WELL POSEDNESS AND REGULARITY FOR THE ELASTICITY EQUATION WITH MIXED BOUNDARY CONDITIONS ON POLYHEDRAL DOMAINS AND DOMAINS WITH CRACKS

By

Anna Mazzucato

and

Victor Nistor

IMA Preprint Series # 2142

(November 2006)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455–0436

Phone: 612-624-6066   Fax: 612-626-7370
URL: http://www.ima.umn.edu

WELL POSEDNESS AND REGULARITY FOR THE ELASTICITY EQUATION WITH MIXED BOUNDARY CONDITIONS ON POLYHEDRAL DOMAINS AND DOMAINS WITH CRACKS

ANNA MAZZUCATO AND VICTOR NISTOR

Dedicated to Michael E. Taylor on occasion of His Sixtieth Birthday

Abstract. We prove a regularity result for the anisotropic elasticity equation $Pu := \operatorname{div}\big(\mathbf{C}\cdot\nabla\mathbf{u}\big) = \mathbf{f}$, with mixed (displacement and traction) boundary conditions L_k on a curved polyhedral domain $\Omega \subset \mathbb{R}^3$ in weighted Sobolev spaces $\mathcal{K}^m_a(\Omega)$, for which the weight given by the distance to the set of edges. In particular, we show that there is no loss of $\mathcal{K}^m_a$–regularity. Our curved polyhedral domains are allowed to have cracks. We establish a well-posedness result when there are no neighboring traction boundary conditions and $|a| < \eta$, for some small $\eta > 0$ that depends on P and L_k and the domain Ω. Our results extend to other strongly elliptic systems.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Then it is well known (see for example [8, 14, 26, 66]) that the equation

$$\Delta u = f \in H^{m-1}(\Omega), \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

has a unique solution $u \in H^{m+1}(\Omega)$. In particular, u will be smooth on $\overline{\Omega}$ if f is smooth on $\overline{\Omega}$. This well-posedness result is especially useful in practice for the numerical approximation of the solution u of Equation (1), see again [8, 10, 14] among the extensive literature on the subject.

In practice, however, it is rarely the case that Ω is smooth. For instance, if $\partial\Omega$ is *not* smooth, then the smoothness of f on $\overline{\Omega}$ does not imply that the solution u of Equation (1) is also smooth on $\overline{\Omega}$. Therefore there is a *loss of regularity* for elliptic problems on non-smooth domains. Wahlbin [69] (see also [7, 43, 70]) has shown that this loss leads to some inconvenience, namely that a quasi-uniform sequence of triangulations on Ω will *not* give optimal rates of convergence for the Galerkin approximations u_h of the solution of (1).

The loss of regularity can be avoided, however, if one removes the singular points. A conformal change of metric will achieve that by "sending the singular points to infinity." It can be proved then that the resulting Sobolev spaces are the "Sobolev spaces with weights" considered for instance in [8, 9, 10, 34, 18, 22, 40, 50, 52, 62] and in many other papers. If $f > 0$ is a smooth function on a domain Ω, we define the *mth Sobolev space with weight f* by

$$\mathcal{K}^m_a(\Omega; f) := \{u,\ f^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \ |\alpha| \le m\}, \quad m \in \mathbb{Z}_+,\, a \in \mathbb{R}. \tag{2}$$

Date: November 10, 2006.

We can then extend the regularity result for Equation (1) to polyhedral domains $\Omega \subset \mathbb{R}^3$, and more generaly to domains $\Omega \subset \mathbb{R}^3$ with a "polyhedral structure", which will be defined in Section 3, provided the usual Sobolev spaces replaced by the Babuška–Kondratiev spaces $\mathcal{K}_a^m(\Omega) := \mathcal{K}_a^m(\Omega; \vartheta)$. Here ϑ is the distance to the edges. In fact, in order to define a suitably large class of smooth functions and differential operators on Ω, we will need to replace ϑ with a weight r_Ω, defined in Equation (14), which behaves like the distance function to the edges ϑ, but is sufficiently smooth at the singular points. The resulting class of smooth functions and operators are exactly the class of C^∞ functions and differential operators on a suitable desingularization of Ω denoted $\Sigma\Omega$, but can be characterized directly, without us having to define $\Sigma\Omega$ explicitly.

A domain $\Omega \subset \mathbb{R}^3$ with a "polyhedral structure" is a polyhedral domain whose faces can be subdivided into smaller faces by introducing artificial vertices and artificial edges. The resulting smaller faces are the faces of this polyhedral structure and are all essentially polygonal domains. Furthermore, there can be interior faces or "cracks," (this happens exactly when $\partial\Omega \neq \partial\overline{\Omega}$). Since we allow for cracks in our domains, the Sobolev spaces at the boundary need to take into account whether Ω is on one side or on both sides of the boundary. To this end, we introduce the *oriented boundary* $\partial^\nu\Omega$ to consist of the *inward unit normal vectors* at the smooth points of Ω. The connected components of $\partial^\nu\Omega$ are called the oriented faces of Ω. Then the natural map $\kappa : \partial^\nu\Omega \to \partial\Omega$ is an at most two-to-one cover of the set $\partial\Omega \smallsetminus \partial_{\text{sing}}\Omega$ of smooth boundary points of Ω. (The map κ is two-to-one exactly at the points at which Ω is on both sides of the boundary, namely at the "crack points.") Then the spaces $\mathcal{K}_a^s(\partial^\nu\Omega) := \mathcal{K}_a^s(\partial^\nu\Omega; \vartheta)$ on the boundary are defined similarly for $s \in \mathbb{Z}_+$; for $s \in \mathbb{R}_+$ they are defined using partitions of unity. The usual trace theorems hold for these spaces at the boundary.

The introduction of domains with "a polyhedral structure" is motivated by our desire to consider mixed boundary value problems. For this reason, we decompose the oriented boundary $\partial^\nu\Omega$ of Ω as a disjoint union $\partial^\nu\Omega = \partial_D\Omega \cup \partial_N\Omega$ such that both $\partial_N\Omega$ and $\partial_D\Omega$ consist of a union of open faces of (the polyhedral structure on) Ω. We impose Dirichlet or "displacement" boundary conditions on $\partial_D\Omega$, and Neumann or "traction" boundary conditions on $\partial_N\Omega$. We collectively refer to the boundary operator on the face D_k as L_k. In particular, our results include the case where different types of boundary conditions are imposed on each side of an interior face, a "crack", via a non-tangential limit. Situations that require mixed boundary conditions on domains with cracks arise in engineering problems, when for instance only part of a face of Ω is externally loaded (traction boundary condition), while the other part is mechanically fixed (zero displacement boundary condition).

Our main focus is the system of classical linear elasticity $P = \operatorname{div}(\mathbf{C} \cdot \nabla)$. However, our regularity and well-posedness results apply more generally to a differential operators $P : \mathcal{C}^\infty(\Omega)^\mu \to \mathcal{C}^\infty(\Omega)^\mu$ (or $\mu \times \mu$ system) with smooth coefficients on a neighborhood of Ω, under certain conditions. We will consider regularity and well-posedness for the following boundary value problem:

$$Pu = f \text{ in } \Omega, \quad u = g_D \text{ on } \partial_D\Omega, \quad \text{and} \quad D_P^\nu = g_N \text{ on } \partial_N\Omega. \tag{3}$$

Above, D_ν^P is the Neumann boundary operator associated to P by the Green's formula (see Lemma 6.1). It is well defined at every point of the oriented boundary $\partial^\nu\Omega$, *i. e.,* away from the singular points, the corners and edges of Ω, and acts on $\mathcal{K}_a^m(\partial^\nu\Omega)$ for every $a \in \mathbb{R}$ and $m \geq 0$. If $m = 0$ the boundary value problem (3) must be understood in a weak, or variational sense. When $P = \Delta$, the Laplace operator, D^P_ν is the usual directional derivative in the direction of the unit outer normal. We remark that our domains may not be Lipschitz or even extension domains, although they can always be decomposed into a finite union of connected, Lipschitz subdomains.

1.1. **Main results.** Our first main result is that the boundary value problem (3) is regular on $\mathcal{K}^m_a(\Omega)$, for all $a \in \mathbb{R}$ and $m \geq 0$, when P and the boundary operators satisfy an *assumption of regularity upon freezing the coefficients* on each face of Ω (Definition 6.6). This regularity assumption holds, for example, if P and L_k satisfy a coercivity condition.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a polyhedral structure and let $P : \mathcal{C}^\infty(\Omega)^\mu \to \mathcal{C}^\infty(\Omega)^\mu$ be a uniformly strongly elliptic operator with coefficients in $\mathcal{C}^\infty(\overline{\Omega})$. We assume that on each oriented face of Ω we are given either Dirichlet or Neumann boundary conditions. Let $m \in \mathbb{Z}$, $m \geq 1$, and $a \in \mathbb{R}$. If P and the boundary conditions satisfy the assumption of regularity upon freezing the coefficients then*

$$\|u\|_{\mathcal{K}^{m+1}_{a+1}(\Omega)} \leq C\big(\|Pu\|_{\mathcal{K}^{m-1}_{a-1}(\Omega)} + \|u|_{\partial_D\Omega}\|_{\mathcal{K}^{m+1/2}_{a+1/2}(\partial_D\Omega)} + \|D^P_\nu u\|_{\mathcal{K}^{m-1/2}_{a-1/2}(\partial_N\Omega)} + \|u\|_{\mathcal{K}^0_{a+1}(\Omega)}\big). \tag{4}$$

for a constant $C = C(\Omega) > 0$ independent of u.

In particular, if $u \in \mathcal{K}^1_{a+1}(\Omega)^\mu$, $Pu \in \mathcal{K}^{m-1}_{a-1}(\Omega)^\mu$, $u|_{\partial_D\Omega} \in \mathcal{K}^{m+1/2}_{a+1/2}(\partial_D\Omega)^\mu$, and $D^P_\nu u \in \mathcal{K}^{m-1/2}_{a-1/2}(\partial_N\Omega)^\mu$, then $u \in \mathcal{K}^{m+1}_{a+1}(\Omega)^\mu$.

The proof of this result employs a characterization of the weighted Sobolev spaces in terms of special partitions of unity near the boundary, and rescaling, which blows up the singularity. The proof also gives the result for $m = 0$ (using a weak formulation). Since the faces of Ω are not necessarily straight, dilations are used in the tangent bundle of $\overline{\Omega}$ near the singular points.

We prove that the regularity theorem applies to *coercive* operators P, such as the Laplace operator and operator of anisotropic elasticity (under the usual assumptions on the elasticity tensor). Let B_P the canonical bilinear form associated to P, Equation (28). By a coercive operator, we mean an operator P for which $B_P(u,u) \geq C_1\|u\|^2_{\mathcal{K}^1_1(\Omega)} - C_2\|r_\Omega^{-1}u\|^2_{L^2(\Omega)}$, $u \in \mathcal{K}^1_1(\Omega)^\mu$, $u = 0$ on $\partial_D\Omega$. This condition allows us to bootstrap regularity using the well-known Nirenberg trick. We would like to stress that Theorem 7.2 does *not* constitute a Fredholm (or "normal solvability") result, because the inclusion $\mathcal{K}^{m+1}_{a+1}(\Omega) \to \mathcal{K}^0_{a+1}(\Omega)$ is *not compact* for all m and a [1]. For example, if Ω is a polygon, then $P = -\Delta$ with Dirichlet boundary conditions is Fredholm precisely when a is different from $k\pi/\alpha$, where $k \in \mathbb{Z}$, $k \neq 0$, and α ranges through the angles of the polygon [40, 41].

We need to impose a stricter condition on P to obtain well-posedness, that is, existence and uniqueness of solutions of the problem (3). We will say that P is *strictly positive* if the Dirichlet form $B_P(u,v)$ associated to P with the given boundary conditions is positive definite (Definition 6.11). For example, the Laplacian $P = -\Delta$, and the elasticity operator $P = -\operatorname{div}(\mathbf{C}\cdot\nabla)$ when the elasticity tensor $\mathbf{C}$ is positive definite (see Section 10.1), are both strictly positive with mixed Dirichlet-Neumann boundary conditions, provided $\partial_D\Omega \neq \emptyset$ and $\partial_N\Omega$ contains no adjacent faces. In order to prove this result, we first establish in Sections 9.1 and 10 weighted forms of the Poincaré and Korn's inequalities for domains with polyhedral structure. These inequalities, together with Theorem 1.1 lead to the following well posedness result.

Theorem 1.2. *Define $\tilde{P}(u) = (Pu,\, u|_{\partial_D\Omega},\, D^P_\nu u|_{\partial_N\Omega})$. Assume that P is strictly positive and all the assumptions of Theorem 1.1 are satisfied. Then there exists $\eta > 0$ such that*

$$\tilde{P} : \mathcal{K}^{m+1}_{a+1}(\Omega)^\mu \to \mathcal{K}^{m-1}_{a-1}(\Omega)^\mu \oplus \mathcal{K}^{m+1/2}_{a+1/2}(\partial_D\Omega)^\mu \oplus \mathcal{K}^{m-1/2}_{a-1/2}(\partial_N\Omega)^\mu$$

is an isomorphism for $m \in \mathbb{Z}_+$ and $|a| < \eta$.

1.2. **Earlier results.** Theorem 7.2 was known in two dimensions, *i. e.,* for polygonal domains, [40, 41, 47]. In two dimensions, the weight ϑ is taken to be the distance to the vertices of the polygonal domain considered. In three dimensions, related results were obtained before by Mazya and Rossmann [48] using properties of the Green function. See also [3, 5, 16, 18, 23, 38, 39, 49, 56], to mention just a few papers. An elementary proof of the regularity theorem, Theorem 1.1 for $P = -\Delta$ and the Dirichlet boundary conditions was given in [2]. There is an extensive literature concerning linear elasticity and cracks for domains with corners. Results related to ours can be found in [4, 58, 61, 20, 21, 32, 33, 24, 42], mainly for the Lamé systems in polygonal domains. The Finite Element Method provides ample motivation for our research [6, 19, 22, 28, 27, 57]. A treatment of boundary value problems in the presence of cracks can be found also in [37] using the edge pseudo-differential calculus. In contrast with our case, there a crack is a smooth immersed manifold of codimension 1. The observation that $P =$the elasticity operator with suitable mixed displacement-traction boundary conditions is strictly positive is, however, a crucial new element in this paper.

Other distincitive features of our paper is that we obtain well posedness results for domains with cracks in full generality and we consider curved faces. In order to deal with curved faces, we introduce for each face D a desingularization ΣD so that the outer unit normal vector function ν extends by continuity to a smooth function $\nu : \Sigma D \to S^2 \subset \mathbb{R}^3$. The set ΣD is a smooth manifold with corners and is compact if Ω is bounded. In case D is not curved (*i. e.,* it is contained in a plane), we can replace ΣD with $\overline{D}$ in our considerations. More pathological domains will be considered in the plane in [44].

Contents of the paper. The paper is organized as follows. In Section 2, we consider the special case of convex polyhedral domains, as an introduction and warm-up for the general case of domains with polyhedral structure, which is treated at length in Section 3. Section 4 deals with the appropriate class of smooth functions and differential operators on $\overline{\Omega}$. In Section 5, we define the Babuška-Kondratiev spaces $\mathcal{K}^m_a(\Omega)$ and their boundary counterparts $\mathcal{K}^m_a(\partial\Omega)$, we study their properties and stablish a trace theorem. Section 6 discusses the boundary value problems, the assumptions on the operator and boundary conditions, and introduces the notion of regularity upon freezing the coefficients, coercivity and positivity. Section 7 is devoted to the proof of the regularity theorem, while Section 8 addresses our main result, that is, the well-posedness of the boundary value problem (3) for positive operators. Finally, Sections 9 and 10 contains applications to the Laplace and elasticity operators. The weighted Poincaré and Korn's inequalities, used to establish positivity of these operators with mixed boundary conditions, are proved there.

Notation and conventions. Here are some of the most important notations and conventions that will apply throughout this paper: $\Omega \subset \mathbb{R}^3$ will be a fixed domain with a polyhedral structure; in all our results, we shall further assume either that Ω is bounded or that it is a cone (a wedge is a cone); $B^k \subset \mathbb{R}^k$ will denote the open unit ball (disk if $k = 2$) in $\mathbb{R}^k$ and by S^{k-1} we will denote its boundary. Our Hilbert and Banach spaces are taken to be real, unless mentioned otherwise. Even when we need to consider complex Hilbert spaces and operators between such spaces, they will arise by complexification. By C we denote a generic constant that may be different at each occurrence. Also $\mathbb{Z}_+ = \{0, 1, \ldots\}$.

Acknowledgements: We would like to thank Constantin Bacuta, Francesco Costanzo, Alan Demlow, and Ludmil Zikatanov for useful discussions.

2. Weights and smooth functions in the case Ω convex

Let $\Omega \subset \mathbb{R}^3$ be an open subset. In this section, we shall introduce two basic constructions, the "weight function" r_Ω and the function space $\mathcal{C}^\infty(\Sigma\Omega)$, in the case when our domain Ω is a polytope (the convex hull of finitely many points). These two basic constructions are significantly more difficult in the general case, so it is reasonable to treat first the case of a polytope, which will give the reader the necessary intuition. The reader can thus skip the general constructions of the next section until after a first reading.

A central role in our analysis is played by the "canonical weight function" $r_\Omega : \overline{\Omega} \to [0, \infty)$. To define it, we need to introduce first some auxiliary functions ρ_Q and $\tilde{r}_e$, where Q ranges through the vertices of Ω and e ranges through the edges of Ω. We first define

$$\rho_Q(x) = \text{the distance from } x \text{ to } Q. \tag{5}$$

(The definition is the same in the general case.) Let next $r_e(x)$ be the distance to the *line* containing $e = [AB]$. We then define

$$\tilde{r}_e = (\rho_A \rho_B)^{-1} r_e. \tag{6}$$

Finally, we define the canonical weight r_Ω as the product

$$r_\Omega(x) = \prod_Q \rho_Q(x) \times \prod_e \tilde{r}_e(x), \tag{7}$$

where Q ranges through the vertices of Ω and e ranges through the edges of Ω. Both the set of vertices and edges of Ω are finite because, we recall, in this section Ω is assumed to be the convex hull of a finite set of non-coplanar points.

An important general property of the canonical weight function is the following. Let $\vartheta(x)$ be the distance from $x \in \overline{\Omega}$ to the union of all edges of Ω. Then $\vartheta(x)/r_\Omega(x)$ extends to a continuous function on $\overline{\Omega}$, and hence there exists a constant $C > 0$ such that

$$C^{-1}\vartheta(x) \le r_\Omega(x) \le C\vartheta(x), \tag{8}$$

for all $x \in \Omega$. (This will be proved in the general case in Lemma 4.3.)

We now turn to the definition of the space $\mathcal{C}^\infty(\Sigma\Omega)$, which is a space of smooth functions on Ω containing $\mathcal{C}^\infty(\overline{\Omega})$. Let us choose, for each edge e of Ω, a plane $\mathcal{P}_e$

containing one of the open faces D of Ω such that $e \subset \overline{D}$. If x is not on the line defined by e, we then define θ_e to be the angle in a cylindrical coordinates system (r_e, θ_e, z) determined by the edge e and the plane $\mathcal{P}_e$. More precisely, let $q \in e$ be the foot of the perpendicular from x to e. Then $\theta_e(x)$ is the angle between xq and $\mathcal{P}_e$. Similarly, for each vertex Q and edge e adjacent to Q, we define $\phi_{Q,e}(p)$ to be the angle between the segment xQ and the edge e.

Since Ω is convex in this section, the functions θ_e and $\phi_{P,e}$ are defined and smooth on Ω (in this paper, by Ω we shall always mean an open subset of $\mathbb{R}^3$, possibly with additional properties). We shall denote by $\theta = (\theta_{e_1}, \ldots, \theta_{e_r})$ the vector variable that puts together all the θ_e functions. Similarly, we shall denote by $\phi = (\phi_{Q_1,e_1}, \ldots, \phi_{Q_p,e_p})$ the vector variable that puts together all the $\phi_{Q,e}$ functions. We then introduce the space $\mathcal{C}^\infty(\Sigma\Omega)$ as the space of functions $u : \Omega \to \mathbb{C}$ of the form

$$u(x, y, z) = f(x, y, z, \theta, \phi) = f(x, y, z, \theta_{e_1}, \ldots, \theta_{e_r}, \phi_{P_1,e_1}, \ldots, \phi_{P_p,e_p}),\\ f \in \mathcal{C}^\infty(\overline{\Omega} \times [0, \pi]^r \times [0, \pi/2]^p).$$

The point of introducing the space $\mathcal{C}^\infty(\Sigma\Omega)$ is that, for example, θ_e is a smooth function on Ω which is not in $\mathcal{C}^\infty(\overline{\Omega})$ but $\theta_e \in \mathcal{C}^\infty(\Sigma\Omega)$. By definition, $\mathcal{C}^\infty(\Sigma\Omega)$ consists of smooth, bounded functions on Ω.

One can show as in [1] that there exists a smooth manifold with corners $\Sigma\Omega$, canonically associated to Ω, such that $\mathcal{C}^\infty(\Sigma\Omega)$ consists of the smooth function on $\Sigma\Omega$. The relevant properties of the function space $\mathcal{C}^\infty(\Sigma\Omega)$, especially in relation to the Babuška–Kondratiev spaces $\mathcal{K}^m_a(\Omega)$ will be discussed in the general framework of domains with a polyhedral structure in Section 5.

3. Domains with polygonal and polyhedral structures

In this section, we shall introduce the class of domains Ω with polyhedral structure and will extend to this class of spaces the definitions of the weight functions ρ_A, $\tilde{r}_e$, and r_Ω, as well as the definition of the space $\mathcal{C}^\infty(\Sigma\Omega) \subset \mathcal{C}^\infty(\Omega)$.

The class of domains with a polyhedral structure is a class of domains that extends the class of polyhedral domains. The more general class of domains with a polyhedral structure is needed in order to be able to deal with domains with cracks (or slits) and with mixed boundary conditions when Dirichlet boundary conditions are used on a part of a face and Neumann (or natural) boundary conditions are used on the remaining part of that face.

3.1. **Domains with a polygonal structure.**

The definition of a "domain with a polyhedral structure" is based on that of a "domain with a polygonal structure." Recall that throughout this paper, we shall denote by $B^k \subset \mathbb{R}^k$ the open unit ball (disk if $k = 2$) in $\mathbb{R}^k$ and by S^{k-1} we shall denote its boundary. Thus $S^0 = \{-1, 1\}$, S^1 is the unit circle, and S^2 is the unit sphere.

Definition 3.1. A *domain with polygonal structure* D in a two dimensional smooth manifold M is an open subset $D \subset M$ together with a distinguished finite subset $\mathcal{V} \subset \partial D$ such that, for each $x \in \partial\Omega$, we are given a neighborhood V'_x of x in M satisfying

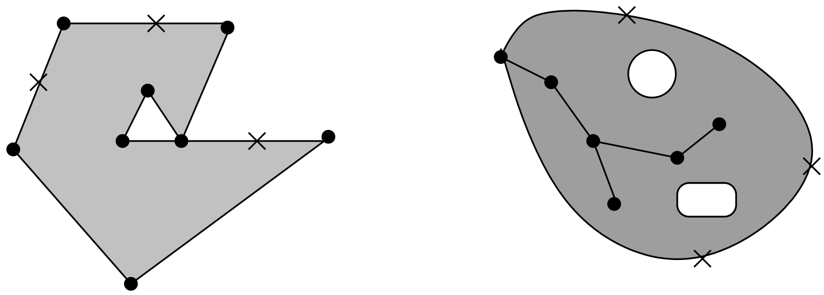

FIGURE 1. Domains with polygonal structures

(i) there is an open subset $\omega_x' \subset S^1$, $\omega_x' \neq S^1$, not necessarily connected, and a diffeomorphism $\phi_x' : V_x' \to B^2$ such that, in polar coordinates (r, θ), we have

$$\phi_x'(V_x') = \{(r,\theta),\ r \in (0,1),\ \theta \in \omega_x'\};$$

(ii) $\phi_x'(x) = 0$;

(iii) if $x \notin \mathcal{V}$, then ω_x' consists of one or two intervals of length π .

In this paper, we will mostly need the case when $M = \mathbb{R}^2$ or $M = S^2$. We continue with some simple remarks and some examples.

Remark 3.2. The points in $\mathcal{V}$ are called the *vertices* of D. Note that the set D does *not* determine its polygonal structure, because we can always increase the set $\mathcal{V}$. However, if there is a polygonal structure on D, then the one with the minimum set of vertices is unique. These are the *true vertices of* D. The other vertices of D will be called *artificial vertices*. The true vertices are the ones for which ω_x' is *not* an half circle. The artificial vertices, and polygonal structures in general, are useful for the study of mixed boundary value problems. Our domains with a polygonal structure are not required to be bounded or connected.

Here are some examples.

Example 3.3. A polygonal domain (in the usual sense of the term) is a typical example of a domain with a polygonal structure. It follows from our definition that each component of $\partial D \smallsetminus \mathcal{V}$ is a smooth curve γ without self-intersections such that $\overline{\gamma} \subset \gamma \cup \mathcal{V}$. The curves γ will be called the *open sides* of D. For example, a domain with smooth boundary in $\mathbb{R}^2$ is a domain with a polygonal structure if we set $\mathcal{V} = \emptyset$ and the sides are the connected components of the boundary.

See Figure 1 for the pictures of two domains with a polygonal structure. The true vertices are represented by thick points whereas the artificial vertices are represented by a cross (*i. e.,* $\times$).

In order to treat curved domains in three dimensions, we need to introduce the desingularization ΣD of a domain D with a polygonal structure. In case only straight faces are considered (*i. e.,* each face is contained in a plane), then we can replace ΣD with $\overline{D}$, and the following construction of ΣD becomes unnecessary. Let us consider first $\omega = (a_1, b_1) \cup \ldots \cup (a_k, b_k) \subset S^1$. We define $\Sigma\omega$ to be the *disjoint* union $[a_1, b_1] \sqcup \ldots \sqcup [a_k, b_k]$, that is, the disjoint union of the closures of the intervals comprising ω (by $\sqcup$ we denote the disjoint union). If $\omega = S^1 \smallsetminus \{e^{it}\}$, then $\Sigma\omega := [t, t + 2\pi] \subset \mathbb{R}$.

We now turn to the construction of ΣD. The main idea is that we separate the edges of the cracks and, for any vertex $x \in \mathcal{V} \subset \partial D$, we use the diffeomorphisms $\phi_x = \phi'_x$ of Definition 3.1 to attach $[0,\epsilon) \times \Sigma\omega_x$ to $\overline{D}$. More precisely, we first define $\partial^\nu D$ to be the set of *inner normal unit vectors. The set* $\partial^\nu D$ *maps canonically to* $\partial D \smallsetminus \mathcal{V}$ *and the map is one-to-one, except at the crack points, where it is two-to-one. We define* $D^\nu := D \cup \partial^\nu D$. *If there are no crack points, then* $D^\nu = \overline{D} \smallsetminus \mathcal{V}$ *(we removed the vertices).* Let S be the disjoint union of the sets D^ν and $[0,\epsilon) \times \Sigma\omega_x$ for x ranging through the set $\mathcal{V}$ of vertices of D. We assume that ϵ is chosen small enough so that the sets $\phi_x^{-1}((0,\epsilon] \times \omega_x)$ are disjoint. The map ϕ_x^{-1} extends to a continuous map $(0,\epsilon) \times \Sigma\omega_x \to D^\nu$. We use ϕ_x^{-1} to identify the points of $(0,\epsilon) \times \Sigma\omega_x$ to their image in $\overline{D} \smallsetminus \mathcal{V}$. This defines an equivalence relation on the set S. The quotient set with respect to this equivalence relation is the desired desingularization ΣD. The space ΣD has a natural structure of a manifold with corners (of codimension at most two) and we obtain a canonical map

$$\kappa : \Sigma D \to \overline{D}. \tag{9}$$

In case D is contained in a plane, we can replace ΣD with $\overline{D}$ in our considerations below. The desingularization ΣD was used already in the framework of manifolds with corners by Melrose [52]. See also [1, 2, 15].

3.2. **Domains with a polyhedral structure.** We now introduce domains with a polyhedral structure. The simplest example is that of a convex polyhedron, but in general we allow curved boundaries and cuts (or cracks).

Recall that a *smooth stratification* $S_0 \subset S_1 \subset \ldots \subset X$ of a topological space X is an increasing sequence of closed sets $S_j = S_j(X)$ such that each point of X has a neighborhood that meets only finitely many of the sets S_j, S_0 is a discrete subset, $S_{j+1} \smallsetminus S_j$, $j \geq 0$, is a disjoint union of smooth manifolds of dimension $j+1$, and $X = \cup S_j$. A continuous map $f : X \to Y$ of stratified spaces is a continuous map satisfying $f(S_j(X)) \subset S_j(Y)$. If X is a stratified space and Y is a manifold of dimension k, then $X \times Y$ is stratified canonically by $S_{j+k}(X \times Y) := S_j(X) \times Y$.

If X is a stratified space, we define $CX := \big([0,1) \times X\big)/\big(\{0\} \times X\big)$, with stratification $S_{j+1}(CX)$ being defined as the image of $[0,1) \times S_j(X)$ in CX and with $S_0(X)$ consisting of the point of CX that is obtained by collapsing $\{0\} \times X$.

We shall need the following examples of stratified spaces. If $\omega \subset S^1$ is a disjoint union of finitely many open intervals, then we stratify $\overline{\omega}$ by $S_0(\overline{\omega}) = \partial\omega$ and $S_1(\overline{\omega}) = \overline{\omega}$. Similarly, in two dimension, if D is a domain with a polygonal structure, then $\overline{D}$ is given the natural stratification $\mathcal{V} \subset \partial D \subset \overline{D}$.

We are now ready to introduce the "domains with a polyhedral structure."

Definition 3.4. A *domain with a polyhedral structure* is an open subset $\Omega \subset \mathbb{R}^3$ together with a smooth stratification $\partial_0\Omega \subset \partial_1\Omega \subset \partial_2\Omega := \partial\Omega$ satisfying:

(i) for each $x \in \partial\Omega$, we are given a fixed neighborhood V_x of x in $\mathbb{R}^3$;
(ii) for each $x \in \partial_j\Omega \smallsetminus \partial_{j-1}\Omega$, we are given an open, non-empty subset $\omega_x \subset S^{2-j}$;
(iii) we are given diffeomorphisms $\phi_x : V_x \to B^j \times B^{3-j}$, $\phi_x(x) = 0$, such that
$$\phi_x(V_x \cap \Omega) = \{(y, rx'),\ y \in B^j,\ r \in (0,a),\ x' \in \omega_x\} \simeq B^j \times (0,a) \times \omega_x,$$
with $a \in (0,+\infty]$;
(iv) if $j = 0$ (*i. e.,* if $x \in \partial_0\Omega$), then $\omega_x \subset S^2$ is domain with a polygonal structure;
(v) if $j = 1$ (*i. e.,* if $x \in \partial_1\Omega \smallsetminus \partial_0\Omega$), then ω_x is an open subset of S^1 with finitely many components;

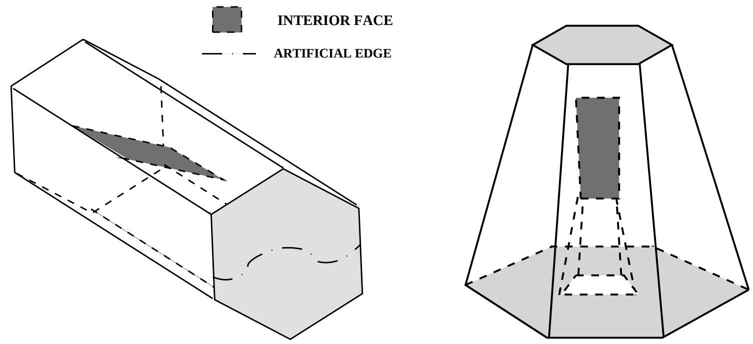

FIGURE 2. Domains with polyhedral structure

(vi) the inverse of ϕ_x extends to a continuous map $B^j \times C\overline{\omega}_x \to \overline{\Omega}$ of stratified spaces that is a homeomorphism onto its image.

From now on and throughout this paper, $\Omega \subset \mathbb{R}^3$ will be a fixed domain with a "polyhedral structure" that is either bounded or is an infinite cone (a wedge is a cone).

The set $\partial_0\Omega$ consists of the *vertices* of Ω. The connected components of $\partial_1\Omega \setminus \partial_0\Omega$ are the (open) *edges* of Ω and the connected components of $\partial_2\Omega \setminus \partial_1\Omega$ are the (open) *faces* of Ω. The set $\partial_2\Omega \smallsetminus \partial_1\Omega$ is the set of *smooth boundary points* and $\partial_{\text{sing}}\Omega = \partial_1\Omega$ is the set of *singular* boundary points. Our definition is very closely related to that of Whitney stratified spaces [71].

Although Ω is not necessarily a Lipschitz domain, it can be written as a finite disjoint union of Lipschitz domains, as shown in the next lemma in the case of a bounded domain. The same proof applies for an infinite cone or wedge.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with polyhedral structure, then there exist finitely many disjoint Lipschitz domains Ω_j such that $\overline{\Omega} = \bigcup_{j=1}^N \overline{\Omega}_j$.*

Proof. Clearly, the interior of Ω can be triangulated, that is, divided into a finite number of tetrahedra with disjoint interiors. We need only cover a neighborhood of the boundary. By definition, Ω has a finite number of vertices Q and edges e. Therefore, there exists a finite number of neighborhoods V_X of Definition 3.4 that covers $\partial\Omega$. The neighborhoods $V_X \cap \Omega$ are diffeomorphic images (through the diffeomorphisms ϕ_X^{-1}) of

$$B^j \times [0, a) \times \omega_X, \qquad j = 0, 1, 2,$$

where ω_X is a domain with polygonal structure on S^{2-j}. If X is a smooth point, then $j = 2$ and $\omega_X = \{1\}$ or $\omega_X = \{-1, 1\}$, so that $V_X \cap \Omega$ is the disjoint union of at most two connected Lipschitz domains. If X is a vertex Q, then $j = 0$ and ω_X is a domain on the sphere S^2. We identify B^0 with $\{1\}$. Using spherical coordinates (θ, ϕ) for ω_X, it is easy to see that each connected component of $V_X \cap \Omega$ is a Lipschitz domain, provided $0 < \theta < \pi$ (for example for a crack, ω_X can be S^2 with a segment removed, so that $0 < \theta < 2\pi$, and the $V_X \cap \Omega$ is the union of two domains). By splitting the interval $(0, 2\pi)$ into disjoint subintervals of length less than π , we again can write $V_X \cap \Omega$ as a finite disjoint union of Lipschitz components. Finally, if X belong to en edge e, then $j = 1$ and ω_X is an open interval in S^1, while we identify $B^1 = [-1, 1]$ Again $V_X \cap \Omega$ can be triangulated by construction. $\square$

Remark 3.6. Let Ω be a domain with a polyhedral structure and $x \in \partial\Omega \smallsetminus \partial_1\Omega$. Then the boundary of Ω is smooth *near* x. We distinguish two possibilities: Ω is only on one side of the boundary close to x or Ω is on both sides of the boundary close to x. We have $x \in \partial\overline{\Omega}$ in the first case and $x \notin \partial\overline{\Omega}$ in the second case. There will be *one* inward unit normal to $\partial\Omega$ at x in the first case and *two* inward unit normals to $\partial\Omega$ at x in the second case. We shall denote by $\partial^\nu\Omega$ the *set of inward unit normal vectors* to the smooth part of the boundary of Ω and call it the *oriented boundary* of Ω. We shall also denote by

$$\kappa : \partial^\nu\Omega \to \partial\Omega \smallsetminus \partial_{\mathrm{sing}}\Omega, \tag{10}$$

the canonical projection from the *oriented* boundary of Ω to the boundary of Ω. This map is one-to-one at the points where Ω is on one side of the boundary and two-to-one at the points where Ω is on both sides of the boundary. An *oriented face* of Ω is a connected component of $\partial^\nu\Omega$. Thus an oriented face of Ω is a face of Ω together with a choice of an inward unit vector. Then we can talk of a unique *outer normal unit vector* $\nu(w)$ at any point $w \in \partial^\nu\Omega$:

$$\nu : \partial^\nu\Omega \to S^2, \quad \nu(w) = -w,$$

(recall that, by definition, w is an *inward* normal unit vector at the boundary).

As in the case of domains with a polygonal structure, we define the set

$$\Omega^\nu := \Omega \cup \partial^\nu\Omega$$

and endow it with the natural topology that makes it a smooth manifold with boundary (usually non-compact). We have $\partial^\nu\Omega = \partial\Omega \smallsetminus \partial_1\Omega$ if Ω has no cracks.

Let D be an oriented (open) face of our fixed domain with a polyhedral structure Ω. It is not true in general that D is a domain with a polygonal structure, because there may be no two dimensional manifold containing the closure of D. However, it is not difficult to check that there exists a domain with a polygonal structure $D' \subset M$, for some manifold M and a diffeomorphism $D \to D'$. This also allows us to define $\Sigma D = \Sigma D'$, up to a diffeomorphism.

We need to define ΣD for D an oriented face of Ω for the following reason. The outer unit normal $\nu : D \to S^2$ extends by continuity to a smooth function on ΣD. (It does not extend to a continuous function on $\overline{D}$, though.)

4. Differential operators and $\mathcal{C}^\infty(\Sigma\Omega)$

In this section we introduce the relevant space of functions and differential operators on Ω. It turns out that the space of smooth functions on $\overline{\Omega}$ is too small, whereas $\mathcal{C}^\infty(\Omega)$ is to big for our purposes. We want a space of smooth functions that is small enough, but contains the analogues of the polar, spherical, and cylindrical coordinates functions.

4.1. **The space $\mathcal{C}^\infty(\Sigma\Omega)$ in general.** Let us first recall the diffeomorphisms $\phi_x : V_x \to B^j \times B^{3-j}$ of Definition 3.4 satisfying

$$\phi_x(V_x \cap \Omega) = \{(y, rx'),\ y \in B^j,\ r \in (0, a),\ x' \in \omega_x\} \simeq B^j \times (0, a) \times \omega_x.$$

Also recall that for $j = 0, 1$, the sets $\Sigma\omega_x$ were already defined ($\omega_x \subset S^{2-j}$). For $j = 2$, the set ω_x consists of one or two points, and then we define $\Sigma\omega_x = \omega_x$.

Finally, given a smooth function $f : \Omega \to \mathbb{R}$, we shall say that $f \in \mathcal{C}^\infty(\Sigma D)$ if, and only if, $f \circ \phi_x^{-1}$ extends to a smooth function on $B^j \times [0, a) \times \Sigma\omega_x$, for all x.

To better explain this definition, let us fix a vertex Q of Ω and look closer at the part of the neighborhood V_Q close to a fixed edge e that contains Q. Let $\omega_Q \subset S^2$ be the domain with a polygonal structure introduced in Definition 3.4 (so $\phi_Q : V_Q \simeq (0, \epsilon) \times \omega_Q$). Let A be the vertex of ω_Q corresponding to the fixed edge e. Then, according to the definition of 3.1, we can find a diffeomorphism $\phi_A : V_A \to (0, \epsilon) \times \omega_A$, $\omega_A \subset S^1$. We denote by $(\phi, \theta) \in (0, \epsilon) \times \omega_A$ the corresponding coordinates. Let $\rho_Q(x)$ denote the distance from x to Q. Then, in the open set corresponding to $(0, \epsilon) \times (0, \epsilon) \times \omega_A$, the condition that $f \in \mathcal{C}^\infty(\Sigma\Omega)$ is equivalent to the fact that $f(x, y, z)$, when written in spherical coordinates ($x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$), extends to a smooth function on $(\rho, \phi, \theta) \in [0, \epsilon) \times [0, \epsilon) \times \Sigma\omega_A$. (If Ω coincides with a straight polyhedral domain in a neighborhood of Q, then the diffeomorphisms ϕ_Q and ϕ_A become the identity, and then we are simply requiring that f extend to a smooth function of the usual spherical coordinates $(\rho, \phi, \theta) \in [0, \epsilon) \times [0, \epsilon) \times [0, \alpha]$, where $\omega_A = (0, \alpha)$.) By definition, $\mathcal{C}^\infty(\Sigma\Omega)$ is closed under addition and multiplication (*i. e.,* it is an algebra).

Let us also notice that if f is smooth in the neighborhood V_x of some $x \in \partial\Omega$, then $f \circ \phi_x^{-1}$ is already a smooth function on closure of the set $(0, a) \times \omega_x$, and hence it trivially extends to a smooth function on $[0, a) \times \Sigma\omega_x$.

If $U \subset \Omega$ is an open set and $f : U \to \mathbb{R}$ is smooth, we shall say that $f \in \mathcal{C}^\infty(\Sigma U)$ if it is the restriction of a function in $\mathcal{C}^\infty(\Sigma\Omega)$. The definition of $\mathcal{C}^\infty(\Sigma U)$ depends on Ω, although this is not shown in the notation.

4.2. **The canonical weight function.** As mentioned earlier, we shall need a canonical weight function $r_\Omega \in \mathcal{C}^\infty(\Sigma\Omega)$ that has the same type of growth as the function ϑ given by

$$\vartheta(x) := \text{the distance from } x \in \Omega \text{ to the singular points of } \partial\Omega. \tag{11}$$

The reason we want to replace in some reasonings the function ϑ with r_Ω is that the function ϑ is not smooth enough. (This is the case both in two and in three dimensions.)

Let us treat first the case of a domain $D \subset \mathbb{R}^2$ with a polygonal structure, which is easier. In that case $\vartheta(x)$ is the distance from x to the set of vertices of D. The smoothed version is then defined by

$$r_D(y) = \prod_{Q \in \mathcal{V}} \rho_Q(y), \tag{12}$$

where Q ranges through the vertices of D. In particular, if D is an angle with vertex A, then $r_D = \rho_A$. Also, the definition of r_D shows that r_D/ϑ extends to a continuous function $\overline{D} \to (0, \infty)$.

We now turn to the case of domains in space. Recall that in this paper, $\Omega \subset \mathbb{R}^3$ is a fixed domain with a polyhedral structure that is either bounded or a cone. Also, recall that $\rho_Q(y)$ denotes the distance from y to Q. The constructions below are simpler in the case of straight polyhedral domains, which is the case that is typically considered in the literature .

Let e be an open edge of Ω. We first want to extend the definition of the function $\tilde{r}_e$ of Equation (6) from the case of a convex polyhedral domain to the general case. For this purpose, we shall assume that either Ω is bounded or that it is a cone of the form $\{rx', x' \in \omega, 0 < r < \infty\}$, for some domain $\omega \subset S^2$ with a polygonal structure, $\omega \neq \emptyset$, $\omega \neq S^2$.

If $\Omega = \{rx', x' \in \omega, 0 < r < \infty\}$, that is, if Ω is a cone that is *not* a wedge, let A be the vertex of ω to which e corresponds and let $\rho_A^\omega(x')$ be the euclidean distance from $x' \in S^2$ to A (measured using straight lines). Then we define

$$\tilde{r}_e(rx') := \rho_A^\omega(x') \quad (e \text{ corresponds to } A), \quad x' \in S^2. \tag{13}$$

Note that we do not require ω to be connected. On the other hand, if $\Omega = \{(r,\theta,z), 0 < \theta < \alpha\}$ in polar coordinates, that is, if Ω is a wedge with edge e, then we define $\tilde{r}_e = r$, the distance to e.

For a bounded Ω, we want our definition of $\tilde{r}_e$ to coincide around each vertex with the definition given above for a cone. Away from the vertices but close to the edge, we want $\tilde{r}_e$ to be equivalent to the distance to the edge. Otherwise, we want $\tilde{r}_e$ to be smooth and bounded away from zero. This can be achieved easily using a partition of unity.

We can now define the *canonical weight function* r_Ω by the same formula as for a convex polyhedral domain, namely

$$r_\Omega(y) = \prod_Q \rho_Q(y) \times \prod_e \tilde{r}_e(y), \tag{14}$$

where Q ranges through the vertices of Ω and e ranges through the edges of Ω. These products make sense since we have assumed that Ω is either a cone or a bounded set. In particular, when $\Omega = \{(r,\theta,z), 0 < \theta < \alpha\}$ in polar coordinates (that is, if Ω is a wedge), we have $r_\Omega = r$, the distance to the unique edge of Ω. On the other hand, Ω is a cone with a smooth base and vertex Q, we have $r_\Omega = \rho_Q$.

We next prove that the various functions that we have introduced are in $\mathcal{C}^\infty(\Sigma\Omega)$.

Lemma 4.1. *Let Q be a vertex of Ω, then $\rho_Q \in \mathcal{C}^\infty(\Sigma\Omega)$.*

Proof. Let (ρ_Q, x'), $x'(y) = y/\rho_Q(y) \in S^2$, be the generalized polar coordinates around Q. Let $\phi_Q : V_Q \to B^j \times B^{3-j} = B^3$ be the given diffeomorphism of Definition 3.4 (in this case $j = 0$, because Q is a vertex). Then ϕ_Q^{-1} extends to a differentiable map $S^2 \times [0,\epsilon) \to \partial B_Q^3(1) \times [0,1)$, for some small $\epsilon > 0$. Consequently, $\rho_Q \circ \phi_Q^{-1}$ is differentiable. Finally, let $b \in \overline{\Omega}$, with $b \neq Q$. Then Q will not belong to V_b and hence ρ_Q is smooth on V_b. Consequently, $\rho_Q \circ \phi_b^{-1}$ extends to differentiable function on $[0,a) \times \overline{\omega_Q}$. □

We also have the following.

Lemma 4.2. *Let e be an edge of Ω, then $\tilde{r}_e \in \mathcal{C}^\infty(\Sigma\Omega)$. In particular, $r_\Omega := \prod_e \tilde{r}_e \times \prod_Q \rho_Q \in \mathcal{C}^\infty(\Sigma\Omega)$.*

For the purpose of the following proof, let us say that two functions $f_1, f_2 : U \to [0,\infty)$, defined on some open subset $U \subset \Omega$ are *equivalent* if the quotients f_1/f_2 and f_2/f_1 extend to functions in $\mathcal{C}^\infty(\Sigma U)$, *i. e.,* are restrictions of some functions in $\mathcal{C}^\infty(\Sigma\Omega)$.

Proof. We shall use ideas and notation that will be used in other proofs as well.

Let $\phi_x : V_x \to B^j \times B^{3-j}$ be as in definition 3.4. Let $U_x := V_x \cap \Omega$. Lemma 4.1 tells us that it is enough to show what $\tilde{r}_e \in \mathcal{C}^\infty(\Sigma U_x)$ for all x. (Recall that $\mathcal{C}^\infty(\Sigma U)$ denotes the set of restrictions to U of functions in $\mathcal{C}^\infty(\Sigma\Omega)$.) Let $\phi_x : V_x \to B^j \times \{rx', r \in (0,a), x' \in \omega_x\}$ be as in Definition 3.4.

Assume first that $x = Q$ is a vertex. Fix spherical coordinates (ρ, θ, ϕ) in a neighborhood of an edge e of Ω corresponding to a vertex A of ω_x. The vertex Q then corresponds to $\rho = 0$ and the edge e corresponds to $\phi = 0$. For x' in is a sufficiently small neighborhood of A, $r = \rho \sin\phi$ is equivalent to r_e. More precisely, the quotients r_e/r and r/r_e are non-zero, smooth functions of (ρ, θ, ϕ), for ϕ small enough. (For the purpose of later proofs, the set defined by x' close enough to the vertices will be denoted W_1.)

For x' away from the edges, we have that r_Ω and ρ are equivalent, in the same sense. Similarly, ρ_x and ρ are equivalent. It follows that r_e/ρ_A and $r/\rho = \sin\phi$ are equivalent. Since $\sin\phi$ is in $\mathcal{C}^\infty(\Sigma U_x)$, it follows that r_e/ρ_A is in $\mathcal{C}^\infty(V_x)$. If A is the only end point of e, then $\tilde{r}_e = r_e/\rho_A$ is in $\mathcal{C}^\infty(V_x)$. Let then B be the other end point of e. Then ρ_B is smooth on V_x. Consequently, $\tilde{r}_e := (r_e/\rho_A)/\rho_B$ is in $\mathcal{C}^\infty(V_x)$ as well. (For the purpose of later proofs, the set defined by x' far from the vertices will be denoted W_2.)

Assume now that x is an edge point. This case is slightly easier. Again, let us chose $\phi_x : V_x \to B^1 \times B^2$ as in definition 3.4. Let (z, r, θ) be the corresponding cylindrical coordinates (z parametrizes $B^1 = (-1, 1)$). Then r_e is equivalent to r on V_x in the same sense as above (their quotients are smooth functions of (z, r, θ)). Since ρ_A and ρ_B are smooth function on V_x, it follows that $\tilde{r}_e := r_e/(\rho_A \rho_B)$ is in $\mathcal{C}^\infty(\Sigma U_x)$. The case when there are fewer end points is completely similar.

This is enough to complete the proof since there are no restrictions at the smooth boundary points of $\partial\Omega$. $\square$

Lemma 4.3. *Let $\vartheta(p)$ be the distance from p to the union of the edges of Ω. Then there exists $C > 0$ such that*

$$C^{-1}\vartheta(y) \le r_\Omega(y) \le C\vartheta(y)$$

for all $y \in \Omega$.

Proof. Using the notation and the proof of the previous lemma. Let x be a vertex. We have seen in the proof of the previous lemma that, up to equivalence, we can replace r_Ω with $\rho_x r_{\omega_x}$ in each of the sets V_x. In the new coordinate system given by the diffeomorphism ϕ_x, the function ϑ is replaced with a function ϑ' such that the quotients ϑ/ϑ' and ϑ'/ϑ are bounded. We can therefore replace ϑ with ϑ'. But the quotient $\rho_x r_{\omega_x}/\vartheta'$ is homogeneous of degree zero. This reduces the problem to the case of the domain with a polygonal structure ω_x. It is clear then from the definitions that the quotients r_D/ϑ' and ϑ'/r_D are bounded on $\omega_x \subset S^2$.

The case of an edge point is completely similar. Since away from the edges the quotients r_Ω/ϑ and ϑ/r_Ω are continuous and $\overline{\Omega}$ can be covered with finitely many sets of the form V_x, the proof is complete. $\square$

4.3. **Rescaled tangent bundle.** For $x \in \overline{\Omega}$, we shall denote by

$$\alpha_x(z) = x + r_\Omega(x)(z - x) \tag{15}$$

the dilation of center x and ratio $r_\Omega(x)$, where r_Ω is the canonical weight. Recall that $\partial^\nu\Omega$ denotes the set of *inner normal unit vectors* y to Ω and we have a canonical map $\kappa : \partial^\nu\Omega \to \partial\Omega$.

Let us fix a connected component $D \subset \partial^\nu\Omega$ (*i. e.,* an oriented face of Ω). Let $\kappa : \partial^\nu\Omega \to \partial\Omega$ be the canonical projection of Equation (10). Let also $T_y D$ be tangent space to D at y, which we identify with $T_x\partial\Omega$, the tangent space of $\partial\Omega$ at $x = \kappa(y)$. Then the dilation α_x^{-1} of Equation (15) maps $T_y D$ to itself. We consider the orthogonal projection $\tilde{\pi}_y$ of $\kappa(D)$ onto T_yD. We shall denote by $B^2_y(r)$, the ball of radius r and center 0 in T_yD.

By $\pi_y : D \to T_yD \simeq \mathbb{R}^2$ we shall denote the composition $\alpha_x^{-1} \circ \tilde{\pi}_y \circ \kappa$, the *rescaled projection* of D onto its tangent plane (where $x = \kappa(y)$ as before). We consider these maps for all faces D of Ω.

Lemma 4.4. *Assume $\overline{\Omega}$ is compact. Then there exists $r > 0$ such that, for all faces D and all $y \in D$, the rescaled projection $\pi_y := \alpha_x^{-1} \circ \tilde{\pi}_y : D \to T_yD$, $x = \kappa(y)$, is a diffeomorphism from a small neighborhood of y in D to $B^2_y(r)$, the ball of radius r and center 0 in T_yD.*

Proof. For every $y \in D$, there exists a small $r_y > 0$ such that the rescaled projection $\pi_y := \alpha_x^{-1} \circ \tilde{\pi}_y : D \to T_yD$ is a diffeomorphism from a small neighborhood of x in D to $B^2_y(r_y)$. Let us chose for every $y \in D$ the largest r_y with this property. We need to show that there exists $r > 0$ such that $r_y > r$. By continuity, r_y is lower semi-continuous (*i. e.,* the set $r_y > \epsilon$ is open for every $\epsilon > 0$). It is therefore enough to show that r_y is bounded from below close to the boundary of D.

To this end, let us notice that the property that r_y be bounded from below is invariant under any change of coordinates, even non-linear change of coordinates (they can change however the lower bound), because $\overline{\Omega}$ is compact. Using the diffeomorphisms ϕ_x of Definition 3.4, we therefore see that we can reduce ourselves to the case of the dihedral angle

$$D_\alpha := \{(r, \theta, z),\, 0 < \theta < \alpha\}$$

or of the cone

$$\mathcal{C}_\omega := \{tx',\, t \in (0, \infty),\, x' \in \omega\}.$$

By Lemma 4.3, we can replace r_Ω with ϑ. Then $r_y = 1$ and hence it is bounded away from 0. This completes the proof. □

The following lemma is very important for our treatment of curved faces.

Corollary 4.5. *Assume that $\overline{\Omega}$ is compact. Let $\pi_y^{-1} : B^2_r(y) \to D$ be the inverse of the diffeomorphism of Lemma 4.4. Then, after decreasing $r > 0$, if needed, the map $\chi_y : B^2_y(r) \times [0, r) \to \Omega^\nu := \Omega \cup \partial^\nu\Omega$*

$$\chi_y(z, t) := \pi_y^{-1}(z) + t\, r_\Omega(x) y = \pi_y^{-1}(z) - t\, r_\Omega(x)\nu(y), \quad x = \kappa(y),$$

defines a diffeomorphism onto a neighborhood W_y of y in Ω^ν such that $B^2_y(r) \times \{0\}$ maps to the boundary $\partial^\nu\Omega$ of Ω^ν and $B^2_y(r) \times (0, r)$ maps to Ω.

Let $\mathbb{H}_y$ be the half-space determined by the unit vector $y \in \partial^\nu\Omega$. Let D be the oriented face of Ω containing y. Then $\partial\mathbb{H}_y = T_yD$. The set $B^2_y(r) \times [0, r)$ will be regarded, in what follows, as a subset of the (closure of) $\mathbb{H}_y$.

4.4. **Differential operators.** In this subection we introduce the differential operators that will be used in this paper. We continue to denote by r_Ω the canonical weight function introduced in (14). These constructions are a direct extension of the results in [1, 2], so we shall omit the proofs that are very close to those in these two papers.

In this section, by x, y, and z we shall denote the Euclidean coordinates of the point $(x, y, z) \in \mathbb{R}^3$. We shall need the following differentiability properties.

Lemma 4.6. *The functions $\partial_x r_\Omega$, $\partial_y r_\Omega$, and $\partial_z r_\Omega$ are in $\mathcal{C}^\infty(\Sigma\Omega)$. If $u \in \mathcal{C}^\infty(\Sigma\Omega)$, then the functions $r_\Omega \partial_x u$, $r_\Omega \partial_y u$, and $r_\Omega \partial_x u$ are also in $\mathcal{C}^\infty(\Sigma\Omega)$.*

Proof. Using the notation introduced in the proof of Lemma 4.2, it is enough to prove that the resulting functions are smooth functions of (ρ, θ, ϕ) on a set of the form W_1, that they are smooth functions of (ρ, x'), $x' \in S^2$, on a set of the form W_2, and that they are smooth functions of $(r, \theta, z) \in [0, a) \times \omega_x \times B^1$ on a set of the form V_p with p and edge point.

Assume first that we are on a set of the form W_1. The equations

$$\begin{aligned} r\partial_x &= \cos\theta\cos\phi\,(\sin\phi\,\partial_\phi) + \cos\theta\sin\phi(\rho\sin\phi\,\partial_\rho) - \sin\theta\,\partial_\theta \\ r\partial_x &= \sin\theta\cos\phi\,(\sin\phi\,\partial_\phi) + \sin\theta\sin\phi(\rho\sin\phi\,\partial_\rho) + \cos\theta\,\partial_\theta \\ r\partial_z &= -\sin\phi\,(\sin\phi\,\partial_\phi) + \cos\phi(\rho\sin\phi\,\partial_\rho), \end{aligned} \tag{16}$$

together with $r = \rho \sin\phi$, show that $\partial_x r$, $\partial_y r$, $\partial_z r$, $r\partial_x u$, $r\partial_y u$, and $r\partial_x u$ are also in $\mathcal{C}^\infty(\Sigma W_1)$. The result for sets of form W_1 then follows from the fact that r and r_Ω are equivalent on W_1.

Assume next that we are on a set of the form W_2. Let p be an arbitrary point in W_2. Our statement is independent of linear changes of coordinates, so we can assume that p is on the positive Oz semi-axis. Let us assume first that $r_\Omega = z$. Then $\partial_x z$, $\partial_y z$, and $\partial_z z$ are smooth functions, which take care of the first part of our statement. Next, we have $u \in \mathcal{C}^\infty(\Sigma W)$ if, and only if, $u(x, y, z) = \tilde{u}(x/z, y/z, z)$ for some smooth function $\tilde{u}$ (smooth in the usual sense). Therefore $z\partial_x u$, $z\partial_y u$, and $z\partial_z u$ are in $\mathcal{C}^\infty(\Sigma W)$ as well. The result then follows from the fact that ρ and z are equivalent on W, because then we can replace z with r_Ω. Since the point p was chosen arbitrarily, this completes the proof on a set of the form W_2.

For a set of the form V_x, with x an edge point, the proof is completely similar (but easier) using the relations

$$\begin{aligned} r\partial_x &= (\cos\theta)\, r\partial_r - (\sin\theta)\,\partial_\theta \\ r\partial_y &= (\sin\theta)\, r\partial_r + (\cos\theta)\,\partial_\theta. \end{aligned} \tag{17}$$

□

Let us denote by $\mathrm{Diff}_0^m(\Omega)$ the differential operators of order m on Ω linearly generated by differential operators of the form

$$u(r_\Omega\partial)^\alpha := u(r_\Omega\partial_x)^{\alpha_1}(r_\Omega\partial_y)^{\alpha_2}(r_\Omega\partial_z)^{\alpha_3}, \quad |\alpha| := \alpha_1+\alpha_2+\alpha_3 \le m,\ u \in \mathcal{C}^\infty(\Sigma\Omega).$$

For $m = 0$ we agree that $\mathrm{Diff}_0^m(\Omega) := \mathcal{C}^\infty(\Sigma\Omega)$. We shall denote by $\mathrm{Diff}_0^\infty(\Omega) := \bigcup_m \mathrm{Diff}_0^m(\Omega)$.

We have the following simple, but fundamental result.

Lemma 4.7. *Let $\lambda \in \mathbb{R}$ and let ∂_j and ∂_k stand for either of ∂_x, ∂_y, or ∂_z. Then $r_\Omega^{-\lambda}(r_\Omega\partial_j)r_\Omega^\lambda - r_\Omega\partial_j = \lambda\partial_j(r_\Omega) \in \mathcal{C}^\infty(\Sigma\Omega)$, and $[r_\Omega\partial_j, r_\Omega\partial_k] \in \mathrm{Diff}_0^1(\Omega)$.*

The proof of this lemma, as that of the following proposition, are as in [2].

Proposition 4.8. *We have $\mathrm{Diff}_0^k(\Omega)\mathrm{Diff}_0^m(\Omega) \subset \mathrm{Diff}_0^{k+m}(\Omega)$ and hence $\mathrm{Diff}_0^\infty(\Omega)$ is an algebra. If P is a differential operator of order m with smooth coefficients, then $r_\Omega^m P \in \mathrm{Diff}_0^m(\Omega)$.*

In particular,

$$r_\Omega^m \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3} - (r_\Omega\partial_x)^{\alpha_1}(r_\Omega\partial_y)^{\alpha_2}(r_\Omega\partial_z)^{\alpha_3} \in \mathrm{Diff}_0^{m-1}(\Omega), \quad |\alpha| = m. \tag{18}$$

5. Function spaces on Ω

In this section, we shall introduce and study the Babuška–Kondratiev spaces $\mathcal{K}_a^m(\Omega)$ and $\mathcal{K}_a^m(\partial^\nu\Omega) := \mathcal{K}_a^m(\partial^\nu\Omega; r_\Omega) = \mathcal{K}_a^m(\partial^\nu\Omega; \vartheta)$. These spaces are weighted Sobolev spaces with weight given by the canonical weight r_Ω, the distance to the set of edges of Ω, as in Equation (2). Therefore these spaces depend on the choice of polyhedral structure on Ω. This is significant for the study of mixed boundary value problems.

5.1. **The Babuška–Kondratiev spaces.** We define as usual

$$W_{BK}^{k,p,a}(\Omega) = \{u : \Omega \to \mathbb{R},\ r_\Omega^{|\alpha|-a}\partial^\alpha u \in L^p(\Omega),\ \text{for all } |\alpha| \leq k\}, \quad k \in \mathbb{Z}_+. \tag{19}$$

If $p = 2$, we denote $\mathcal{K}_a^k(\Omega) := W_{BK}^{k,2,a}(\Omega)$, which coincides with the definition in the Introduction (Equation 2).

Let D be an oriented face of Ω, then

$$W_{BK}^{m,p,a}(D) = \{\, u : D \to \mathbb{R},\ r_\Omega^{k-a} P u \in L^p(D),\ P \text{ a differential operator of order } k \leq m \text{ on } D\,\},$$

$m \in \mathbb{Z}_+$. We let $\mathcal{K}_a^k(\partial^\nu\Omega) := \bigoplus W_{BK}^{m,2,a}(D_k)$ for all faces D_k of Ω. Note that we require *no compatibility conditions* for the resulting functions on the faces D_k.

Lemma 4.3, Lemma 4.7, and Equation 18 give immediately the following lemma.

Lemma 5.1. *We have $\mathcal{K}_a^m(\Omega) = \{u,\ \vartheta^{-a} P u \in L^2(\Omega),\ \text{for all } P \in \mathrm{Diff}_0^k(M)\}$. A similar result holds for $\mathcal{K}_a^m(\partial^\nu\Omega)$ and for $W_{BK}^{k,p,a}(\Omega)$.*

Next,Lemma 4.7 and Equation 18 together with a straightforward calculation, show the following.

Lemma 5.2. *The multiplication map $W_{BK}^{m,\infty,b} \times \mathcal{K}_a^m(\Omega) \ni (u,f) \to uf \in \mathcal{K}_{a+b}^m(\Omega)$ is continuous. We also have $\mathcal{C}^\infty(\Sigma\Omega) \subset W_{BK}^{m,\infty,0}(\Omega)$ and $r_\Omega^b \in W_{BK}^{m,\infty,b}(\Omega)$, and hence the map $\mathcal{K}_a^m(\Omega) \ni u \to r_\Omega^b u \in \mathcal{K}_{a+b}^m(\Omega)$ is a continuous isomorphism of Banach spaces. The same result is true if we replace Ω with $\partial^\nu\Omega$.*

From this lemma we obtain right away the following result.

Proposition 5.3. *Let $k \geq m$. Each $P_0 \in \mathrm{Diff}_0^m(\Omega)$ defines a continuous map $P_0 : \mathcal{K}_a^k(\Omega) \to \mathcal{K}_a^{k-m}(\Omega)$. The family $r_\Omega^{-\lambda} P_0 r_\Omega^\lambda$ is a family of bounded operators $\mathcal{K}_a^k(\Omega) \to \mathcal{K}_a^{k-m}(\Omega)$ depending continuously on λ.*

Similarly, if P is a differential operator with smooth coefficients on Ω, then $r_\Omega^{-\lambda} P r_\Omega^\lambda$ defines a continuous family of bounded operators $\mathcal{K}_a^k(\Omega) \to \mathcal{K}_{a-m}^{k-m}(\Omega)$.

We define the spaces $\mathcal{K}_a^{-k}(\Omega)$, $k \in \mathbb{Z}_+$, by duality. More precisely, let $\overset{\circ}{\mathcal{K}}{}_a^k(\Omega)$ be the closure of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{K}_a^k(\Omega)$. Then we define $\mathcal{K}_{-a}^{-k}(\Omega)$ to be the dual of $\overset{\circ}{\mathcal{K}}{}_a^k(\Omega)$, the duality pairing being an extension of the bilinear form $(u,v) \mapsto \int_\Omega uv\, dx$. With this definition, we can drop the requirement that $k \geq m$ in Proposition 5.3.

We define $\mathcal{K}_{-a}^{-k}(\partial^\nu\Omega)$ as the dual of $\mathcal{K}_a^k(\partial^\nu\Omega)$. The spaces $\mathcal{K}_a^s(\partial^\nu\Omega)$, $s \notin \mathbb{Z}$ will be defined using partitions of unity.

5.2. **Partitions of unity.** We now introduce a smooth partition of unity $\mathcal{P}_\Omega$ on $\Omega^\nu := \Omega \cup \partial^\nu \Omega$, where either Ω is a cone or $\overline{\Omega}$ is compact. The partition of unity $\mathcal{P}_\Omega$ will be constructed out of a uniformly locally finite family of non-negative functions $\mathcal{F}_\Omega$ by the usual procedure: let $g := \sum f$, $f \in \mathcal{F}_\Omega$, which is defined since the family $\mathcal{F}_\Omega$ is locally finite. Then we consider the family

$$\mathcal{P}_\Omega := \{f/g,\ f \in \mathcal{F}_\Omega\}, \tag{20}$$

which will be defined in our case since $g > 0$.

We need to consider first the same construction for the case of a domain $D \subset S^2$ with a *polygonal* structure, yielding a uniformly locally finite family $\mathcal{F}_D$ and then, by the procedure of Equation (20), a partition of unity $\mathcal{P}_D$. Let $\mathcal{V} \subset \partial D$ be the set of vertices of D. For each $A \in \mathcal{V}$, let $\phi_A : V_A \to B^2$ be the diffeomorphism of definition 3.1 and ρ_A be the distance function to A. Let $\psi : \mathbb{R} \to [0,1]$ be a smooth function such that $\psi = 1$ on $[0,1]$ and ψ has support in $[-1,2]$. Then for each vertex we consider the family of functions $\psi_n(y) := \psi(\log_2 \rho_A(y) - n)$, for all values of n for which these functions are supported in V_A. We also consider a smooth partition of unity ξ_k of ω_A consisting of at least two functions. Let $\mathcal{F}_A := \{(\xi_k \circ \theta \circ \phi_A)\, \psi_n\}$.

If D is in fact an angle with vertex A, then the family $\mathcal{F}_A$ will suffice. Otherwise, let $\mathcal{F}_0$ be a smooth partition of unity on $\overline{D}$ defined in a neighborhood of $\overline{D}$ (we require that the functions $\phi \in \mathcal{F}_0$ be smooth functions defined on $\mathbb{R}^2$ with compact support that add up to 1 on $\overline{D}$). We may assume the family $\mathcal{F}_0$ to be finite and that the supports of the functions in $\mathcal{F}_0$ be small enough so that if a vertex A is in the support of some $\psi \in \mathcal{F}_0$, then the support of ψ is completely contained in V_A. Then we define $\mathcal{F}_D$ to consist of all the functions in $\mathcal{F}_A$ for all A and all the functions in $\mathcal{F}_0$ that are zero in the neighborhood of all vertices.

Lemma 5.4. *Assume that $\overline{D}$ is compact. Then there exists a constant $\kappa_D > 0$ such that no point $y \in \overline{D}$ belongs to the support of more than κ_D of the functions $\psi \in \mathcal{F}_D$.*

Proof. It is enough to prove this result for each of the sets $\mathcal{F}_A$, because the family $\mathcal{F}_0$ is finite. By refining the partition of unity ξ_k on the components of $\omega_A \subset S^1$, we can assume that no more than two supports of the functions ξ_k overlap (at any given point). Then, by construction, a point $y \in V_A$ can belong to at most 6 of the supports of the functions $\psi \in \mathcal{F}_A$. □

We will denote by $\mathcal{P}_D$ the partition of unity associated to the family $\mathcal{F}_D$ as explained in the beginning of this subsection (see Equation (20) and the discussion surounding it). Let r_D be the canonical weight function of D, Equation (12).

Lemma 5.5. *There exists a constant $C > 0$ such that*

$$|r_D^{|\alpha|} \partial^\alpha \psi(y)| \le C \text{ for all } y \in D \text{ and all } \psi \in \mathcal{F}_D \text{ or } \psi \in \mathcal{P}_D. \tag{21}$$

A point $y \in D$ belongs to at most κ_D of the supports of the functions ψ in our partition of unity $\mathcal{P}_D$. Moreover, for any $\psi \in \mathcal{P}_D$, the support of ψ contains no vertex, intersects at most one side of D, and has diameter $\le C r_D(y)$, for any y in the support of ψ and a constant C independent of ψ.

Proof. Only the estimate (21) needs a proof. In view of Lemma 5.4, it is enough to prove the statement for $\psi \in \mathcal{F}_D$. We need only check that (21) holds close to a vertex Q. Let us fix polar coordinates (r, θ) in a neighborhood W of Q. The functions r_D and r are equivalent in this neighborhood of Q in the usual sense that their quotients are in $\mathcal{C}^\infty(\Sigma W)$, and hence we can assume that $r_D = r$. Then we use the fact that $(r\partial_r)^i\partial_\theta^j\psi$ are bounded, by construction, and the relations (17), which allow us to express $r^{|\alpha|}\partial^\alpha\psi$ as a combination of $(r\partial_r)^i\partial_\theta^j\psi$ with coefficients that are smooth functions in r and θ (up to $r = 0$). □

For a polyhedral domain, we proceed in a similar way. On a set of the form $B^j \times (0, a) \times \omega_x$, we consider a product type family, coming from a partition of unity ζ_k on a neighborhood of B^j, a dyadic (infinite) partition of unity $\psi_n = \psi(\log_2 r - n)$ for $r \in (0, a)$, and a partition of unity ξ_l of ω_x. The partition of unity on ω_x is as before: at least two function if ω_x is an interval, and a partition of unity of the form $\mathcal{F}_{\omega_x}$ if ω_x is a domain with a polygonal structure. Only the former case must be considered if Ω is a dihedral angle and only the later case must be considered if Ω is a cone. Let $\phi_x : V_x \to B^j \times (0, a) \times \omega_x$ be as in Definition 3.4. This leads to the family $\mathcal{F}_x := \{(\zeta_k \otimes \phi_n \otimes \xi_l) \circ \phi_x\}$, which is a partition of unity subordinated to the component of V_x corresponding to x. By slightly decreasing the sets V_x, we can assume that all derivatives of ϕ_x are bounded on the closure of V_x.

Assume now that $\overline{\Omega}$ is compact (recall that we always assume this, unless Ω is an infinite cone or a dihedral angle), then $\partial\Omega$ can be covered by finitely many open sets of the form V_x. We fix such a covering corresponding to a finite set of points $x_j \in \partial\Omega$.

As in the case of a polygonal domain D earlier, we now consider a smooth partition of unity $\mathcal{F}_0$ on $\overline{\Omega}$ defined in a neighborhood of $\overline{\Omega}$. We may assume the family $\mathcal{F}_0$ to be finite and that the support of the functions in $\mathcal{F}_0$ be small enough so that, if an edge e intersects the support of some $\psi \in \mathcal{F}_0$, then the support of ψ is completely contained in one of the neighborhoods V_{x_j}. Finally, we let $\mathcal{F}_\Omega$ be the union of all the sets $\mathcal{F}_{x_j}$ and of the set of funtions in $\mathcal{F}_0$ that do not intersect the edges.

Lemma 5.6. *Assume that $\overline{\Omega}$ is compact. Then there exists a constant $\kappa_\Omega > 0$ such that no point $y \in \overline{\Omega}$ belongs to the support of more than κ_Ω of the functions $\psi \in \mathcal{F}_\Omega$.*

Proof. It is enough to prove this for each of the sets $\mathcal{F}_x$, with $x \in \partial\Omega$ mapping in $\partial\Omega$ to one of the finitely many points x_j. Assume x is a vertex and let κ_x be the constant bounding the multiplicity of the partition of unity associated to ω_x used in the construction of $\mathcal{F}_x$. Such a constant exists by Lemma 5.4. Then we can take κ (for V_x) to be $3\kappa_x$. The proof when x is an edge point is completely similar (in fact, slightly easier). □

We again see that the family $\mathcal{F}_\Omega$ is locally finite and hence $g := \sum \psi$, $\psi \in \mathcal{F}_\Omega$ is well defined and smooth. By decreasing the supports of the functions in $\mathcal{F}_0$, if necessary, we can assume that $g > 0$. Then we replace ψ with ψ/g to obtain a partition of unity that will be denoted by $\mathcal{P}_\Omega$. Let r_Ω be the canonical weight associated to Ω, Equation (14).

Lemma 5.7. *There exists a constant $C > 0$ such that*

$$|r_\Omega^{|\alpha|}\partial^\alpha\psi(y)| \leq C \text{ for all } y \in \Omega \text{ and all } \psi \in \mathcal{F}_\Omega \text{ or } \psi \in \mathcal{P}_\Omega. \tag{22}$$

A point $y \in \Omega$ belongs to at most κ_Ω of the supports of the functions ψ in our partition of unity $\mathcal{P}_D$. Moreover, for any $\psi \in \mathcal{P}_\Omega$, the support of ψ intersects no edge, intersects at most one face of D, and has diameter $\leq Cr_\Omega(y)$, for any y in the support of ψ and a constant C independent of ψ.

Proof. Again, we need only prove the estimate (22). In view of Lemma 5.6, it is enough to prove the statement for $\psi \in \mathcal{F}_\Omega$. Let $x \in \partial\Omega$ map to one of the finitely many fixed points x_j in $\partial\Omega$. If x is not a vertex, the proof is the same as that of Lemma 5.5.

Assume now that x is a vertex. Let $\phi_x : V_x \to B^t \times B^{3-t}$ be the diffeomorphism of Definition 3.4. Since all the derivatives of ϕ_x (for all x) are bounded on the closure of V_{x_j}, we can replace the function ψ with $\psi \circ \phi_x^{-1} = \zeta_k \otimes \phi_n \otimes \xi_l$. Moreover, on $W := (0,a) \times \omega_x$, the function $r_\Omega \circ \phi_x^{-1}$ and $\rho r_{\omega_x}(x')$ are equivalent (*i. e.,* their quotients are in $\mathcal{C}^\infty(\Sigma W)$). We can therefore replace the first function with the later and assume that $r(tx') = tr_D(x')$. We can also assume that Ω is the cone $\mathcal{C} := \{rx', r \in (0,\infty), x' \in \omega_x\}$.

We shall proceed as in the proof of Lemma 4.6. Let us first fix spherical coordinates $y = \rho x' = (\rho, \theta, \varphi)$ in a neighborhood of an edge e of Ω corresponding to a vertex A of ω_x. The vertex x then corresponds to $\rho = 0$ and the edge e corresponds to $\varphi = 0$. For x' is a sufficiently small neighborhood of A, $r = \rho \sin\varphi$ is equivalent to r_D. We then use Equation (16) that expresses $r\partial_x$, $r\partial_y$, and $r\partial_z$ in spherical coordinates to see that $(r\partial)^\alpha \psi$ is bounded uniformly.

Finally, let us assume that our point $y = \rho x' \in \mathcal{C} := \{rx', r \in (0,\infty), x' \in \omega_x\}$ is such that x' is away from the vertices. Then, as in the proof of Lemma 4.6, we introduce a coordinate system with x' in the direction of the positive axis. □

5.3. **Definition of Sobolev spaces using partitions of unity.** As in [1], it is important to define the spaces $\mathcal{K}_m^a(\Omega)$ using partitions of unity. Similar constructions were used in [17, 63, 64, 67]. This construction is possible because the spaces $\mathcal{K}_{3/2}^m(\Omega)$ are the usual Sobolev spaces associated to the metric $r_\Omega^{-2} g_E$, where g_E is the Euclidean metric.

We will need to use the partition of unity $\mathcal{P}_\Omega$ defined in the previous subsection. The partition of unity $\mathcal{P}_\Omega$ is a particular case of a construction that, in the case of non-compact manifolds, goes back to Aubin. It was subsequently used by Gromov and in [1, 63, 64, 67] and other papers.

In order to introduce the spaces $\mathcal{K}_a^m(\Omega)$ using partitions of unity, we shall use the diffeomorphisms χ_y defined in Corollary 4.5 for any $y \in \partial^\nu\Omega$. Let us fix y and denote by D the oriented face of Ω containing y. We agree to rotate T_yD to identify it with $\mathbb{R}^2$ when computing the Sobolev norms. The same applies to the half space $\mathbb{H}_y$ defined by y (this half space has T_yD as a boundary and y as an *inner* unit normal vector. These identifications are not unique, but differ by orthogonal transformations that do not change the Sobolev norms.

Let J be the set of indices j such that the support of ψ_j intersects $\partial^\nu\Omega$. Let us chose a point y_j in the support of ψ_j for each j. If $j \in J$, then we shall chose $y_j \in \partial^\nu\Omega$. We can assume that the supports of the functions ψ_j are small enough so that they are contained in the range W_{y_j} of the diffeomorphism χ_{y_j} of Corollary 4.5.

We let $x_j = \kappa(y_j)$. We shall denote by $\theta_j := \vartheta(x_j)$ and define $\alpha_j(x) = \alpha_{x_j}(x) := \theta_j(x - x_j) + x_j$ the dilation of ratio θ_j and center x_j, where ϑ is the distance to the edges of Ω. Also, we let $\chi_j = \chi_{y_j}$ and $\mathbb{H}_j = \mathbb{H}_{y_j}$.

We are now in a position to characterize the spaces $\mathcal{K}_m^a$ via partition of unity. (Note the half spaces $\mathbb{H}_i$ below.)

$$\begin{aligned}(23)\quad \nu_{m,a}(u)^2 &:= \sum_j \theta_j^{3-2a} \|(\psi_j u)\circ \alpha_j\|_{H^m}^2 \\ &:= \sum_{j\notin J} \theta_j^{3-2a} \|(\psi_j u)\circ \alpha_j\|_{H^m(\mathbb{R}^3)}^2 + \sum_{j\in J} \theta_j^{3-2a} \|(\psi_j u)\circ \chi_j^{-1}\|_{H^m(\mathbb{H}_j)}^2.\end{aligned}$$

We agree that $\|(\psi_j u)\circ\alpha_j\|_{H^m(\mathbb{H}_j)} = \infty$ if $(\psi_j u)\circ \alpha_j \notin H^m(\mathbb{R}^3)$ (or if $(\psi_j u)\circ\chi_j^{-1} \notin H^m(\mathbb{H}_j)$, respectively).

Proposition 5.8. *We have $u \in \mathcal{K}_a^m(\Omega)$, $m \in \mathbb{Z}$, if, and only if, $\nu_{m,a}(u) < \infty$. Moreover, $\nu_{m,a}(u)$ defines an equivalent norm on $\mathcal{K}_a^m(\Omega)$.*

For polyhedral domains, this characterization is well-known and the proof is standand (see [11, Lemma 2.4], [1, 2], or [67]); for $m < 0$ one also has to check that both definitions are compatible with duality. We sketch a proof for general domains with polyhedral structure.

We proceed in the same way to study the spaces $\mathcal{K}_a^s(\partial^\nu\Omega)$, $s \in \mathbb{R}$. Let again J be the set of indices j for which $x_j \in \partial\Omega$. We set

$$(24)\qquad \mu_{s,a}(u)^2 := \sum_{j\in J} \theta^{2-2a} \|(\psi_j u)\circ\chi_j^{-1}\|_{H^s(\partial\mathbb{H}_j)}^2, \quad s \in \mathbb{R}_+.$$

Then we have an analogous description of the spaces $\mathcal{K}_a^s(\partial^\nu\Omega)$, $s \in \mathbb{Z}$. We have $u \in \mathcal{K}_a^s(\partial^\nu\Omega)$ if, and only if, $\mu_{s,a}(u) < \infty$. Moreover, $\mu_{s,a}(u)$ defines an equivalent norm on $\mathcal{K}_a^s(\partial^\nu\Omega)$, $s\in\mathbb{Z}$. Therefore, we define

$$(25)\qquad \mathcal{K}_a^s(\partial^\nu\Omega) := \{u,\ \mu_{s,a}(u) < \infty\}, \quad s \in \mathbb{R},$$

with the induced norm.

Recall that $\partial^\nu\Omega$ is the set of inward unit normal vectors. It is an at most two-to-one cover of the smooth part of $\partial\Omega$. Its connected components are called the *oriented* faces of Ω. Recall that the set $\Omega^\nu := \Omega\cup\partial^\nu\Omega$ is naturally endowed with the structure of a smooth manifold with boundary $\partial^\nu\Omega$. However, a smooth function on Ω^ν may not extend to a smooth function on $\overline{\Omega}\smallsetminus\partial_{\text{sing}}\Omega$, because a smooth function on Ω^ν may have different limit values at the crack points, depending on the side from which we approach these points.

Let $S \subset \partial\Omega$ be a union of oriented faces D of Ω. Then we let

$$\mathcal{K}_a^m(S) := \oplus_{D\subset S}\mathcal{K}_a^m(D), \quad D\subset S,\ D \text{ an oriented face.}$$

Let $\partial_N\Omega \subset \partial^\nu\Omega$ and $\partial_D\Omega\subset\partial^\nu\Omega$ be a unions of *oriented* faces of Ω such that $\partial_N\Omega\cap\partial_D\Omega = \emptyset$. The proof of this result is as in [1, 2].

Theorem 5.9. *The space $\mathcal{C}_c^\infty(\Omega^\nu)$ is dense in $\mathcal{K}_a^m(\Omega)$, $m\in\mathbb{Z}_+$. Then the restriction to $\partial_D\Omega$ extend to a continuous, surjective map*

$$\mathcal{K}_a^m(\Omega) \ni u \to u|_{\partial_D\Omega} \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_D\Omega)$$

for $m\geq 1$. The kernel of this map is the closure of $\mathcal{C}_c^\infty(\Omega^\nu\smallsetminus\partial_D\Omega)$ in $\mathcal{K}_a^m(\Omega)$.

6. The boundary value problem

Let P be a second order operator on Ω and L_k be a boundary operator on the face D_k of Ω, which is either the restriction (Dirichlet) of the Neumann operator D_ν^P associated to P, defined for each k in Equation (29) below. We assume that P and L_k have smooth coefficients that extend to smooth functions in a neighborhood of $\overline{\Omega}$.

We will prove several regularity and well-posedness results for the boundary value problem

$$Pu = f \text{ in } \Omega, \quad L_k u = g_k \text{ on } D_k.$$

These results hold under different types of additional assumptions on P and on the boundary operators L_k. We begin this section by formulating these various additional assumptions on P, L, and Ω, in increasing generality. As usual, we concentrate on domains $\Omega \subset \mathbb{R}^3$ with polyhedral structure.

Recall that we are assuming Ω to be bounded or a cone. Some of these result extend to the more general case when $\overline{\Omega}$ has a *finite* covering with neighborhoods of the form V_p.

6.1. The three types of assumptions on P and L_k: an informal statement.

In addition to assuming that P is a second order operator with smooth coefficients on $\overline{\Omega}$ and L_k is Dirichlet or Neumann, we shall consider increasingly stringent assumptions on the operator P and the boundary operators L_k. We shall mostly consider the following three conditions, which we formulate informally first:

- The *regularity upon freezing the coefficients* assumption: P is uniformly elliptic and at any point of ΣD_k the pair $(P; L_k)$ satisfies an H^{m+1}–regularity estimate on the half-space determined by that point;
- The *coercivity* assumption: P is a second order operator, the boundary conditions are the natural (*i. e.,* Neumann) boundary conditions on $\partial_N\Omega$ and the Dirichlet boundary conditions on $\partial_D\Omega := \partial\Omega \smallsetminus \big(\partial_N\Omega \cup \partial_{\text{sing}}\Omega\big)$, and the Dirichlet form B_P associated to P satisfies the following Gårding inequality

$$B_P(u,u) \geq C_1\|u\|^2_{\mathcal{K}^1_1(\Omega)} - C_2\|r_\Omega^{-1}u\|^2_{L^2(\Omega)} \tag{26}$$

  for all u satisfying Dirichlet boundary conditions on $\partial_D\Omega$ and some constants $C_1 > 0$, C_2 independent of u;
- The *"positivity assumption"*: the operators P and L_k satisfy the same conditions as in the "coercivity assumption" and, in addition, the form B_P is positive-definite, that is, (26) holds with $C_2 = 0$.

We shall prove full regularity in the space $\mathcal{K}^m_a(\Omega)$, $m \geq 1$ and any $a \in \mathbb{R}$, under the regularity upon freezing the coefficients assumption. Then we show that the coercivity assumption implies the regularity upon freezing the coefficients assumption. Clearly, the positivity assumption is then the strongest, and we shall establish a well-posedness result under this assumption (positivity), provided $|a| < \eta$, for some small enough $\eta > 0$.

In the next subsection, we shall explain in detail these assumptions. We begin by fixing notation and by making some preliminary remarks.

6.2. **The form of the operator.** Let D_k be the faces of Ω. We shall consider a differential operator $P : \mathcal{C}_c^\infty(\Omega)^\mu \to \mathcal{C}_c^\infty(\Omega)^\mu$. Thus, for $\mu > 1$, we obtain a system. Let $P = [P_{pq}]$, $1 \le p, q \le \mu$, be the matrix notation for P, with

$$P_{pq} = -\sum_{i,j=1}^{3} \partial_j a_{pq}^{ij} \partial_i + \sum_{i=1}^{3} b_{pq}^{i} \partial_i + c_{pq}. \tag{27}$$

We assume that all the coefficients of the differential operators P_{pq} are smooth *real valued* functions in $x \in \overline{\Omega}$ (that is, these coefficients extend to smooth functions in a neighborhood of $\overline{\Omega}$). Our results also extend to some operators with complex coefficients, but that is not needed for the case of the elasticity operator, which our main interest. We also assume, for simplicity, that $a_{pq}^{ij} = a_{pq}^{ji}$. We shall denote by $B_P(u, v)$ the bilinear form:

$$B_P(u, v) := \int_\Omega a_{pq}^{ij} \partial_i u_p \partial_j u_q dx + \int_\Omega b_{pq}^{i} \partial_i u_p u_q dx + \int_\Omega c_{pq} u_p u_q dx, \tag{28}$$

$1 \le p, q \le \mu$, $1 \le i, j \le 3$, where Einstein's summation convention (summation over repeated indices), was used. We will also denote by D_ν^P the Neumann operator associated to P:

$$(D_\nu^P u)_p := \nu_i a_{pq}^{ij} \partial_j u_q, \tag{29}$$

where $\nu = (\nu_i)$ is the *outer unit normal vector* to $\partial\Omega$. Then, P, B_P, and D_ν^P are related as usual by the following divergence formula.

Lemma 6.1. *We have* $(Pu, v)_{L^2(\Omega)} = B_P(u, v) - (D_\nu^P u, v)_{L^2(\partial^\nu\Omega)}$ *for any real valued* $u \in H^2(\Omega)^\mu$ *and* $v \in H^1(\Omega)^\mu$.

Proof. We partition Ω into a finite number of disjoint Lipschitz domains by Lemma 3.5, to which the usual Gauss-Green formula applies [51, 72]. □

A typical example is $P = -\Delta$, in which case $B_P(u, v) := \int_\Omega \nabla u \cdot \nabla v dx$ and $D_\nu^P u = \partial_\nu u$ (normal derivative).

6.3. **The boundary conditions.** We assume that, for any oriented face D_k of our polyhedral structure on Ω, we are given $m_k \in \{0, 1\}$. We then let

$$L_k : \mathcal{C}^\infty(\overline{\Omega})^\mu \to \mathcal{C}^\infty(D_k)^\mu \tag{30}$$

to be $L_k(u) = u|_{D_k}$, if $m_k = 0$, and $L_k(u) = D_\nu^P u$, if $m_k = 1$. For notational simplicity, we shall assume that $m_1 = \ldots = m_r = 1$ and $m_{r+1} = \ldots = m_N = 0$.

We shall denote by $\partial_N\Omega$ the union of the open oriented faces D_k for which $m_k = 1$ and by $\partial_D\Omega$ the union of the open faces D_k for which $m_k = 0$. Then we shall denote $\mathcal{K}_a^s(\partial_N\Omega) := \oplus_{m_k=1}\mathcal{K}_a^s(D_k)$ and, similarly, $\mathcal{K}_a^s(\partial_D\Omega) := \oplus_{m_k=0}\mathcal{K}_a^s(D_k)$.

The operator P and the boundary conditions L_k give rise, for any $m \ge 1$ and any $a \in \mathbb{R}$, to continuous, linear maps

$$\begin{aligned} [P; L_k]_m : \mathcal{K}_{a+1}^{m+1}(\Omega)^\mu &\to \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu \oplus \Big(\bigoplus_k \mathcal{K}_{a+1/2-m_k}^{m+1/2-m_k}(D_k)^\mu \Big) \\ &=: \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)^\mu \oplus \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D\Omega)^\mu, \\ [P; L_k]_m &:= (Pu, L_1 u, \ldots, L_N u) = (Pu, D_\nu^P u|_{\partial_N\Omega}, u|_{\partial_D\Omega}). \end{aligned} \tag{31}$$

In order to deal also with the weak formulation of our boundary value problems, we will need to consider the map $[P; L_k]_m$ for $m = 0$ as well. This is achieved as follows. Let

$$\mathcal{H}_a := \{u \in \mathcal{K}^1_{a+1}(\Omega)^\mu, \ u = 0 \ \text{ on } \ \partial_D\Omega\}. \tag{32}$$

The bilinear form B_P defines a continuous, linear map $B^*_P : \mathcal{K}^1_{a+1}(\Omega)^\mu \to \mathcal{H}^*_{-a}$ by the equation

$$\langle B^*_P u, v\rangle := B_P(u, v). \tag{33}$$

This allows us to define

$$\begin{aligned} &[P; L_k]_0 : \mathcal{K}^1_{a+1}(\Omega)^\mu \to \mathcal{H}^*_{-a} \oplus \mathcal{K}^{m+1/2}_{a+1/2}(\partial_D\Omega)^\mu, \\ &\qquad\qquad [P; L_k]_0(u) := (B^*_P(u), u|_{\partial_D\Omega}u), \quad a \in \mathbb{R}. \end{aligned} \tag{34}$$

Then $[P; L_k]_m$ extends $[P; L_k]_{m+1}$ for all $m \geq 0$. For $m \geq 1$ this is directly seen from the definition. For $m = 0$, the meaning in which $[P; L_k]_0$ extends $[P; L_k]_1$ is the following. Let us define

$$nat : \mathcal{K}^0_{a+1}(\Omega)^\mu \oplus \mathcal{K}^{1/2}_{a-1/2}(\partial_N\Omega)^\mu \to \mathcal{H}^*_{-a}$$

by $nat(f, u)(v) = \int_\Omega fv dV + \int_{\partial\Omega} uv dS$ for any $v \in \mathcal{H}_{-a}$. The extension of nat to a map $\mathcal{K}^0_{a+1}(\Omega)^\mu \oplus \mathcal{K}^{1/2}_{a-1/2}(\partial_N\Omega)^\mu \oplus \mathcal{K}^{3/2}_{a+1/2}(\partial_D\Omega)^\mu \to \mathcal{H}^*_{-a} \oplus \mathcal{K}^{1/2}_{a+1/2}(\partial_D\Omega)^\mu$ with still be denoted nat (instead of $nat \oplus id$). Then nat is continuous and Lemma 6.1 gives

$$nat \circ [P; L_k]_1 = [P; L_k]_0.$$

If $u \in \mathcal{K}^1_a(\Omega)$ is such that $B^*_P(u) = nat(f, g)$, then we shall write $Pu = f$, $D^P_\nu u = g$.

We shall write (P, L_k) when we refer to the pair consisting of P and a fixed L_k (for instance, in the regularity condition upon freezing the coefficients). On the other hand, we shall write $\{P; L_k\}$ when we refer to the family consisting of P and *all* boundary conditions L_k (for instance, in Equation (31) and in the following definition).

Definition 6.2. We say that $\{P; L_k\}$ *satisfies a regularity estimate on* $\mathcal{K}^{m+1}_{a+1}(\Omega)$ (or, simply, that $\{P; L_k\}$ is $\mathcal{K}^{m+1}_{a+1}$–regular), $m \geq 1$, if there exists $C > 0$ such that

$$\|u\|_{\mathcal{K}^{m+1}_{a+1}(\Omega)} \leq C\big(\|Pu\|_{\mathcal{K}^{m-1}_{a-1}(\Omega)} + \sum_k \|L_k u\|_{\mathcal{K}^{m+1/2-m_k}_{a+1/2-m_k}(D_k)} + \|u\|_{\mathcal{K}^m_{a+1}(\Omega)}\big), \tag{35}$$

for all $u \in \mathcal{K}^1_{a+1}(\Omega)$. For $m = 0$, we require instead

$$\|u\|_{\mathcal{K}^1_{a+1}(\Omega)} \leq C\big(\|B^*_P(u)\|_{\mathcal{H}^*_{-a}} + \|u|_{\partial_D\Omega}\|_{\mathcal{K}^{1/2}_{a+1/2}(\partial_D\Omega)} + \|u\|_{\mathcal{K}^0_{a+1}(\Omega)}\big). \tag{36}$$

Remark 6.3. This definition shall be understood in the sense that the undefined terms are taken to be ∞. More precisely, assume that $u \in \mathcal{K}^1_{a+1}(\Omega)$, $Pu \in \mathcal{K}^{m-1}_{a-1}(\Omega)$, and $L_k u \in \mathcal{K}^{m+1/2-m_k}_{a+1/2-m_k}(D_k)$ for each oriented face D_k of Ω. Our definition then states that $u \in \mathcal{K}^{m+1}_{a+1}(\Omega)$ and satisfies Equation (35). All the relations similar to Equation (35) will be understood in this sense in what follows.

Remark 6.4. Suppose that $\{P; L_k\}$ satisfies a regularity estimate on $\mathcal{K}_{a+1}^{j+1}(\Omega)$, $0 \leq j \leq m$, then Equation (35) together with its analogue for $m = 0$ and the continuity of the map *nat* show, by induction, that there exists $C > 0$ such that

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \leq C\big(\|Pu\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)} + \sum_k \|L_k u\|_{\mathcal{K}_{a+1/2-m_k}^{m+1/2-m_k}(D_k)} + \|u\|_{\mathcal{K}_{a+1}^{0}(\Omega)}\big),$$

for all $u \in \mathcal{K}_{a+1}^{1}(\Omega)$ (the last term was replaced with a weaker one).

If Ω is a smooth domain and the family L_k reduces to a single boundary condition L (with only Dirichlet or only Neumann boundary conditions), we shall simply say that $(P, L) = (P, L_k)$ satisfies a regularity estimate on $H^{m+1}(\Omega)$ if $\{P; L_k\}$ satisfies a regularity estimate on $\mathcal{K}_a^{m+1}(\Omega) = H^{m+1}(\Omega)$. (In this case the weight is 1, so the spaces $\mathcal{K}_a^{m+1}(\Omega)$ are independent of a.) Then (P, L) satisfies a regularity estimate on $H^{m+1}(\Omega)$ if, and only if, there exists a constant $C > 0$ such that

$$\|u\|_{H^{m+1}(\Omega)} \leq C\big(\|Pu\|_{H^{m-1}(\Omega)} + \|Lu\|_{H^{m+1/2-m_L}(\partial^\nu\Omega)} + \|u\|_{H^m(\Omega)}\big), \tag{37}$$

for all $u \in H^{m+1}(\Omega)$ [45, 66]. In this paper, this condition will be used for $\Omega = \mathbb{R}^3_+$ with the usual (Euclidean) metric. (The term "coercive estimate" is often used instead of our term, "regularity estimate." We feel however that the term "coercive" is overused, and hence we prefer "regularity estimate" for our paper.)

Definition 6.2 and Proposition 5.3 allow us to make the following simple remark.

Remark 6.5. The condition that $\{P, L_k\}$ satisfy a regularity estimate on $\mathcal{K}_{a+1}^{m+1}(\Omega)$ is independent of perturbations of P of the form $r_\Omega^{2-k}Q$, with Q a differential operator with smooth coefficients of order $k = 0, 1$. In particular, the condition that $\{P, L_k\}$ satisfy a regularity estimate on $\mathcal{K}_{a+1}^{m+1}(\Omega)$ is independent of a.

6.4. **The "regularity upon freezing the coefficients" assumption.** Recall that $\partial^\nu\Omega$, the oriented boundary of Ω, is the set of *inner* normal unit vectors to the smooth part of the boundary of Ω. In particular, we have a natural map $\kappa : \partial^\nu\Omega \to \partial\Omega$ ofEquation (10), which assigns to a normal unit vector y its starting point. This map is one-to-one at the points where Ω is on only one side of the boundary, and two-to-one otherwise. The identity map $\nu : \partial^\nu\Omega \to S^2$, $\nu(y) = -y$, extends to a smooth map $\nu : \Sigma D \to \mathbb{R}^3$ for any oriented face D of Ω. The map κ also extends to the continuous map $\kappa : \Sigma D \to \overline{D} \subset \overline{\Omega}$ defined in Equation (9).

Let $D_k \subset \partial^\nu\Omega$ be an arbitrary oriented face of Ω and $\nu : D_k \to S^2$ be the outer unit normal function. Then ν extends to a smooth function $\Sigma D_k \to S^2$. This allows us to associate to each $y \in \Sigma D_k$ the half-space $\mathbb{H}_y \subset \mathbb{R}^3$ with outer unit normal vector $\nu(y) = -y$ and the Neumann operator $D^P_{\nu(y)}$ by the formula (29). We then obtain the boundary differential operator

$$L_{ky} : H^{m+1}(\mathbb{H}_y) \to H^{m-1/2-m_k}(\partial\mathbb{H}_y),$$

$L_{ky}u = u|_{\partial\mathbb{H}_y}$, if $m_k = 0$, and $L_{ky}u = D^P_y = D^P_{\nu(y)}u$ if $m_k = 1$. The norms on the Sobolev spaces on $\mathbb{H}_y$ and on $\partial\mathbb{H}_y$ are determined by the Euclidean structure on $\mathbb{R}^3 \supset \mathbb{H}_y$. Let $\kappa : \Sigma D_k \to \overline{D}$ be the canonical map as above and $x = \kappa(y)$. As before, by freezing the coefficients of P at x and dropping the lower order terms, we obtain the operators $P_y = P_x$, $x = \kappa(y)$, for any $y \in \Sigma D_k$. (Unlike $\mathbb{H}_y$ and L_{ky}, the operators P_y depend only on $x = \kappa(y)$.)

Recall that P is called *uniformly elliptic* if there exists a constant $C_P > 0$ such that

$$\|(\zeta_p)\| \geq C_P \|(\eta_q)\|, \tag{38}$$

where $\zeta_p := a_{pq}^{ij}(x)\xi_i\xi_j\eta_q$, $x \in \overline{\Omega}$, $\| \ \|$ is the norm on $\mathbb{R}^\mu$, (ξ_i) is an arbitrary vector in $\mathbb{R}^3$ and (η_p) is an arbitrary vector in $\mathbb{R}^\mu$. Similarly, P is called *uniformly strongly elliptic* if there exists a constant $C_P > 0$ such that

$$a_{pq}^{ij}(x)\xi_i\xi_j\eta_q\eta_p \geq C_P \sum_q \eta_q^2, \tag{39}$$

where $x \in \overline{\Omega}$, (ξ_i) is an arbitrary vector in $\mathbb{R}^3$ and (η_p) is an arbitrary vector in $\mathbb{R}^\mu$.

Definition 6.6. Let Ω be a domain with a polyhedral structure. We say that $\{P; L_k\}$ satisfies the *regularity assumption upon freezing the coefficients* if P is uniformly elliptic and, for any $m \in \mathbb{Z}_+$, any k, and any $y \in \Sigma D_k$, the pair (P_y, L_{ky}) satisfies a $H^{m+1}(\mathbb{H}_y)$–regularity estimate.

In other words, $\{P; L_k\}$ satisfies the regularity assumption upon freezing the coefficients if P is uniformly elliptic and, for any $m \in \mathbb{Z}_+$, any k, and any $y \in \Sigma D_k$, there is $C_y > 0$ such that

$$\|v\|_{H^{m+1}(\mathbb{H}_y)} \leq C_y \big(\|P_y v\|_{H^{m-1}(\mathbb{H}_y)} + \\ \|(L_{ky}u)v\|_{H^{m+1/2-m_k}(\partial\mathbb{H}_y)} + \|v\|_{H^m(\mathbb{H}_y)} \big), \tag{40}$$

for all $v \in H^1(\mathbb{H}_y)$.

Recall the diffeomorphism $\chi_y : B_y^2(r) \times [0, r) \to W_y \subset \Omega^\nu := \Omega \cup \partial^\nu\Omega$ of Corollary 4.5, $B_y^2(r) \times [0, r) \subset \mathbb{H}_y$, the closed half-space determined by the vector $\nu(y) = -y$. We define the Sobolev spaces on $\mathbb{H}_y$ and $T_y\partial^\nu\Omega = \partial\mathbb{H}_y$ using the Euclidean structure (metric) on these spaces.

The following lemma is crucial for proving our regularity results in the $\mathcal{K}_a^m$–spaces.

Lemma 6.7. *Assume that $\overline{\Omega}$ is compact and that the family $\{P; L_k\}$ satisfies the regularity assumption upon freezing the coefficients. Then there exists $C_P > 0$ independent of y such that*

$$\|u \circ \chi_y\|_{H^{m+1}(\mathbb{H}_y)} \leq C_P \Big(r_\Omega(y)^2 \|(Pu) \circ \chi_y\|_{H^{m-1}(\mathbb{H}_y)} + \\ r_\Omega(y)^{m_k} \|(L_k u) \circ \chi_y\|_{H^{m+1/2-m_k}(\partial\mathbb{H}_y)} + \|u \circ \chi_y\|_{H^m(\mathbb{H}_y)} \Big),$$

for any $y \in D_k$ and any u with support in $W_y := \chi_y\big(B_y^2(r) \times [0, r)\big)$.

Proof. The diffeomorphism $\chi_y : B_y^2(r) \times [0, r) \to W_y \subset \Omega^\nu$ of Corollary 4.5 allows us to define a second order differential operator Q_y on $B_y^2(0; r) \times (0, r)$ and the boundary condition M_y by the formulas

$$Q_y(u \circ \chi_y) := r_\Omega(x)^2 (Pu) \circ \chi_y \quad \text{and} \quad M_y(u \circ \chi_y) := r_\Omega(y)^{m_k} (L_k u) \circ \chi_y,$$

where $x = \kappa(y)$.

The family Q_y depends continuously on $y \in \Sigma D_k$. Let $\xi^2 + \zeta^2 = 1$ be two real valued smooth functions with compact support and $\tilde{Q}_y := \xi Q_y \xi + \zeta\Delta\zeta$, which is defined if $\xi = 0$ on the complement of $B^2(0; r) \times (0, r)$. We can also assume that

$\xi = 1$ on $B^2(0;r') \times (0,r')$, for some $0 < r' < r$. Changing coordinates preserves regularity, and hence

$$\|v\|_{H^{m+1}(\mathbb{H}_y)} \leq C_y\big(\|\tilde{Q}_y v\|_{H^{m+1}(\mathbb{H}_y)} + \|M_y v\|_{H^{m+1/2-m_k}(\partial\mathbb{H}_y)} + \|v\|_{H^m(\mathbb{H}_y)}\big).$$

The continuity of the family $(\tilde{Q}_y, M_y)$ shows that C_y is lower semi-continuous. Since ΣD_k is compact, if follows that C_y is bounded from below by a constant $C > 0$. The result then follows since $\tilde{Q}_y(u \circ \chi_y) = r_\Omega(x)^2(Pu) \circ \chi_y$ and $M_y(u \circ \chi_y) = r_\Omega(y)^{m_k}(L_k u) \circ \chi_y$ for u with support in $\chi_y(B_0^2(r') \times (0,r'))$. □

6.5. **Coercivity and positivity.** We continue to assume that $\partial^\nu\Omega = \partial_N\Omega \cup \partial_D\Omega$, where $\partial_N\Omega$ and $\partial_D\Omega$ are unions of oriented open faces of Ω. (Recall that $\partial^\nu\Omega = \partial\Omega \smallsetminus \partial_{\text{sing}}\Omega$ if Ω has no cracks.) We also assume that no two adjacent faces can belong both to $\partial_N\Omega$. Furthermore, we assume that $\partial_D\Omega \neq \emptyset$. We set $Lu = u = 0$ on $\partial_D\Omega$ and $Lu = D_\nu^P u = 0$ on $\partial_N\Omega$.

Definition 6.8. We say that the operator P is *coercive* if there exist positive constants C_1, C_2 such that

$$B_P(u,u) \geq C_1\|u\|^2_{\mathcal{K}_1^1(\Omega)} - C_2\|r_\Omega^{-1}u\|^2_{L^2(\Omega)}, \quad u \in \mathcal{K}_1^1(\Omega)^\mu, \quad u = 0 \ \text{ on } \ \partial_D\Omega,$$

where B_P is the bilinear form of equation (28).

Note that our definition of coercive operators is weaker than the usual definition: $B_P(u,u) \geq C_1\|u\|^2_{H^(\Omega)} - C_2\|u\|^2_{L^2(\Omega)}$, $u = 0$ on $\partial_D\Omega$, because of the factor r_Ω^{-1}.

We have the following basic result. Recall that for any oriented face D of Ω, ΣD denotes the desingularization of that face.

Proposition 6.9. *Assume P is coercive. Then P is uniformly strongly elliptic and satisfies the assumption of regularity upon freezing coefficients with a constant C_y independent of $y \in \Sigma D$.*

Proof. Let us fix an arbitrary oriented face D_k of Ω and let $y \in \Sigma D_k$. Let also (P_y, L_{ky}) be as in the assumtion of regularity upon freezing the coefficients (Definition 6.6. Assume first that $y \in D_k$ and let $\chi_y : B_x^2(r) \times [0,r) \to \Omega^\nu := \Omega \cup \partial^\nu\Omega$ be the diffeomorphism defined in Corollary 4.5. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{H}_y)$ and $\phi_t(z) = t^{3/2}\phi(tz)$.

We use Proposition 5.8, to conclude that P satisfies regularity upon freezing coefficients at x for $m = 0$ first.

Since we are on a half-space, the classical Nirenberg argument (see e.g. [29, 54, 59, 66]) shows that regularity upon freezing the coefficients can be bootstrapped so it holds for any $m > 0$. □

Remark 6.10. If P is second order strongly elliptic and the L_k are the restrictions to the boundary (*i. e.,* Dirichlet boundary conditions), then P satisfies a Garding type inequality, and therefore regularity upon freezing the coefficients [59, 65, 68].

Other examples of coercive operators are the Laplace operator Δ and the elasticity operator $\operatorname{div}(\mathbf{C} \cdot \nabla)$ with mixed boundary conditions (with $\mathbf{C}$ positive definite).

A stronger condition than coercivity will be needed in general to establish well-posedness.

Definition 6.11. We say that the operator P is *strictly positive* if the bilinear form B_p is positive-definite on $\mathcal{K}^1_1$, that is, there is a $\gamma > 0$ such that

$$B_P(u,u) \geq \gamma \|u\|^2_{\mathcal{K}^1_1(\Omega)}.$$

For example, if the elasticity tensor $\mathbf{C} = (\mathsf{C}^{ij}_{pq})$ is positive definite on symmetric matrices, then the elasticity operator $\operatorname{div}(\mathbf{C} \cdot \nabla)$ is positive in the sense of the above definition, with Dirichlet or Neumann boundary conditions provided that no two adjacent faces are endowed with Neumann boundary conditions and $\partial_D \Omega \neq \emptyset$. This result follows by an application of Korn's inequality, which will be discussed later (in Section 10.2). The Laplace operator $-\Delta$ turns out to be positive under the same conditions: no two adjacent faces are endowed with Neumann boundary conditions and $\partial_D \Omega \neq \emptyset$.

7. Proof of the regularity theorem

We include in this section the proof of Theorem 1.1. Its proof is reduced to the Euclidean case using the partition of unity $\mathcal{P}_\Omega$ of Lemma 5.7 and the diffeomorphisms χ_k of Corollary 4.5.

We continue to assume that P is a second order differential operator as in Equation (27) and that L_k are either Dirichlet or Neumann boundary conditions. We also continue to assume that $\partial^\nu \Omega$ is written as a disjoint $\partial^\nu \Omega = \partial_N \Omega \cup \partial_D \Omega$, with $\partial_N \Omega$ and $\partial_D \Omega$ union of oriented faces of Ω.

Let $x \in \Omega$ and α_x be the dilation with center x and ration r_Ω, as defined in Equation (15). We have the following regularity result.

Lemma 7.1. *Assume that $\overline{\Omega}$ is compact. Then there exists a constant $C > 0$ such that*

$$\|u \circ \alpha_x\|_{H^{m+1}(\mathbb{R}^3)} \leq C\big(r_\Omega^2 \|(Pu) \circ \alpha_x\|_{H^{m-1}(\mathbb{R}^3)} + \|u \circ \alpha_x\|_{H^m(\mathbb{R}^3)}\big),$$

for any $x \in \Omega$ and any $u \in \mathcal{C}^\infty_{\mathrm{c}}(B^3(x, \vartheta(x)) \cap \Omega)$, where $B^3(x, \vartheta(x)) \subset \mathbb{R}^3$ is the ball of radius $\vartheta(x)$ and center x.

Proof. We have that $(Pu) \circ \alpha_x = r_\Omega(x)^{-2} Q_x(u \circ \alpha_x)$, for

$$(Q_x)_{p,q} = -\sum_{i,k=1}^{3} \big(a^{ij}_{pq} \circ \alpha_x\big)\, \partial_j \partial_i, 1 \leq, p, q, \leq \mu. \tag{41}$$

Therefore elliptic regularity applied to the operator Q_x shows that there exists $C_x > 0$ such that

$$\|u \circ \alpha_x\|_{H^{m+1}(\mathbb{R}^3)} \leq C_x\big(r_\Omega(x)^2 \|(Pu) \circ \alpha_x\|_{H^{m+1}(\mathbb{R}^3)} + \|u \circ \alpha_x\|_{H^m(\mathbb{R}^3_+)}\big).$$

Let us chose for each x the least C_x with this property. We only need to show that we can chose C_x independent of x.

Since $\partial_l(a^{ik}_{pq} \circ \alpha_x) = r_\Omega(x)(\partial_l a_{ik}) \circ \alpha_x$, the family of functions $\{a^{ik}_{pq} \circ \alpha_x\}$ depends continuously on $x \in \overline{\Omega}$ in $\mathcal{C}^\infty(B^2)$. Hence the family Q_x of differential operators on $\mathcal{C}^\infty_{\mathrm{c}}(B^2)^\mu$ depends continuously on $x \in \overline{\Omega}$. Since the function $x \to C_x$ is upper semi-continuous and $\overline{\Omega}$ is compact, it follows that the function C_x is bounded. This completes the proof. $\square$

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a polyhedral structure. Assume that the family $\{P, L_k\}$ satisfies the regularity assumption upon freezing the coefficients. Let $m \in \mathbb{Z}$, $m \geq 1$, and $a \in \mathbb{R}$. Assume that $u \in \mathcal{K}^1_{a+1}(\Omega)^\mu$. Then there exists $C > 0$ such that*

$$\|u\|_{\mathcal{K}^{m+1}_{a+1}(\Omega)} \leq C\big(\|Pu\|_{\mathcal{K}^{m-1}_{a-1}(\Omega)} + \sum_k \|L_k u\|_{\mathcal{K}^{m+1/2-m_k}_{a+1/2-m_k}(D_k)} + \|u\|_{\mathcal{K}^m_{a+1}(\Omega)}\big).$$

In particular, if $u \in \mathcal{K}^1_{a+1}(\Omega)^\mu$, $Pu \in \mathcal{K}^{m-1}_{a-1}(\Omega)^\mu$, and $L_k u|_{\partial_D \Omega} \in \mathcal{K}^{m+1/2-m_k}_{a+1/2-m_k}(D_k)^\mu$ for all oriented faces D_k of Ω then $u \in \mathcal{K}^{m+1}_{a+1}(\Omega)^\mu$.

Some applications of this result for the Laplace operator with mixed boundary conditions were given in [12]. By contrast, it is known that in the framework of the usual Sobolev spaces $H^m(\Omega)$, the smoothness of the solution of (1) is bounded [22, 31, 33, 35, 53].

Proof. By Remark 6.5, it is enough to assume that P is homogeneous of degree 2:

$$P = -\sum_{i,j=1}^{3} a^{ij}_{pq}\,\partial_j \partial_i, 1 \leq p, q \leq \mu.$$

(The boundary conditions are already homogeneous.) We shall extend the coefficients of P to the whole plane to satisfy the same estimates uniformly on $\mathbb{R}^3$. This extension is possible because the coeffcients of P are assumed defined and smooth on a neighborhood of Ω,

Recall the partition of unity $\mathcal{P}_\Omega = \{\psi_j\}$ of Lemma 5.7. Also, recall that we have fixed x_j in the support of each ψ_j, with x_j in the boundary if possible. Let $\theta_j = \vartheta(x_j)$, with ϑ the distance to the set of edges. We denote again by α_j the dilation $\theta_j x + x_j$, and by $\chi_j = \chi_{y_j}$, $x_j = \kappa(y_j)$ the boundary diffeomorphism of Corollary 4.5, whenever $x_j \in \partial^\nu \Omega$. By decreasing the supports of the initial choices defining the partition of unity $\mathcal{P}_\Omega$, we can assume that the functions $u_j = (\psi_j\, u) \circ \alpha_j$, if $x_j \in \Omega$, or $(\psi_j\, u) \circ \chi_j$, if $x_j \in \partial^\nu \Omega$, are all supported in the fixed ball $B^3 = B(0; 1)$ (of radius 1 and center the origin). We implicitly assume that the half space $\mathbb{H}_j$ has been rotated and translated to agree with $\mathbb{R}^3_+$.

We next bound the commutator $[P, \psi_j]$ using Lemma 5.7. We have $[P, \psi_j] = Q_j$, where Q_j is a differential operator of order 1 with coefficients that depend on the first and second derivatives of ψ_j, so that $\theta_j^2\, \alpha_j, Q_j \alpha_j^{-1} : H^m(B^3) \to H^{m-1}(B^3)$ is uniformly bounded in j. (Here, $\alpha_j\, Q_j \alpha_j^{-1}(v) = [Q_j\,(v \circ \alpha_j^{-1})] \circ \alpha_j$.) Consequently, for $m \in \mathbb{Z}_+$,

$$\|([P, \psi_j]\, u) \circ \alpha_j\|_{H^{m-1}(B^3)} = \|(\alpha_j\, Q_j\, \alpha_j^{-1})\, u_j\|_{H^{m-1}(B^3)} \leq C\, \theta_j^{-2}\, \|u_j\|_{H^m(B^3)}, \tag{42}$$

with C independent of j, since $\{\psi_j \circ \alpha_j\}$ is uniformly bounded in $\mathcal{C}^\infty_{\mathrm{c}}(B^2)$ again by Lemma 5.7. A similar estimate holds for $\chi_j\, Q_j \chi_j^{-1}$, using also Corollary 4.5.

If the support of ψ_j does not intersect the boundary of Ω, we conclude using also Lemma 7.1 that

$$\begin{aligned}\theta_j^{3/2-a-1}\|(\psi_j\, u)\circ\alpha_j\|_{H^{m+1}(B^3)} &\le C\,\theta_j^{3/2-a-1}\Big(\theta_j^2\|(P\,\psi_j\, u)\circ\alpha_j\|_{H^{m-1}(B^3)}\\ &\quad + \theta_j^2\,\|([\psi_j,P]\,u)\circ\alpha_j\|_{H^{m-1}(B^3)} + \|u\circ\alpha_j\|_{H^m(B^3)}\Big)\\ &\le C\,\theta_j^{3/2-a-1}\Big(\theta_j^2\,\|(\psi_j P\,u)\circ\alpha_j\|_{H^{m-1}(B^3)} + \|u\circ\alpha_j\|_{H^m(B^3)}\Big)\\ &\le C\,\theta_j^{3/2-a-1}\Big(\theta_j^2\,\|(\psi_j P\,u)\circ\alpha_j\|_{H^{m-1}(B^3)} + \|u\circ\alpha_j\|_{H^m(B^3)}\Big)\\ &\le C\,\Big(\theta_j^{3/2-(a-1)}\|(\psi_j\, Pu)\circ\alpha_j\|_{H^{m-1}(B^3)} + \theta_j^{3/2-a-1}\|u\circ\alpha_j\|_{H^m(B^3)}\Big)\end{aligned}$$

with constants C independent of j. By squaring and using $(a+b)^2\le 2(a^2+b^2)$ we obtain

$$\begin{aligned}\text{(43)}\quad &\theta_j^{1-2a}\|(\psi_j\, u)\circ\alpha_j\|^2_{H^{m+1}(B^3)}\\ &\qquad\le C\,\Big(\theta_j^{3-2(a-1)}\|(\psi_j\, Pu)\circ\alpha_j\|^2_{H^{m-1}(B^3)} + \theta_j^{1-2a}\|u\circ\alpha_j\|^2_{H^m(B^3)}\Big).\end{aligned}$$

On the other hand, if the support of ψ_j does intersect the boundary of Ω, we can estimate the commutators of ψ_j with L_k, similarly to (42), to obtain for $m_k=1$

$$\theta^{m_k}\|([L_k,\psi_j]u)\circ\chi_y\|_{H^{m+1/2-m_k}(\mathbb{R}^2)} \le \|u\circ\chi_j\|_{H^m(\mathbb{R}^3_+)}.$$

(For $m_k=0$ the commutator vanishes, so the analogous equation is trivially true.)

Lemma 6.7 gives

$$\begin{aligned}\text{(44)}\quad \theta_j^{3/2-a-1}\|(\psi_j u)\circ\chi_j\|_{H^{m+1}(\mathbb{R}^3_+)} &\le C\theta_j^{3/2-a-1}\big(\theta_j^2\|(P\,\psi_j u)\circ\chi_j\|_{H^{m-1}(\mathbb{R}^3_+)}+\\ &\quad \theta^{m_k}\|(L_k\,\psi_j u)\circ\chi_j\|_{H^{m+1/2-m_k}(\mathbb{R}^2)} + \|u\circ\chi_j\|_{H^m(\mathbb{R}^3_+)}\big)\\ &\le C\theta_j^{3/2-a-1}\big(\theta_j^2\|(\psi_j Pu)\circ\chi_j\|_{H^{m-1}(\mathbb{R}^3_+)} + \theta_j^2\|([P,\psi_j]u)\circ\chi_j\|_{H^{m-1}(\mathbb{R}^3_+)}\\ &\quad + \theta^{m_k}\|(\psi_j L_k u)\circ\chi_j\|_{H^{m+1/2-m_k}(\mathbb{R}^2)} + \theta^{m_k}\|([L_k,\psi_j]u)\circ\chi_j\|_{H^{m+1/2-m_k}(\mathbb{R}^2)}\\ &\quad + \|u\circ\chi_j\|_{H^m(\mathbb{R}^3_+)}\big)\\ &\le C\theta_j^{3/2-a-1}\big(\theta_j^2\|(\psi_j Pu)\circ\chi_j\|_{H^{m-1}(\mathbb{R}^3_+)} + \theta^{m_k}\|(\psi_j L_k u)\circ\chi_j\|_{H^{m+1/2-m_k}(\mathbb{R}^2)}\\ &\quad + \|u\circ\chi_j\|_{H^m(\mathbb{R}^3_+)}\big)\end{aligned}$$

We now observe that $r_\Omega(x)$ is comparable to θ_j for x in the support of ψ_j, so that combining (43) with (44) (after we similarly square that relation), we conclude by Proposition 5.8 that

$$\|u\|_{\mathcal{K}^{m+1}_{a+1}(\Omega)} \le C\,\Big(\|P\,u\|_{\mathcal{K}^{m-1}_{a-1}(\Omega)} + \sum_k \|L_k u\|_{\mathcal{K}^{m+1/2-m_k}_{a-1}(D_k)} + \|u\|_{\mathcal{K}^m_{a+1}(\Omega)}\Big).$$

for any $u\in\mathcal{K}^1_{a+1}(\Omega)$ and a constant C that depends on m, a, and Ω. □

Remark 7.3. The last theorem could be improved in the following direction. Sometimes a vertex Q touches a smooth part of the boundary, like in the left picture of Figure 1. In that case, we could take the canonical weight (r_D or r_Ω) to be $O(1)$ on the smooth side of the boundary near Q. This additional generality, to be treated in a forthcoming paper [44], would make our presentation significantly more complicated with only minimum gain.

Proposition 6.9 and Theorem 7.2 give right away the following corollary.

Corollary 7.4. *Assume P is coercive and Ω is compact, then P satisfies $\mathcal{K}_a^m(\Omega)$ regularity for all $m \geq 1$ and $a \in \mathbb{R}$.*

8. Well-posedness for strictly positive operators

Let $\Omega \subset \mathbb{R}^3$ be a domain with a polyhedral structure. We assume that we have decomposed the oriented boundary $\partial^\nu\Omega$ of Ω as a disjoint union $\partial^\nu\Omega = \partial_N\Omega \cup \partial_D\Omega$, with $\partial_N\Omega$ and $\partial_D\Omega$ unions of oriented faces of Ω. In this section, we shall assume that P is strictly positive (see Definition 6.11).

We shall consider for each non-negative integer m the equation $[P; L_k]_m(u) = (f, g_N, g_D)$. For $m \geq 1$, this is simply the boundary value problem

$$
\begin{cases} Pu \;=\; f \in \mathcal{K}_{a-1}^{m-1}(\Omega) & \text{in } \Omega \\ u \;=\; g_D \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D\Omega) & \text{on } \partial_D\Omega \\ D_\nu^P u \;=\; g_N \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_D\Omega) & \text{on } \partial_N\Omega. \end{cases} \tag{45}
$$

Recall that for $m = 0$, $[P; L_k]_m(u)$ is defined in a weak sense. We have the following result.

Theorem 8.1. *Let $m \in \mathbb{Z}_+$ and let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a polyhedral structure. Assume P is strictly positive. Then there exists $\eta > 0$ such that the boundary value problem (45) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)^\mu$ for any $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu$, any $g_D \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D\Omega)^\mu$, any $g_N \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)^\mu$, and any $|a| < \eta$. This solution depends continuously on f, g_D, and g_N.*

If $m = a = 0$ and $g_D = 0$, this solutions is the solution of the associated variational problem, which is obtained from equation (33).

Proof. First, let us notice that the surjectivity of the trace map (Theorem 5.9) allows us to reduce the proof to the case when $g_D = 0$.

Recall that $\mathcal{H}_a := \{u \in \mathcal{K}_{a+1}^1(\Omega), \quad u = 0 \text{ on } \partial_D\Omega\}$. Thus we will look for solutions $u \in \mathcal{H}_a$. The maps $[P; L_k]_m$ of Equations (31) and (34) restrict to maps $\mathcal{K}_{a+1}^{m+1}(\Omega)^\mu \cap \mathcal{H}_a \to \mathcal{K}_{a-1}^{m-1}(\Omega)^\mu \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)$, respectively $\mathcal{H}_a \to \mathcal{H}_{-a}^*$ for $m = 0$. We shall denote all these maps by $\tilde{P}_{m,a}$.

Our assumption that P is strictly positive implies that there exists $\gamma > 0$ such that

$$\big(Pu, u\big) = B_P(u, u) \geq \gamma \|u\|^2_{\mathcal{K}_1^1(\Omega)}$$

for any $u \in \mathcal{H}_0$. In particular, B_P satisfies the assumptions of the Lax-Milgram lemma, and hence $\tilde{P}_{0,0} = B_P^* : \mathcal{H}_0 \to \mathcal{H}_0^*$ is an isomorphism. This proves the result for $m = 0$ and $a = 0$.

Recall the function r_Ω introduced in 14. The operators $r_\Omega^{-a}\tilde{P}_{m,a}r_\Omega^a$ will all act on the same space and by Proposition 5.3, these operators depend continuously on a. Let $m = a = 0$. Then $r_\Omega^{-a}\tilde{P}_{m,a}r_\Omega^a = P_{0,0}$ is an isomorphism, and hence we can conclude that there exists $\eta > 0$ such that $\tilde{P}_{0,a}$ is an isomorphism for any $|a| < \eta$.

We now prove the result for $|a| < \eta$ and $m \in \mathbb{Z}_+$ arbitrary. Corollary 7.4 then gives that

$$\tilde{P} : \mathcal{K}_{a+1}^{m+2}(\Omega)^\mu \cap \mathcal{H}_a \to \mathcal{K}_{a-1}^m(\Omega)^\mu$$

is surjective. Since this map is also continuous (Proposition 5.3) and injective (from the case $m = 0$), it is an isomorphism by the open mapping theorem. The proof is now complete. $\square$

9. Poincaré 's inequality and applications to the Laplace operator

In this section we will establish a weighted Poincaré inequality from which we will obtain solvability of some mixed boundary value problems for the Laplace operator Δ. We continue to assume that $\Omega \subset \mathbb{R}^3$ is a domain with a polyhedral structure, and we continue to decompose the oriented boundary of Ω into two disjoint sets

$$\partial^\nu \Omega = \partial_D \Omega \cup \partial_N \Omega \tag{46}$$

with $\partial_D \Omega$ a union of closed faces. In particular, we assume that $\partial_D \Omega \neq \emptyset$.

Precisely, we will show that $P = \alpha^2/r_\Omega(x)^2 - \Delta$, $\alpha > 0$ large enough, with arbitrary mixed (Dirichlet or Neumann) boundary conditions, or $P = -\Delta$ in the case that $\partial_N \Omega$ contains no adjacent faces, satisfy the conditions of Theorem 8.1. The fact that P is strictly positive will be a consequence of a weighted form of Poincaré inequality.

9.1. **Poincaré inequality.** We begin with some prelinaries lemmas that settle the case of domains with polygonal structure.

With a little abuse of notation, we shall write $u(r, \theta) := u(r\cos\theta, r\sin\theta)$ for a function $u(x_1, x_2)$ expressed in polar coordinates. Below, $dx = dx_1 dx_2 \dots dx_n$, $n = 2$ or 3. The following two lemmas are standard. See [12] or [55], for example.

Lemma 9.1. *Let $\mathcal{C} = \mathcal{C}_{R,\alpha} := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2, 0 < r < R, 0 < \theta < \alpha\}$, $0 < \alpha \geq 2\pi$. Then*

$$\int_{\mathcal{C}} \frac{|u|^2}{r^2}\, dx \leq \left(\frac{2\alpha}{\pi}\right)^2 \int_{\mathcal{C}} \frac{|\partial_\theta u|^2}{r^2}\, dx \leq \left(\frac{2\alpha}{\pi}\right)^2 \int_{\mathcal{C}} |\nabla u|^2\, dx$$

for any u, $\nabla u \in L^2(\mathcal{C})$ satisfying $u(r, \theta) = 0$ if $\theta = 0$, in the trace sense.

Observe that no condition is imposed on $u(r, \alpha)$ in the above lemma and that we allow $\alpha = 2\pi$, but we distinguish the limit $\theta \to 0^+$ from the limit $\theta \to 2\pi^-$ in order to consider domains with cracks.

In [12], a weighted Poincaré inequality for a curvilinear polygon was derived from Lemma 9.1. In fact, that proof applies to a general domain D with polygonal structure provided the oriented boundary $\partial^\nu D$ is used in place of the usual boundary ∂D.

Lemma 9.2. *Let $D \subset \mathbb{R}^2$ be a domain with a polygonal structure. Let $r_D(z)$ be the canonical weight function on D and let $\partial_D D$ be a non-empty closed subset of $\partial^\nu D$ such that $\partial_N D := \partial^\nu D \setminus \partial_D D$ is a union of oriented open sides of D, no two of which are adjacent. Then there exists a constant $C_D > 0$ such that*

$$\|u\|^2_{\mathcal{K}^0_1(D)} := \int_D \frac{|u(z)|^2}{r_D(x)^2}\, dz \leq C_D \int_D |\nabla u(z)|^2 dz$$

for any $u \in H^1(D)$ satisfying $u = 0$ on $\partial_D D$.

We will clarify further the need for the oriented boundary in the proof of Poincaré inequality for domains in $\mathbb{R}^3$ in Theorem 9.6.

We will also need a version of Lemma 9.1 in spherical coordinates. The proof exploits the usual, unweighted Poincaré's inequality on domains on the unit sphere.

We include a proof, since domains with polygonal structure are not extension domains for the standard Sobolev spaces when cracks are present. This observation holds also for domains with polyhedral structure. Recall that a domain $\Omega \in \mathbb{R}^n$ is called an *extension domain* for $H^s(\Omega)$ if there exists a bounded extension operator from $H^s(\Omega)$ to $H^s(\mathbb{R}^n)$ (see e.g. [72]).

For simplicity, we state and prove Poincaré's inequality for planar domains in $\mathbb{R}^2$. The same proof applies for domains on the sphere, where ∇ is to be intended as the covariant derivative.

Lemma 9.3. *Let $D \subset \mathbb{R}^2$ be a connected, compact domain with polygonal structure. Let $u \in H^1(D)$ be such that $u \equiv 0$ in trace sense on a set A of positive measure in ∂D. Then, there exists a positive constant $c = c_D$ independent of u, such that*

$$\|u\|_{L^2(D)} \leq c_D \, \|\nabla u\|_{L^2(D)}. \tag{47}$$

Proof. We begin by establishing a version of Rellich's Theorem for domains with a polygonal structure that will be used in the course of the proof, that is, we show that the imbedding $H^1(D) \hookrightarrow L^2(D)$ is compact. To this effect, we write D as a finite union of Lipschitz domains $D_1, \dots, D_N$, following Lemma 3.5. Let $\{u_n\} \subset H^1(D)$, $\|u_n\|_{H^1(D)} \leq 1$, $\forall n$. On each subdomain D_j, Rellich's Theorem applies. Therefore, by a diagonal argument, we can construct a subsequence u_{n_k} such that its restriction to D_j converges to a function u_j strongly in $L^2(D_j)$, $j = 1, \dots, N$. By the uniqueness of the limit, $u_j = u_k$ on $D_k \cap D_j$, so that the u_j extend to a function u on D such that $u_{n_k} \to u$ strongly in $L^2(D)$ as $k \to \infty$.

Next, we let $\mathcal{A} = \{u \in H^1(D),\ u|_A \equiv 0\}$. By changing the trace of u on sets of measure zero in ∂D, we can always set $u(x) = 0$, $\forall x \in A$. Assume by contradiction that (47) does not hold. Then, for each $n \in \mathbb{N}$, there exists $u_n \in \mathcal{A}$, $\|u_n\|_{H^1(D)} = 1$, such that

$$\|\nabla u_n\|_{L^2(D)} \leq \frac{1}{n} \|u_n\|_{L^2(D)} \leq \frac{1}{n}. \tag{48}$$

By the Banach-Alaoglu Theorem, there exists a subsequence u_{n_k} converging weakly to u in $H^1(D)$. In particular, $\nabla u_{n_k} \to \nabla u$ weakly in $L^2(D)$. By Rellich's Theorem, $u_{n_k} \to u$ strongly in $L^2(D)$. From (48), it follows then that $\|\nabla u_{n_k}\|_{L^2(D)}$ must converge to 0. Therefore, $\nabla u = 0$ in $L^2(D)$, so that u is constant in D. As $\|u_n\|_{H^1(D)} = 1$ by construction, $\|u\|_{L^2} \neq 0$. Finally, since the trace operator is weakly continuous in H^1, we must also have $u|_{\partial D} = \lim_{k\to\infty} u_{n_k}|_{\partial D}$ weakly in $L^2(\partial D)$. In particular, $u|_A \equiv 0$. But $u \in H^1(D)$ and constant on D, hence $u \equiv 0$ on the whole of D, a contradiction. □

We can now state and prove our last preliminary lemmas. We will write $x = (\rho, x')$, $x' \in S^2 = \partial B^3$, in spherical coordinates in $\mathbb{R}^3$, and again with abuse of notation set $u(\rho, \omega) = u(x_1, x_2, x_3)$. Recall that the canonical weight is comparable to ρ near a vertex away from any edge, and it is comparable to the distance to the edge close to it. We address these cases in two separate lemmas.

Lemma 9.4. *Let ω be a domain with polygonal structure on the unit sphere. Let $\mathcal{C} = \mathcal{C}_{R,\omega} = \{(\rho, x'),\ 0 < \rho < R,\ x' \in \omega\}$. Then, if $u, \nabla u \in L^2(\mathcal{C})$, and $u(R, \omega) = 0$ on a subset of positive measure on $\partial\omega$ in trace sense, then*

$$\int_{\mathcal{C}} \frac{|u|^2}{\rho^2} dV \leq C \int_{\mathcal{C}} |\nabla_{x'} u(\rho, x')|^2 dS d\rho \leq C \int_{\mathcal{C}} |\nabla u(x)|^2 dV,$$

where $\nabla_{x'}$ is the covariant derivative on S^2.

Proof. Let (φ, θ) be spherical coordinates on ω. Then, the standard Poincaré inequality on the sphere gives:

$$\int_\omega |u|^2\, dS \leq C \int_\omega \left(u_\varphi^2 + \frac{1}{\sin^2 \varphi} u_\theta^2 \right) \sin\varphi\, d\varphi d\theta := C \int_\omega |\nabla_{x'} u|^2\, dS.$$

Then, we have

$$\begin{aligned} (49) \qquad \int_{\mathcal{C}} \frac{|u|^2}{\rho^2} dV &= \int_0^R \int_\omega |u|^2 dS d\rho \leq C \int_0^R \int_\omega \left(u_\varphi^2 + \frac{1}{\sin^2\varphi} u_\theta^2 \right) dS d\rho \\ &\leq C \int_0^R \int_\omega \left(u_\rho^2 + \frac{1}{\rho^2} u_\varphi^2 + \frac{1}{\rho^2 \sin^2\varphi} u_\theta^2 \right) \rho^2 dS d\rho = C \int_{\mathcal{C}} |\nabla u(x)|^2 dV, \end{aligned}$$

since $|\nabla u|^2 = u_\rho^2 + \dfrac{1}{\rho^2} u_\varphi^2 + \dfrac{1}{\rho^2 \sin^2\varphi} u_\theta^2.$ □

We observe that in the most extreme case, ω is the sphere S^2 with a segment removed. Its oriented boundary consists of two copies of this segment, and it is enough for u to vanish in trace sense on one copy. More precisely, it is enough that u vanishes on a one-sided, non-tangential approach to the boundary.

We last replace ρ in the above lemma with the canonical weight, which is comparable to the distance from the singular set, as it is this function that appears in the definition of the weighted Sobolev spaces.

Lemma 9.5. *Assume that the hypotheses of Lemma 9.4 hold, and assume further that $0 < R < \epsilon$, where ϵ is sufficiently small. Then, if $r_{\mathcal{C}}$ is the canonical weight associated to $\mathcal{C} = \mathcal{C}_{R,\omega}$,*

$$(50) \qquad \|u\|_{\mathcal{K}_1^0(\mathcal{C})} := \int_{\mathcal{C}} \frac{|u|^2}{r_{\mathcal{C}}^2} dV \leq C \int_{\mathcal{C}} |\nabla u(x)|^2 dV.$$

Proof. We continue to denote by (ρ, φ, θ) spherical coordinates on $\mathcal{C}$. For each vertex A of ω, we construct an open set $U_A \subset \mathcal{C}$ as follows. By the definition of a domain with polygonal structure, there is an open set $V_A \subset S^2$ and a diffeomorphism ϕ_A such that $\phi_A(\omega \cap V_A) = \{(r, \theta),\ r \in (0, \delta),\ \theta \in \varpi_A\}$, where ϖ_A is a disjoint, finite union of intervals in $[-1, 1]$, and δ is independent of A. We let then $U_A = \{(\rho, x'),\ 0 < \rho < R,\ x' \in \omega \cap V_A\}$, and write $U = \mathcal{C} \setminus \bigcup_A U_A$. On U, we observe that $C^{-1}\rho \leq r_{\mathcal{C}} \leq C\rho$ for some positive constant C, since ϵ is small, and use Lemma 9.4:

$$\int_U \frac{|u|^2}{r_{\mathcal{C}^2}} dV \leq C \int_{\mathcal{C}} \frac{|u|^2}{\rho^2} dV \leq C \int_{\mathcal{C}} |\nabla u(x)|^2 dV.$$

Next we parameterize each U_A by the coordinates (ρ, r, θ), $0 < \rho < R$, $0 < r < \delta$, $\theta \in \varpi_A$. If $\epsilon > \delta$ are chosen small enough, then on U_A we have $C^{-1} r\rho \leq r_{\mathcal{C}} \leq C\, r\rho$, with C uniform in A, so that applying Lemma 9.2 to the domain $\phi_A(\omega \cap V_A)$

$$\int_{U_A} \frac{|u(x)|^2}{r_\Omega^2}\, dV \le C \int_0^R \int_{\phi_A(\omega\cap V_A)} \frac{|u(\rho, r, \theta)|^2}{\rho^2 r^2}\, drd\theta\, d\rho$$
$$\le C \int_0^R \int_{\phi_A(\omega\cap V_A)} \frac{|\nabla_{(r,\theta)} u|^2}{\rho^2}\, drd\theta\, d\rho$$
$$\le C' \int_0^R \int_{\omega\cap V_A} \frac{1}{\rho^2}\left(u_\phi^2 + \frac{1}{\sin^2\phi} u_\theta^2\right) d\phi d\theta\, d\rho$$
$$\le C' \int_0^R \int_{\omega\cap V_A} \left[u_\rho^2 + \frac{1}{\rho^2}\left(u_\varphi^2 + \frac{1}{\sin^2\varphi} u_\theta^2\right)\right] d\varphi d\theta\, d\rho$$
$$\le C' \int_0^R \int_{\omega\cap V_A} |\nabla u|^2\, dV.$$

By summing all the inequality for U_A and for U, we finally obtain (50). □

We now turn to the proof of the desired weighted Poincaré inequality in three dimensions. The proof reduces to the case covered by Lemmas 9.1, 9.4 and 9.5, by applying the diffeomorphism ϕ_p of Definition 3.4.

Again, we denote with $\partial_D\Omega$ the part of the oriented boundary $\partial^\nu\Omega$ with Dirichlet conditions, and recall that a dihedral angle is a cone in our notation.

Theorem 9.6. *Assume that $\overline{\Omega}$ is compact or a cone. Assume also that $\partial_D\Omega \neq \emptyset$ and that $\partial_N\Omega = \partial^\nu\Omega \setminus \partial_D\Omega$ is a union of open oriented faces of Ω, no two of which have an edge in common. Then there exists C_Ω, depending only on Ω, such that*

$$\|u\|^2_{\mathcal{K}_1^0(\Omega)} := \int_\Omega \frac{|u(x)|^2}{r_\Omega^2}\, dx \le C_\Omega \int_\Omega |\nabla u(x)|^2\, dx\,,$$

for any $u \in H^1(\Omega)$ that satisfies $u = 0$ on $\partial_D\Omega$.

Proof. The idea of the proof is to cover the domain Ω with a finite number of open sets $\widetilde{\Omega}$ on which the integration simplifies and we can reduce our proof to the usual Poincaré inequality. The result will follow by adding the corresponding inequalities. Also, by Theorem 5.9 and standard density arguments, we can assume that u is a smooth function.

We shall write $dx = dV = dx_1dx_2dx_3$ for the volume element. Also, recall that r_Ω is the canonical weight function, which is comparable to $\vartheta(x)$, the distance from $x \in \overline{\Omega}$ to the edges of Ω.

Let us consider the inequality

$$\|u\|^2_{\mathcal{K}_1^0(\widetilde{\Omega})} := \int_{\widetilde{\Omega}} \frac{|u(x)|^2}{r_\Omega(x)^2}\, dV \le C \int_{\widetilde{\Omega}} |\nabla u(x)|^2\, dV, \quad u = 0 \text{ on } \partial_D\Omega \tag{51}$$

for subdomains $\widetilde{\Omega} \subset \Omega$. The statement of the theorem is exactly the inequality (51) for $\widetilde{\Omega} = \Omega$. The proof of our inequality for Ω will be obtained by adding the inequality (51) for suitable subdomains $\widetilde{\Omega}$. These domains will be either of the form $V_p \cap \Omega$ or a single interior domains that is at a positive distance from the edges. On the domain that is away from the edges, we can use the usual Poincaré inequality. If $\tilde{\Omega} = V_p \cap \Omega$, we notice that the inequality (51) is preserved under the change of coordinates ϕ_p with constants uniform in p. V_p is the open neighborhood of a point $p \in \partial^\nu\Omega$ of Definition 3.4

Depending whether p is a vertex Q or belongs to an edge e, we denote

$$\Omega_Q = \phi_Q(V_Q \cap \Omega), \qquad \Omega_e = \phi_p(V_p \cap \Omega).$$

With abuse of notation, we still denote a point in the transformed space by $x = (x_1, x_2, x_3)$. By modifying slightly the definition of ϕ_p, we can take Ω_Q to satisfy all the hypotheses of Lemma 9.5 uniformly in Q, while Ω_e can be characterized in suitable cylindrical coordinates, by

$$\Omega_e = \{(r, \theta, z),\ 0 < r < \delta,\ 0 < \theta < \theta_e,\ 0 < z < z_e := |e| - 2\epsilon\},$$

where $|e|$ is the length of the curve $\phi_p(e)$, and ϵ, δ are chosen small enough uniformly in e. Moreover, r is the distance to the singular set on Ω_e. In these cylindrical coordinates, the image of the edge e corresponds to the the z-axis (in particular, $r = 0$ on e).

For $\widetilde{\Omega}_Q = \phi_Q^{-1}(\Omega_Q)$ we use Lemma 9.5. In fact, by hypothesis $u = 0$ on at least one side of the oriented boundary $\partial^\nu \widetilde{\Omega}_Q$, so that $u \circ \phi_Q^{-1} = 0$ on on at least one side of $\partial^\nu \Omega_Q$. Therefore, (50) holds for Ω_Q, and hence for $\widetilde{\Omega}_Q$ with uniform constants, by making a change of variables.

We next prove the inequality (51) for $\widetilde{\Omega}_e = \phi_p^{-1}(\Omega_e)$. We write $\Omega_e = W_\delta \times (0, z_e)$, where $W_\delta = \{0 < r < \delta,\ 0 < \theta < \theta_e\}$ in suitable cylindrical coordinates (r, θ, z). The hypothesis on $\partial\Omega \backslash \partial_D \Omega$ imply that in trace sense $u(r, 0, z) = 0$ or $u(r, \theta_e, z) = 0$, and we can always arrange the coordinate system locally in Ω_e so that $u(r, 0, z) = 0$. It is also crucial that on $\widetilde{\Omega}_e$, $c^{-1} r \leq r_\Omega \leq C r$, with C uniform in e, since r is exactly the distance to the singular set on Ω_e. By Fubini's Theorem and Lemma 9.1, we have:

$$\begin{aligned}
&\int_{\widetilde{\Omega}_e} \frac{|u|^2}{r_\Omega}\, dV \leq C_\Omega \int_{\Omega_e} \frac{|u \circ \phi_p^{-1}|^2}{r^2}\, dx_1 dx_2 dx_3 \\
&= C_\Omega \int_0^{z_e} \int_{W_\delta} \frac{|u \circ \phi_p^{-1}|^2}{r}\, dr d\theta dz \leq C_\Omega \int_0^{z_e} \int_{W_\delta} \left(\frac{|\partial_\theta (u \circ \phi_p^{-1})|^2}{r} \right) dr d\theta dz \\
&\leq C'_\Omega \int_0^{z_e} \int_{W_\delta} |\nabla_{x_1, x_2} (u \circ \phi_p^{-1})|^2\, dx_1 dx_2 \qquad (52) \\
&\leq C'_\Omega \int_0^{z_e} \int_{W_\delta} \left(|\nabla_{x_1, x_2} (u \circ \phi_p^{-1})|^2 + |\partial_z (u \circ \phi_p^{-1})|^2 \right) dx_1 dx_2 dz \\
&= C'_\Omega \int_{\Omega_e} |\nabla (u \circ \phi_p^{-1})|^2 dx_1 dx_2 dx_3 \leq C''_\Omega \int_{\widetilde{\Omega}_e} |\nabla u|^2 dV,
\end{aligned}$$

which is exactly (51) with $\widetilde{\Omega}_e = \widetilde{\Omega}$.

Finally, we add the inequalities (52) for all $\widetilde{\Omega} = \widetilde{\Omega}_e$ and the inequalities (50) for all $\widetilde{\Omega} = \widetilde{\Omega}_Q$, and the usual Poincaré inequality for

$$U = \Omega \setminus \bigcup_{e,Q} (\Omega_Q \cup \Omega_e),$$

with ϵ and δ replaced by $\epsilon/2$,$\delta/2$, where Q ranges over all the vertices and e ranges over all edges of Ω. This concludes the proof for u smooth, and hence by density for $u \in H^1(\Omega)$. $\square$

From the above Poincaré type inequalities we obtain the following corollary from [12].

Corollary 9.7. *Let Ω be a domain with a polyhedral structure and assume that $\partial_D\Omega \neq \emptyset$, that $\partial_N\Omega \subset \partial^\nu\Omega$ is a union of open oriented faces of Ω no two of which have an edge in common. Then the norms $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_{\mathcal{K}^1_1(\Omega)}$, and the seminorm $|\cdot|_{H^1(\Omega)}$ are equivalent on $H^1_D(\Omega) = \{u \in \mathcal{K}^1_1(\Omega), u = 0 \text{ on } \partial_D\Omega\}$. In particular, $H^1_0(\Omega) = \overset{\circ}{\mathcal{K}^1_1}(\Omega)$.*

Proof. The weighted Poincaré inequality of Theorem 9.6 immediately gives that for $u \in H^1_D(\Omega)$, $\|u\|_{\mathcal{K}^1_1} \leq C\,|u|_{H^1} \leq C\,\|u\|_{H^1}$. The reverse inequality follows observing that $r_\Omega(x) \geq 1/\alpha$, for some large enough constant $\alpha > 0$, so that the $\mathcal{K}^1_1$ norm is stronger than the H^1 norm. □

9.2. **Well-posedness and regularity for the Laplace operator.** Let Δ be the Laplace operator on $\mathbb{R}^3$. Here, we illustrate an application of the results of the previous sections to the solvability of the mixed boundary value problem (45) ($Pu = f$ in Ω, $u = g_D$ on $\partial_D\Omega$ and $u = g_N$ on $\partial_N\Omega$) for $P = -\Delta$ and $P = \alpha^2/r_\Omega(x)^2 - \Delta$. Recall that $r_\Omega(x)$ is comparable to $\vartheta(x)$, but $r_\Omega \in C^\infty(\Sigma\Omega)$.

We first observe that the Laplace operator with Dirichlet or Neumann boundary conditions satisfies the assumption of regularity upon freezing the coefficients of Definition 6.6 (see, e.g. [65] Proposition 11.12), and therefore full regularity in weighted Sobolev spaces follows from Theorem 7.2.

We now set $P = \alpha^2/r_\Omega(x)^2 - \Delta$, with α a positive constant such that

$$\alpha/r_\Omega(x) \geq 1, \qquad \forall x \in \overline{\Omega}, \tag{53}$$

(in fact, we can always arrange that $r_\Omega \leq 1$) and impose arbitrary mixed (Dirichlet or Neumann) boundary conditions on the faces D_k of Ω. Then,

$$B_P(u,u) = \|u/r_\Omega\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \equiv \|u\|_{\mathcal{K}^1_1(\Omega)},$$

so that P is positive on $\mathcal{K}^1_1(\Omega)$, which correspond in our notation to $a = 0$.

Next, we set $P = -\Delta$ and impose again Dirichlet boundary conditions on $\partial_D\Omega$ and Neumann boundary conditions on $\partial_N\Omega$, but we assume in addition that $\partial_N\Omega$ contains no adjacent faces. Then, from the weighted Poincaré inequality 51 we obtain that

$$B_P(u,u) = \|\nabla u\|^2_{L^2(\Omega)} \geq C(\Omega)\,\|u/r_\Omega\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \equiv \|u\|_{\mathcal{K}^1_1(\Omega)},$$

that is, P is positive on $\mathcal{K}^1_1(\Omega)$.

Therefore, by applying Theorem 8.1, we obtain solvability of the boundary value problem (45) on $\mathcal{K}^{a+1}_{m+1}$ for any $m \geq 0$ and $|a|$ small enough. We summarize this result in the following theorem.

Theorem 9.8. *Let $m \in \mathbb{Z}_+$ and let $\Omega \subset \mathbb{R}^3$ be a domain with a polyhedral structure. Let $P = \alpha^2/r_\Omega(x)^2 - \Delta$ with arbitrary mixed (Dirichlet or Neumann) boundary conditions or $P = -\Delta$, but such that $\partial_D\Omega \neq \emptyset$ and that $\partial_N\Omega$ contains no adjacent faces. Then there exists $\eta > 0$ such that the boundary value problem* (3) *has a unique solution $u \in \mathcal{K}^{m+1}_{a+1}(\Omega)^\mu$ for any $f \in \mathcal{K}^{m-1}_{a-1}(\Omega)^q$, any $g_D \in \mathcal{K}^{m+1/2}_{a+1/2}(\partial_D\Omega)^q$, any $g_N \in \mathcal{K}^{m-1/2}_{a-1/2}(\partial_N\Omega)^q$, and any $|a| < \eta$. This solution depends continuously on f, g_D, and g_N.*

This result is in contrast to the case of (unweighted) Sobolev spaces $H^m(\Omega)$, for which solvability of the boundary value problem holds only for a finite range of values of m. For example, if Ω is a polygon in $\mathbb{R}^2$, the Poisson's equation with homogeneous Dirichlet boundary conditions and L^2 data admits a solution in $H^2(\Omega)$ if Ω is convex, but the solution may fail to be H^2 otherwise. For a proof we refer in the vast literature on the subject to [36] and also to the book by Grisvard [31].

10. A weighted Korn's inequality and applications to linear elasticity

The purpose of this section is to highlight the applicability of our main result Theorem 8.1 to systems of uniformly elliptic operators, specifically those appearing in linear elasticity. Singular domains, and especially domains with cracks, arise naturally in elasticity, as a result of large applied stresses.

We first derive a weighted form of Korn's inequality, which will imply that under certain conditions on the elasticity parameters, the matrix operator $P = \operatorname{div}(\mathbf{C} \cdot \nabla)$ of linear elasticity is positive on $\mathcal{K}_1^1(\Omega)$. The assumptions on Ω and its oriented boundary $\partial^\nu \Omega$ of the previous section continue to hold here. We then study the regularity and solvability of the boundary value problem (45) on $\mathcal{K}_{m+1}^{a+1}(\Omega)$.

We begin by introducing the equations of linear elasticity and the elasticity tensor.

10.1. **Linear elasticity.** Linear elastostatics is modeled by the following 3×3 system of equations:

$$(P_{\mathbf{C}}\, u)_i := \sum_{j,k,l=1}^{3} \partial_{x_j} \big(\mathsf{C}_{ijkl}\, \partial_{x_l} u_k \big) := \operatorname{div}(\mathbf{C} \cdot \nabla u)_i = f_i, \quad i = 1, 2, 3, \tag{54}$$

for an unknown vector field $u = (u_1, u_2, u_3)$ on Ω, where f are volume forces. For convenience, throughout this section, we use the convention of summation over repeated indices.

Above, Ω represents a bounded elastic body in $\mathbb{R}^3$ and u is the displacement at the point x under the elastic deformation of Ω by the given volume forces f. $\mathbf{C} = [\mathsf{C}_{ijkl}]$ is a fourth-order tensor field, called the *elasticity tensor*, with coefficients $\mathsf{C}_{ijkl} \in C^\infty(\overline{\Omega})$ and symmetries:

$$\mathsf{C}_{ijkl} = \mathsf{C}_{klij} = \mathsf{C}_{jikl} = \mathsf{C}_{ijlk}.$$

The components of $\mathbf{C}$, sometimes referred to as *elastic moduli*, encode the elastic response of the medium to deformations. For example, if a material is isotropic, then $\mathbf{C}$ has the simple form:

$$\mathsf{C}^{iso}_{ijkl} \;=\; \lambda(x)\delta_{ij}\delta_{kl} \;+\; \mu(x)\big[\delta_{ik}\delta_{jl} \;+\; \delta_{il}\delta_{jk}\big]$$

with λ, μ the so-called *Lamé parameters*, and δ is the Kronecker symbol $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. In general, there can be up to 21 independent elastic moduli.

The system (54) is obtained from the laws of balance of energy and momentum by linearising around an unperturbed or natural state of the elastic body, and therefore holds in the regime of small deformations. (For a derivation, we refer for example to [46]). The principal part of $P_{\mathbf{C}}$ is the matrix operator

$$P = \mathbf{C}[\partial, \partial] = [\mathsf{C}_{ijkl}\, \partial_{x_j} \partial_{x_k}],$$

We will consider two types of boundary conditions: Dirichlet or *displacement* boundary conditions $u = g$ on $\partial_D \Omega$, and natural or *traction* boundary conditions

$(\partial_\nu^P u)_i = (\nu \cdot (\mathsf{C} \cdot \nabla u)_i := \nu_j\, \mathsf{C}^{ijkl}\, \partial_l u_k = (g_N)_i$, $i = 1, 2, 3$, on $\partial_N^\nu(\Omega)$. The traction $\nu \cdot (\mathsf{C} \cdot \nabla u)$ is the normal vector component of the stress at the boundary.

Remark 10.1. In domains with cracks, linear elastostatics can model small perturbations around cracks in equilibrium. If the crack is a material surface, and not just a surface of discontinuity for the displacement and the stress, then the laws of balance of momentum implies that the traction has to be continuous across the crack . Therefore, the most physically motivated boundary problems in this case are transmission problems, which however we do not consider in this paper.

Definition 10.2. The elasticity tensor C is called (uniformly) strongly elliptic if there is a constant $c > 0$ such that, for any $x \in \Omega$, $\mathsf{C}_{ijkl}(x) V^i W^j V^k W^l \geq c|\mathbf{V}|^2|\mathbf{W}|^2$ for all vectors $\mathbf{V}, \mathbf{W}$ in the tangent space $T_x\Omega$. Similarly, C is called positive-definite if , for any $x \in \Omega$, $\mathsf{C}_{ijkl}(x) E^{ij} E^{kl} \geq 0$ for any *symmetric* two-tensor $E = [E^{ij}]$ in $T_x^2\Omega$.

In the isotropic case, C is strongly elliptic if, and only if, $\mu > 0$ and $\lambda + 2\mu > 0$, while C is positive definite if, and only if, $\mu > 0$ and $2\mu + 3\lambda > 0$. In practice, it is not too restrictive to assume that C is positive definite, since this assumption implies that the elastic stored energy function is positive and convex in the strain (see again [46]). If C is a positive-definite elasticity tensor, then the elasticity operator $\mathsf{C}[\partial, \partial]$ with mixed displacement-traction boundary conditions is positive in the sense of Definition 6.11, provided $\partial_N(\Omega)$ does not contains two adjiacent faces. This result is a consequence of Korn's inequality and will be proved in Lemma 10.5 below.

10.2. **A weighted Korn's inequality.** For $u \in H^1_{\text{loc}}(\Omega)^3$, we shall denote by $\epsilon(u)$ the *deformation tensor*

$$\epsilon_{ij}(u) := (\partial_i u_j + \partial_j u_i)/2. \tag{55}$$

We have $\epsilon(u) := (\epsilon_{ij}(u)) \in L^2_{\text{loc}}(\Omega)^9$ and set $\|\epsilon(u)\|^2_{L^2(\Omega)} := \sum_{ij} \int_\Omega |\epsilon_{ij}(u)|^2 dx$.

The usual Korn's inequality states that as a system $\epsilon(u)$ is H^1- regular (that is, coercive) or even positive, depending on the boundary conditions. It is in fact a special case of Gärding inequality for systems.

If $\partial\Omega$ is smooth, this result is classical (see e.g., [30], [25], and again [46] with references therein). We will use here Korn's inequality for Lipschitz domains [60], [13].

Proposition 10.3. *Let Ω be a connected, bounded, Lipschitz domain and $u \in H^1_{\text{loc}}(\Omega)^3$. There exists $C > 0$, depending only on Ω, such that*

$$\|u\|^2_{H^1(\Omega)} \leq C\big(\|\epsilon(u)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big) \tag{56}$$

for all $u \in H^1(\Omega)^3$. Assume $\partial_D\Omega \neq \emptyset$. Then there exists $C > 0$, again depending only on Ω, such that

$$\|u\|^2_{H^1(\Omega)} := \sum_{ij} \int_\Omega |\partial_i u_j|^2\, dx \leq C\|\epsilon(u)\|^2_{L^2(\Omega)} \tag{57}$$

for all $u \in H^1_{\text{loc}}(\Omega)^3$ that satisfy $u = 0$ on $\partial_D\Omega$.

Inequality (57) is referred to as "first Korn's inequality", while (56) is "second Korn's inequality". Second Korn's inequality can be equivalently formulated as

$$|u|_{H^1(\Omega)} \leq C\|\epsilon(u)\|_{L^2(\Omega)},$$

for all $u \in H^1(\Omega)^3$ that satisfy $\int_\Omega (\partial_i u_j - \partial_j u_i) dx = 0$ for all i, j.

Combining Proposition 10.3 with the weighted Poincaré inequality of Theorem 9.6, we obtain the following corollary.

Corollary 10.4. *Let Ω be a domain with polyhedral structure. Assume $\overline{\Omega}$ is compact, $\partial_D \Omega \neq \emptyset$, and $\partial_N \Omega$ is a union of open faces, no two of which are adjacent. Then, there exists $C = C(\Omega) > 0$ such that*

$$\begin{aligned} \|u\|^2_{\mathcal{K}^1_1(\Omega)} := \int_\Omega \frac{|u(x)|^2}{r_\Omega(x)^2}\, dx + \int_\Omega |\nabla u(x)|^2\, dx + \\ \leq C\, \|\epsilon(u)\|^2_{L^2(\Omega)} \end{aligned} \tag{58}$$

for all $u \in \mathcal{K}^1_1(\Omega)^3$ that satisfies $u = 0$ on $\partial_D \Omega$.

Proof. We begin by writing Ω as a finite disjoint union of connected, bounded Lipschitz domains Ω_j, $j = 1, \ldots, N$, as in Lemma 3.5. In each domain Ω_j, inequality (57) holds. Then, from the weighted Poincaré inequality we have:

$$\begin{aligned} \|u\|^2_{\mathcal{K}^1_1(\Omega)} &\leq C\, \|\nabla u\|^2_{L^2(\Omega)} \\ &\leq C\, \|u\|^2_{H^1(\Omega)} = \sum_j \|u\|^2_{H^1(\Omega_j)} \\ &\leq \widetilde{C} \sum_j \|\epsilon(u)\|^2_{L^2(\Omega_j)} = \widetilde{C}\, \|\epsilon(u)\|^2_{L^2(\Omega)}, \end{aligned}$$

since the domains Ω_j are disjoint. □

10.3. **Solvability and regularity for the elasticity system.** In this section, we apply the weighted Korn's inequality to study the existence, uniqueness, and regularity of solutions to the elasticity system (54) with mixed traction-displacement boundary conditions.

Lemma 10.5. *If C is a positive-definite elasticity tensor, the bilinear form B_P of equation* (28)

$$B_P(u, v) = \int_\Omega \nabla u \cdot \mathsf{C} \cdot \nabla v \text{ dvol} = \int_\Omega \mathsf{C}^{ijkl}(x) \partial_j\, u_i(x)\, \partial_l\, u_k(x)\, dx,$$

is coercive on $H^1(\Omega)$, more precisely:

$$B_P(u, u) \geq C_\Omega\, \|\epsilon(u)\|^2_{L^2(\Omega)}. \tag{59}$$

Proof. We fix $x \in \Omega$, and let $F = (\nabla u)(x)$, a 3×3 matrix. We denote by E its symmetric part: $E = (\nabla u(x) + \nabla u(x)^T)/2$, a symmetric matrix. From the symmetry properties of C and the assumption that C is positive definite, we have

$$\mathsf{C}^{ijkl}(x)\, F_{ij} F_{lk} = \mathsf{C}^{ijkl}(x)\, E_{ij} E_{lk} \geq c\, |E|^2.$$

But, by hypothesis c is uniform in $x \in \Omega$, so that (59) follows by integrating over Ω. □

By combining the above lemma with the inequality (58), we immediately have that then the elasticity operator $-\mathsf{C}[\partial, \partial]$ with mixed-boundary conditions is positive in the sense of Definition 6.11, provided no two adjacent faces of Ω have traction boundary condtions:

$$B_P(u, u) \geq C_\Omega\, \|\epsilon(u)\|^2_{L^2(\Omega)} \geq C'_\Omega \|u\|^2_{\mathcal{K}^1_1(\Omega)}, \qquad u = 0 \text{ on } \partial_D \Omega \neq \emptyset.$$

Similarly, by combining the above lemma with the inequality (56) instead, we can conclude that $\alpha^2/r_\Omega(x)^2 - \operatorname{div}(\mathbf{C}\cdot\nabla)$, where α satisfies again (53), is positive in the sense of Definition 6.11, with arbitrary displacement-traction boundary conditions:

$$\alpha^2 \|u/r_\Omega(x)\|^2_{L^2(\Omega)} + B_P(u,u) \geq C_\Omega \|\epsilon(u)\|^2_{L^2(\Omega)} + \|u/r_\Omega(x)\|_{L^2(\Omega)}$$
$$\geq C_\Omega \|\nabla(u)\|^2_{L^2(\Omega)} + \|u/r_\Omega(x)\|^2_{L^2(\Omega)} \leq C'_\Omega \|u\|^2_{\mathcal{K}^1_1(\Omega)}.$$

Above we have used that $\alpha^2 \|u/r_\Omega(x)\|_{L^2(\Omega)} \geq \|u\|_{L^2(\Omega)}$ and that the bilinear form of equation (28) for the operator $1/r_\Omega(x)^2 - \operatorname{div}(\mathbf{C}\cdot\nabla)$ is exactly $\|u/r_\Omega(x)\|^2_{L^2(\Omega)} + B_P(u,u)$, provided $\partial_D\Omega \neq \emptyset$.

Finally, solvability and regularity for the boundary value problem follows by applying Theorem 8.1. We summarize these results in the following theorem.

Theorem 10.6. *Let $m \in \mathbb{Z}_+$ and let $\Omega \subset \mathbb{R}^3$ be a domain with a polyhedral structure. Let $\mathbf{C}$ be a positive-definite elasticity tensors. Set $P = 1/r_\Omega(x)^2 - \operatorname{div}(\mathbf{C}\cdot\nabla)$ with arbitrary mixed (displacement or traction) boundary conditions or $P = -\operatorname{div}(\mathbf{C}\cdot\nabla)$, but such that $\partial_D\Omega \neq \emptyset$ and that $\partial_N\Omega$ contains no adjacent faces. Then there exists $\eta > 0$ such that the boundary value problem (3) has a unique solution $u \in \mathcal{K}^{m+1}_{a+1}(\Omega)^\mu$ for any $f \in \mathcal{K}^{m-1}_{a-1}(\Omega)^q$, any $g_D \in \mathcal{K}^{m+1/2}_{a+1/2}(\partial_D\Omega)^q$, any $g_N \in \mathcal{K}^{m-1/2}_{a-1/2}(\partial_N\Omega)^q$,and any $|a| < \eta$. This solution depends continuously on f, g_D, and g_N.*

References

[1] B. Ammann, A. Ionescu, and V. Nistor. Sobolev spaces on Lie manifolds and regularity for polyhedral domains. Preprint, math.ap/0402321, MSRI, February 2004. New title. Old title: *Sobolev spaces on Lie manifolds and on polyhedral domains*.

[2] B. Ammann and V. Nistor. Weighted sobolev spaces and regularity for polyhedral domains. IMN Preprint, January 2006.

[3] T. Apel and S. Nicaise. The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges. *Math. Methods Appl. Sci.*, 21(6):519–549, 1998.

[4] Thomas Apel, Anna-Margarete Sändig, and Sergey I. Solov′ev. Computation of 3D vertex singularities for linear elasticity: error estimates for a finite element method on graded meshes. *M2AN Math. Model. Numer. Anal.*, 36(6):1043–1070 (2003), 2002.

[5] D. Arnold and R. Falk. Well-posedness of the fundamental boundary value problems for constrained anisotropic elastic materials. *Arch. Rational Mech. Anal.*, 98(2):143–165, 1987.

[6] D. N. Arnold, R. S. Falk, and R. Winther. Multigrid preconditioning in $H(\mathrm{div})$ on non-convex polygons. *Comput. Appl. Math.*, 17(3):303–315, 1998.

[7] I. Babuška. The rate of convergence for the finite element method. *SIAM J. Numer. Anal.*, 8:304–315, 1971.

[8] I. Babuška and A. K. Aziz. Survey lectures on the mathematical foundations of the finite element method. In *The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972)*, pages 1–359. Academic Press, New York, 1972. With the collaboration of G. Fix and R. B. Kellogg.

[9] I. Babuška and M. Rosenzweig. A finite element scheme for domains with corners. *Numer. Math.*, 20:1–21, 1972/73.

[10] I. Babuška and S. L. Sobolev. Optimization of numerical processes. *Apl. Mat.*, 10:96–129, 1965.

[11] I. Babuška and V. Nistor. Boundary value problems in spaces of distributions and their numerical investigation. Work in progress.

[12] Constantin Bacuta, Victor Nistor, and Ludmil T. Zikatanov. Improving the rate of convergence of high-order finite elements on polyhedra. I. A priori estimates. *Numer. Funct. Anal. Optim.*, 26(6):613–639, 2005.

[13] James H. Bramble. A proof of the inf-sup condition for the Stokes equations on Lipschitz domains. *Math. Models Methods Appl. Sci.*, 13(3):361–371, 2003. Dedicated to Jim Douglas, Jr. on the occasion of his 75th birthday.

[14] S. Brenner and R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2002.

[15] C. Băcuţă, V. Nistor, and L. Zikatanov. Boundary value problems and regularity on polyhedral domains. IMA preprint #1984, August 2004.

[16] A. Buffa, M. Costabel, and M. Dauge. Anisotropic regularity results for Laplace and Maxwell operators in a polyhedron. *C. R. Math. Acad. Sci. Paris*, 336(7):565–570, 2003.

[17] M. Cheeger, J. Gromov and M. Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17(1):15–53, 1982.

[18] M. Costabel and M. Dauge. Singularities of electromagnetic fields in polyhedral domains. *Arch. Ration. Mech. Anal.*, 151(3):221–276, 2000.

[19] M. Costabel, M. Dauge, and S. Nicaise. Corner singularities of Maxwell interface and eddy current problems. In *Operator theoretical methods and applications to mathematical physics*, volume 147 of *Oper. Theory Adv. Appl.*, pages 241–256. Birkhäuser, Basel, 2004.

[20] Martin Costabel and Monique Dauge. Computation of corner singularities in linear elasticity. In *Boundary value problems and integral equations in nonsmooth domains (Luminy, 1993)*, volume 167 of *Lecture Notes in Pure and Appl. Math.*, pages 59–68. Dekker, New York, 1995.

[21] Martin Costabel and Monique Dauge. Crack singularities for general elliptic systems. *Math. Nachr.*, 235:29–49, 2002.

[22] M. Dauge. *Elliptic boundary value problems on corner domains*, volume 1341 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988. Smoothness and asymptotics of solutions.

[23] M. Dauge. Singularities of corner problems and problems of corner singularities. In *Actes du 30ème Congrès d'Analyse Numérique: CANum '98 (Arles, 1998)*, volume 6 of *ESAIM Proc.*, pages 19–40 (electronic). Soc. Math. Appl. Indust., Paris, 1999.

[24] R. Duduchava and D. Natroshvili. Mixed crack type problem in anisotropic elasticity. *Math. Nachr.*, 191:83–107, 1998.

[25] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics*. Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[26] L.C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[27] Richard S. Falk. Nonconforming finite element methods for the equations of linear elasticity. *Math. Comp.*, 57(196):529–550, 1991.

[28] Richard S. Falk and John E. Osborn. Remarks on mixed finite element methods for problems with rough coefficients. *Math. Comp.*, 62(205):1–19, 1994.

[29] G. Fichera. *Linear elliptic differential systems and eigenvalue problems*, volume 8 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1965.

[30] K. O. Friedrichs. On the boundary-value problems of the theory of elasticity and Korn's inequality. *Ann. of Math. (2)*, 48:441–471, 1947.

[31] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[32] P. Grisvard. Singularités en elasticité. *Arch. Rational Mech. Anal.*, 107(2):157–180, 1989.

[33] P. Grisvard. *Singularities in boundary value problems*, volume 22 of *Research in Applied Mathematics*. Masson, Paris, 1992.

[34] B. Guo and I. Babuška. Regularity of the solutions for elliptic problems on nonsmooth domains in $\mathbf{R}^3$. I. Countably normed spaces on polyhedral domains. *Proc. Roy. Soc. Edinburgh Sect. A*, 127(1):77–126, 1997.

[35] D. Jerison and C.E. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130(1):161–219, 1995.

[36] David S. Jerison and Carlos E. Kenig. Boundary value problems on Lipschitz domains. In *Studies in partial differential equations*, volume 23 of *MAA Stud. Math.*, pages 1–68. Math. Assoc. America, Washington, DC, 1982.

[37] David Kapanadze and B.-Wolfgang Schulze. *Crack theory and edge singularities*, volume 561 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 2003.

[38] B Kellogg and M. Stynes. Corner singularities and boundary layers in a simple convection-diffusion problem. *J. Differential Equations*, 213(1):81–120, 2005.
[39] R. Kellogg and J. Osborn. A regularity result for the Stokes problem in a convex polygon. *J. Functional Analysis*, 21(4):397–431, 1976.
[40] V. A. Kondrat′ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Transl. Moscow Math. Soc.*, 16:227–313, 1967.
[41] V. Kozlov, V. Maz′ya, and J. Rossmann. *Spectral problems associated with corner singularities of solutions to elliptic equations*, volume 85 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.
[42] V. A. Kozlov and V. G. Maz′ya. On stress singularities near the boundary of a polygonal crack. *Proc. Roy. Soc. Edinburgh Sect. A*, 117(1-2):31–37, 1991.
[43] P. Laasonen. On the degre of convergence of discrete approximations for the solutions of the Dirichlet problem. *Ann. Acad. Sci. Fenn. Ser. A. I.*, 1957(246):19, 1957.
[44] H. Li, A. L. Mazzucato, and V. Nistor. Some issue related to the analysis and implementation of the finite element methods on more pathological polygonal domains. In preparation.
[45] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1.* Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.
[46] Jerrold E. Marsden and Thomas J. R. Hughes. *Mathematical foundations of elasticity.* Dover Publications Inc., New York, 1994. Corrected reprint of the 1983 original.
[47] W. G. Mazja, S. A. Nasarow, and B. A. Plamenewski. *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten. I*, volume 82 of *Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien [Mathematical Textbooks and Monographs, Part II: Mathematical Monographs].* Akademie-Verlag, Berlin, 1991. Störungen isolierter Randsingularitäten. [Perturbations of isolated boundary singularities].
[48] V. Maz'ya and J. Roßmann. Weighted L_p estimates of solutions to boundary value problems for second order elliptic systems in polyhedral domains. *ZAMM Z. Angew. Math. Mech.*, 83(7):435–467, 2003.
[49] V. Maz'ya and J. Roßmann. Regularity in polyhedral cones. Preprint, 2004.
[50] R. Mazzeo. Elliptic theory of differential edge operators. I. *Commun. Partial Differ. Equations*, 16(10):1615–1664, 1991.
[51] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.
[52] R.B. Melrose. *Geometric scattering theory.* Stanford Lectures. Cambridge University Press, Cambridge, 1995.
[53] M. Mitrea and M. Taylor. Boundary layer methods for Lipschitz domains in Riemannian manifolds. *J. Funct. Anal.*, 163(2):181–251, 1999.
[54] C. B. Morrey, Jr. Second-order elliptic systems of differential equations. In *Contributions to the theory of partial differential equations*, Annals of Mathematics Studies, no. 33, pages 101–159. Princeton University Press, Princeton, N. J., 1954.
[55] S. Nazarov and B. Plamenevsky. *Elliptic problems in domains with piecewise smooth boundaries*, volume 13 of *de Gruyter Expositions in Mathematics.* Walter de Gruyter & Co., Berlin, 1994.
[56] S. Nicaise. Regularity of the solutions of elliptic systems in polyhedral domains. *Bull. Belg. Math. Soc. Simon Stevin*, 4(3):411–429, 1997.
[57] S. Nicaise and A.-M. Sändig. Transmission problems for the Laplace and elasticity operators: regularity and boundary integral formulation. *Math. Models Methods Appl. Sci.*, 9(6):855–898, 1999.
[58] Serge Nicaise and Anna-Margarete Sändig. Transmission problems for the Laplace and elasticity operators: regularity and boundary integral formulation. *Math. Models Methods Appl. Sci.*, 9(6):855–898, 1999.
[59] L. Nirenberg. Estimates and existence of solutions of elliptic equations. *Comm. Pure Appl. Math.*, 9:509–529, 1956.
[60] J. A. Nitsche. On Korn's second inequality. *RAIRO Anal. Numér.*, 15(3):237–248, 1981.
[61] Anna-Margarete Sändig, Uwe Richter, and Rainer Sändig. The regularity of boundary value problems for the Lamé equations in a polygonal domain. *Rostock. Math. Kolloq.*, (36):21–50, 1989.

[62] E. Schrohe and B.-W. Schulze. Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities. I. In *Pseudo-differential calculus and mathematical physics*, volume 5 of *Math. Top.*, pages 97–209. Akademie Verlag, Berlin, 1994.

[63] M.A. Shubin. Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, 207:5, 35–108, 1992. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).

[64] L. Skrzypczak. Mapping properties of pseudodifferential operators on manifolds with bounded geometry. *J. London Math. Soc. (2)*, 57(3):721–738, 1998.

[65] M. Taylor. *Pseudodifferential operators*, volume 34 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1981 xi+452.

[66] M. Taylor. *Partial differential equations I, Basic theory*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995.

[67] H. Triebel. Characterizations of function spaces on a complete Riemannian manifold with bounded geometry. *Math. Nachr.*, 130:321–346, 1987.

[68] M. I. Vishik. Les opérateurs différentiels et pseudo-différentiels à une infinité de variables, les problèmes elliptiques et paraboliques. In *Colloque International CNRS sur les Équations aux Dérivées Partielles Linéaires (Univ. Paris-Sud, Orsay, 1972)*, pages 342–362. Astérisque, 2 et 3. Soc. Math. France, Paris, 1973.

[69] L. Wahlbin. On the sharpness of certain local estimates for $\mathring{H}^1$ projections into finite element spaces: influence of a re-entrant corner. *Math. Comp.*, 42(165):1–8, 1984.

[70] L. Wahlbin. Local behavior in finite element methods. In *Handbook of numerical analysis, Vol. II*, Handb. Numer. Anal., II, pages 353–522. North-Holland, Amsterdam, 1991.

[71] Hassler Whitney. Tangents to an analytic variety. *Ann. of Math. (2)*, 81:496–549, 1965.

[72] William P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

Pennsylvania State University, Math. Dept., University Park, PA 16802, USA
E-mail address: `mazzucat@math.psu.edu`

Pennsylvania State University, Math. Dept., University Park, PA 16802, USA
E-mail address: `nistor@math.psu.edu`