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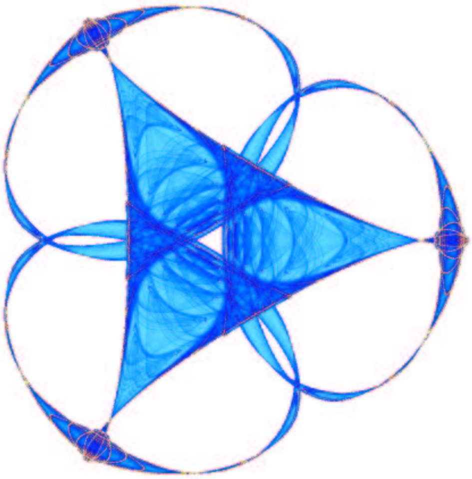

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Calculation of the Number of Tunnels

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Abstract. The paper considers 2-regions of grid cubes and proposes an algorithm for calculating the number of tunnels of such a region. The graph-theoretical algorithm proceeds layer by layer, a proof of its correctness is provided, and its time complexity is also given.

1 Introduction

Calculations of numbers of tunnels (i.e., of the first Betti number β_1) have been a subject of interest in 3D digital topology in the context of image analysis (see, e.g., [11, 12, 18]), in graph theory (see, e.g.,[23]), or in computational modeling of 3D forms (see, e.g., [6]),

The Euler characteristic $\chi(K) = \alpha_0 - \alpha_1 + \alpha_2$ of a bounded 3D set K is defined by numbers of cells in a (not uniquely defined) Euclidean surface complex; α_0 is the number of 0-cells (vertices), α_1 the number of 1-cells (edges), and α_2 the number of 2-cells (faces). The Poincaré formula [17]

$$\chi(K) = \beta_0 - \beta_1 + \beta_2$$

allows a calculation based on Betti numbers β_0 (number of connected components), β_1 (see above), and β_2 (number of closed surfaces). (For details, see, for example, chapter 6 in [9].) The Euler characteristic is used for characterizing the topology of cold dense matter at subnuclear density [22] (using integral geometry), in material sciences [21] (using estimates per volume unit), or for studying 3D biomedical structures such as cancellous bone architecture [19]. The number of tunnels is of interest in digital topology; see, e.g., [8, 13, 14, 20]. For a review on different algorithms for calculating the Euler number, and variations in selecting the adjacency model, see [15]. See also Section 5.3.3 in [9].

Obviously, a known Euler number does not yet specify numbers β_1 or β_2, even if $\beta_0 = 1$ can be assumed.

This paper proposes a particular algorithm for calculating β_1, where characterizing graphs are calculated layer by layer, and finally combined into one "3D" graph structure, characterizing the topology of the given 2-region of grid cubes. The algorithm is based on the idea of utilizing linear skeletons.

Figure 1 illustrates the complexity of calculating tunnels. A set K of the complexity of an office tower (left) would be considered 'floor by floor' by our algorithm (assuming an approximating cuboidal model). There can be 'hierarchies of cavities' in such a set K; 'columns' or 'openings between adjacent floors'

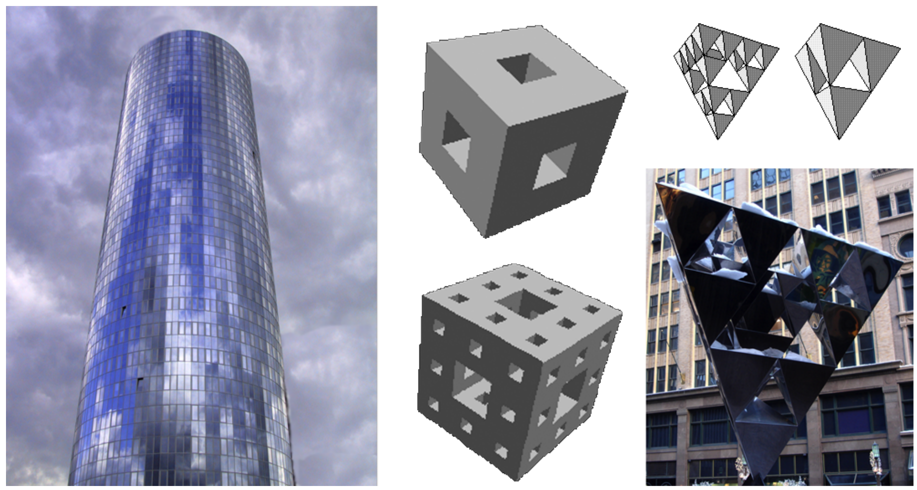

Fig. 1. Left: office tower (note: both open windows contribute to β_1) has $\beta_0 = 1$, $\beta_1 > 0$, and $\beta_2 > 0$. Middle: the Menger sponge has $\beta_0 = 1$ and $\beta_2 = 0$ for any iteration, $\beta_1 = 9$ tunnels at the start (top), and $\beta_1 = 209$ tunnels at the first iteration. Right: Sierpinski tetrahedron (start and first iteration), and its sculptural representation.

contribute to the number of tunnels. In contrast, the Menger sponge (already in appropriate cuboidal structure) does only have one closed surface (i.e., $\beta_2 = 0$). Here, local counts of surface cells (e.g., as in Section 5.3.3 of [9]) are sufficient for identifying the Euler number, and, thus, also the number of tunnels. At the start of the Menger sponge iterations, we (can) have $\alpha_0 = 40$, $\alpha_1 = 78$, and $\alpha_2 = 30$ (note that a Euclidean surface complex is not uniquely defined), and at the first iteration $\alpha_0 = 680$, $\alpha_1 = 1398$, and $\alpha_2 = 510$. Consequently, we have $\beta_1 = 9$ (start) or $\beta_1 = 209$ (first iteration) tunnels.[1] The Sierpinski tetrahedron also has $\beta_2 = 0$, for any iteration, and its cuboidal representation (e.g., by Gauss digitization) would allow similar calculations.

The paper is organized as follows: the next section recalls basic definitions, and Section 3 provides theoretical results which constitute the proof of correctness. The next two sections are about the algorithm and experiments, illustrating its use. Finally we provide a discussion of time complexity. Our conclusions end the paper.

2 Definitions

We apply the grid cell model [9], using 0-, 1-, 2-, or 3-cells in 3D space, which are vertices, grid edges, grid squares, or grid cubes, respectively. For example, two

[1] In the notation of [9], [1] defined a *gap* by a *separator*. A 2-region of grid cubes has a tunnel iff it has a 2-gap. Besides this, numbers of tunnels or of 2-gaps are not related because the first is a topologic, and the second a geometric property.

3-cells (*cubes*, for short) c_1 and c_2 are *2-adjacent* iff $c_1 \neq c_2$ and the intersection $c_1 \cap c_2$ contains a 2-cell.

In general, let S be a countable set and A an adjacency relation on S, defining an *adjacency structure*. If p and q are adjacent (in formal notation: pAq) then p, q are an *adjacency pair*. For any $p \in S$, the set $A(p) = \{q : pAq\}$ is the *adjacency set* of p. Maximal connected subsets of S are *components* of S. An adjacency structure $[S, A]$ is called an *adjacency graph* iff it has the following properties: $A(p)$ is finite for any $p \in S$; S is connected with respect to A; and any finite subset $M \subseteq S$ has at most one infinite complementary component. Any finite component of an adjacency graph is called a *region*. The set $A(M)$ of all nodes adjacent to $M \subseteq S$ is called the *adjacency set* of M. If $[S, A]$ is an adjacency graph, two disjoint subsets M_1 and M_2 of S are called *adjacent* iff $A(M_1) \cap M_2 \neq \phi$, denoted by $M_1 \ \mathcal{A} \ M_2$ or $(M_1, \ M_2) \in \mathcal{A}$. Let $\mathcal{R}$ be a partition of S into regions without including the infinite background component. The undirected graph $[\mathcal{R}, \mathcal{A}]$ is the *region adjacency graph* of $[S, A]$. For example, 2-adjacency of cubes defines 2-regions and related adjacency graph of regions of cubes.

Following Section 4.2.2 of [9], the *merging of two adjacent nodes* p and q in a graph $G = [S, A]$ is defined by replacing p and q with a new node r that is adjacent to every node in $S \setminus \{p, q\}$ to which p or q was originally adjacent. A finite sequence of merging operations is called a *contraction*; see Figure 2. If graph G_1 is contracted into G_2, then G_1 and G_2 are *homeomorphic*.

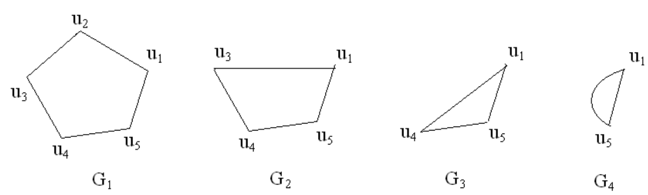

Fig. 2. G_1 is contracted into G_4; G_4 can be further contracted into a loop, but not into a single node (without any edge).

The *linear skeleton* (see Section 6.3.3 of [9]) of a set $M \subseteq \mathbb{E}^3$ is defined by continuos contractions. We transform a linear skeleton into a graph by having a node at each of its singular points (see Section 7.1.3), and arcs between singular points define the edges.

Definition 1. *Let S be a 2-region of cubes and G a graph. G is a* homotopic graph *of S iff G is homeomorphic to the linear skeleton of S.*

In Figure 3, G_1 and G_2 are both homotopic graphs of S_1, and G_3 and G_4 are both homotopic graphs of S_2. – Two relational structures $[S_1, R_1]$ and $[S_2, R_2]$ are called *isomorphic* iff there exists a one-to-one mapping f from S_1 onto S_2 such that pR_1q iff $f(p)R_2f(q)$, for all $p, q \in S_1$. f is called an *isomorphism*.

Definition 2. *Let S be a 2-region and G a homotopic graph of S. S is called a* minimal *cubical set iff G is not isomorphic to any homotopic graph of a 2-region S', where S' is obtained by removing a cube from S.*

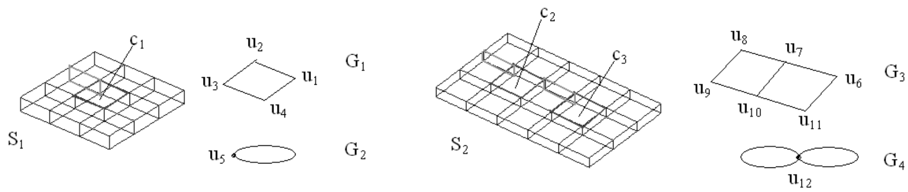

Fig. 3. Two 2-regions of cubes with homotopic graphs. S_1 does not contain cube c_1, and S_2 does not contain cubes c_2 and c_3.

A *subregion* is a (non-empty) connected subset of a region. Assume a region adjacency graph and a partition of involved regions into subregions; these subregions define a *subregion adjacency graph.*

Definition 3. *Let L be a set of cubes and H a homotopic graph of L. Let $\mathcal{R}$ be a partition of L into subregions without including the infinite background component. Let $[\mathcal{R}, \mathcal{A}]$ be the subregion adjacency graph of $\mathcal{R}$. If $[\mathcal{R}, \mathcal{A}]$ is isomorphic to H, then $[\mathcal{R}, \mathcal{A}]$ is called a* homotopic subregion adjacency graph *of L with respect to H, denoted by $\bar{H}$.*

Let f be the isomorphism; for every $v \in V(H)$, $f(v)$ is denoted by $\bar{v}$, and for every $V \subseteq V(H)$, $f(V)$ is denoted by $\bar{V}$. Figure 4 shows a layer L of cubes. H is a homotopic graph of L. Let $\bar{H}$ be a subregion adjacency graph of L, and $\bar{V}$ the vertices of $\bar{H}$. Each vertex $\bar{v} \in \bar{V}$ is a set of cubes.

In the rest of this paper, we denote a graph by $G = [V, E]$, where $V = V(G)$ is the set of nodes and E is the set of edges. If the relation $\mathcal{A}$ (or set E) is obviously defined for a homotopic subregion adjacency graph $\bar{H} = [\mathcal{R}, \mathcal{A}] = [V, E]$ then we denote $\bar{H}$ just by $\bar{V}(H)$ or $\bar{V}$.

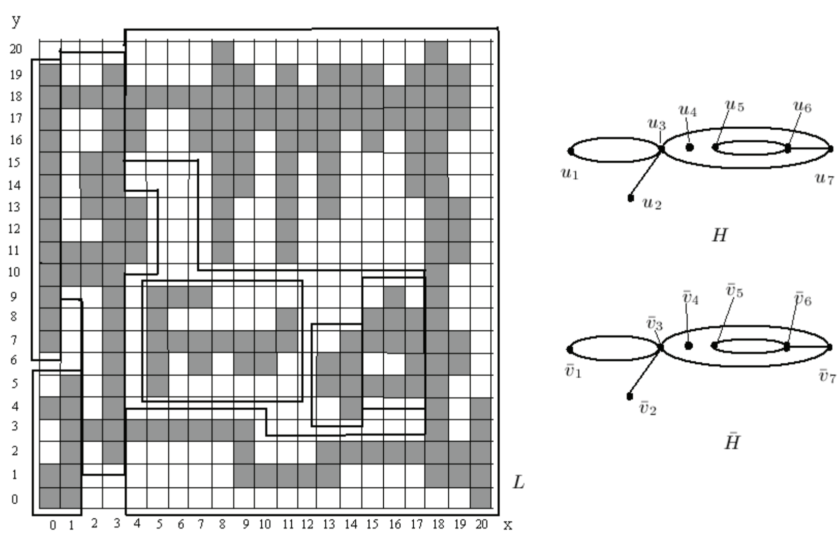

Fig. 4. A layer of cubes and a homotopic subregion adjacency graph with respect to a homotopic graph of it.

Let $G = [V, E]$ be a graph and $V_1 \subseteq V$. Subgraph $[V_1, E_1]$ is called the *restriction* of G on V_1, where $E_1 = \{v_1 v_2 : v_1 \in V_1 \wedge v_2 \in V_1\}$. – The following definition is a special case of Definition 2.64 in [10].

Definition 4. *Let S be a family of cubical sets in kD space, where $k = 0, 1, 2, 3$. Let X be the set of i-cells contained in S, with $i \leq k$. Let p, $q \in X$ such that p is a j-cell, q is a $(j-1)$-cell and p is the unique j-cell in X incident with q, $j \leq k$. Let $X' = X \backslash \{p, q\}$ and $S' = \cup_{\{c \in X'\}} c$. Then S' is called* obtained from S via an elementary collapse of p by q.

3 Basics

Let S be a family of cubical sets in kD space, where $k = 0, 1, 2, 3$. Let p be a j-cell, q a $(j-1)$-cell, for $j \leq k$. Let S' be obtained from S via an elementary collapse of p by q. Let $H_*(S')$ or $H_*(S')$ be the homology group (see Section 6.4.6 in [9]) of S or S', respectively. Then, by Theorem 2.68 in [10], it follows that

$$H_*(S') \sim H_*(S) \tag{1}$$

where $\sim$ is the homology (equivalence) relation.

Lemma 1. *Let $G = [V, E]$ be a connected finite graph. There exists a minimal cubical set S such that G is a homotopic graph of S.*

Proof. Let $\mathrm{card}\, V = n$ and $V = \cup_{i=1}^{n} \{v_i\}$. Let d_i be the degree of $v_i \in V$, and $h_i = 2 \sum_{j=1}^{i} d_j - 1$, for $i = 1, 2, \ldots, n$.

For each v_i, construct a simple arc of length $h_i + 2$, denoted by $g_i = \cup_{j=1}^{h_i+2} \{(2i-1, j, 1)\}$, where $(2i-1, j, 1)$ are the coordinates of the center of a grid cube, for $i = 1, 2, \ldots, n$. Suppose there are $l(i, j)$ edges between v_i and v_j. Without loss of generality, assume that $i < j$. If $l(i, j) \geq 1$ then construct $l(i, j)$ simple arcs of length $2(j - i) + 3$ as follows:

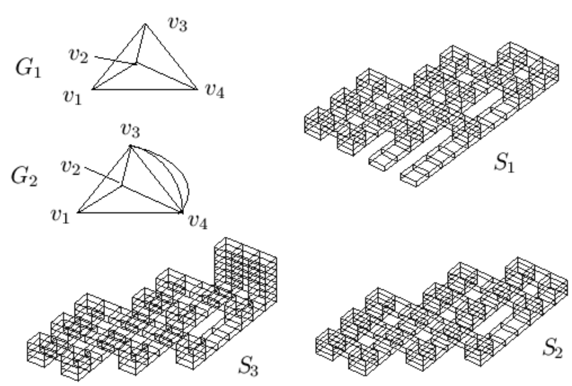

Fig. 5. Illustration for Lemma 1.

Let $A(v_i) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{d_i}}\}$ be the adjacency set of node v_i, with $i_1 < i_2 < \ldots < i_{d_i}$. Assume that $j = i_r$, where $r \in \{1, 2, \ldots, d_i\}$. Let $s(i,j) = h_{i-1} + 2r + 1$ and $g(i,j,z) = \cup_{x=2i-1}^{2j-1}\{(x, s(i,j), z)\} \cup \{(2i-1, s(i,j), z-1), (2j-1, s(i,j), z-1)\}$, where z is an integer.

If $l(i,j) \geq 1$ let $g(i,j) = \cup_{z=3}^{l(i,j)+2} g(i,j,z)$. Otherwise let $g(i,j) = \phi$. Let $S = \cup_{i=1}^{n}\{g_i\} \cup (\cup_{1\leq i<j\leq n}^{n} g(i,j))$. If there is a cube c in S such that there is only one cube c' in S such that $c \cap c'$ is a 2-cell then remove c from S. Repeat this operation until S does not contain such a cube c, and S is as required. □

In Figure 5, S_2 is obtained from S_1 after removing 6 cubes. S_2 is a minimal cubical set and G_1 is a homotopic graph of S_2. S_3 is a minimal cubical set and G_2 is a homotopic graph of S_3. – From homology (1) and Lemma 1, we obtain the following:

Theorem 1. *If a cubical 2-region S is minimal such that a given graph G is a homotopic graph of S, then we have $\beta_1(S) = \beta_1(G)$.*

We also recall the following theorem by J. W. Alexander (see Theorem 6.8 in [9]).

Theorem 2. *Let S_1 and S_2 be two Euclidean complexes defined by partitions of polyhedra $\cup K_1$ and $\cup K_2$. If $\cup K_1$ and $\cup K_2$ are homeomorphic then K_1 and K_2 have the same Betti numbers.*

Corollary 1. *If S is a 2-connected cubical set and G a homotopic graph of S. Then the first Betti number of S is equal to the Betti number of G.*

Proof. Let S' is a minimal cubical 2-region such that G is a homotopic graph of S. Then S' is homeomorphic to S. Theorems 1 and 2 prove the corollary. □

We recall that a sequence $(p_0, p_1, \cdots, p_n)$ of 2-cells is a *1-path of 2-cells* iff p_{i+1} is 1-adjacent to p_i, for $0 \leq i \leq n-1$. p_0, p_n are the *endnodes* of this path.

Let S be a 2-region of cubes with nonempty intersections with grid layers L_i, for $i = 1, \ldots, i_{max}$. Let S_i be a component of the intersection of S with layer L_i, H_i a homotopic graph of S_i, $\bar{H}_i$ be a homotopic subregion adjacency graph of S_i with respect to H_i, and let $\bar{u}_i, \bar{v}_i, \bar{w}_i \in \bar{H}_i$, for $i = 1, \ldots, i_{max}$.

Lemma 2. *If there exists a 1-path of 2-cells $(p_0, p_1, \cdots, p_n)$ such that p_0 is contained in $\bar{u}_1 \cap \bar{u}_2$ and p_n is contained in $\bar{w}_1 \cap \bar{w}_2$, then the set of cubes $(\bar{u}_1 \cup \bar{u}_2) \cup (\bar{w}_1 \cup \bar{w}_2)$ is simply connected.*

Proof. Let g be a closed curve contained in $\bar{u}_1 \cup \bar{u}_2 \cup \bar{w}_1 \cup \bar{w}_2$. Let P_2 be an arc of g such that P_2 is contained in $\bar{u}_2 \cup \bar{w}_2$ and $P_2 \cap (p_0, p_1, \cdots, p_n) = \{A, B\}$. Since $\bar{u}_2 \cup \bar{w}_2$ is simply connected then (see Figure 6) arc AP_2B can be continuously contracted into line segment $AP_1B \subset p_0, p_1, \cdots, p_n$. Therefore, within $\bar{u}_1 \cup \bar{u}_2 \cup \bar{w}_1 \cup \bar{w}_2$, g can be continuously contracted into a line segment and then into a single point in $(p_0, p_1, \cdots, p_n)$. □

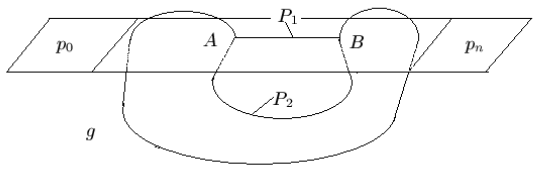

Fig. 6. Illustration to the proof of Lemma 2.

Now consider a pair (v_1, v_2), where $v_i \in V(H_i)$ and $\bar{v}_i \in V(\bar{H}_i)$, for $i = 1, 2$. Let c be the number of components of $\bar{v}_1 \cap \bar{v}_2$ (It is a set of 2-cells.).

Operation 1. If $c \geq 1$ then add c edges between v_1 and v_2 to graph $H_1 \cup H_2$. Let E be the set of all edges added. Let V_i be the set of such $v_i \in V(H_i)$. In other words, $V_1 = \{v_1 : v_1 \in V(H_1) \wedge v_1v_2 \in E\}$ and $V_2 = \{v_2 : v_2 \in V(H_2) \wedge v_1v_2 \in E\}$. Then $V_i \subseteq V(H_i)$, for $i = 1, 2$.

Operation 2. Let $\bar{U}_i$ be the restriction of $\bar{H}_i$ on $\bar{V}_i$, for $i = 1, 2$. For each pair $(\bar{u}_i, \bar{w}_i)$, where $\bar{u}_i$, $\bar{w}_i \in \bar{U}_i$, let g be the number of 1-paths of 2-cells $(p_0, p_1, \cdots, p_n)$ such that p_0 is contained in $\bar{U}_1 \cap \bar{U}_2$ and p_n is contained in $\bar{w}_1 \cap \bar{w}_2$. See Figure 7 for an example. (Note that $\bar{u}_1 \cap \bar{u}_2$ and $\bar{w}_1 \cap \bar{w}_2$ are also

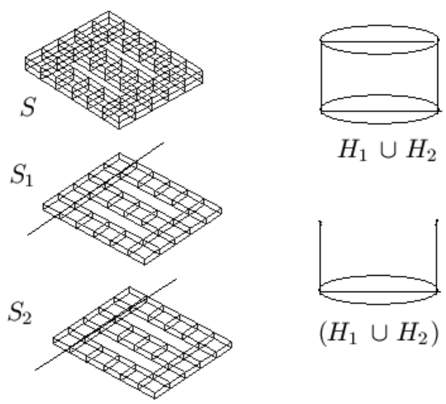

Fig. 7. Example with $g = 3$.

sets of 2-cells.) Delete p edges between u_1 and w_1 in H_1.

Let $(H_1 \cup H_2)$ be the resulting graph after applying Operations 1 and 2 on $H_1 \cup H_2$. Then we have

Lemma 3. *$(H_1 \cup H_2)$ is a homotopic graph of the set $S_1 \cup S_2$ of cubes.*

Proof. Since H_i is a homotopic graph of S_i, it follows that S_i can be continuously contracted into H_i, where $i = 1, 2$. The edges, added between H_1 and H_2, are not redundant because when S_i is contracted continuously, the subset of cubes $\bar{v}_1 \cup \bar{v}_2$ must also be contracted or extracted continuously. Therefore, the only possible redundant edges (because of duplicating) may be edges such as $u_i w_i$ in H_i, for $i = 1, 2$. Together with Lemma 2, this proves the lemma. □

Let $H = \cup_{i=1}^{l} H_i$. Repeat Operations 1 and 2 on $H_i \cup H_{i+1}$ until Stop (this is guaranteed to happen, after a finite number of steps), where $i = 1, 2, \ldots, l$ - 1. Let the resulting graph be (H). Then, by Lemma 3, we have the following:

Theorem 3. *(H) is a homotopic graph of S.*

The following algorithm is justified by this theorem.

4 Subprocess and Algorithm

At first we present an algorithm for producing a homotopic subregion adjacency graph and a homotopic graph, for any given layer of cubes within the given 2-region. This is then used to describe our main algorithm.

4.1 Preparation: Runs

A layer of cubes can be decomposed into a number of columns (of cubes); cubes in the same column have the same x-coordinate. A column of cubes can then be decomposed into a number of runs (i.e., *runs*) of cubes. Cubes in the same run have consecutive y-coordinates (i.e., each run is a simple cube-arc). Each run can be represented by its two endcubes.

A run I_1 is *left-(right-)adjacent* to a run I_2 if the x-coordinate of I_1 is less (greater) than that of I_2, and there exists a cube $c_i \in I_i$ such that $c_1 \cap c_2$ is a 1-cell.

Each left- or right-adjacent run of run I is called a *neighbor* of I. A *run-path* is a sequence of runs $(i_0, i_1, \cdots, i_n)$ such that i_{j+1} is the unique right neighbor of i_j ($0 \leq j \leq n-1$ and $n \geq 1$); i_0 and i_n are called the *endruns* of the path. The endrun with smaller (larger) x-coordinate is called *the first (second)* endrun. A *maximal* run-path is a run-path such that the number of left neighbors of its first endrun is not one, and the number of right neighbors of its second endrun is also not one. If the number the neighbors of the second endrun of a run-path p_I is greater than one, then it is called the *head-run* of p_I, denoted by h_{p_I}. Otherwise the first endrun of p_I is called the head-run of p_I. In both cases, we say p_I belongs to h_{p_I}. The second endrun of p_I, different to h_{p_I}, is called the *tail-run* of p_I. We remove the tail-run from the run-path if the number of its the neighbors is greater than one. The resulting run-path is called the *reduced* maximal run-path. Thus, if the number of neighbors of the tail-run is one, then the reduced maximal run-path of a run-path coincides with itself.

In Figure 4, the run in column 18 (the cubes in this run have x-coordinate 18 and y-coordinates between 1 and 20) has 3 left neighbors and 4 right neighbors. There are two maximal run-paths from column 3 to column 18. The run in column 3 is the first end- and tail-run of both, and the run in column 18 is the second end- and the head-run of both. With the tail-run removed, they induce two reduced maximal run-paths from column 4 to column 18. There is also a maximal run-path from column 5 to 11 (i.e., a disconnected "island" in the largest of the three holes). The run in column 5 is the first end- and head-run of it, and the run in column 11 is the second end- and tail-run of it. It coincides with its reduced maximal run-path (The tail-run in column 11 is not removed because it has only one neighbor).

4.2 Single-Layer Subprocess

For a single layer of cubes, do the following:

1. Get the set of all runs, denoted by S_I.
2. For each run in S_I, get and set its left and right neighbors.
3. Get the set of all maximal run-paths, denoted by S_P.
4. For each run-path in S_P, get and set its head- and tail-runs.
5. Get the set of all reduced maximal run-paths from S_P, denoted by S_{RP}.

6. Get the set of all runs, denoted by S_H such that for each run h in S_H, the number of neighbors of h is greater than two, or the number of left neighbors of h equals two and the number of right neighbors of h equals zero, or the number of left neighbors of h equals zero and the number of right neighbors of h equals two.

7. Get the set of head-runs of all run-paths in S_{RP}, denoted by S_{PH}.

8. For each h in $S_H \cup S_{PH}$, reset its neighbors as follows:

8.1. For each pair h_1, h_2 in $S_H \cup S_{PH}$, if they have a common neighbor I, remove I in the set of neighbors of h_i where the index i satisfies $x_i < x_{i+1}$ (mod 2), where x_i is the x-coordinate of h_i, for $i = 1, 2$.

8.2. For each h in S_H, remove its neighbor if this neighbor is contained in a run-path in S_{RP} (Note that 8.2 and 8.1 are independent of each other because, by definition, a run-path has at least two runs).

9. For each head-run h in $S_H \cup S_{PH}$, let $C_h = C \cup C_N \cup C_p$, where C is the set of all cubes contained in h, C_N is the set of all cubes contained in neighbors of h, and C_p is the set of all cubes contained in reduced maximal run-paths belonging to h (i.e., $C_P = \{c : c \in p \in S_{RP} \wedge p \text{ belongs to } h\}$).

10. Get a homotopic subregion adjacency graph $\bar{H}$ by $S_H \cup S_{PH}$, where each subregion is a set C_h as obtained in Step 9;

11. Get a homotopic graph H from $\bar{H}$.

Section 5.1 illustrates this algorithm by means of an example. We use now this single-layer subprocess for describing the main procedure, which computes the number of tunnels of a 2-region of cubes.

4.3 Algorithm

Assume a 2-region S of cubes. We proceed as follows:

1. Decompose S into layers, defined by all those z-coordinates with cells in S. (Each *layer* is thus the subset of S defined by an identical z-coordinate.) Let

L_i be the i-th layer of S, where i = 1, 2, ..., m, and m is the number of layers of S.

2. By the single-layer subprocess, construct a homotopic graph H_i for each layer L_i, and a homotopic subregion adjacency graph, $\bar{H}_i$, with respect to H_i.

3. Without loss generality, let L_1 and L_2 be any two consecutive layers. Let H_1 or H_2 be homotopic graphs of L_1 or L_2, respectively. Let $\bar{H}_1$ or $\bar{H}_2$ be homotopic subregion adjacency graphs with respect to H_1 or H_2, respectively. For each pair (v_1, v_2), where $v_1 \in V(H_1)$, $v_2 \in V(H_2)$, $\bar{v}_1 \in V(\bar{H}_1)$, and $\bar{v}_2 \in V(\bar{H}_2)$, do the following:

3.1. Apply the FILL algorithm (see, e.g., [9]) to find the number of components of $\bar{v}_1 \cap \bar{v}_2$ (which is a set of 2-cells), denoted by c.

3.2. If $c \geq 1$, then add c edges between v_1 and v_2 in graph $H_1 \cup H_2$. Let E be the set of all edges added. Let V_i be the set of these $v_i \in V(H_i)$. In other words, $V_1 = \{v_1 : v_1 \in V(H_1) \wedge v_1v_2 \in E\}$ and $V_2 = \{v_2 : v_2 \in V(H_2) \wedge v_1v_2 \in E\}$. Then $V_i \subseteq V(H_i)$, for i = 1, 2.

3.3. Let $\bar{U}_i$ be the restriction of $\bar{H}_i$ on $\bar{V}_i$, where i = 1, 2. For each pair $(\bar{u}_i, \bar{w}_i)$, where $\bar{u}_i$, $\bar{w}_i \in \bar{U}_i$, apply the FILL algorithm to find the number of 1-paths of 2-cells $(p_0, p_1, \cdots, p_n)$ such that p_0 is contained in $\bar{u}_1 \cap \bar{u}_2$, and p_n is contained in $\bar{w}_1 \cap \bar{w}_2$ (where both $\bar{u}_1 \cap \bar{u}_2$ and $\bar{w}_1 \cap \bar{w}_2$ are again sets of 2-cells), denoted by p. Delete p edges between u_1 and w_1 in H_1 (see Lemma 2).

4. Apply Step 3 to every pair L_i and L_{i+1} of two consecutive layers, for $i = 1, 2, \ldots, i_{max} - 1$. Let G be the resulting graph. Then the number of tunnels of S equals $E(G)$ - $V(G)$ +1, see Equation (2).

For this final step we make use of a basic result in combinatorial graph theory (see, for example, [4, 23]). Let G be a graph with n nodes, m edges and k connected components. Then we have

$$\beta_1(G) = m - n + k \tag{2}$$

A proof of this equation is straightforwardly by mathematical induction on the number of edges: a new edge either increments the number of 1-cycles or decrements the number of connected components.

5 Examples

5.1 An Example for the Single-Layer Subprocess

For an implementation in C++ or Java, S_I, S_P, S_{RP}, S_H, S_{PH}, and neighbors of a run can be represented by *Vector Objects*. Each run is represented by a *Class*. We discuss the the layer of cubes as shown in Figure 4.

Step 1. Column 18 has only one run (with 20 cubes). Column 0 has three runs. There are 62 runs altogether.

Step 2. A Vector Object in C++ or Java solves this tasks straightforwardly.

Step 3. Simply go through each run I in S_I to get the maximal run-path which contains I, then take the union of these maximal run-paths to get S_P. There are seven elements in S_P. Apart from the three already described in Section 4, there are two more maximal run-paths from column 0 to 3 and another two from column 14 to 17.

Step 4. For each run-path, its head- and tail-runs can be found and set by checking the number of neighbors of its endruns. The two maximal run-paths from column 0 to 3 have their head-run in column 3 and tail-run in column 0 while the two from column 14 to 17 have their head-run in column 17 and tail-run in column 14.

Step 5. S_{RP} (a Vector Object) can remove an element easily if necessary. Corresponding to the four elements in S_P from Step 4, we remove two reduced maximal run-paths, from column 1 to 3, with tail-run in column 0. Another two reduced maximal run-paths from column 15 to 17 have their head-run in column 17, with tail-run in column 14 removed (because it has 3 neighbors; so it must be removed).

Step 6. Each run in S_H will later correspond to, or induce a subregion. A run is put into S_H iff it has more than 3 neighbors, or it is "the end of a cycle" of runs.

Step 7. Each run in S_{PH} will also later correspond to, or induce a subregion.

Step 8. This step ensures that each run must be a neighbor of only one other run, or contained in only one run-path of only one element in $S_H \cup S_{PH}$. The run in column 2 with y-coordinate 3 is now a left neighbor of the run in column 3. It is no longer a right neighbor of the run in column 1. Moreover, the run in column 4 with y-coordinate 3 is now contained in a reduced maximal run-path with the head-run in column 18. It is no longer a right neighbor of the run in column 3.

Step 9. In this step we get all those cubes "represented" (or "dominated") by an element in $S_H \cup S_{PH}$. If we think of these cubes as being "supporters" of an element in $S_H \cup S_{PH}$, then the head-run in column 0 (with y-coordinates from 7 to 19) has 13 supporters, the head-run in column 1 (with y-coordinates from 0 to 5) has 9 supporters, and the head-run in column 3 (with y-coordinates from 2 to 19) has 31 supporters. Each cube must be a supporter, and can only support one head-run.

Step 10. Each of the polygons shown in Figure 4 contains cubes of one C_h obtained in Step 9. For any pair C_{h_1} and C_{h_2}, obtained in Step 9, let k be the number of 2-cells in $C_{h_1} \cap C_{h_2}$. Then there will be k edges between C_{h_1} and C_{h_2} in the homotopic subregion adjacency graph $\bar{H}$ shown in Figure 4, where $\bar{v}_i$ represents C_{h_i}, for $i = 1, 2, \ldots, 7$.

Step 11. See H in Figure 4.

5.2 Examples for the Main Algorithm

Each cube is represented (in the grid-point model) by the coordinates (x, y, z) of its center, where x, y, and z are integers. Consequently, each 2-cell is represented by the coordinates (x, y, z) of its center, where two of them are integers, and the other one is a half-integer $i.5$. Figure 8 shows the following five sets of cubes:

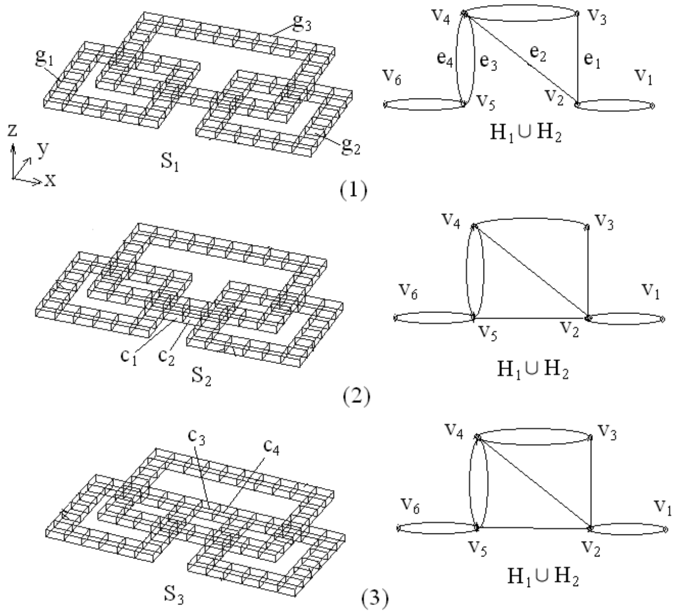

Fig. 8. Three input examples, used for illustrating the main algorithm.

Let g_1 = { (1, 0, 0), (2, 0, 0), (3, 0, 0), (4, 0, 0), (5, 0, 0), (6, 0, 0), (6, 1, 0), (6, 2, 0), (6, 3, 0), (6, 4, 0), (1, 1, 0), (1, 2, 0), (1, 3, 0), (1, 4, 0), (1, 5, 0), (2, 5, 0), (3, 5, 0), (4, 5, 0), (5, 5, 0), (6, 5, 0) }. g_1 is a simple 2-curve of 20 cubes in layer 2.

Let g_2 = { (9, 0, 0), (10, 0, 0), (11, 0, 0), (12, 0, 0), (13, 0, 0), (14, 0, 0), (14, 1, 0), (14, 2, 0), (14, 3, 0), (14, 4, 0), (9, 1, 0), (9, 2, 0), (9, 3, 0), (9, 4, 0), (9, 5, 0), (10, 5, 0), (11, 5, 0), (12, 5, 0), (13, 5, 0), (14, 5, 0) }. g_2 is also a simple 2-curve of 20 cubes in layer 2.

Let g_3 = { (3, 2, 1), (4, 2, 1), (5, 2, 1), (6, 2, 1), (7, 2, 1), (8, 2, 1), (9, 2, 1), (10, 2, 1), (11, 2, 1), (12, 2, 1), (12, 3, 1), (12, 4, 1), (12, 5, 1), (12, 6, 1), (12, 7, 1), (12, 8, 1), (3, 3, 1), (3, 4, 1), (3, 5, 1), (3, 6, 1), (3, 7, 1), (3, 8, 1), (4, 8, 1), (5, 8, 1), (6, 8, 1), (7, 8, 1), (8, 8, 1), (9, 8, 1), (10, 8, 1), (11, 8, 1) }. g_3 is a simple 2-curve of 30 cubes in layer 1.

Let g_4 = { (7, 2, 0), (8, 2, 0) } = { c_1, c_2 }. g_4 is a simple 2-arc of two cubes in layer 2. and

Let g_5 = { (7, 5, 0), (8, 5, 0) }= { c_3, c_4 }. g_5 is also a simple 2-arc of two cubes in layer 2.

Example 1. We consider $S_1 = L_1 \cup L_2$, where $L_1 = g_3$ and $L_2 = g_1 \cup g_2$. $\bar{H}_1 = [\bar{V}_1, \bar{E}_1]$, where $\bar{V}_1 = \{\bar{v}_3, \bar{v}_4\}$; $\bar{H}_2 = [\bar{V}_2, \bar{E}_2]$, where $\bar{V}_2 = \{\bar{v}_1, \bar{v}_2, \bar{v}_5, \bar{v}_6\}$, with

$\bar{v}_1$ = { (10, 0, 0), (11, 0, 0), (12, 0, 0), (13, 0, 0), (14, 0, 0), (14, 1, 0), (14, 2, 0), (14, 3, 0), (14, 4, 0), (14, 5, 0) }

$\bar{v}_2$ = { (9, 0,0), (9, 1, 0), (9, 2, 0), (9, 3, 0), (9, 4, 0), (9, 5, 0), (10, 5, 0), (11, 5, 0), (12, 5, 0), (13, 5, 0) }

$\bar{v}_5$ = { (6, 0, 0), (6, 1, 0), (6, 2, 0), (6, 3, 0), (6, 4, 0), (6, 5, 0), (2, 5, 0), (3, 5, 0), (4, 5, 0), (5, 5, 0) }

$\bar{v}_6$ = { (1, 0, 0), (2, 0, 0), (3, 0, 0), (4, 0, 0), (5, 0, 0), (1, 1, 0), (1, 2, 0), (1, 3, 0), (1, 4, 0), (1, 5, 0) }

Sets $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_5$ and $\bar{v}_6$ are simple 2-arcs, each of ten cubes in layer 2. Furthermore

$\bar{v}_3$ = { (4, 8, 1), (5, 8, 1), (6, 8, 1), (7, 8, 1), (8, 8, 1), (9, 8, 1), (10, 8, 1), (11, 8, 1), (12, 8, 1), (12, 2, 1), (12, 3, 1), (12, 4, 1), (12, 5, 1), (12, 6, 1), (12, 7, 1) }

$\bar{v}_4$ = { (3, 2, 1), (3, 3, 1), (3, 4, 1), (3, 5, 1), (3, 6, 1), (3, 7, 1), (3, 8, 1), (4, 2, 1), (5, 2, 1), (6, 2, 1), (7, 2, 1), (8, 2, 1), (9, 2, 1), (10, 2, 1), (11, 2, 1) }

Sets $\bar{v}_3$ and $\bar{v}_4$ are simple 2-arcs, each containing 15 cubes in layer 1.

For Step 3.2 of the algorithm, see Figure 8 or the simple calculation $\bar{v}_4 \cap \bar{v}_2$ = { (9, 2, 0.5) }. This implies that the number of components of $\bar{v}_4 \cap \bar{v}_2$ equals one. So we add one edge between v_4 and v_2 in $H_1 \cup H_2$.

Analogously, $\bar{v}_4 \cap \bar{v}_5$ = { (3, 5, 0.5), (6, 2, 0.5) }, which implies that the number of components of $\bar{v}_4 \cap \bar{v}_5$ equals 2. So we add two edges between v_4 and v_5 in $H_1 \cup H_2$.

Let E = { e_1, e_2, e_3, e_4 } be the set of added edges, and V_1 = { v_3, v_4 } and V_2 = { v_2, v_5 } the sets of vertices.

Step 3.3. The restriction of $\bar{H}_1$ on $\bar{V}_1$ is $\bar{U}_1 = \bar{H}_1$, and that of $\bar{H}_2$ on $\bar{V}_2$ is $\bar{U}_2$ = { $\bar{v}_2$, $\bar{v}_5$ }.

By Figure 8 or a simple calculation, we have $\bar{v}_3 \cap \bar{v}_1$ = { (12, 5, 0.5) }. It follows that the number of components of $\bar{v}_3 \cap \bar{v}_1$ equals one, and we add one edge between v_3 and v_1 in $H_1 \cup H_2$.

Analogously, $(\bar{v}_2 \cup \bar{v}_5) \cap (\bar{v}_3 \cup \bar{v}_4)$ does not contain a 1-path of 2-cells $(p_0, p_1, \cdots, p_n)$ such that p_0 is contained in $\bar{v}_2 \cap \bar{v}_3$ and p_n is contained in $\bar{v}_5 \cap \bar{v}_4$. Thus, we do not have to delete edge v_3v_4 in $H_1 \cup H_2$.

Step 3.4. $H_1 \cup H_2$ is shown on the top of Figure 8 [case (1)].

Step 4. By Figure 8, the number of edges of the homotopic graph $H_1 \cup H_2$ equals ten and that of the vertices equals six; consequently, the number of tunnels of S_1 equals 10 - 6 + 1 = 5.

Example 2. Let $S_2 = L_1 \cup L_2 \cup g_4 = S_1 \cup g_4$. Step 3.2 is exactly the same as in Example 1.

Step 3.3, by examining Figure 8 or simple calculation, we have $(\bar{v}_2 \cup \bar{v}_5) \cap (\bar{v}_3 \cup \bar{v}_4)$ contains a 1-path of 2-cells ((9, 2, 0.5), (8, 2, 0.5), (7, 2, 0.5), (6, 2, 0.5)) such that (9, 2, 0.5) is contained in $\bar{v}_3 \cap \bar{v}_2$ ((9, 2, 0.5) = (9, 2, 1) $\cap$ (9, 2, 0)) and (6, 2, 0.5) is contained in $\bar{v}_4 \cap \bar{v}_5$ ((6, 2, 0.5) = (6, 2, 1) $\cap$ (6, 2, 0)). Thus, we have to delete edge v_3v_4 in $H_1 \cup H_2$.

Step 3.4. $H_1 \cup H_2$ is shown in the middle of Figure 8 [case (2)].

Step 4. The number of tunnels of S_2 equals 10 - 6 + 1 = 5.

Example 3. Now consider $S_3 = L_1 \cup L_2 \cup g_5 = S_1 \cup g_5$. Step 3.2 is exactly the same as in Example 1.

Step 3.3. By examining Figure 8, we can see that $(\bar{v}_2 \cup \bar{v}_5) \cap (\bar{v}_3 \cup \bar{v}_4)$ does not contain any 1-path of 2-cells $(p_0, p_1, \cdots, p_n)$ such that p_0 is contained in $\bar{v}_3 \cap \bar{v}_2$ and p_n is contained in $\bar{v}_4 \cap \bar{v}_5$. Thus, we do not have to delete edge v_3v_4 in $H_1 \cup H_2$.

Step 3.4. $H_1 \cup H_2$ is shown at the bottom of Figure 8 [case (3)].

Step 4. The number of tunnels of S_3 equals 11 - 6 + 1 = 6.

6 Computational Complexity

6.1 Single- Layer Subprocess

Let l be the number of cubes in the considered layer L, and r or c the number of rows or columns in L, respectively.

Step 1 can be computed in time $\mathcal{O}(l)$, because $|S_I| \leq l$. Each run can have at most $\lfloor r/2 \rfloor$ left (right) neighbors, so Step 2 can be computed in $\mathcal{O}(lr)$. Since each run-path has at most c runs and $|S_P| \leq |S_I| \leq l$, it follows that Step 3 can be done in $\mathcal{O}(lc)$. Analogous to Step 1, Steps 4, 5, 6 and 7 can be completed in $\mathcal{O}(l)$. In Step 8.1, the combination number of pairs (h_1, h_2) equals

$$\binom{l}{2}$$

because $|S_H \cup S_{PH}| \leq |S_I| \leq l$. Furthermore, since each h_i has at most r neighbors, it follows that Step 8.1 can be computed in $\mathcal{O}(l^2 r)$. In Step 8.2, there are at most lr ways to take a neighbor of a run in S_H, and there are also at most lr ways to take a run-path in S_{RP}. Thus, this step can be finished in $\mathcal{O}((lr)^2)$. For Step 9, note that $|S_H \cup S_{PH}| \leq |S_I| \leq l$, $|C| \leq r$, $|C_N| \leq r$, and $|C_P| \leq rc$. That means that Step 9 has time complexity $\mathcal{O}(lrc)$. Finally, it is clear that Step 10 has complexity $\mathcal{O}(l^2)$, and this immediately also follows for Step 11. In summary, a run of the single-layer subprocess has time complexity $\mathcal{O}((lr)^2)$.

6.2 Main Algorithm

Let S be our input, a 2-region of cubes, and i_{max} the number of layers in S. Let l_i be the number of cubes in the ith layer of S, denoted by L_i. Let r_i or c_i be the numbers of rows or columns in layer L_i, respectively.

Obviously, Step 1 can be done in $\mathcal{O}(i_{max})$. Step 2 is discussed in Section 6.1. For Step 3.1, the labeling algorithm (as, for example, in [9]) has complexity $\mathcal{O}(r_i c_i)$. Note that $|V(\bar{H}_i)| \leq l_i$, for i = 1, 2. It follows that the combination number of pairs $(\bar{v}_1, \bar{v}_2)$ equals $l_1 l_2$; thus Step 3.1 has time complexity $\mathcal{O}(l_i^2 r_i c_i)$. For Step 3.2, note that $|V_i| \leq l_i$, consequently $|E| \leq l_1 l_2$. Therefore, Step 3.2 can be computed in $\mathcal{O}(l_i^2)$. In Step 3.3, the combination number of pairs $(\bar{u}_i, \bar{w}_i)$ equals

$$\binom{l_i}{2}$$

and the number of 1-paths $(p_0, p_1, \cdots, p_n)$ of 2-cells, such that p_0 is contained in $\bar{u}_1 \cap \bar{u}_2$ and p_n is contained in $\bar{w}_1 \cap \bar{w}_2$, is at most $\lfloor r_i/2 \rfloor$, where $i = 1, 2$. [The labeling algorithm has complexity $\mathcal{O}(r_i c_i)$.] Thus, this step can be completed in $\mathcal{O}(l_i^2 r_i^2 c_i)$. Step 4, finally, recalls the subprocess for all i_{max} layers, and delivers the final result. Let

$$l = \max_{1 \leq i \leq i_{max}} l_i\,, \quad r = \max_{1 \leq i \leq i_{max}} r_i\,, \quad \text{and} \quad c = \max_{1 \leq i \leq i_{max}} c_i$$

The main algorithm has time complexity $\mathcal{O}(i_{max} l^2 r^2 c)$.

7 Concluding Remarks

[18] computes the number of tunnels in a $3 \times 3 \times 3$ neighborhood of any cube of a given 2-region. It does not deal with the problem of computing the number of tunnels for the whole 2-region.

[2, 3] discuss the computational complexity of homology (as being 'large'). Both [5] and [7] have developed software to compute homology. However, they are (so far) unable to inform about the computational complexity of their algorithms, and so are [10, 16]. The latter two compute homology for any finite k-dimensional cubical set (no restrictions on dimensions or shapes). [5] suggests a search for improved algorithms and for new approaches to the task of homology computation while [10] leaves the problem of improving the efficiency of their algorithms to the future. [7] states that time complexity of the proposed algorithm may grow 'horrifically'.

We have presented our algorithm and analyzed its computational complexity, which is relatively low compared to other options. We also believe that the graph-theoretical nature of our algorithm is of general interest. Finally, note that connectedness is also defined for cubical sets of any dimension, and digitization schemes for arbitrary sets in nD space (see [9]). Our graph-theoretical approach can be generalized to compute homology for digitized sets in arbitrary dimensions.

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