BOUNDARY AND LOCAL NULL CONTROLLABILITY OF STRUCTURALLY DAMPED ELASTIC SYSTEMS

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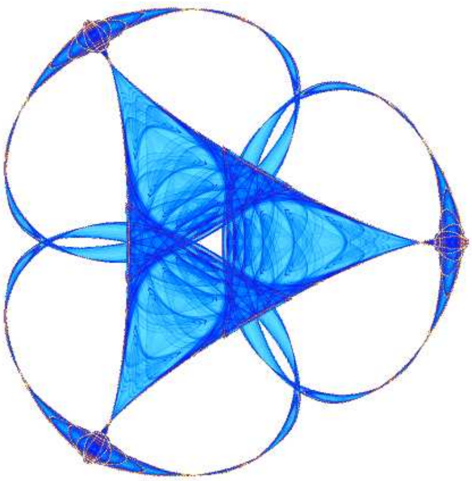

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Boundary and Local Null Controllability of Structurally Damped Elastic Systems

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Abstract

In this paper, we consider the problem of null controllability for an elastic operator under square root damping. It is now well-known that such partial differential equation models are described by analytic semigroups on the basic space of finite energy. Thus because of this underlying parabolicity, the null controllability problem is appropriate for consideration. In particular, we will show that the solution variables can be steered to the zero state by means of iterations of finite dimensional controls. In this work, key usage is made of the diagonalization of the spatial operators which is available in the case of hinged boundary conditions. Moreover, the control strategy in [3] is critically adapted to our present needs. In particular, this strategy hinges upon the availability of a Carleman's estimate for linear combinations of eigenfunctions of the Dirichlet Laplacian. Finally, in order to inherit a boundary controllability result from local controllability, the analyticity of the structurally damped system is brought to bear.

1 Introduction and Statement of Main Results

Throughout, Ω will denote an open, connected subset of $\mathbb{R}^2$, with C^∞ boundary. Given terminal time $0 < T < \infty$, we will consider the following boundary *controlled* partial differential equation:

$$\begin{aligned} y_{tt} &= -\Delta^2 y + \rho \Delta y_t \quad \text{in } Q \equiv (0,T) \times \Omega \\ y|_{\partial\Omega} &= u_1 \quad \text{on } \Sigma \equiv (0,T) \times \partial\Omega \\ \Delta y|_{\partial\Omega} &= u_2 \quad \text{on } \Sigma \equiv (0,T) \times \partial\Omega \\ [y(0), y_t(0)] &= [y_0, y_1] \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega). \end{aligned} \tag{1}$$

As given, $y(t)$ is the solution of a beam equation with inserted "square root" structural damping, and which is under the influence of "control" functions $[u_1, u_2]$ (in to be specified spaces). The positive parameter ρ quantifies the extent of damping; it will be assumed here that $\rho \neq 2$. It is wellknown that for $\rho > 0$ the solution of the corresponding free dynamics–i.e., $u_1 = u_2 = 0$–can be associated with the generator of an *analytic* semigroup on the appropriate Hilbert space H of wellposedness.

In fact, let

$$H = [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega).$$

Moreoever, denote $S : D(S) \subset L^2(\Omega) \to L^2(\Omega)$ to be the following positive definite, self-adjoint operator:

$$Sy_0 = -\Delta y_0, \;\; D(S) = [H^2(\Omega) \cap H_0^1(\Omega)]. \tag{2}$$

On Ω, the biharmonic operator with hinged boundary conditions can then be characterized in terms of the realization S: That is,

$$\text{That is, } \Delta^2 y_0 = S^2 y_0, \;\; \left\{ y_0 \in H^4(\Omega) \cap H_0^1(\Omega) : \quad y_0|_{\partial\Omega} = \Delta y_0|_{\partial\Omega} = 0 \right\} = D(S^2)$$

Subsequently, we can define the "elastic" operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -S^2 & -\rho S \end{bmatrix}; \quad D(\mathcal{A}) = [y_0, y_1] \in D(S^2) \times D(S). \tag{3}$$

It was once a longstanding conjecture that $\mathcal{A}$ generates an *analytic* C_0-semigroup of contractions $\left\{ e^{\mathcal{A}t} \right\}_{t \geq 0}$ on H (see [4]); this conjecture was eventually borne out in [5]. Thus, the structurally damped system (1) exhibits *parabolic-like* behavior, in the sense that the semigroup $\left\{ e^{\mathcal{A}t} \right\}_{t \geq 0}$ passes on a smoothing effect to its solutions. For example, if $u_1 = u_2 = 0$ in (1), then from [6] and [9], we infer that solution $[y, y_t]$ is not only in $C([0, T]; H)$, as expected from classical semigroup theory, but also in $L^2(0, T; D(S^{\frac{3}{2}}) \times D(S^{\frac{1}{2}}))$. So by the characterization in [7], of the fractional powers of S in terms of familiar Sobolov spaces, $[y, y_t] \in L^2 \left(0, T; H^3(\Omega) \times H^1(\Omega) \right)$.

Because of this underlying infinite speed of propagation of signals associated to (square root) structurally damped systems, it then makes sense, for a given control to terminal state map, to consider the *null controllability* problem. In the context of our boundary controlled system, our problem may be formulated as follows: "The PDE (1) is said to be *null controllable* in time $T > 0$, within the class of controls $\mathcal{U}$ (to be specified below), if given any initial data $[y_0, y_1] \in H$, there exists $[u_1, u_2] \in \mathcal{U}$ such that the corresponding solution $[y, y_t]$ of (1) satisfies the terminal condition $[y(T), y_t(T)] = [0, 0]$.

By way of addressing said boundary controllability problem (in particular, addressing the choice of controls $\mathcal{U}$), we will study the following structurally damped system under *locally distributed* control, this problem certainly being of independent interest:

$$\begin{aligned} y_{tt} + \Delta^2 y - \rho \Delta y_t &= \chi_\omega(x) u \;\; \text{in } Q \equiv (0, T) \times \Omega \\ y|_{\partial\Omega} &= \Delta y|_{\partial\Omega} = 0 \;\; \text{on } \Sigma \equiv (0, T) \times \partial\Omega \\ [y(0), y_t(0)] &= [y_0, y_1] \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega). \end{aligned} \tag{4}$$

Above, $\omega \subset \Omega$ is a nonempty subdomain, and $\chi_\omega(\cdot)$ its characteristic function. If $\omega = \Omega$, it was shown in [12] and [1] (see also [22]) that the system (4) is null controllable at given time $T > 0$, within the class of controls $L^2(0, T; L^2(\Omega))$; that is, given arbitrary initial data $[y_0, y_1] \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$ and time $T > 0$, there is a control $u \in L^2(0, T; L^2(\Omega))$ which steers the solution $[y(t), y_t(t)]$ of (4) to rest at time $t = T$. In this paper, we will be concerned with the case that the support of the control ω is strictly contained in Ω (although it will turn out that the observability estimates derived in [1] and [22], for the fully distributed case, are quite critical here).

In order to clearly formulate our results, we first recall the notion of "minimal energy": Supposing the null controllability property holds true for (4), we can subsequently define the minimal null control $u^*(T; Y^0)$ ($Y^0 \equiv [y_0, y_1]$) by

$$\left\|u^*(T; Y^0)\right\|_{\mathcal{U}} = \min_{e^{\mathcal{A}T}Y^0 + \mathcal{L}_T u = 0} \|u\|_{\mathcal{U}},$$

where control→terminal state map $\mathcal{L}_T : \mathcal{U} \to H$ is given by

$$\mathcal{L}_T u \equiv \int_0^T e^{\mathcal{A}(T-s)} \begin{bmatrix} 0 \\ \chi_\omega u \end{bmatrix} ds.$$

By the classical convex analysis, the minimizer $u^*(T; Y^0)$ is certainly well-defined, and in fact admits of an explicit representation (see e.g., Appendix B of [11] and [13]). Subsequently, we can define the *minimal energy function* $\mathcal{E}_{\min}(T)$ by

$$\mathcal{E}_{\min}(T) = \sup_{\|Y^0\|_H = 1} \left\|u^*(T; Y^0)\right\|_{\mathcal{U}}. \tag{5}$$

Assuming the null controllability property for (4) to hold true for any terminal time, then $\mathcal{E}_{\min}(T)$ will be bounded on $(0, T]$. In line with the classic studies undertaken since at least the late 1960's (see e.g., [17]), and in addition to establishing the question of local and boundary null controllability, we are interested in ascertaining the precise rate of blowup for $\mathcal{E}_{\min}(T)$.

Our results in this connection are as follows:

Theorem 1. *Let positive parameter $\rho \neq 2$ in (4). Then the system (4) is locally null controllable, within the class of controls $\mathcal{U} = L^2((0,T) \times \omega)$. Moreover, $\mathcal{E}_{\min}(T) = \mathcal{O}\left(\exp(\frac{C}{T^{1+\epsilon}})\right)$, for all $\epsilon > 0$. In particular, there is a positive constant C_ϵ, which depends on ϵ but not on T, such that $\mathcal{E}_{\min}(T) \leq C_\epsilon \exp(\frac{C}{T^{1+\epsilon}})$.*

In turn, we can appeal to the "embedding" technique of Seidman in [19], valid for parabolic equations, so as to obtain the following:

Theorem 2. *Let positive parameter $\rho \neq 2$ in (1). Then the system (1) is locally null controllable, within the class of controls $\mathcal{U} = L^2((0,T) \times \partial\Omega) \times L^{2-\epsilon_0}(0, T; L^2(\Gamma))$, where $\epsilon_0 > 0$ is arbitrarily small. Moreover, $\mathcal{E}_{\min}(T) = \mathcal{O}\left(\exp(\frac{C}{T^{1+\epsilon}})\right)$, for all $\epsilon > 0$.*

Remark 3. *In view of what is known for blowup rates of minimal norm controls, relative to boundary null controllability of parabolic systems, our estimate for $\mathcal{E}_{\min}(T)$ of (1) (or (4)) is seemingly "unsharp by ϵ". (See e.g., [19], [20]). In particular, see Theorem 2 of [21] which gives the sharp blowup estimate for $\mathcal{E}_{\min}(T)$, in the case that the geometry Ω is the canonical unit square, and the structurally damped beam is under the free boundary conditions.*

Remark 4. *Here, our imposition that $\rho \neq 2$ allows for a "diagonalization" of the elastic generator $\mathcal{A}$ (see Section 2 below), which will be exploited below. On the other hand, for the special case $\rho = 2$, $\mathcal{A}$ admits of a "Jordan canonical form" (see Remark 2.2 of [12]), which in principle would allow for the same analysis as that undertaken in the present paper, although this work for $\rho = 2$ has not yet been done. These factorizations are a direct consequence of the hinged boundary conditions which are being enforced for the beam.*

As we said earlier, the question of quantifying the extent of minimal norm control blowup is a classical one in control theory, be it in the finite or infinite dimensional setting. For the case of structurally damped systems under *fully* distributed interior control, the asymptotics of $\mathcal{E}_{\min}(T)$ for the (infinite dimensional) structurally damped operator will actually behave like those for controlled finite dimensional systems; in fact it was shown in [1] and [22] that the minimal energy function manifests a *rational* rate of singularity (see also [18] which initially solved the blowup problem for finite dimensions). Conversely, when dealing with PDE controllability problems involving locally distributed or boundary controls, the singularity of the associated $\mathcal{E}_{\min}(T)$ will inevitably obey an *exponential* blowup law; see [21] for an interesting history of observability and blowup estimates in the control of PDEs.

In theme this paper aims, like the classic works [17], [15] and [19], to obtain results of null controllability and observation for parabolic equations. For, as we said at the outset, it is known that the elastic operator (3), under square root structural damping exhibits *parabolic-like* dynamics (see [5]). Concerning controllability results in the literature dealing with structurally damped operators, we note the paper [21], wherein a result of boundary observability is obtained, in the case that Ω is a rectangular region, and with the hinged boundary control conditions in (1) being replaced by $\frac{\partial \Delta y}{\partial \nu} = u$ and $\frac{\partial y}{\partial \nu} = 0$ on Σ. For this scenario, the "expected" asymptotics $\mathcal{E}_{\min}(T) = \mathcal{O}(e^{\frac{C}{T}})$ are obtained in [21], by making use of the explicit spectral information available for the canonical geometry under consideration. See also [10] for the observation and null controllability of the thermoelastic plate under boundary control–another "non-classical" parabolic system–in a special geometry. In addition, the paper [3] is also concerned with the null controllability problem for said thermoelastic plate (with no treatment therein for attaining the sharp observability inequality).

As we said, we will consider the boundary control system (1) as a follow-up to an analysis of (4). Our particular *modus operandi* in this paper is as follows :

1. We initially consider the null controllability of the locally distributed controlled PDE system (4). Using the similarity transformation to a diagonal "matrix", which is available for the elastic operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ under hinged boundary conditions (as first noted in [12]), we will be able to express the solution $[\phi, \phi_t]$ of the homogenous adjoint problem–adjoint with respect to (4); see system (7) below–in terms of an infinite series of the form $\sum \alpha_i \Phi_i$, with the basis functions Φ_i involving the eigenfunctions of the aforesaid positive definite, self-adjoint operator $S : D(S) \subset L^2(\Omega) \to L^2(\Omega)$.

2. Using finite dimensional truncations of the aforesaid series, we consider the null controllability problem (4) in the context of the finite dimensional space $H_l = Span\{\Phi_1, \Phi_2, ..., \Phi_l\}$, and proceed to obtain the necessary observability inequality for terminal time T_l, where $T = \sum_l^\infty T_l$. In this step we are using the main idea of the paper [3]. Such an approach allows for a critical invocation of a Carleman estimate in [8], which is applicable to *finite* linear combinations of eigenfunctions of the Dirichlet Laplacian $S : D(S) \subset L^2(\Omega) \to L^2(\Omega)$. In addition, the known observability estimates in [1] and [22] for the case of *fully* distributed control will be indispensible here. Building the infinite dimensional control u from the respective (minimal norm) finite dimensional controls $\{u^l\}$, we will show that the null controllability property for (4) is attained as l tends to ∞.

3. Subsequently, we proceed to measure the singularity of the null controller u devised in Step

2. This is done by appealing to the fact that the rate of blowup of the minimal norm control is of order $\mathcal{O}(C_T)$, where C_T is the sharp observability constant for the associated dual homogeneous problem (see (13) below); see also [2]. We will actually apply this fact to each finite dimensional control u^l; in this way, we will end up with an infinite series to estimate, this being of the form $\sum_l^\infty C(T_l)$. During the course of this estimation work, we will see that the exponential rate of singularity for $\mathcal{E}_{\min}(T)$, given in Theorem 1, is due to the first l^* terms of the series; the contribution of the "tail end" of the series $\sum_{l^*+1}^\infty C(T_l)$ will be found to be essentially benign.

4. Having dealt with the null controllability of the structurally damped beam with locally distributed control, we now turn our attention to the boundary controlled model (1). The key result which underpins this work is the fact that the elastic operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ generates an *analytic* contraction C_0-semigroup (see [5]). Because of this underlying parabolicity, we can appeal to the classical "embedding technique" of T. I. Seidman, outlined in [19] and [16]. In fact, given the PDE system (1), we will take our controls u_1 and u_2 to be the restriction (or traces") to $\partial\Omega$ of the solution $[y^e, y_t^e]$ of a locally controlled problem (4) on $(0,T) \times \Omega_e$, where Ω_e is an "extended" domain which contains Ω. In justifying the well-definition of the controls $[u_1, u_2]$, we will use critically the well-quantified regularity of analytic semigroups (see [9]).

2 Abstract Formulation for the Dynamics

The PDE (1) can be written as the first order ODE system on $(0,T)$

$$\begin{cases} \begin{bmatrix} y \\ y_t \end{bmatrix}_t = \begin{bmatrix} 0 & I \\ -S^2 & -\rho S \end{bmatrix} \begin{bmatrix} y \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{X}_W u \end{bmatrix} & \\ [y, y_t](0) = [y_0, y_1] & \text{on } \Omega \end{cases} \tag{6}$$

We set $\mathcal{A} = \begin{bmatrix} 0 & I \\ -S^2 & -\rho S \end{bmatrix}$ and $B_W = \begin{bmatrix} 0 \\ \mathcal{X}_W \end{bmatrix}$ to determine the corresponding adjoint system. With the inner product on H given by

$$\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle_H = (Sx_1, Sx_2)_{L^2(\Omega)} + (y_1, y_2)_{L^2(\Omega)} \text{ for } [x_1, y_1], [x_2, y_2] \in H,$$

we find $\mathcal{A}^* = \begin{bmatrix} 0 & -I \\ S^2 & -\rho S \end{bmatrix}$.

The backward adjoint system is given by

$$\begin{cases} \begin{bmatrix} \phi \\ \phi_t \end{bmatrix}_t = -\mathcal{A}^* \begin{bmatrix} \phi \\ \phi_t \end{bmatrix} = \begin{bmatrix} \phi_t \\ -S^2\phi + \rho S \phi_t \end{bmatrix} & \text{in } Q \\ [\phi, \phi_t](T) = [\phi_0, \phi_1] & \text{in } \Omega^2 \\ S\phi|_{\partial\Omega} = \phi_t|_{\partial\Omega} = 0 & \text{on } \Sigma \end{cases}$$

and has solution $\begin{bmatrix}\phi(t)\\ \phi_t(t)\end{bmatrix} = e^{A^*(T-t)}\begin{bmatrix}\phi_0\\ \phi_1\end{bmatrix}$. After the change of variables $t := T - t$, the forward adjoint system is given by

$$\begin{cases} \begin{bmatrix}\phi\\ \phi_t\end{bmatrix}_t = \mathcal{A}\begin{bmatrix}\phi\\ \phi_t\end{bmatrix} := \begin{bmatrix}0 & I\\ -S^2 & -\rho S\end{bmatrix}\begin{bmatrix}\phi\\ \phi_t\end{bmatrix} & \text{in } Q\\ [\phi, \phi_t](0) = [\phi_0, -\phi_1] & \text{in } \Omega^2\\ S\phi|_{\partial\Omega} = \phi_t|_{\partial\Omega} = 0 & \text{on } \Sigma \end{cases} \tag{7}$$

With $B_W = \begin{bmatrix}0\\ \mathcal{X}_W\end{bmatrix}$ as before, $B_W^* = \begin{bmatrix}0 & \mathcal{X}_W\end{bmatrix}$. Then the solution to the system (7) can be written as

$$\begin{bmatrix}\phi(t)\\ \phi_t(t)\end{bmatrix} = e^{\mathcal{A}^*t}\begin{bmatrix}\phi_0\\ -\phi_1\end{bmatrix} \implies B_W^* e^{\mathcal{A}^*t}\begin{bmatrix}\phi_0\\ -\phi_1\end{bmatrix} = \mathcal{X}_W\phi_t(t).$$

Null-controllability to the fully distributed system ($W = \Omega$) follows if there exists a constant $C_T > 0$ such that

$$\left\| \begin{bmatrix}\phi(T)\\ \phi_t(T)\end{bmatrix} \right\|_H^2 \le C_T \int_0^T \|\phi_t(t)\|^2_{L^2(\Omega)} dt \tag{8}$$

for any solution $[\phi, \phi_t]$ of the system (7). In fact, this inequality follows by [1] with $C_T = \mathcal{O}(T^{-3})$ as $T \searrow 0$.

To obtain null-controllability for locally distributed controls, we first look at the truncation of $\mathcal{A}$ on the span of finitely many eigenfunctions.

In the following work, we will also make use of the eigenpairs of $\mathcal{A}$. Let $\{\mu_n, e_n\}_{n=1}^{\infty}$ be the eigenpairs of S with $0 < \mu_n \le \mu_{n+1}$ for all $n \in \mathbb{N}$. Let $z_1 = \frac{-\rho+\sqrt{\rho^2-4}}{2}$ and $z_2 = \frac{-\rho-\sqrt{\rho^2-4}}{2}$. Having in mind the diagonalization of $\mathcal{A}$ used in [12], let Π be the linear mapping on $L^2(\Omega) \times L^2(\Omega)$ given by $\Pi = \begin{bmatrix}I & I\\ z_1 I & z_2 I\end{bmatrix}$. $\mathcal{A}$ can be diagonalized by

$$\begin{bmatrix}z_1 S & 0\\ 0 & z_2 S\end{bmatrix} = \Pi^{-1}\begin{bmatrix}S & 0\\ 0 & I\end{bmatrix}\mathcal{A}\begin{bmatrix}S^{-1} & 0\\ 0 & I\end{bmatrix}\Pi.$$

With this setup, we find that $\mathcal{A}$ has eigenpairs

$$\left\{\left(z_1\mu_n, \begin{bmatrix}S^{-1} & 0\\ 0 & I\end{bmatrix}\Pi\begin{bmatrix}e_n\\ 0\end{bmatrix}\right), \left(z_2\mu_n, \begin{bmatrix}S^{-1} & 0\\ 0 & I\end{bmatrix}\Pi\begin{bmatrix}0\\ e_n\end{bmatrix}\right)\right\}_{n\in\mathbb{N}}.$$

This can be written more concisely as

$$\{\lambda_{n,j}, \Phi_{n,j}\}_{n\in\mathbb{N}, j=1,2} := \left\{z_j\mu_n, c_j\begin{bmatrix}e_n/\mu_n\\ z_j e_n\end{bmatrix}\right\}_{n\in\mathbb{N}, j=1,2}$$

(here $c_j, j = 1, 2$ is chosen so that $\|\Phi_{n,j}\|_H = 1$).

Define $H_l = \text{span}\{\Phi_k^j : 1 \le k \le l, j = 1, 2\}$.

3 A Technical Lemma

We shall here prove the following:

Lemma 5. *Let ω be a nonempty, open subset of Ω. Then for $T_l > 0$ and $Y_0 \in H$, there exists a control $u_l \in L^2((0,T_l)\times\omega)$ so that $[y, y_t](T_l) \in H_l^\perp$ and $\|u_l(T_l, Y^0)\|^2_{L^2((0,T_l)\times\omega)} \le C_{T_l} e^{C\sqrt{\mu_l}} \|Y^0\|^2_H$.*

Proof of Lemma (5). Again it is our goal for appropriate initial data in H_l, to verify the observability inequality. I.e. for all solutions $[\phi, \phi_t]$ to (3), there exists $C_{T_l} > 0$ so that

$$\left\| \begin{bmatrix} \phi(T_l) \\ \phi_t(T_l) \end{bmatrix} \right\|_H^2 \le C_{T_l} \int_0^{T_l} \|\phi_t(t)\|^2_{L^2(\omega)} dt.$$

Since $\mathcal{A}$ is H_l-invariant, let $[\phi_0, -\phi_1] \in D(\mathcal{A}) \cap H_l$. By the observability inequality from [1] we have that,

$$\left\| \begin{bmatrix} \phi(T_l) \\ \phi_t(T_l) \end{bmatrix} \right\|_H^2 \le C_{T_l} \int_0^{T_l} \|\phi_t(t)\|^2_{L^2(\Omega)} dt \tag{9}$$

again with $C_{T_l} = \mathcal{O}(T_l^{-3})$ as $T_l \searrow 0$.

We can diagonalize the system to solve the adjoint problem. The solution to this problem can be expressed as

$$\begin{bmatrix} \phi \\ \phi_t \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} \phi_0 \\ -\phi_1 \end{bmatrix} = \begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \Pi \begin{bmatrix} e^{z_1 S t} & 0 \\ 0 & e^{z_2 S t} \end{bmatrix} \Pi^{-1} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \phi_0 \\ -\phi_1 \end{bmatrix}$$

Using the expansion for ϕ_0 and $-\phi_1$ in terms of the eigenfunctions for S, we have that

$$\begin{bmatrix} \phi \\ \phi_t \end{bmatrix}(t) = \begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \Pi \sum_{n=1}^{l} \begin{bmatrix} e^{z_1 \mu_n t}(\Pi_{11}^{-1} \mu_n (\phi_0, e_n) + \Pi_{12}^{-1}(-\phi_1, e_n)) e_n \\ e^{z_2 \mu_n t}(\Pi_{21}^{-1} \mu_n (\phi_0, e_n) + \Pi_{22}^{-1}(-\phi_1, e_n)) e_n \end{bmatrix}$$

This allows us to write ϕ_t as

$$\begin{aligned} \phi_t(t) &= \sum_{n=1}^{l} a_n(t) e_n \\ &:= \sum_{n=1}^{l} \left[\frac{z_1}{z_2 - z_1} e^{z_1 \mu_n t}(z_2 \mu_n \phi_0 + \phi_1, e_n) + \frac{z_2}{z_2 - z_1} e^{z_2 \mu_n t}(-z_1 \mu_n \phi_0 - \phi_1, e_n) \right] e_n. \end{aligned}$$

By Parseval's relation we have now,

$$\|\phi_t(t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{l} |a_n(t)|^2. \tag{10}$$

Jerison and Lebeau, in [8], give an estimate for sums of eigenfunctions of S using a Carleman inequality. Namely,

$$\|\phi_t(t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{l} |a_n(t)|^2 \leq Ce^{C\sqrt{\mu_l}} \int_\omega \left| \sum_{n=1}^{l} a_n(t) e_n(x) \right|^2 dx = Ce^{C\sqrt{\mu_l}} \|\phi_t(t)\|^2_{L^2(\omega)}. \tag{11}$$

Combining relations (9) - (11), we have that

$$\left\| \begin{bmatrix} \phi(T_l) \\ \phi_t(T_l) \end{bmatrix} \right\|^2_H \leq C_{T_l} e^{C\sqrt{\mu_l}} \int_0^{T_l} \|\phi_t(t)\|^2_{L^2(\omega)} dt. \tag{12}$$

This inequality gives the existence of a control u_l for initial data Y^0 in H_l with $u_l(T_l, Y^0)$ satisfying

$$\|u_l(T_l, Y^0)\|^2_{L^2((0,T_l)\times\omega)} \leq C_{T_l} e^{C\sqrt{\mu_l}} \|Y^0\|^2_H \tag{13}$$

and $C_{T_l} = \mathcal{O}(T_l^{-3})$ as $T_l \searrow 0$. □

4 Proof Proper Theorem 1

Now that null-controllability has been established on H_l, we can use this to establish a strategy, as in [3] to steer arbitrary initial data in H to zero. Let $\alpha \in (0, 1/2)$, $T_l = K2^{-l\alpha}$ where $K = \frac{T(2^\alpha - 1)}{2}$ is chosen so that $2\sum_{l=1}^\infty T_l = T$. Also let $a_0 = 0$ and $a_l = a_{l-1} + 2T_l$ for $l \in \mathbb{N}$.

Define the control → state map $\mathcal{L}_{t_0,t}(z, f)$ by having for all $\{z, f\} \in H \times L^2(0, t; L^2(\omega))$,

$$\mathcal{L}_{t_0,t}(\xi, f) = e^{\mathcal{A}(t-t_0)}\xi + \int_{t_0}^{t} e^{\mathcal{A}(t-s)} B_\omega f(s) ds.$$

Recall that $B_\omega = \begin{bmatrix} 0 \\ \mathcal{X}_\omega \end{bmatrix}$. Moreover, for any index j, let $u_{2^j}(T_j, \xi)$ denote the control which steers initial data $\xi \in H$ to $H^\perp_{2^j}$ at time T_j. The existence of such a control is assured by Lemma(5).

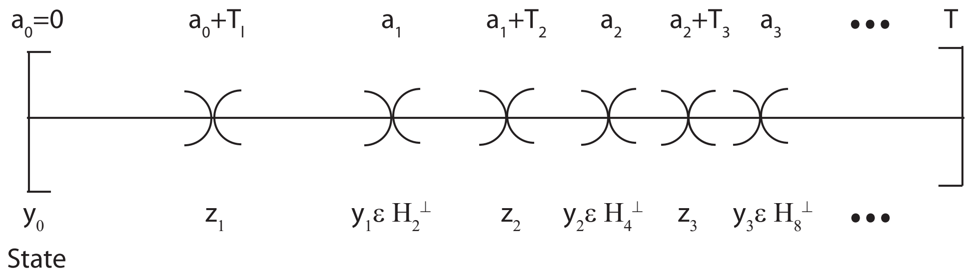
With quantities $\mathcal{L}_{t_0,t}(\cdot,\cdot)$ and $u_{2^\cdot}(\cdot,\cdot)$ in hand, we now define the iteration scheme upon which we will build our null steering control: Set $y^0 = [y_0, y_1] \in H$ to be the given initial data in (1). For $l = 1, 2, ...,$

$$z^l = e^{\mathcal{A}T_l} y^{l-1}, \text{ and then } y^l = \mathcal{L}_{a_{l-1}+T_l, a_l}\left(z^l, u_{2^l}(\cdot - a_{l-1} - T_l; T_l, z^l)\right).$$

This constructs a control u given by

$$u(t) = \begin{cases} 0, & a_{l-1} \leq t < a_{l-1} + T_l,\ l \in \mathbb{N} \\ u_{2^l}(t - (a_{l-1} + T_l); T_l, z^l), & a_{l-1} + T_l \leq t < a_{l-1} + 2T_l = a_l,\ l \in \mathbb{N} \end{cases} \tag{14}$$

The following diagram should help illustrate the strategy.



To show that the state z goes to zero, note that for each l, we first estimate $\|y^l\|_H$ in terms of $\|z^l\|_H$. Using that $\mathcal{A}$ generates a semigroup of contractions and inequality (13), we have that

$$
\begin{aligned}
\|y^l\|_H &\leq \|e^{AT_l} z^l\|_H + \left\| \int_{a_{l-1}+T_l}^{a_l} e^{A(a_l - s)} B_\omega u_{2^l}(s - (a_{l-1} + T_l); T_l, z^l) ds \right\|_H \\
&\leq \|z^l\|_H + \|u_{2^l}(T_l, z^l)\|_{L^2((0,T_l)\times\omega)} \\
&\leq \left(1 + \sqrt{C_{T_l}} e^{C\sqrt{\mu_{2^l}}}\right) \|z^l\|_H
\end{aligned} \tag{15}
$$

We now provide the crucial estimate for $\|z^l\|_H$ in terms of $\|y^{l-1}\|_H$. Recall that $z^l = e^{\mathcal{A}T_l} y^{l-1}$. Rather than using the contraction property of $\mathcal{A}$, we take advantage of the fact that $y^{l-1} \in H^{\perp}_{2^{l-1}}$. Before diagonalizing again, consider the following argument:

Note that for $\tilde{y} \in H_{2^l}^{\perp}$, $\Pi^{-1}\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}\tilde{y} \in H_{2^l}^{\perp}$. Write

$$
\begin{aligned}
&\Pi^{-1}\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}\tilde{y} = \sum_{n=2^l+1}^{\infty}\begin{bmatrix} \xi_n^{(1)} e_n \\ \xi_n^{(2)} e_n \end{bmatrix} \\
\implies &\left\| \begin{bmatrix} e^{z_1 S T_l} & 0 \\ 0 & e^{z_2 S T_l} \end{bmatrix} \Pi^{-1}\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}\tilde{y} \right\|_{(L^2(\Omega))^2}^2 = \left\| \begin{bmatrix} e^{z_1 S T_l} & 0 \\ 0 & e^{z_2 S T_l} \end{bmatrix} \sum_{n=2^l+1}^{\infty}\begin{bmatrix} \xi_n^{(1)} e_n \\ \xi_n^{(2)} e_n \end{bmatrix} \right\|_{(L^2(\Omega))^2}^2 \\
&= \left\| \sum_{n=2^l+1}^{\infty}\begin{bmatrix} e^{z_1 \mu_n T_l}\xi_n^{(1)} e_n \\ e^{z_2 \mu_n T_l}\xi_n^{(2)} e_n \end{bmatrix} \right\|_{(L^2(\Omega))^2}^2 \\
&= \left\| \sum_{n=2^l+1}^{\infty} e^{z_1 \mu_n T_l}\xi_n^{(1)} e_n \right\|_{L^2(\Omega)}^2 + \left\| \sum_{n=2^l+1}^{\infty} e^{z_2 \mu_n T_l}\xi_n^{(2)} e_n \right\|_{L^2(\Omega)}^2 \\
&= \sum_{n=2^l+1}^{\infty}\left(\left\| e^{z_1 \mu_n T_l}\xi_n^{(1)} e_n \right\|_{L^2(\Omega)}^2 + \left\| e^{z_2 \mu_n T_l}\xi_n^{(2)} e_n \right\|_{L^2(\Omega)}^2 \right) \\
&\le e^{2\mathrm{Re}(z_1)\mu_{2^l+1}T_l} \sum_{n=2^l+1}^{\infty}\left(\left\| \xi_n^{(1)} e_n \right\|_{L^2(\Omega)}^2 + \left\| \xi_n^{(2)} e_n \right\|_{L^2(\Omega)}^2 \right) \\
&= e^{2\mathrm{Re}(z_1)\mu_{2^l+1}T_l} \left(\left\| \sum_{n=2^l+1}^{\infty} \xi_n^{(1)} e_n \right\|_{L^2(\Omega)}^2 + \left\| \sum_{n=2^l+1}^{\infty} \xi_n^{(2)} e_n \right\|_{L^2(\Omega)}^2 \right) \\
&= e^{2\mathrm{Re}(z_1)\mu_{2^l+1}T_l} \left\| \sum_{n=2^l+1}^{\infty}\begin{bmatrix} \xi_n^{(1)} e_n \\ \xi_n^{(2)} e_n \end{bmatrix} \right\|_{(L^2(\Omega))^2}^2 \\
&\le e^{2\mathrm{Re}(z_1)\mu_{2^l+1}T_l} \left\| \sum_{n=2^l+1}^{\infty}\begin{bmatrix} \xi_n^{(1)} e_n \\ \xi_n^{(2)} e_n \end{bmatrix} \right\|_{(L^2(\Omega))^2}^2
\end{aligned}
$$

Since $y^{l-1} \in H_{2^{l-1}}^{\perp}$, we can use the above argument to estimate $\|z^l\|_H$.

$$
\begin{aligned}
\|z^l\|_H &= \left\| \begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \Pi \begin{bmatrix} e^{z_1 S T_{l-1}} & 0 \\ 0 & e^{z_2 S T_{l-1}} \end{bmatrix} \Pi^{-1}\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} y^{l-1} \right\|_H \\
&= \left\| \Pi \begin{bmatrix} e^{z_1 S T_{l-1}} & 0 \\ 0 & e^{z_2 S T_{l-1}} \end{bmatrix} \Pi^{-1}\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} y^{l-1} \right\|_{(L^2(\Omega))^2} \\
&\le \|\Pi\| e^{\mathrm{Re}(z_1)\mu_{2^{l-1}+1}T_{l-1}} \|\Pi^{-1}\| \left\| \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} y^{l-1} \right\|_{(L^2(\Omega))^2} \\
&= C e^{\mathrm{Re}(z_1)\mu_{2^{l-1}+1}T_{l-1}} \|y^{l-1}\|_H
\end{aligned} \tag{16}
$$

Combining inequalities (15) and (16) and using the explicit order of C_T given in [1], we have

$$\|y^l\|_H \le e^{\mathrm{Re}(z_1)\mu_{2^{l-1}+1}T_{l-1}} \left(1 + \sqrt{C_{T_l}} e^{C\sqrt{\mu_{2^l}}}\right) \|y^{l-1}\|_H. \tag{17}$$

Weyl's formula states that we can estimate large eigenvalues of S by $\mu_l \sim C(\Omega)l$ as $l \to \infty$. This implies that for positive constants C' and C, we have the estimates

$$\mathrm{Re}(z_2)\mu_{2^{l-1}+1} \le -C'(2^l) \text{ and } \sqrt{\mu_{2^l}} \le C2^{l/2}. \tag{18}$$

Applying estimates (15)-(17), we have that

$$\begin{aligned}
\|y^l\|_H &\le \left(e^{-C'(2^l)}\right)^{T_l} \left(1 + \sqrt{C_{T_l}} e^{C\sqrt{\mu_{2^l}}}\right) \|y^{l-1}\|_H \\
&\le e^{-C'2^l T 2^{-\alpha l}} C\sqrt{(T2^{-\alpha l})^{-3}} e^{C\sqrt{2^l}} \|y^{l-1}\|_H \\
&= CT^{-3/2}2^{3\alpha l/2} \exp\left(-C'T2^{(1-\alpha)l} + C2^{l/2}\right) \|y^{l-1}\|_H
\end{aligned}$$

Iterating this estimate, we obtain now that

$$\begin{aligned}
\|y^l\|_H &\le C^l T^{-3l/2} 2^{\frac{3\alpha}{2}\sum_{j=0}^l j} \exp\left(-C'T\sum_{j=0}^{l} 2^{(1-\alpha)j} + C\sum_{j=0}^{l} 2^{j/2}\right) \|y^0\|_H \\
&= C^l T^{-3l/2} 2^{3\alpha l(l+1)/4} \exp\left(-C'T\frac{2^{(1-\alpha)(l+1)} - 1}{2^{1-\alpha} - 1} + C\frac{2^{(l+1)/2} - 1}{\sqrt{2} - 1}\right) \|y^0\|_H \\
&\le \exp\left(-C'T2^{(1-\alpha)l} + C2^{l/2} + l\ln C - \frac{3l\ln T}{2} + \frac{3\alpha l(l+1)\ln 2}{4}\right) \|y^0\|_H
\end{aligned} \tag{19}$$

For each fixed T, $0 < \alpha < 1/2$ ensures that the dominant term in the exponent is $-C'T2^{(1-\alpha)l}$ as $l \to \infty$. This shows, as in [1], that the state is null-controllable since taking the limit in (19) gives

$$\lim_{l\to\infty} \|y^l\|_H = 0$$

with exponential decay.

To estimate u, we have

$$
\begin{aligned}
\|u\|_{L^2((0,T)\times\omega)} &= \sum_{l=1}^{\infty} \|u_{2^l}(\cdot - (a_{l-1}+T_l); T_l, z^l)\|_{L^2((a_{l-1}+T_l, a_l)\times\omega)} \\
&= \sum_{l=1}^{\infty} \|u(T_l, z^l)\|_{L^2((0,T_l)\times\omega)} \\
&\leq \sum_{l=1}^{\infty} \sqrt{C_{T_l}} e^{\sqrt{\mu_{2^l}}} \|y^l\|_H \\
&\leq C_\alpha \sum_{l=1}^{\infty} T^{-3/2} 2^{3\alpha l/2} e^{C2^{l/2}} \|y^l\|_H \text{ after using (9) and (18)} \\
&\leq C_\alpha \sum_{l=1}^{\infty} T^{-3/2} 2^{3\alpha l/2} e^{C2^{l/2}} \exp\left(-C'T2^{(1-\alpha)l} + C2^{l/2} + l\ln C - \tfrac{3l\ln T}{2} + \tfrac{3\alpha l(l+1)\ln 2}{4}\right) \|y^0\|_H \\
&= C_\alpha T^{-3/2} \sum_{l=1}^{\infty} \exp\left(-C'T2^{(1-\alpha)l} + C2^{l/2} + l\ln C - \tfrac{3l\ln T}{2} + \tfrac{3\alpha l(l+1)\ln 2}{4} + \tfrac{3\alpha l \ln 2}{2}\right) \|y^0\|_H \\
&\leq C_\alpha T^{-3/2} \sum_{l=1}^{\infty} \exp\left(-C'T2^{(1-\alpha)l} + C2^{l/2} - \tfrac{3l\ln T}{2}\right) \|y^0\|_H
\end{aligned}
$$

The last inequality above follows since, as $l \to \infty$, $C2^{(1-\alpha)l}$ dominates terms in the exponential that are positive and independent of T. Again, for fixed T, the sum converges by the same argument as in estimating $\|y^l\|_H$ as its terms are order $e^{-C'2^{(1-\alpha)l}}$ for $l \to \infty$. This argument again uses that $0 < \alpha < 1/2$.

To obtain a bound for $\|u\|$ as $T \searrow 0$, we continue

$$
\begin{aligned}
\|u\|_{L^2((0,T)\times\omega)} &\leq C_\alpha T^{-3/2} \sum_{l=1}^{\infty} \exp\left(-C'T2^{(1-\alpha)l} + C2^{l/2} - \frac{3l\ln T}{2}\right) \|y^0\|_H \\
&= C_\alpha T^{-3/2} \sum_{l=1}^{\infty} \exp\left(-C'T2^{(1-\alpha)l}\left(1 - \tfrac{C2^{(\alpha-1/2)l}}{C'T} + \tfrac{3l2^{(\alpha-1)l}\ln T}{2C'T}\right)\right) \|y^0\|_H \qquad (20)
\end{aligned}
$$

To bound this series, we first break the sum into two parts where the tail is composed of terms where

$$
1 - \frac{C2^{(\alpha-1/2)l}}{C'T} + \frac{3l2^{(\alpha-1)l}\ln T}{2C'T} \geq \frac{1}{2}. \qquad (21)
$$

That is, when l is large enough so that

$$
\frac{C'T}{2} \geq C2^{(\alpha-1/2)l} - \frac{3l2^{(\alpha-1)l}\ln T}{2}.
$$

To do this we first consider the function $g(x) = \frac{3x2^{-x/2}}{2}$ for $x > 0$. g is bounded above by

$\delta := \frac{3}{e\ln 2}$. Thus, for $T > 0$ small,

$$-\delta \ln T \geq \frac{-3l2^{-l/2}\ln T}{2}$$
$$\iff (C - \delta\ln T)2^{(\alpha-1/2)l} \geq C2^{(\alpha-1/2)l} - \frac{3l2^{(\alpha-1)l}\ln T}{2}$$

Then for

$$l > l^* = \frac{\ln(2C - 2\delta\ln T) - \ln(C'T)}{(1/2-\alpha)\ln 2}, \tag{22}$$

we have that

$$\frac{C'T}{2} \geq (C - \delta\ln T)2^{(\alpha-1/2)l} \geq C2^{(\alpha-1/2)l} - \frac{3l2^{(\alpha-1)l}\ln T}{2}. \tag{23}$$

In for $l > l^*$, we have the inequality (21). In arriving at estimate (23), we are using the fact that $(C - \delta\ln T)2^{(\alpha-1/2)l}$ is a decreasing function in l. The tail of the sum (20) will involve terms $l > l^*$. To estimate this series, we will use the following estimate:

For $s > 0$,

$$\ln s < s \implies (1-\alpha)l\ln 2 < 2^{(1-\alpha)l} \text{ taking } s = 2^{(1-\alpha)l} \tag{24}$$

An application of the mean value theorem gives that for $\beta > 0$ and $0 < T < 1$,

$$\frac{1}{1-\exp(-\beta T)} < \frac{\exp\beta}{\beta T} \tag{25}$$

We now apply the estimates (21), (24) and (25) to the tail of the series in (20).

$$\begin{aligned}
\sum_{l=l^*+1}^{\infty} \exp\left(-C'T2^{(1-\alpha)l} + C2^{l/2} - \tfrac{3l\ln T}{4}\right) &\leq \sum_{l=l^*+1}^{\infty} \exp\left(-\tfrac{C'T}{2}2^{(1-\alpha)l}\right) \\
&\leq \sum_{l=l^*+1}^{\infty} \exp\left(-\frac{C'T(1-\alpha)l\ln 2}{2}\right) \\
&= \frac{\exp\left(-\frac{C'T(1-\alpha)\ln 2(l^*+1)}{2}\right)}{1-\exp\left(-\frac{C'T(1-\alpha)\ln 2}{2}\right)} \\
&< \frac{1}{1-\exp\left(-\frac{C'T(1-\alpha)\ln 2}{2}\right)} \\
&< \frac{2\exp\left(\frac{C'(1-\alpha)\ln 2}{2}\right)}{C'T(1-\alpha)\ln 2} \\
&\leq CT^{-1}
\end{aligned} \tag{26}$$

We break the remaining finite sum into two pieces using the inequality $ab \leq \frac{a^2+b^2}{2}$.

$$
\begin{aligned}
&\sum_{l=1}^{l^*} \exp\left(-C'T2^{(1-\alpha)l} + C2^{l/2} - \frac{3l\ln T}{2}\right) \\
&\qquad \leq \frac{1}{2}\left(\sum_{l=1}^{l^*} \exp\left(-2C'T2^{(1-\alpha)l} + 2C2^{l/2}\right) + \sum_{l=1}^{l^*} \exp\left(-3l\ln T\right)\right) \qquad (27)
\end{aligned}
$$

To deal with the first term, consider the following function on $\mathbb{R}$

$$
G_T(x) = -2C'T2^{(1-\alpha)x} + 2C2^{x/2}
$$

As fixed $\alpha \in (0, 1/2)$, then G_T enjoys the properties that $\lim_{x\to-\infty} G_T(x) = 0$, $\lim_{x\to\infty} G_T(x) = -\infty$ and $\frac{dG_T}{dx}$ has exactly one zero. Therefore G_T has a global maximum at $x^* = \frac{1}{(1/2-\alpha)\ln 2} \ln\left(\frac{C}{2C'T(1-\alpha)}\right)$. Further

$$
\begin{aligned}
G_T(x^*) &= -2C'T\exp\left(\frac{1-\alpha}{1/2-\alpha}\ln\left(\frac{C}{2C'T(1-\alpha)}\right)\right) + 2C\exp\left(\frac{\ln\left(\frac{C}{2C'T(1-\alpha)}\right)}{1-2\alpha}\right) \\
&\leq 2C\left(\frac{C}{2C'T(1-\alpha)}\right)^{(1-2\alpha)^{-1}} \\
&\leq CT^{(-1+2\alpha)^{-1}},
\end{aligned}
$$

giving the following estimate

$$
\begin{aligned}
\sum_{l=1}^{l^*} \exp(G_T(l)) &\leq l^* \exp(G_T(x^*)) \\
&\leq l^* \exp\left(CT^{(-1+2\alpha)^{-1}}\right) \qquad (28)
\end{aligned}
$$

The other finite sum can be dealt with in a similar way. For this estimate we need that

$$
l^* = \frac{\ln(2C - 2\delta\ln T) - \ln(C'T)}{(1/2-\alpha)\ln 2} \leq \frac{2C - 2\delta\ln T - \ln(C'T)}{(1/2-\alpha)\ln 2} \leq -C\ln T \qquad (29)
$$

and the estimate that $s^2 < \exp\left(\frac{s}{1-2\alpha}\right)$ for $s >> 1 \iff$ via $s = -\ln T$ we have $(\ln T)^2 < T^{(-1+2\alpha)^{-1}}$ where $0 < T << 1$. Therewith, we have

$$
\begin{aligned}
\sum_{l=1}^{l^*} \exp(-3l\ln T) &\leq l^* \exp(-3l^*\ln T) \\
&\leq l^* \exp\left(C(\ln T)^2\right) \\
&\leq l^* \exp\left(CT^{(-1+2\alpha)^{-1}}\right) \qquad (30)
\end{aligned}
$$

Combining (20) and (26) - (30), we have that

$$
\begin{aligned}
\|u\|_{L^2(0,T)\times\omega} \le C_\alpha T^{-3/2} & \left(\sum_{l=1}^{l^*} \exp\left[-C'T2^{(1-\alpha)l} + \sum_{l=l^*+1}^{\infty} C2^{l/2} - \frac{3l\ln T}{2} \right] \right. \\
& \left. + \exp\left[-C'T2^{(1-\alpha)l} + C2^{l/2} - \frac{3l\ln T}{2} \right] \right) \|y^0\|_H \\
& \le C_\alpha T^{-3/2} \left[-C\ln T \exp\left(CT^{(-1+2\alpha)^{-1}} \right) + CT^{-1} \right] \|y^0\|_H.
\end{aligned}
$$

Noting that the dominant term on the right hand side is $\exp\left(CT^{(-1+2\alpha)^{-1}}\right)$, we have that

$$\|u\|_{L^2((0,T)\times\omega)} \le C_\alpha \exp\left(CT^{(-1+2\alpha)^{-1}}\right) \|y^0\|_H.$$

Taking $\alpha = \frac{\epsilon}{2(1+\epsilon)}$ and taking the supremum over $y^0 \in H$ with $\|y^0\|_H = 1$ gives that

$$\mathcal{E}(T) = e^{\mathcal{O}(C/T^{1+\epsilon})} \text{ as } T \searrow 0$$

where the definition of the minimal energy is as given in (5). This completes the proof of Theorem 1.

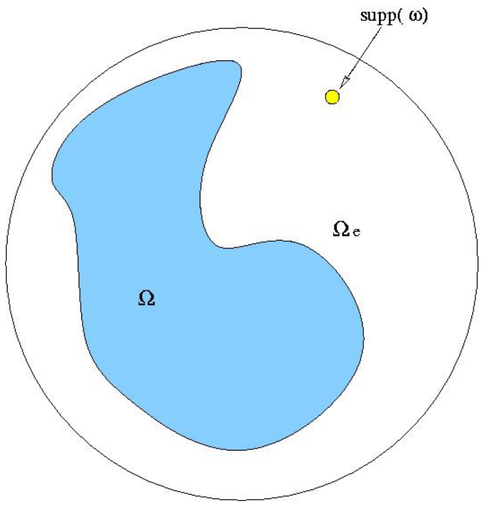
5 Proof of Theorem 2

We have just shown locally distributed null controllability of the following system:

$$
\begin{cases}
y^e_{tt} = -\Delta^2 y^e + \Delta y^e_t + \chi_\omega(x)u \quad \text{on} \quad (0,T)\times\Omega_e \\
y^e|_{\partial\Omega} = \Delta y^e|_{\partial\Omega} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega_e \\
[y^e(0), y^e_t(0)] = [y^e_0, y^e_1] \in H \equiv \left[H^2(\Omega_e)\cap H^1_0(\Omega_e)\right]\times L^2(\Omega_e),
\end{cases}
\tag{31}
$$

where Ω_e is a bounded open set in $\mathbb{R}^2$ with smooth boundary. In turn, we can use this result to show the null controllability property for the *boundary* controlled system (1).

The theme of the proof is ostensibly classical. Namely, we shall invoke the "embedding technique" alluded to in [15] (see also [19]). Given the smooth, bounded domain Ω, we surround this geometry with a larger domain Ω_e, as depicted in the following figure.



Subsequently, given initial data $[y_0, y_1] \in \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L^2(\Omega)$, we denote $[y_0^e, y_1^e] \in \left(H^2(\Omega_e) \cap H_0^1(\Omega_e)\right) \times L^2(\Omega_e)$ to be an extension of this data onto all of Ω_e; viz., $y_0^e|_{\Omega_e} = y_0$ and $y_1^e|_{\Omega_e} = y_1$. For this geometry Ω_e, we now consider the locally distributed problem (31), with $[y_0^e, y_1^e]$ being the aforesaid extensions of $[y_0, y_1]$. Also, importantly, in (31) the region of control support ω is configured so that $\bar{\Omega}\cap$ supp$(\omega) = \emptyset$.

As we said, we wish to employ the embedding technique of Seidman. To this end. we must derive a requisite regularity result, which is a direct consequence of the parabolic-like behavior of elastic operators under Kelvin-Voight damping.

Lemma 6. *Let $\{[w_0, w_1], f\} \in H \times L^2((0,T) \times \Omega_e)$. Define the function $[w, w_t] \in C([0,T]; H)$ by*

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \int_0^t e^{\mathcal{A}(t-s)} \begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds.$$

Then with the geometry Ω_e as pictured in Figure 1, we have the following "traces" for $w(t)$:

$$w|_{\partial\Omega} \in L^2(0,T; L^2(\partial\Omega)); \tag{32}$$

$$\Delta w|_{\partial\Omega} \in L^{2-\epsilon}(0,T; L^2(\partial\Omega)), \text{ where } \epsilon > 0. \tag{33}$$

Moreover,

$$\| w|_{\partial\Omega}\|_{L^2((0,T)\times\partial\Omega)} + \| \Delta w|_{\partial\Omega}\|_{L^{2-\epsilon}(0,T;L^2(\partial\Omega))} \leq C_T \left(\|[w_0, w_1\|_{\left[H^2(\Omega_e)\cap H_0^1(\Omega_e)\right]\times L^2(\Omega_e)} + \|f\|_{L^2((0,T)\times\Omega_e)} \right).$$

Assume for the time being the validity of Lemma 6. Then one can straightforwardly establish the reachability property of Theorem 2. In fact, let $u \in L^2((0,T) \times \omega)$ in (31) be the controller which drives the (extended) initial data $[y_0^e, y_1^e]$ to the zero state, in accord with Theorem 1. Then by Lemma 6, the corresponding trajectory $[y^e, y_t^e]$ satisfies the "trace" regularity

$$y^e|_{\partial\Omega} \in L^2(0,T; L^2(\partial\Omega)); \quad \Delta y^e|_{\partial\Omega} \in L^{2-\epsilon}(0,T; L^2(\partial\Omega)).$$

Since supp$(\omega) \cap \bar{\Omega}$ is empty, then uniqueness of the mixed Cauchy problem in will yield that the PDE system (1) can be steered to zero, by taking therein,

$$u_1 = y^e|_{\partial\Omega} \in L^2(0,T; L^2(\partial\Omega)) \text{ and } u_2 = \Delta y^e|_{\partial\Omega} \in L^{2-\epsilon}(0,T; L^2(\partial\Omega)).$$

Thus, the proof of Theorem 2 will be completed once Lemma 6 is shown to be true. This, we now proceed to do.

Proof of Lemma 6: In what follows, $H = \left[H^2(\Omega_e) \cap H_0^1(\Omega_e)\right] \times L^2(\Omega_e)$; and the realizations $S : D(S) \subset L^2(\Omega_e) \to L^2(\Omega_e)$, $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ are the same as those in defined in Section 1 (only being applied now to geometry Ω_e, rather than Ω).

Here we will use critically the fact that $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ generates an *analytic* semigroup of contractions on H; see [23]. We set

$$\begin{aligned} \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} &= \begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} + \begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix}, \\ \text{where } \begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} &= e^{\mathcal{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}; \quad \begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix} = \int_0^t e^{\mathcal{A}(t-s)} \begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds \end{aligned} \tag{34}$$

First, by the regularity associated with analytic generators–see e.g., [9]–we have the continuity of the following map:

$$f \to \int_0^{(\cdot)} e^{\mathcal{A}(\cdot - s)} \begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds \in \mathcal{L}\left(L^2(\Omega_e), L^2(0,T; D(\mathcal{A})\right). \tag{35}$$

To use this abstraction, we recall the following characterization of the fractional powers of (analytic) $\mathcal{A}$ in [6] (see specifically, Theorem 1.1 of [6]): *Let* $\frac{1}{2} \leq \theta \leq 1$. *Then,*

$$D(\mathcal{A}^\theta) = \left\{ \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H : w_0 \in D(S^{1+\theta}) \text{ and } w_1 \in D(S^\theta) \right\}. \tag{36}$$

Thus, combining (35) and (36), we have that the variable z defined in (34) satisfies

$$z \in L^2(0,T; D(S^2)).$$

By the definition of $S : D(S) \subset L^2(\Omega_e) \to L^2(\Omega_e)$ in (2), we have then,

$$z \in L^2(0,T; H^4(\Omega_e)), \text{ with continuous dependence on the data } f. \tag{37}$$

Moreover, making further use of the regularizing effect of analytic semigroups (see [9]), we have

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \to e^{\mathcal{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{L}\left(H, L^2(0,T; D(\mathcal{A}^{\frac{1}{2}})\right). \tag{38}$$

Combining this boundedness with the characterization in (36), we have then for the variable v defined in (34):

$$v \in L^2(0,T; D(S^{\frac{3}{2}})). \tag{39}$$

Moreover, from the characterization of the fractional powers of the self-adjoint operator $S : D(S) \subset L^2(\Omega_e) \to L^2(\Omega_e)$ we have, in terms of familiar Sobolev space,

$$D(S^{\frac{\theta}{2}}) \subset H^\theta(\Omega_e), \text{ for all } \theta > 0 \tag{40}$$

(see [7]). Applying this characterization to (39) yields now

$$v \in L^2(0,T; H^3(\Omega_e)), \text{ with continuous dependence on the data } [w_0, w_1]. \tag{41}$$

Now combining (34) with (37) and (41), we infer that

$$w \in L^2(0,T; H^3(\Omega_e)), \text{ with continuous dependence on the data } f \text{ and } [w_0, w_1].$$

Subsequently, from the Sobolev Embedding Theorem (as dimension $n = 2$), we have that $w \in L^2(0,T; C(\bar{\Omega}_e))$. We can now estimate $w|_{\partial\Omega}$ in the L^2-norm:

$$\begin{aligned} \int_0^T \|w|_{\partial\Omega}\|^2_{L^2(\partial\Omega)}\, dt &= \int_0^T \int_{\partial\Omega} \left(w(x,t)|_{\partial\Omega}\right)^2 dx dt \\ &\le C_{\partial\Omega} \int_0^T \|w(t)\|^2_{L^\infty(\Omega_e)}\, dt \le C \int_0^T \|w\|^2_{H^3(\Omega_e)}\, dy \\ &\le C_T \left(\|[w_0, w_1\|^2_H + \|f\|^2_{L^2((0,T)\times\Omega_e}\right), \end{aligned} \tag{42}$$

This work establishes (32).

The work to establish (33) is a bit more subtle. Indeed, from (34) and (41), we see that $\Delta w = \Delta v + \Delta z$, where $\Delta v \in L^2(0,T; H^1(\Omega_e))$, only. In consequence we cannot establish (spatially) pointwise values of $\Delta w(t)$ by means of the Soboloev Embedding Theorem, and so cannot proceed as before to estimate $\Delta w|_{\partial\Omega}$. Instead, we again make use of the underlying analyticity of the generator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$. In particular, we use the fact that, in regard to the semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\ge 0}$, there is the availability of the following pointwise estimate (see [23] and Theorem 6.13, p. 74, of [14]):

$$\left\| \mathcal{A}^\alpha e^{\mathcal{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_H \le \frac{C_\alpha}{t^\alpha} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_H \quad \text{for all } t > 0 \text{ and } \alpha \ge 0. \tag{43}$$

With this in mind, we again invoke the characterization (36), with $\theta \equiv \frac{1}{2} + \frac{\epsilon}{2}$ therein. This gives,

$$D(\mathcal{A}^{\frac{1}{2}+\frac{\epsilon}{2}}) = D(S^{\frac{3}{2}+\frac{\epsilon}{2}}) \times D(S^{\frac{1}{2}+\frac{\epsilon}{2}}) \subset H^{3+\epsilon}(\Omega_e) \times H^{1+\epsilon}(\Omega_e), \tag{44}$$

where in the last containment, we have reused the classical characterization in (40). Armed with (43) and (44), we revisit the component $v(t)$ of (34): For all $t > 0$,

$$\begin{aligned} \|v(t)\|_{H^{3+\epsilon}(\Omega_e)} &\le C \|v(t)\|_{D(S^{\frac{3}{2}+\frac{\epsilon}{2}})} \\ &\le \left\| \mathcal{A}^{\frac{1}{2}+\frac{\epsilon}{2}} e^{\mathcal{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_H \\ &\le \frac{C}{t^{\frac{1}{2}+\frac{\epsilon}{2}}} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_H. \end{aligned}$$

In turn, this estimate and the Sobolev Embedding Theorem yield, for all $t > 0$,

$$\|\Delta v(t)\|_{C(\bar{\Omega}_e)} \le C \|\Delta v(t)\|_{H^{1+\epsilon}(\Omega_e)} \le \frac{C}{t^{\frac{1}{2}+\frac{\epsilon}{2}}} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_H. \tag{45}$$

We are now in a position to estimate $\Delta v(t)|_{\partial\Omega}$ in the asserted norm: Using the estimate (45), and taking $\epsilon \equiv \frac{\delta}{4-\delta}$ therein, where $1 > \delta > 0$ is arbitrarily small, we have

$$\begin{aligned}
\int_0^T \| \Delta v(t)|_{\partial\Omega} \|^{2-\delta}_{L^2(\partial\Omega)} \, dt &= \int_0^T \left(\int_{\partial\Omega} (\Delta v(x,t)|_{\partial\Omega})^2 \right)^{\frac{2-\delta}{2}} dx dt \\
&\leq C_{\partial\Omega} \int_0^T \|\Delta v(t)\|^{2-\delta}_{L^\infty(\Omega_e)} \, dt \leq C_{\partial\Omega} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|^{2-\delta}_H \int_0^T \frac{dt}{\left(t^{\frac{1}{2}+\frac{\epsilon}{2}}\right)^{2-\delta}} \\
&= C_{\partial\Omega} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|^{2-\delta}_H \int_0^T \frac{dt}{t^{\frac{4-2\delta}{4-\delta}}} = C_\delta \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|^{2-\delta}_H . \qquad (46)
\end{aligned}$$

For the component z of (34): Since $z \in L^2(0,T;H^4(\Omega_e))$, from (37), then $\Delta z(t) \in L^2(0,T;H^2(\Omega_e))$. Thus, by the Sobolev Embedding Theorem and (37) we have,

$$\begin{aligned}
\Delta z &\in L^2(0,T;C(\bar{\Omega}_e)); \\
\|\Delta z\|_{L^2(0,T;C(\bar{\Omega}_e))} &\leq C \|f\|_{L^2((0,T)\times\Omega_e)} .
\end{aligned}$$

Subsequently,

$$\begin{aligned}
\int_0^T \| \Delta z(t)|_{\partial\Omega} \|^2_{L^2(\partial\Omega)} \, dt &= \int_0^T \left(\int_{\partial\Omega} (\Delta z(x,t)|_{\partial\Omega})^2 \right) dx dt \\
&\leq C \int_0^T \|\Delta z(t)\|^2_{L^\infty(\Omega_e)} \, dt \\
&\leq C \|f\|^2_{L^2((0,T)\times\Omega_e)} . \qquad (47)
\end{aligned}$$

The fact that $\Delta w(t)|_{\partial\Omega} \in L^{2-\delta}(0,T;L^2(\partial\Omega))$, with continuous dependence on the data, now follows from (34), (46) and (47). This completes the proof of Lemma 6, and so too of Theorem 2.

References

[1] G. Avalos and I. Lasiecka, *Optimal Blowup Rates for the Minimal Energy Null Control for the Strongly Damped Abstract Wave Equation* Annali della Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche, Serie V. Vol. II. Fasc. 3 (2003).

[2] G. Avalos and I. Lasiecka, *The null controllability of thermoelastic plates and singularity of the associated minimal energy function*", Journal of Mathematical Analysis and its Applications, 294 (2004), pp. 34-61.

[3] A. Benabdallah and M. Naso, *Null controllability of a thermoelastic plate*, Abstract & Applied Analysis, **7** (2002), pp. 585-599.

[4] G. Chen and D.L. Russell, *A mathematical model for linear elastic systems with structural damping*", Quart. Appl. Math., January (1982), pp. 433-454.

[5] , S. Chen and R. Triggiani, *Proof of two conjectures of G. Chen and D.L. Russell on structural damping for elastic systems: The case* $\frac{1}{2} \leq \alpha \leq 1$, Pacific Journal of Mathematics **136**, N1 (1989(a)), pp. 15-55.

[6] S. Chen and R. Triggiani, *Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications*, J. Diff. Eqn., **88** (1990(b)), pp. 279-293.

[7] P. Grisvard, *Caracterization de quelques espaces d'interpolation*, Arch. Rational Mech. Anal. **25** (1967), pp. 40-63.

[8] D. Jerison and G. Lebeau, *Nodal sets of sums of eigenfunctions*, Harmonic Analysis and Partial Differential Equations (Chicago, 1996), Lectures in Math. University, Chicago Press, Illinois (1999), pp. 223-239.

[9] I. Lasiecka, *Unified Theory for abstract parabolic boundary problems: A semigroup approach*, Applied Mathematics and Optimization, **6** (1980(b)), pp. 287-333.

[10] I. Lasiecka and T. I. Seidman, *Thermal boundary control of a thermoelastic system* preprint (2004).

[11] I. Lasiecka and R. Triggiani, *Exact controllability of the wave equation with Neumann boundary control*, Appl. Math. Optimi., **28** (1993), pp. 243-290.

[12] I. Lasiecka and R. Triggiani, *Exact null controllability of structurally damped and thermoelastic parabolic models*, Rendiconti Lincei, Matematica e Applicazione, Serie IX, Volume IX, Fasicolo 1 (1998), pp. 43-69.

[13] I. Lasiecka and R. Triggiani, "Control Theory for Partial Differential Equations: Continuous and Approximation Theories", Cambridge University Press, New York (2000).

[14] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[15] D. L. Russell, *A uniform boundary controllability theory for hyperbolic and parabolic partial differential equations*, Studies in Appl. Math. **52**, 3 (1973), pp. 189-211.

[16] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Review, Vol. 20, No. 6 (October 1978), pp. 639-739.

[17] V. J. Mizel and T. I. Seidman, *Observation and prediction for the heat equation*, Journal of Mathematical Analysis and Applications **38** (1972), pp. 149-166.

[18] T. I. Seidman, *How violent are fast controls?*, Math. of Control, Signals, Syst. **1** (1988), pp. 89-95

[19] T. I. Seidman, *Observation and prediction for the heat equation III*, Journal of Differential Equations, Volume 20, Issue 1, (January 1976), pp. 18-27).

[20] T. I. Seidman, *The coefficient map for certain exponential sums*, Nederl. Akad. Wetensch. Proc. Ser. A, **89** (1986), pp. 463-478.

[21] T. I. Seidman, *On uniform nullcontrollability and blow-up estimates*, Control Theory of Partial Differential Equations (O. Imanuvilov, G. Leugering, R. Triggiani, B. Y. Zhang, editors) Chapman and Hall/CRC, Boca Raton, Florida (2005), pp. 215–227.

[22] R. Triggiani, *Optimal estimates of norms of fast controls in exact null controllability of two non-classical abstract parabolic systems*, Advances in Differential Equations, Volume 8, No. 2 (2003), pp. 189-229.

[23] S. Chen and R. Triggiani, *Proof of extension of two conjectures on structural damping for elastic systems: The case* $\frac{1}{2} \leq \alpha \leq 1$, Pac. J. Math., **136**, No. 1 (1989(a)), pp. 15-55.