

ON THE STABILITY RADIUS OF GENERALIZED STATE-SPACE SYSTEMS

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Abstract. The concept of “distance to instability” of a system matrix is generalized to system *pencils* which arise in descriptor (semi-state) systems. Difficulties arise in the case of singular systems, because the pencil can be made unstable by an infinitesimal perturbation. It is necessary to measure the distance subject to restricted, or structured, perturbations. In this paper a suitable measure for the stability radius of a generalized state-space system is defined, and a computable expression for the distance to instability is derived for regular pencils of index less than or equal to one. For systems which are strongly controllable it is shown that this measure is related to the sensitivity of the poles of the system over all feedback matrices assigning the poles.

Key words. control, singular systems, descriptor systems, semi-state, stability, matrix pencil.

1. Introduction. Robustness, or insensitivity to perturbations, is an essential property of a control system design. In frequency response analysis the concept of “stability margin” has long been in use as a measure of the size of plant disturbances or model uncertainties that can be tolerated before a system loses stability. Recently a state-space approach has been established for measuring the “distance to instability” or “stability radius” of a linear multivariable system, and numerical methods for computing this measure have been derived [2,7,8,11]. This measure has also been related to the sensitivity of the poles of the system and to the “margin of stability” defined in the frequency domain [10].

In this paper we extend the concept of distance to instability to system *pencils* which arise in descriptor or generalized state-space systems described by implicit differential-algebraic equations. In Section 2 the distance measure is defined and notation is presented. A computable expression for the distance is derived in Section 3 for regular pencils of index less than or equal to one. Detailed proofs of the results which we previously reported in [3] are given. In Section 4 it is shown that for systems which are strongly controllable, the distance measure is related to the sensitivity of the poles of the system with respect to all feedback matrices assigning the poles. In Section 5 different classes of perturbations are discussed, and conclusions are given.

2. Distance to Instability – Definitions and Notations. We consider the linear time-invariant system

$$(1) \quad E\dot{x} = Ax + Bu$$

where $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $\text{rank}(B) = p$, and $q \equiv \text{rank}(E) \leq n$. The system (1) is said to be *solvable* if and only if there exists a unique solution for any given sufficiently differentiable control function $u(t)$ and any given admissible initial conditions corresponding to an admissible $u(t)$ [4,13]. It has been shown [4] that systems (1) is *solvable* if and only if the system pencil $(\alpha A - \beta E)$ is *regular*, that is, for some $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$

$$(2) \quad \det(\alpha A - \beta E) \neq 0.$$

(A pencil which is not regular is called *singular*).

For a regular system pencil, the solutions to (1) can be characterized in terms of the eigenstructure of the pencil. The generalized eigenvalues are defined by the pairs $(\alpha_j, \beta_j) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

$$(3) \quad \det(\alpha_j A - \beta_j E) = 0, \quad j = 1, 2, 3, \dots$$

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Without loss of generality, it can be assumed that $|\alpha_j|^2 + |\beta_j|^2 = 1$. If $\alpha_j \neq 0$, then $\lambda_j = \beta_j/\alpha_j$ is a *finite* eigenvalue and if $\alpha_j = 0$, then $\lambda_j \sim \infty$ is an *infinite* eigenvalue of the system. The right and left generalized eigenvectors and principal vectors are given by the columns of the non-singular matrices $X = [X_r, X_\infty]$ and $Y = [Y_r, Y_\infty]$ (respectively) which transform the pencil into the Kronecker Canonical Form (KCF)

$$(4) \quad Y^T A X = \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix}, Y^T E X = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}$$

where J is the $r \times r$ Jordan matrix associated with the $r \leq q$ finite eigenvalues of the pencil and N is the nilpotent Jordan matrix corresponding to the $n - r$ infinite eigenvalues [5]. The degree of nilpotency of N , that is, the smallest non-negative integer m such that $N^m = 0$, is called the *index* of the system. (The index $m = 0$ if N is empty.)

A simple example of a regular index one system is given by the differential-algebraic equations

$$(5) \quad \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u,$$

where A_{22} and $E_{11} - E_{12}A_{22}^{-1}A_{21}$ are square and of full rank. The first block row of equations describes the dynamic behaviour of the system, while the second block row gives algebraic constraints on the states. Such systems arise, for example, where path constraints are imposed on the dynamic response.

For a regular system, the solution to (1) is given explicitly in terms of the KCF [4,13] by

$$(6) \quad x(t) = X \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = X_r z_1(t) + X_\infty z_2(t),$$

where

$$\begin{aligned} z_1(t) &= e^{tJ} z_1(0) + \int_0^t e^{(t-s)J} Y_r^T B u(s) ds, \\ z_2(t) &= - \sum_{i=0}^{m-1} d^i (N^i Y_\infty^T B u) / dt^i \end{aligned}$$

and $u(t) \in \Omega$, the set of admissible controls. Here Ω is the set of real-valued, measurable p -dimensional vector functions $u(t)$ such that $d^i (N^i Y_\infty^T B u) / dt^i$ exists and is continuous for $i = 0, 1, \dots, m-1$.

The set of admissible initial conditions for the system (1) is defined to be

$$(7) \quad \varphi(A, E) \equiv \left\{ X \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \left| z_1 \in \mathbb{R}^r, z_2 = - \sum_{i=0}^{m-1} d^i (N^i Y_\infty^T B u) / dt^i \right|_{t=0}, u \in \Omega \right\}.$$

The set φ contains the null vector $\underline{0}$ and is a subspace. It has been shown [13] that the complete set of reachable states of the system (1) from the set φ of initial conditions is equal to φ . Thus, for any control $u(t) \in \Omega$, if the initial state of the system is admissible, then the response of the system $x(t)$ remains in $\varphi \forall t$. We refer to $\varphi(A, E)$ as the solution space of the system (1).

From (6) it can be seen that with $u \equiv 0 \forall t$, the response $x(t)$ of the system converges to a position of stable equilibrium at the origin, i.e. $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for *any* admissible $x(0)$, if and only if the finite eigenvalues of the system all lie in the left half of the complex plane. We make the following definition.

DEFINITION 2.1. *If the pencil $(\alpha A - \beta E)$ is regular and its $r \leq q$ finite eigenvalues $\lambda_j = \beta_j/\alpha_j$ satisfy $Re(\lambda_j) < 0$, $j = 1, 2, 3, \dots, r$, then the pencil is (asymptotically) stable. Otherwise it is unstable.*

For a *standard* stable system (with $E = I$), the distance to instability, or radius of stability, is measured in terms of the minimum perturbation δA to the matrix A required to make the perturbed system unstable [2,7,8,11]. For descriptor systems this definition is not immediately applicable. If we consider perturbations $(\delta A, \delta E)$ to the system pair (A, E) , it is easy to see from (4) that an infinitesimal perturbation to the nilpotent part of the pencil can change its eigenstructure and the solution space of the system.

To illustrate this, consider the system (of type (5))

$$(8) \quad \begin{aligned} \dot{x}_1 &= -2x_1 + x_3 \\ \dot{x}_2 &= -x_2 + x_4 \\ x_3 &= -u_1 \\ x_4 &= -u_2 \end{aligned}$$

with system matrices

$$(9) \quad A = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The system (8) is regular, index one and stable with two finite eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$. If we introduce the perturbations $(\delta A, \delta E)$, where

$$(10) \quad \delta A = 0, \quad \delta E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & \epsilon_2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\epsilon_1 > 0$, $\epsilon_2 = 0$, then the perturbed pencil $(\alpha(A + \delta A) - \beta(E + \delta E))$ is still regular and index one, but it has *three* finite eigenvalues, $\lambda_1 = -2$, $\lambda_2 = -1$ at $\lambda_3 = 1/\epsilon_1 > 0$ and is clearly unstable for any positive value of ϵ_1 . The solution of the perturbed system has more degrees of freedom than the original system, and the admissible initial conditions are altered. If the perturbation (10) is introduced, but we let $\epsilon_1 = 0$ and $\epsilon_2 > 0$, then the perturbed pencil remains regular, but has index equal to two. The solution space of the system is altered and the admissible controls must be smoother. In both cases the perturbation causes the algebraic constraint to become differential.

The allowable perturbations are, however, limited in reality by the physical structure of the system. For index one systems of form (5), for example, where the algebraic equations represent path constraints, the zero blocks in the matrix E are structural and are not subject to disturbances or uncertainty. Infinitesimal perturbations which cause the algebraic constraints to become differential thus have no physical interpretation and may be excluded.

For a meaningful definition of the “radius of stability” of a generalized state-space system, it is necessary, therefore, to restrict the allowable set of perturbations. If we exclude perturbations which alter the nilpotent part of the pencil, then the finite eigenvalues of the perturbed pencil depend continuously on $(\delta A, \delta E)$ and the “distance to instability” of the pencil can be measured in terms of the minimum perturbation required for a finite eigenvalue to move to the imaginary axis (compactified by adding the point at infinity), or for the pencil to lose regularity.

In practice, it is reasonable to allow only perturbations such that the system remains solvable for the same fixed class of admissible controls. This is ensured if the nilpotent structure of the

pencil is preserved, or more specifically, if in the KCF (4), the nilpotent Jordan matrix N and the corresponding left invariant space spanned by the rows of Y_∞^T are both preserved under the allowable perturbations. For systems of the type (5) such restrictions exclude perturbations which cause the algebraic constraints to become differential. This is a natural limitation, as we have indicated. Allowable perturbations can, nevertheless lead to systems which are unstable, of different index or not regular. Such perturbations are in general, however, of positive magnitude.

As a brief illustration, we consider the system (8) subject to perturbations $(\delta A, \delta E)$ where

$$(11) \quad \delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \delta E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_1 & \epsilon_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For small values of the parameters $\tau, \epsilon_1, \epsilon_2$, these perturbations do not affect the nilpotent structure of the system. For larger values of the parameters, however, the perturbations can alter the nilpotency of the pencil or cause it to become unstable or to lose regularity. If, for instance, we select $\epsilon_1 = \epsilon - 2 = 0$, then in the limit as $\tau \rightarrow 1$, a finite eigenvalue moves to the imaginary axis and the system becomes unstable. If $\tau = \epsilon_2 = 0$ then as $\epsilon_1 \rightarrow -1$, a finite eigenvalue becomes infinite and, in the limit, the nilpotent structure is changed, although the system is still of index one. Similarly, if $\tau = 0$ and $\epsilon_2 = -\epsilon_1$, then as $\epsilon_1 \rightarrow -1$ a finite eigenvalue becomes infinite, but in this case, the index of the system is increased to two. Finally, if $\epsilon_2 = 0$ and $\epsilon_1 = -\tau$, then in the limit as $\tau \rightarrow 1$, the system loses regularity.

Motivated by these examples we define the distance $\rho(A, E)$ from the pencil $(\alpha A - \beta E)$ to the “nearest” unstable pencil, as the minimum perturbation which causes the pencil to become unstable, to change its nilpotent structure or to lose regularity, measured over a class $\overline{\mathcal{D}}(A, E)$ of allowable perturbations. To make the definition more precise we introduce the following notation. We denote the pencil $(\alpha A - \beta E)$ by (A, E) and the set of *unstable* (complex) pencils by \mathcal{U}_n ; that is

$$(12) \quad \mathcal{U}_n \equiv \left\{ (A, E) \left| \begin{array}{l} A, E \in \mathbb{C}^{n \times n}, (A, E) \text{ is regular, and there exists } \alpha, \beta \in \mathbb{C} \text{ with} \\ \alpha \neq 0 \text{ such that } \operatorname{Re}(\beta/\alpha) \geq 0 \text{ and } \det(\alpha A - \beta E) = 0 \end{array} \right. \right\}.$$

We denote the nilpotent structure of the pencil (A, E) by $\operatorname{nil}(A, E)$, where the nilpotent structure specifically refers to the nilpotent Jordan matrix N of the KCF (4) and to the corresponding left invariant space spanned by the rows of Y_∞^T . We define the set $\mathcal{D} = \mathcal{D}(A, E)$ by

$$(13) \quad \mathcal{D} = \left\{ (\delta A, \delta E) \left| \begin{array}{l} \delta A, \delta E \in \mathbb{C}^{n \times n}, (A + \delta A, E + \delta E) \text{ is regular} \\ \text{and } \operatorname{nil}(A + \delta A, E + \delta E) = \operatorname{nil}(A, E) \end{array} \right. \right\},$$

and define its closure $\overline{\mathcal{D}} = \overline{\mathcal{D}}(A, E)$ to be the set of allowable perturbations. We remark that boundary perturbations $(\delta A, \delta E) \in \overline{\mathcal{D}} \setminus \mathcal{D}$ alter the nilpotency or the regularity (or both) of the pencil. We now define the measure of distance to instability as follows.

DEFINITION 2.2. *The distance to instability or radius of stability of the stable regular pencil (A, E) is given by*

$$\rho(A, E) \equiv \inf_{(\delta A, \delta E) \in \overline{\mathcal{D}}(A, E)} \{ \|\delta A\|_F \|\delta E\|_F \mid (A + \delta A, E + \delta E) \in \mathcal{U}_n \text{ or } (\delta A, \delta E) \notin \mathcal{D}(A, E) \},$$

where $\|\cdot\|_F$ denotes the Frobenius norm and $\overline{\mathcal{D}}(A, E)$ denotes the closure of the set $\mathcal{D}(A, E)$ defined by (13).

It follows from Definition 2.2 that if $(\delta A, \delta E) \in \mathcal{D}(A, E)$ and $\|[\delta A | \delta E]\|_F < \rho(A, E)$, then $(A + \delta A, E + \delta E)$ is *stable*. We remark that $\rho(A, E)$ measures the distance to the nearest *complex* pencil which is unstable, has a different nilpotent structure or is not regular. In practice we may be interested only in *real* perturbations to the pencil. The measure $\rho(A, E)$ gives a lower bound for this case.

We conclude this section by observing that the measure $\rho(A, E)$ of distance to instability given by Definition 2 is invariant under unitary transformations of the system pencil. We have the following.

LEMMA 2.1. *Let P, Q be unitary matrices such that*

$$PAQ = \tilde{A}, \quad PEQ = \tilde{E}.$$

then

$$\rho(A, E) = \rho(\tilde{A}, \tilde{E}).$$

Proof. If $\delta \tilde{A} = P\delta A Q$ and $\delta \tilde{E} = P\delta E Q$, then

$$(\delta A, \delta E) \in \overline{\mathcal{D}}(A, E) \Leftrightarrow (\delta \tilde{A}, \delta \tilde{E}) \in \overline{\mathcal{D}}(\tilde{A}, \tilde{E}).$$

The result then follows from the invariance of the Frobenius norm under unitary transformations. \square

The results of the lemma are valuable in determining estimates for the stability radius. In the next section we derive a computable expression for the distance to instability $\rho(A, E)$ for systems which are of index at most one.

3. Regular Index One Systems. We now assume that (A, E) is a regular pencil of index less than or equal to one. Then (A, E) has precisely $q \equiv \text{rank}(E)$ finite eigenvalues and the nilpotent structure of the system pencil is given by $N = 0$ and $\mathcal{R}_R\{Y_\infty^T\} \equiv \mathcal{N}_L\{E\}$, where $\mathcal{R}_R\{\cdot\}$ and $\mathcal{N}_L\{\cdot\}$ denote the row space and left null spaces respectively.

As indicated in Section 2, regular systems of index one consist essentially of a set of dynamic equations with algebraic constraints on the state variables. For such systems the transient response is completely determined by the *finite* eigenstructure of the system alone, and a unique continuous solution exists for all continuous controls satisfying the initial consistency conditions. Moreover, the system can, in theory, be reduced to a *standard* system of dimension $q \leq n$ by eliminating the algebraic constraints. For higher index systems, if the control is not sufficiently smooth, impulses can arise in the response of the system and the system can lose causality [1, 12]. It is desirable therefore to design systems which are regular and of index at most one. Many descriptor systems can be transformed into systems of this type by state or output feedback. Simple conditions guaranteeing the existence of such feedback controls are given in [1, 9].

In order to derive a computable measure for the radius of stability we must obtain an explicit description of the set $\mathcal{D}(A, E)$ of allowable perturbations. We use the following result from [9] which characterizes regular systems of index at most one.

LEMMA 3.1. *The pencil (A, E) is regular and of index less than or equal to one if and only if*

$$(14) \quad \text{rank} \left(\begin{bmatrix} T_\infty^T A \\ E \end{bmatrix} \right) = n$$

where the rows of T_∞^T give an orthonormal basis for $\mathcal{N}_L\{E\}$.

From this result we can show the following

LEMMA 3.2. *If (A, E) is regular and of index less than or equal to one, then the set $\mathcal{D}(A, E)$ is equal to the set of all complex perturbations $(\delta A, \delta E)$ satisfying*

- (i) $\text{rank}(E + \delta E) = \text{rank}(E)$,
- (ii) $T_\infty^T \delta E = 0$,
- (iii) $\text{rank} \left(\begin{bmatrix} T_\infty^T(A + \delta A) \\ E + \delta E \end{bmatrix} \right) = n$,

where T_∞^T gives an orthonormal basis for $\mathcal{N}_L\{E\}$.

Proof. If (A, E) and $(A + \delta A, E + \delta E)$ are both regular pencils of index one, then $\text{nil}(A, E) = \text{nil}(A + \delta A, E + \delta A)$ if and only if E and $E + \delta E$ share a common left null space, that is

$$(15) \quad \mathcal{R}_R\{T_\infty^T\} \equiv \mathcal{N}_L\{E\} = \mathcal{N}_L\{E + \delta E\}.$$

Now if $(\delta A, \delta E) \in \mathcal{D}$, then by the definition of \mathcal{D} , the perturbed pencil $(A + \delta A, E + \delta E)$ must be regular and of index one and (15) must hold. This implies (i) and (ii), and Lemma 3.1 together with (15) implies (iii). Conversely if (i), (ii) and (iii) hold, then (15) holds and by Lemma 3.1 the perturbed pencil $(A + \delta A, E + \delta E)$ is regular and of index one. It then follows that $(\delta A, \delta E) \in \mathcal{D}$. \square

We have, furthermore, that the measure $\rho(A, E)$ is invariant under unitary transformations of the pencil by Lemma 2.1 and, therefore, it is sufficient to compute $\rho(A, E)$ only for a class of equivalent pencils. We have the following lemma.

LEMMA 3.3. *If (A, E) is regular and of index less than or equal to one, then there exist orthogonal matrices P, Q such that*

$$(16) \quad PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad PEQ = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}$$

where $\text{rank}(E_{11}, E_{12}) = \text{rank}(E) \equiv q$, $\text{rank}(A_{22}) = n - q$ and

$$\rho(A, E) = \rho \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \right).$$

Furthermore, $(\delta A, \delta E) \in \overline{\mathcal{D}}(A, E)$ implies

$$(17) \quad P\delta AQ = \begin{bmatrix} \delta A_{11} & \delta A_{12} \\ \delta A_{21} & \delta A_{22} \end{bmatrix}, \quad P\delta EQ = \begin{bmatrix} \delta E_{11} & \delta E_{12} \\ 0 & 0 \end{bmatrix}.$$

Proof. The proof is by construction. The orthogonal matrix P is obtained by the reduction of E to upper trapezoidal form (with the last $n - q$ rows equal to zero), using the QR algorithm [6]. Since the transformed pencil must also be regular and of index less than or equal to one, it follows from Lemma 3.1 that the last $n - q$ rows of PA must be linearly independent. A column permutation matrix Q can then be constructed to ensure that $\text{rank}(A_{22}) = n - q$. The rest of the proof follows from Lemma 3.2 and Lemma 2.1. \square

We remark that in transforming the system pencil to the reduced form (16) we do *not* alter the state variables of the system, but simply reorder them.

In order to evaluate $\rho(A, E)$ we may assume without loss of generality, that (A, E) is already in the partitioned form (16). We define for $\theta, \omega \in \mathbb{R}$, $\theta^2 + \omega^2 = 1$, the matrix function

$$(18) \quad H(\theta, \omega) = \begin{bmatrix} \theta A_{11} & \theta A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} i\omega E_{11} & i\omega E_{12} \\ 0 & 0 \end{bmatrix}.$$

We can now show that perturbations to $H(\theta, \omega)$ of the form

$$(19) \quad \Delta(\theta, \omega) = \begin{bmatrix} \theta \delta A_{11} & \theta \delta A_{12} \\ \delta A_{21} & \delta A_{22} \end{bmatrix} - \begin{bmatrix} i\omega \delta E_{11} & i\omega \delta E_{12} \\ 0 & 0 \end{bmatrix},$$

can cause the perturbed $H(\theta, \omega) + \Delta(\theta, \omega)$ to lose rank if and only if the perturbations $(\delta A, \delta E)$ (in partitioned form (17)), fail to satisfy the conditions of Lemma 3.2 or else cause the pencil (A, E) to become unstable. We can thus establish the following.

LEMMA 3.4. *If (A, E) is stable, regular and of index less than or equal to one, then*

$$(20) \quad \rho(A, E) = \hat{\rho} \equiv \inf_{\substack{\theta, \omega \in \mathbb{R} \\ \theta^2 + \omega^2 = 1 \\ T_\infty^T \delta E = 0}} \{ \|\delta A | \delta E\|_F | \text{rank}(H(\theta, \omega) + \Delta(\theta, \omega)) < n \}$$

where $H(\theta, \omega)$ is as in (18), $\Delta(\theta, \omega)$ is as in (19) and T_∞^T gives an orthonormal basis for $\mathcal{N}_L\{E\}$.

Proof. Without loss of generality we assume that (A, E) is in partitioned form (16) and that the perturbations $(\delta A, \delta E)$ are in partitioned form (17), that is, $T_\infty^T \delta E = 0$. Since (A, E) is stable, $\det(\lambda E - A) \neq 0$, and hence A is non-singular. Since (A, E) is also regular and has $\text{inex} \leq 1$, by Lemma 3.1

$$\text{rank} \begin{bmatrix} i\omega E_{11} & i\omega E_{12} \\ A_{21} & A_{22} \end{bmatrix} = n \quad \text{for } \omega \neq 0.$$

Therefore, $H(\theta, \omega)$ has full rank $\forall \theta, \omega \in \mathbb{R}$, $\theta^2 + \omega^2 = 1$. Furthermore, by the form of the perturbations, the matrix $H(\theta, \omega) + \Delta(\theta, \omega)$ has full rank if and only if

(i) $\theta = 0$, $\omega = 1$ and $\text{rank}(E + \delta E) = \text{rank}(E)$ and $(A + \delta A, E + \delta E)$ is regular, $\text{index} \leq 1$ (by Lemma 3.1):

or

(ii) $\theta \neq 0$ and

$$\det \left(\begin{bmatrix} I & 0 \\ 0 & \theta I \end{bmatrix} (H + \delta) \right) = \det(\theta(A + \delta A) - i\omega(E + \delta E)) \neq 0.$$

Hence, $H(\theta, \omega) + \Delta(\theta, \omega)$ has full rank for all θ if and only if $(\delta A, \delta E) \in \mathcal{D}(A, E)$ by Lemma 3.2, and $(A + \delta A, E + \delta E)$ has no finite eigenvalue with zero real part. It follows that for $\|\delta A | \delta E\|_F < \hat{\rho}$, $H(\theta, \omega) + \Delta(\theta, \omega)$ has full rank and hence $(\delta A, \delta E) \in \mathcal{D}(A, E)$ and, by continuity of the finite eigenvalues, $(A + \delta A, E + \delta E) \notin \mathcal{U}_n$. Therefore $\rho(A, E) \geq \hat{\rho}$.

To show equality we note that if $H(\theta, \omega) + \Delta(\theta, \omega)$ loses rank for $(\delta A, \delta E)$ and some $\theta \neq 0$ (but not for $\theta = 0$), then $(\delta A, \delta E) \in \mathcal{D}(A, E)$ and $(A + \delta A, E + \delta E)$ has a finite eigenvalue with zero real part; thus $(\delta A, \delta E)$ is an allowable destabilizing perturbation. If $H(\theta, \omega) + \Delta(\theta, \omega)$ loses rank for $(\delta A, \delta E)$ and $\theta = 0$, $\omega = 1$, then there exists a sequence of perturbations $\delta A_n \rightarrow \delta A$, $\delta E_n \rightarrow \delta E$ such that for $(\delta A_n, \delta E_n)$ the matrix $H(0, 1) + \Delta(0, 1)$ has full rank; therefore, $(\delta A_n, \delta E_n) \in \mathcal{D}$ and the limiting perturbation $(\delta A, \delta E) \in \overline{\mathcal{D}} \setminus \mathcal{D}$. In either case $\rho(A, E) \leq \|\delta A | \delta E\|_F$ and hence $\rho(A, E) \leq \hat{\rho}$, which proves the result. \square

Finally we can show that an equivalence exists between a perturbation $\hat{\Delta}$ which causes H to lose rank, and a perturbation $(\delta A, \delta E) \in \overline{\mathcal{D}}(A, E)$ such that $\|\delta A | \delta E\|_F = \|\hat{\Delta}\|_F$.

This gives us the main theorem.

THEOREM 3.1. *If (A, E) is stable, regular and of index less than or equal to one, then*

$$(21) \quad \rho(A, E) = \inf_{\substack{\theta, \omega \in \mathbb{R} \\ \theta^2 + \omega^2 = 1}} \sigma_{\min}\{H(\theta, \omega)\},$$

where $\sigma_{\min}\{\cdot\}$ denotes the smallest singular value, and $H(\theta, \omega)$ is given by (18).

Proof. The proof uses the well-known result [6] that if $\text{rank}(H) = n$, then the perturbed matrix $H + \hat{\Delta}$ has rank less than n only if $\|\hat{\Delta}\|_F \geq \sigma_{\min}\{H\}$, with equality for some $\hat{\Delta}$. (The perturbation $\hat{\Delta}$ which ensures equality can be found from the singular value decomposition of H , as shown in [6].)

We assume that (A, E) is already in partitioned form (16). Then, given any $\theta, \omega \in \mathbb{R}$, $\theta^2 + \omega^2 = 1$, and $\hat{\Delta}$ such that $\det(H(\theta, \omega) + \hat{\Delta}) = 0$, the perturbations

$$\begin{bmatrix} \delta A_{11} & \delta A_{12} \\ \delta A_{21} & \delta A_{22} \end{bmatrix} = \begin{bmatrix} \theta \hat{\Delta}_{11} & \theta \hat{\Delta}_{12} \\ \hat{\Delta}_{21} & \hat{\Delta}_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} \delta E_{11} & \delta E_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} i\omega \hat{\Delta}_{11} & i\omega \hat{\Delta}_{12} \\ 0 & 0 \end{bmatrix}$$

ensure that $T_\infty^T \delta E = 0$, $\text{rank}(H(\theta, \omega) + \Delta(\theta, \omega)) < n$, and $\|[\delta A | \delta E]\|_F = \|\hat{\Delta}\|_F$. This implies that $\hat{\rho}\{H(\theta, \omega)\}$, $\forall \theta, \omega$.

Conversely, given $(\delta A, \delta E)$ and $\theta, \omega \in \mathbb{R}$, $\theta^2 + \omega^2 = 1$ such that $\|[\delta A | \delta E]\|_F < \sigma_{\min}\{H(\theta, \omega)\}$, then, since

$$\begin{aligned} \|\hat{\Delta}\|_F^2 &= \sum_{j=1}^2 \|\theta \delta A_{1j} - i\omega \delta E_{1j}\|_F^2 + \|\delta A_{2j}\|_F^2 \\ &\leq \sum_{j=1}^2 \|[\delta A_{1j} | \delta E_{1j}]\|_F^2 \left\| \begin{bmatrix} \theta I & \\ & -i\omega I \end{bmatrix} \right\|_2^2 + \|\delta A_{2j}\|_F^2 = \|[\delta A | \delta E]\|_F^2, \end{aligned}$$

$H(\theta, \omega) + \Delta(\theta, \omega)$ must be of full rank. This implies $\hat{\rho} \geq \inf \sigma_{\min}\{H(\theta, \omega)\}$ over θ, ω .

The rest of the proof follows directly from Lemma 3.4. \square

A simple, but expensive, method for computing $\rho(A, E)$ is to apply a standard software library program to minimize the one parameter function

$$(22) \quad f(\alpha) \equiv \sigma_{\min}\{H(\cos(\alpha), \sin(\alpha))\}.$$

To illustrate the results of Theorem 3.5, we apply this technique to determine the radius of stability for the system of example (8). We find that $\rho(A, E) = .6180$. The minimizing perturbation, given to five figures by

$$\delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.27638 & 0 & .17082 \\ 0 & 0 & 0 & 0 \\ 0 & -0.44721 & 0 & -.27638 \end{bmatrix}, \quad \delta E = 0,$$

causes the pencil to become unstable with an eigenvalue at the origin.

Further results can be deduced directly from Theorem 3.1. For example, for systems of index zero, where $\text{rank}(E) = n$, the distance to instability is given immediately by the following corollary.

COROLLARY 3.1. *If (A, E) is a stable, regular pencil such that $\text{rank}(E) = n$, then*

$$(23) \quad \rho(A, E) = \inf_{\substack{\theta, \omega \in \mathbb{R} \\ \theta^2 + \omega^2 = 1}} \sigma_{\min}\{\theta A - i\omega E\}.$$

The distance to instability for regular systems of index less than or equal to one subject to perturbations which preserve regularity, index and the *right* null space of E can also be obtained from Theorem 3.1, simply by applying all the arguments to the transposed pencil (A^T, E^T) . This result is relevant in observer design.

4. Robust Stability for Closed Loop Systems. If the system (1), represented by the triple (A, E, B) , is not regular or is of index higher than one, then it is possible to construct a real feedback matrix K such that the *closed loop* system pencil $(A + BK, E)$ is regular and of index one, provided the infinite poles of the system are “controllable” [9]. If the finite poles are also “controllable”, then the feedback K can be selected such that the closed loop system is stable. The finite and infinite poles, respectively, are said to be “controllable” if

$$\mathbf{C1} \text{ rank } [\alpha A - \beta E, b] = n, \quad \forall \alpha, \beta \in \mathbb{C}, \quad \alpha \neq 0$$

$$\mathbf{C2} \text{ rank } [AS_\infty, E, B] = n,$$

where the columns of S_∞ form an orthonormal basis for the right null space $\mathcal{N}_R\{E\}$. Systems which satisfy **C1** and **C2** are called *strongly controllable*, or “impulse controllable” (see, for example, [1,9,12,13]).

In this section we examine the distance to instability $\rho(A + BK, E)$ of a closed loop pencil over all choices of the feedback K which assign a particular set of $Q \equiv \text{rank}(E)$ stable, finite poles to the system. The distance to instability is related to the sensitivity of the poles, and a lower bound on the stability radius is given in terms of a measure of pole robustness. It is assumed that the open loop system is strongly controllable and that for each choice of the feedback K , the closed loop system is regular and of index less than or equal to one. We have the following

THEOREM 4.1. *Suppose the system (A, E, B) satisfies **C1** and **C2**. Let $\mathcal{L} = \{\lambda_j\}_1^q$ be a self-conjugate set of $q \equiv \text{rank}(E)$ stable finite eigenvalues satisfying*

$$\lambda_j = \beta_j / \alpha_j \in \mathbb{C}, \quad \text{Re}(\lambda_j) < 0, \quad \lambda_j \in \mathcal{L} \Rightarrow \bar{\lambda}_j \in \mathcal{L}, \quad j = 1, 2, \dots, q.$$

Let K be any real feedback assigning the eigenvalues $\lambda_j \in \mathcal{L}$, $j = 1, 2, \dots, q$, to the pencil $(A + BK, E)$ such that the pencil is regular and the assigned eigenvalues are non-defective. Let $M \equiv A + BK$. Then

$$(24) \quad \rho(M, E) \geq \frac{1}{\nu(K)} \inf_{\substack{\theta, \omega \in \mathbb{R} \\ \theta^2 + \omega^2 = 1}} \min \left\{ \min_j |\theta \lambda_j - i\omega|, 1 \right\},$$

where

$$\nu(K) \equiv \|Y^T\|_F \|X\|_F.$$

Here Y^T and X are the non-singular model matrices of left and right eigenvectors of the closed loop pencil (M, E) .

Proof. By definition, the non-singular matrices X and Y transform the closed loop system pencil into the KCF

$$Y^T M X = \begin{bmatrix} \lambda & 0 \\ 0 & I \end{bmatrix} \equiv \widetilde{M}, \quad Y^T E X = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \equiv \widetilde{E},$$

where $\Lambda = \text{diag}\{\lambda_j, j = 1, 2, \dots, q\}$. If we define the perturbations

$$\delta \widetilde{M} = Y^T \delta M X, \quad \delta \widetilde{E} = Y^T \delta E X,$$

then it is not difficult to show

$$(i) (\delta M, \delta E) \in \mathcal{D}(M, E) \setminus \mathcal{D}(M, E) \Leftrightarrow (\delta \widetilde{M}, \delta \widetilde{E}) \in \mathcal{D}(\widetilde{M}, \widetilde{E}) \setminus \mathcal{D}(\widetilde{M}, \widetilde{E}).$$

and

$$(ii) (M + \delta M, E + \delta E) \in \mathcal{U}_n \Leftrightarrow (\widetilde{M} + \delta \widetilde{M}, \widetilde{E} + \delta \widetilde{E}) \in \mathcal{U}_n.$$

Hence, $(\delta M, \delta E)$ is an allowable destabilizing perturbation of (M, E) if and only $(\delta \widetilde{M}, \delta \widetilde{E})$ is an allowable destabilizing perturbation of $(\widetilde{M}, \widetilde{E})$. Furthermore

$$\|X\|_2 \|Y\|_2 \|\delta M | \delta E\|_F \geq \left\| Y^T [\delta M | \delta E] \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \right\|_F = \|[\delta \widetilde{M} | \delta \widetilde{E}]\|_F.$$

It follows, by Theorem 3.5 and the standard norm inequalities, that

$$\|X\|_F \|Y\|_F \rho(M, E) \geq \rho(\widetilde{M}, \widetilde{E}) = \inf_{\substack{\theta, \omega \in \mathbb{R} \\ \theta^2 + \omega^2 = 1}} \sigma_{\min} \left\{ \begin{bmatrix} \theta \Lambda - i\omega I & 0 \\ 0 & I \end{bmatrix} \right\},$$

and hence (by the definition of $\sigma_{\min}\{\cdot\}$) the theorem is proved. \square

We can now establish the following.

COROLLARY 4.1. *Let the assumptions of Theorem 4.1 hold. Then*

$$(25) \quad \rho(M, E) \geq \frac{1}{\nu(K)} \min \left\{ \min_j \left(\frac{\operatorname{Re}(\lambda_j)^2 + (\operatorname{Im}(\lambda_j) - \gamma_j)^2}{1 + \gamma_j^2} \right)^{1/2}, 1 \right\}$$

where $\gamma_j = (k_j + \sqrt{k_j^2 + 4\operatorname{Im}(\lambda_j)^2}) / (2\operatorname{Im}(\lambda_j))$ with $k_j = \operatorname{Re}(\lambda_j)^2 + \operatorname{Im}(\lambda_j)^2 - 1$ if $\operatorname{Im}(\lambda_j) \neq 0$, and $\gamma_j = 0$ otherwise.

Proof. To show the result it is necessary to determine the infimum of $\hat{\sigma}_j = |\theta\lambda_j - i\omega|$ over all $\theta, \omega \in \mathbb{R}$ with $\theta^2 + \omega^2 = 1$. In the case $\theta \neq 0$ we write $\xi = \omega/\theta$ and

$$\hat{\sigma}_j^2 = \theta^2 \left| \lambda_j - i\frac{\omega}{\theta} \right|^2 = (\operatorname{Re}(\lambda_j)^2 + (\operatorname{Im}(\lambda_j) - \xi)^2) / (1 + \xi^2).$$

Then $\hat{\sigma}_j^2$ is minimal where $d\hat{\sigma}_j^2/d\xi = 0$ and $d^2\hat{\sigma}_j^2/d\xi^2 > 0$, that is, at the root ξ of

$$\operatorname{Im}(\lambda_j)\xi^2 - k_j\xi - \operatorname{Im}(\lambda_j) = 0$$

such that $2\operatorname{Im}(\lambda_j)\xi - k_j > 0$. The required root is γ_j . In the case $\theta = 0$, we have $w = 1$ and $\hat{\sigma}_j = 1$. The corollary then follows by Theorem 4.1. \square

The function $\nu(K)$ in Theorem 4.1 and Corollary 4.2 measures the sensitivity of the closed loop poles to perturbations in the matrix pencil, as shown in [9]. More specifically, if perturbations of the order of magnitude $0(\epsilon)$ are introduced into the closed loop system pencil, then the eigenvalues $\hat{\lambda}_j$ of the perturbed pencil satisfy

$$\hat{\lambda}_j = \lambda_j + 0(\epsilon c_j), \quad 1 \leq c_j \leq \nu(K),$$

provided the assigned poles λ_j are non-defective. If the poles λ_j are defective, then the perturbation to some eigenvalue is at least an order of magnitude worse. The significance of Theorem 4.1 is that if the sensitivity of the poles is minimized, then a lower bound on the distance to instability is maximized. Algorithms for assigning poles robustly, in the sense of minimizing $\nu(K)$, are described in [9].

5. Extensions. In the previous sections a measure of the distance of a matrix pencil to the “nearest” unstable pencil is established under a natural set of allowable perturbations. For regular systems of index less than or equal to one, the restrictions on the perturbations simply imply that

in the differential-algebraic equations, the perturbations cannot cause the algebraic part to become differential.

For some systems other restrictions on the perturbations might be appropriate. For example, it might be natural to assume that, in addition to the set of admissible controls remaining fixed, the solution space of the system is also preserved. Alternatively, it might be natural in the case of semi-explicit systems to assume that the complete structure of E is preserved. Extensions of the theory to these cases are discussed in the remainder of the paper.

5.1. Restricted Distance to Distribibility. We first consider the assumption that, in addition to the set of admissible controls remaining fixed, the freedom in the selection of the initial conditions is unchanged. This corresponds to assuming that the solution space of the homogeneous differential equations (or uncontrolled system) remains fixed. From (6) it can be seen that the solution space of the homogeneous equations is just $\mathcal{R}\{X_r\}$, the range space of the eigenvectors and principal vectors associated with the $r \leq q \equiv \text{rank}(E)$ finite eigenvalues of the system pencil.

In the case of regular systems of index less than or equal to one, where $r = q$, we find that if T_∞^T gives a basis for $\text{can}_L\{E\}$, then $T_\infty^T A X_r = 0$. Since $\text{rank}(T_\infty^T A) = n - q$ by Lemma 3.1, we have

$$\mathcal{R}\{X_r\} = \mathcal{N}_R\{T_\infty^T A\}.$$

The set of allowable perturbations which preserve $\mathcal{R}\{X_r\}$ is thus given by $\overline{\mathcal{D}}_s(A, E)$, the closure of the set

$$\mathcal{D}_s(A, E) = \{(\delta A, \delta E) \mid (\delta A, \delta E) \in \mathcal{D}(A, E) \text{ and } \mathcal{N}_R\{T_\infty^T(A + \delta A)\} = \mathcal{N}_R\{T_\infty^T A\}\}.$$

The distance to instability over all perturbations in $\overline{\mathcal{D}}_s(A, E)$ is denoted $\rho_s(A, E)$ and is defined as

$$\rho_s(A, E) \equiv \inf_{(\delta A, \delta E) \in \overline{\mathcal{D}}_s(A, E)} \{ \|\delta A\|_F \mid (A + \delta A, E + \delta E) \in \mathcal{U}_n \text{ or } (\delta A, \delta E) \notin \mathcal{D}_s(A, E) \}.$$

We remark that since $\mathcal{D}_s(A, E)$ is contained in $\mathcal{D}(A, E)$, the distance $\rho(A, E) \leq \rho_s(A, E)$.

To find a computable expression for the restricted distance to instability, $\rho_s(A, E)$, in the case of a regular, index one pencil, we assume without loss of generality that A, E are in the partitioned form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix},$$

where $\text{rank}(E_{11}) = \text{rank}(E) \equiv q$ and $\text{rank}(A_{22}) = n - q$, and the perturbations $\delta A, \delta E$ are in the form

$$\begin{bmatrix} \delta A_{11} & \delta A_{12} \\ 0 & \delta A_{22} \end{bmatrix}, \quad \begin{bmatrix} \delta E_{11} & \delta E_{12} \\ 0 & 0 \end{bmatrix},$$

respectively. Since A_{22} is of full rank, the components of A_{21} in (5) can always be eliminated by orthogonal column operations which do not affect the distance measure. By Lemma 3.1, E_{11} is then also of full rank. Moreover, since $(\delta A, \delta E) \in \mathcal{D}(A, E)$, $\mathcal{N}_R\{T_\infty^T(A + \delta A)\} = \mathcal{N}_R\{T_\infty^T A\}$ if and only if $\delta A_{21} = 0$.

We establish the following.

THEOREM 5.1. *If (A, E) is stable, regular and of index less than or equal to one, then*

$$(26) \quad \rho_s(A, E) = \min \left\{ \begin{array}{l} \inf_{\substack{\theta, \omega \in \mathbb{R} \\ \theta^2 + \omega^2 = 1}} \sigma_{\min}\{H_{11}(\theta, \omega)\}, \sigma_{\min}\{A_{22}\} \end{array} \right\},$$

where

$$H_{11}(\theta, \omega) = \theta A_{11} - i\omega E_{11},$$

Proof. We let $H(\theta, \omega)$ and $\Delta(\theta, \omega)$ be defined as in (18) and (19) with $A_{21} = 0$ and $\delta A_{21} = 0$; then by the same arguments as in Lemma 3.4, $\text{rank}(H(\theta, \omega) + \delta(\theta, \omega)) = n$ for all $\theta, \omega \in \mathbb{R}$, $\theta^2 + \omega^2 = 1$ if and only if $(\delta A, \delta E) \in \mathcal{D}_s(A, E)$ and $(A + \delta A, E + \delta E)$ has it no finite pole on the imaginary axis. Moreover, $H(\theta, \omega) + \Delta(\theta, \omega)$ loses rank if and only if $\text{rank}(A_{22} + \delta A_{22}) < n$ and/or $\text{rank}(H_{11}(\theta, \omega) + \Delta_{11}(\theta, \omega)) < n$, where

$$\Delta_{11}(\theta, \omega) = \theta \delta A_{11} - i\omega \delta E_{11}.$$

For any perturbations $\hat{\Delta}_{11}$ which causes $H_{11}(\theta, \omega)$ to lose rank for given θ, ω , we may define $\delta A_{11} = \theta \hat{\Delta}_{11}$, $\delta E_{11} = i\omega \hat{\Delta}_{11}$, $\delta A_{12} = 0 = \delta E_{12}$ and $\delta A_{22} = 0$. Then $\det(H(\theta, \omega) + \Delta(\theta, \omega)) = 0$ and $\|[\delta A|\delta E]\|_F = \|\hat{\Delta}_{11}\|_F \geq \sigma_{\min}(H_{11}(\theta, \omega))$, with equality for some $\hat{\Delta}_{11}$. Similarly for any $\hat{\Delta}_{22}$ such that $\text{rank}(A_{22} + \hat{\Delta}_{22}) < n$, we may define $\delta A_{1j} = 0 = \delta E_{1j}$, $j = 1, 2$ and then $\|[\delta A|\delta E]\|_F = \|\hat{\Delta}_{22}\|_F \geq \sigma_{\min}\{A_{22}\}$, with equality for some $\hat{\Delta}_{22}$.

conversely, if $\delta A, \delta E$ are such that $\|[\delta A|\delta E]\|_F < \min\{\sigma_{\min}\{H_{11}(\theta, \omega)\}, \sigma_{\min}\{A_{22}\}\}$, for given θ, ω , then $\|\hat{\Delta}_{11}\|_F \leq \|[\delta A_{11}|\delta E_{11}]\|_F < \sigma_{\min}\{H_{11}(\theta, \omega)\}$ and $\|[\delta A_{22}]\|_F < \sigma_{\min}\{A_{22}\}$, and hence both $H_{11}(\theta, \omega) + \Delta_{11}(\theta, \omega)$ and $A_{22} + \delta A_{22}$ are of full rank. It follows that $\text{rank}(H(\theta, \omega) + \Delta(\theta, \omega)) = n$. The theorem is then established by the same arguments as in Lemma 3.4 and Theorem 3.1. \square

5.2. Semi-Explicit Systems. In practice many descriptor systems arise naturally in the semi-explicit form

$$(27) \quad \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u.$$

Physically, the matrix E is not subject to perturbations. We are thus interested in the distance to instability under the assumption that $\delta E = 0$. In this case the set of allowable perturbations is the closure of the set $\mathcal{D}_A(A, E)$ of perturbations such that the perturbed pencil $(A + \delta A, E)$ is stable and regular, $\text{nil}(A + \delta A, E) = \text{nil}(A, E)$ and $\delta E = 0$; that is,

$$\mathcal{D}_A(A, E) = \{(\delta A, \delta E) | (\delta A, \delta E) \in \mathcal{D}(A, E) \text{ and } \delta E = 0\}.$$

The distance to instability over all perturbations in $\overline{\mathcal{D}}_A(A, E)$ is denoted $\rho_A(A, E)$ and is defined as

$$\rho_A(A, E) \equiv \inf_{(\Delta A, \delta E) \in \overline{\mathcal{D}}_A(A, E)} \{ \|[\delta A|\delta E]\|_F | (A + \delta A, E + \delta E) \in \mathcal{U}_n \text{ or } (\delta A, \delta E) \notin \mathcal{D}_A(A, E) \}.$$

If the case (A, E) is of index ≤ 1 , to obtain a computable expression for the distance to instability $\rho_A(A, E)$ of the semi-explicit system (26), we assume without loss of generality that A, E are in the partitioned form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix},$$

where $\text{rank}(A_{22}) = n - q$ and the perturbation δA is in the form

$$\begin{bmatrix} \delta A_{11} & \delta A_{12} \\ \delta A_{21} & \delta A_{22} \end{bmatrix}$$

We obtain the following.

THEOREM 5.2. *If (A, E) is a stable, regular, semi-explicit system pencil of form (26) which is of index less than or equal to one, then*

$$(28) \quad \rho_A(A, E) = \min \left\{ \inf_{\lambda \in \mathbb{R}} \sigma_{\min}\{A - i\lambda E\}, \sigma_{\min}\{A_{22}\} \right\}.$$

Proof. The perturbation $(\delta A, 0) \in \mathcal{D}(A, E)$ if and only if

$$\text{rank} \begin{bmatrix} I_q & 0 \\ A_{21} + \delta A_{21} & A_{22} + \delta A_{22} \end{bmatrix} = n.$$

This holds only if $\|\delta A_{22}\|_F < \sigma_{\min}\{A_{22}\}$ and fails to hold for some δA_{22} with $\|\delta A_{22}\|_F = \sigma_{\min}\{A_{22}\}$. The perturbed pencil $(A + \delta A, E)$ has no finite pole on the imaginary axis if and only if $\text{rank}(A + \delta A - i\lambda E) = n \forall \lambda \in \mathbb{R}$. This holds only if $\|\delta A\|_F < \sigma_{\min}\{A - i\lambda E\} \forall \lambda$, and fails to hold for each λ for some δA with $\|\delta A\|_F = \sigma_{\min}\{A - i\lambda E\}$. The result follows directly. \square

6. Conclusion. In summary, we establish here an approach for measuring the distance of a system pencil to the nearest “unstable” pencil. It is assumed that the physical structure of the system restricts the allowable perturbations so that a finite measure of the distance exists. Computable expressions for the distance under various natural restrictions are derived for system pencils of index at most one. Numerical algorithms for evaluating the distance measures by accurate and efficient techniques are now being developed, based on the bisection methods derived in [2] and [7] for standard systems. It is expected that this approach can be extended to obtain computable measures of the distance to instability for systems of higher index.

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