $L^r(Q_T)$ for any $r>1$. □

Now, for $n<10$, we see that $\dfrac{n+2}{2} < \dfrac{4(n+1)}{(n-2)_+}$. So, by Lemma 3.3 and Lemma 4.1, we see that u in $L^r(Q_T)$ for any $r>1$ and u is in $V_2(Q_T)$. Then, the proof of the theorem is trivial and is exactly the same as that of Theorem 1.1 of [16].

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References

[1] H. Amann, *Dynamic theory of quasilinear parabolic equations - I. Abstract evolution equations*, Nonlinear Analysis - Theory, Methods and Applications, Vol. 12, 9(1988), 895-919.

[2] H. Amann, *Dynamic theory of quasilinear parabolic equations - II. Reaction-diffusion systems*, Differential Integral Equations, Vol. 3, 1(1990), 13-75.

[3] H. Amann, *Dynamic theory of quasilinear parabolic systems - III. Global existence*, Math. Z. 202(1989), 219-250.

[4] H. Amann, *Abstract methods in differential equations*, Rev. R. Acad. Cien. Serie A. Mat. Vol. 97, 1(2003), 89-105.

[5] Y.S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion*, Discrete and Continuous Dynamical Systems, Vol. 9, 5(2003), 1193-1200.

[6] Y.S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion*, Discrete and Continuous Dynamical Systems, Vol. 10, 3(2004), 719-730.

[7] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1993.

[8] P. Deuring, *An initial-boundary value problem for certain density-dependent diffusion system*, Math. Z. 194 (1987), 375-396.

[9] J. U. Kim, *Smooth solutions to a quasilinear system of diffusion equations for certain population model,* Nonlinear Analysis, T.M. and A. 8(1984) 1121-1144.

[10] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, Translations of Mathematical Monographs, 23, AMS, 1968.

[11] D. Le, *Global existence for a class of strongly coupled parabolic systems. Ann. Mat. Pura Appl.* (4) 185 (2006), no. 1, 133–154.

[12] D. Le, T. Nguyen, *Global existence for a class of triangular parabolic systems on domains of arbitrary dimension*, Proceedings of AMS, Vol. 133, 7(2005), 1985-1992.

[13] D. Le, L. Nguyen, T. Nguyen, *Shigesada-Kawasaki-Teramoto model on higher dimensional domains*, Electron. J. Differential Equations 2003, No. 72.

[14] Y. Li, C. Zhao, *Global existence of solutions to a cross-diffusion system in higher dimensional domains*, Discrete and Continuous Dynamical Systems, Vol. 12, 2(2005), 185-192.

[15] Y. Lou, W.M. Ni and Y. Wu, *On the global existence of a cross-diffusion system*, Discrete and Continuous Dynamical Systems, Vol. 4, 2(1998), 193-203.

[16] P.V. Tuoc, *Global existence of solutions to Shigesada-Kawasaki-Teramoto cross-diffusion systems on domains of arbitrary dimensions*, to appear.

[17] N. Shigesada, K. Kawasaki and E. Teramoto, *Spatial segregation of interacting species*, J. Theoretical Biology, 79(1979), 83-89.

[18] Seong-A Shim *Uniform boundedness and convergence of solutions to cross-diffusion systems* J. Differential Equations, 185(2002), no. 1, 281-305.

[19] A. Yagi, *Global solution of some quasilinear parabolic systems in population dynamics*, Nonlinear Analysis, T.M. and A. 21(1993), 603-630.

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