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GAMMA LIMIT OF THE NON SELF-DUAL CHERN-SIMONS-HIGGS ENERGY

MATTHIAS KURZKE AND DANIEL SPIRN

Abstract. We consider the Gamma limit of the Abelian Chern-Simons-Higgs energy

$$G_{csh} := \frac{1}{2}\int_U |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{\mu^2}{4}\frac{|\operatorname{curl} A_\varepsilon - h_{ex}|^2}{|u_\varepsilon|^2} + \frac{1}{\varepsilon^2}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2 dx$$

on a bounded, simply connected, two dimensional domain under the $\varepsilon \to 0$ limit. As a first step we study the Gamma limit of

$$E_{csh} := \frac{1}{2}\int_U |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}|u_\varepsilon|^2\left(1-|u_\varepsilon|^2\right)^2 dx.$$

under two different scalings; $E_{csh} \approx |\log\varepsilon|$ and $E_{csh} \approx |\log\varepsilon|^2$. We apply the $|\log\varepsilon|^2$-scaling result to the full Chern-Simons-Higgs energy G_{csh}, and as a consequence we are able to compute the first critical field $H_1 = H_1(U,\mu)$ for the nucleation of a vortex. The method entails estimating in certain weak topologies the Jacobian $J(u_\varepsilon) = \det(\nabla u_\varepsilon)$ in terms of the Chern-Simons-Higgs energy E_{csh}.

1. Introduction

The Abelian Chern-Simons-Higgs (CSH) theory serve as an anyon model [5, 13, 12, 37] and is a classical field theory defined on (2+1) dimensional Minkowski space. Such models have applications to the theory of high temperature superconductivity, quantum Hall effects and carry fractional charge values [5, 37]. The CSH Lagrangian has the form

$$\mathcal{L}_{CSH} = D_\alpha \overline{D^\alpha u} + \frac{\mu^2}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} - \frac{\lambda}{\mu^2}|u|^2\left(1-|u|^2\right)^2$$

where $\epsilon^{\alpha\beta\gamma}$ is the antisymmetric tensor and $\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}$ is the Chern-Simons term. The associated Euler-Lagrange equations are

$$D_\alpha D^\alpha u + \frac{\lambda}{\mu^2}u\left(|u|^2-1\right)\left(3|u|^2-1\right) = 0$$
$$\frac{1}{4}\mu^2\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} + \mathscr{J}^\alpha = 0.$$

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The second of these two equations is very different from the more conventional Maxwell's current equation, $D^\beta F_{\alpha\beta} + \mathscr{J}^\alpha = 0$, found in the more widely studied Maxwell-Higgs model, which says that the change in the matter current is due to the rate of change of the electromagnetic field. In the Chern-Simons case $\frac{1}{4}\mu^2\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} + \mathscr{J}^\alpha = 0$ implies the matter current is **proportional** to the electromagnetic field. This model has been the source of much interest in the physics community; the book of Yang [37] offers an excellent overview of Chern-Simons-Higgs and related theories.

To date most rigorous analysis has been restricted to self-duality which occurs when $\lambda = 1$, as discovered independently by Hong-Kim-Pac and Jackiw-Weinberg in [12, 13]. On the other hand in this paper we consider $\lambda = \frac{\mu^2}{\varepsilon^2} \ll 1$ and $\mu = O(1)$. Since the $\alpha = 0$ refers to time coordinates, we replace D_0 by $\partial_\Phi = \partial_t - i\Phi$ and replace D_α by $\nabla_A = \nabla - iA$ when $\alpha \in \{1,2\}$. The curvature tensor is defined by

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & -h \\ E_2 & h & 0 \end{pmatrix},$$

where $h = \operatorname{curl} A$ and $E_\alpha = \partial_t A_\alpha - \partial_\alpha \Phi$. We write the current $\mathscr{J}^\alpha$ in a more classical notation by setting

$$\mathscr{J}^0 = (iu, \partial_\Phi u) = q \qquad \mathscr{J}^\alpha = (iu, \nabla_{A_\alpha} u) = j_{A_\alpha}$$

for $\alpha \in \{1,2\}$ which are the charge and supercurrent, respectively. Therefore, the current equation reads $\frac{\mu^2}{2}h + q = 0$, $-\frac{\mu^2}{2}E_2 + j_A^1 = 0$, and $\frac{\mu^2}{2}E_1 + j_A^2 = 0$, and in more classical notation we write the CSH equations as:

$$\partial_\Phi^2 u = \nabla_A^2 u + \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right)\left(3|u|^2 - 1\right) \tag{1.1}$$

$$q = -\frac{\mu^2}{2} \operatorname{curl} A \tag{1.2}$$

$$j_A = \frac{\mu^2}{2}\left(E \times e_3\right). \tag{1.3}$$

If we take the curl and div of the equations, then we get the following useful identities

$$\operatorname{div} j_A = \frac{\mu^2}{2} \operatorname{curl} E = \partial_t q = -\frac{\mu^2}{2}\partial_t h \tag{1.4}$$

$$J_A = -\frac{\mu^2}{4} \operatorname{div} E. \tag{1.5}$$

Well-posedness questions for equations (1.1)-(1.3) were studied in [6, 7].

Since $u : \mathbb{R}^2 \to \mathbb{C}$ we can easily induce the formation of topological vortices – regions where $|u| = 0$ and about which the winding number of the phase is nontrivial. Setting $u = \rho e^{i\varphi} \approx e^{i\varphi}$ over $\mathbb{R}^2$ and $\varphi = d\theta$, then $J_A \approx \operatorname{curl}(\nabla\varphi - A) = \operatorname{curl}\nabla\varphi - h$. Formally, if we integrate (1.5) over $\mathbb{R}^2$ then $2\pi d = \int_{\mathbb{R}^2} h dx$. Furthermore, integrating (1.2) over the plane yields

$$d = \frac{1}{2\pi}\int_{\mathbb{R}^2} h dx = -\frac{1}{\mu^2\pi}\int_{\mathbb{R}^2} q dx. \tag{1.6}$$

As in Ginzburg-Landau theory, we see that the current and the magnetic field are quantized about a topological vortex; however, in CSH theory the magnetic field to induces a quantized electric charge, which can have arbitrary values, depending on μ. This quantized electric charge is a fundamental feature of Chern-Simons-Higgs theory.

We look for solutions independent of time; setting $\partial_t u \equiv 0$ then (1.1)-(1.3) become

$$-\Phi^2 u = \nabla_A^2 u + \frac{1}{\varepsilon^2} u\left(1 - |u|^2\right)\left(3|u|^2 - 1\right)$$
$$\Phi|u|^2 = \frac{\mu^2}{2}\left(\operatorname{curl} A - h_{ex}\right) \qquad j_A(u) = \frac{\mu^2}{2}\nabla\Phi \times e_3.$$

Since the CSH equations serve as a model for high temperature superconductors, we include the possible presence of an applied magnetic field h_{ex}. Removing the electric field potential Φ, we are left with an unusual system of coupled elliptic PDE's:

$$-\frac{\mu^2}{4}\frac{\left|\operatorname{curl} A - h_{ex}\right|^2}{|u|^4} u = \nabla_A^2 u + \frac{1}{\varepsilon^2} u\left(1 - |u|^2\right)\left(3|u|^2 - 1\right) \tag{1.7}$$

$$0 = -\frac{\mu^2}{4}\operatorname{curl}\left(\frac{\operatorname{curl} A - h_{ex}}{|u|^2}\right) + j_A(u). \tag{1.8}$$

Taking the curl and div of (1.8) yields two more useful equations

$$J_A(u) = -\frac{\mu^2}{4}\Delta\left(\frac{\operatorname{curl} A - h_{ex}}{|u|^2}\right) \tag{1.9}$$

$$\operatorname{div} j_A(u) = 0. \tag{1.10}$$

Equations (1.7)-(1.8) can be viewed as the Euler-Lagrange equations of the following Chern-Simons-Higgs energy

$$G_{csh}(u, A; h_{ex}) = \frac{1}{2}\int_U \left|\nabla_A u\right|^2 + \frac{\mu^2}{4}\frac{\left|\operatorname{curl} A - h_{ex}\right|^2}{|u|^2} + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 dx \tag{1.11}$$

for an applied magnetic field, h_{ex}, and a bounded, simply connected domain, $U \subset \mathbb{R}^2$. A discussion of the CSH theory on bounded domains can be found in

[10]. Our paper studies (1.11) under different scaling limits of the energy.

Even at this point we can see unusual features of (1.11). Suppose $h_{ex} = 0$ and suppose u has a topological vortex at 0. Then u must vanish at the origin. But the second term of (1.11) implies that $h = \operatorname{curl} A$ must likewise vanish at the origin. On the other hand the quantization relation (1.6) implies there exists a finite mass of magnetic field about this vortex, and consequently the magnetic field concentrates in an annular region about each topological vortex. This is in stark contrast to Ginzburg-Landau vortices, where the magnetic field concentrates at the site of the vortex. The second term proves to greatly increase the difficulty of analyzing (1.7)-(1.10).

Another important feature of (1.11) is that in the $\varepsilon \to 0$ limit, $|u_\varepsilon|$ relaxes to $\mathbb{S}^1 \cup \{0\}$, as opposed to $\mathbb{S}^1$ in the Ginzburg-Landau case. This implies that nontopological vortices (regions where $|u| = 0$ with trivial winding number about the region) are possible and potentially favorable. We show, however, that such regions are of size $O(\varepsilon)$ if $|u_\varepsilon| \geq \frac{1}{2}$ on the boundary, see Corollary 2.4.

1.1. **Results.** Up to now, most attention has focussed on the *self-dual* case where $\varepsilon = \mu$. In this case the CSH equations reduce, following Hong-Kim-Pac and Jackiw-Weinberg [12, 13], to a system of first order PDE's. Solutions can be recovered by solving (after a substitution) a Liouville-type elliptic equation, similar to the Jaffe-Taubes approach to solving self-dual solutions in Ginzburg-Landau theory [14]. It is impossible to give an adequate accounting of the extensive results on self-dual solutions to the Chern-Simons-Higgs equations, but we direct the reader to [5, 8, 10, 12, 13, 27, 35, 37] and the references therein.

We turn our attention to non self-dual Chern-Simons-Higgs theory. The only results to our knowledge for small ε and $\mu = O(1)$ for the CSH functional are those of Han-Kim [11], who studied among other things sequential minimizers $\{u_\varepsilon, A_\varepsilon\}$ of (1.11) with $A_\varepsilon \equiv 0$ and Dirichlet boundary condition $u_\varepsilon = g$ on ∂U with $|g| = 1$. Their proofs are similar in spirit to the methods Bethuel-Brezis-Helein [3] for the simplified Ginzburg-Landau energy (1.19) and rely heavily on the maximum principle for $|u_\varepsilon|$. The maximum principle fails when gauge field $A_\varepsilon \not\equiv 0$, so another approach is needed.

In this paper we allow both $A_\varepsilon \not\equiv 0$ and more general boundary conditions (we still assume that $|u_\varepsilon| \to 1$ on the boundary at a certain rate); our study yields compactness and Γ-convergence results for two scalings of the energy. In particular, our convergence results are true for non-minimizers and indeed even for sequences of functions that are not solutions of the corresponding equations. Our techniques are related to the approach of Jerrard-Soner [17, 18] combined with the vortex ball construction method of Sandier [29]. Similar to their approach, we first study the simplified functional

$$E_{csh}(u) = \frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2(1-|u|^2)^2. \tag{1.12}$$

(In fact our approach is robust enough to deal with more general potentials of the form $\frac{1}{\varepsilon^2}W(|u|^2)$ with $W(s) = s^p(1-s)^q$ for $p \geq 0$ and $q \geq 1$.) The results for (1.12) are then used to analyze the case with a magnetic field h_{ex} and gauge field A.

We have the following results, stated here in the spirit of Γ-convergence; that is, separated into a compactness result combined with a lower bound for the energy and a construction that shows that the lower bound is essentially optimal. For the sake of unified exposition, we only state the two-dimensional results here, although some of the results are also true in higher dimensional domains (see Theorem 4.1).

Our general assumptions are that $U \subset \mathbb{R}^2$ is a bounded, simply connected domain with smooth boundary and that $\{u_\varepsilon\}$ is a sequence of functions in $H^1(U;\mathbb{C})$ whose traces on ∂U satisfy $|u_\varepsilon| \geq 1 - \frac{1}{|\log\varepsilon|}$.

Theorem 1.1 (Compactness and Γ-convergence in the $|\log\varepsilon|$ scaling)**.** *Assume $E_{csh}(u_\varepsilon) \leq K|\log\varepsilon|$ for some constant $K > 0$. Then the Jacobians $J(u_\varepsilon)$ are precompact in the weak $(C^{0,\beta})^*$ topology for every $\beta > 0$, and every cluster point $\overline{J}$ satisfies $\overline{J} = \pi\sum_i d_i\delta_{a_i}$ for some $d_i \in \mathbb{Z}$. Furthermore,*

$$\liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}E_{csh}(u_\varepsilon) \geq \left\|\overline{J}\right\|_{\mathcal{M}} \tag{1.13}$$

Conversely, for every $J = \sum_i d_i\delta_{a_i}$, there exists a sequence $\{u_\varepsilon\}$ with $J(u_\varepsilon) \rightharpoonup J$ and

$$\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}E_{csh}(u_\varepsilon) = \|J\|_{\mathcal{M}}. \tag{1.14}$$

The proof of this theorem is contained in the proofs of Propositions 5.2 and 6.3. The next result concerns energy scalings at which the nucleation of vortices should occur, see [33, 32, 18] in the Ginzburg-Landau case.

Theorem 1.2 (Compactness and Γ-convergence in the $|\log\varepsilon|^2$ scaling)**.** *Assume $E_{csh}(u_\varepsilon) \leq K|\log\varepsilon|^2$, for some constant K.*

Set $v_\varepsilon = \frac{1}{|\log\varepsilon|}j(u_\varepsilon)$ and $w_\varepsilon = \frac{1}{|\log\varepsilon|}J(u_\varepsilon) = \frac{1}{2}\operatorname{curl} v_\varepsilon$. Then $\{w_\varepsilon\}$ is precompact in the weak $(C^{0,\beta})^*$ topology and $\{v_\varepsilon\}$ is bounded in L^p for $1 \leq p < 2$. Furthermore, if $w_\varepsilon \rightharpoonup w = \frac{1}{2}\operatorname{curl} v$ and $v_\varepsilon \rightharpoonup v$, then also $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ in L^2, and the energy satisfies

$$\liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|^2}E_{csh}(u_\varepsilon) \geq \frac{1}{2}\left(\|v\|_{L^2}^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right). \tag{1.15}$$

Conversely, for every $v \in L^2(U;\mathbb{R}^2)$ *such that* $w = \frac{1}{2}\operatorname{curl} v$ *is a Radon measure, there exists a sequence* $\{u_\varepsilon\}$ *in* $H^1(U;\mathbb{C})$ *with* $|u_\varepsilon| = 1$ *on* ∂U *such that* $v_\varepsilon = \frac{1}{|\log\varepsilon|} j(u_\varepsilon) \rightharpoonup v$ *in* L^2 *and* $w_\varepsilon = \frac{1}{|\log\varepsilon|} J(u_\varepsilon) \rightharpoonup w$ *in* $(C^{0,\beta})^*$ *such that the energy satisfies*

$$\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|^2} E_{csh}(u_\varepsilon) = \frac{1}{2}\left(\|v\|_{L^2}^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right). \tag{1.16}$$

The proof of this theorem is given in contained in the proofs of Propositions 5.1 and 6.1.

In the energy scaling of Theorem 1.2, we also have a result with an external magnetic field h_{ex} and gauge field A. For simplicity, we state the result only in Coulomb gauge, which amounts to considering only pairs (u, A) with $\nabla \cdot A = 0$ in U and $A \cdot \nu = 0$ on ∂U. For definiteness, we also assume $\int_U A = 0$. These conditions can always be satisfied by an appropriate gauge transformation replacing (u, A) by $(ue^{i\chi}, A + \nabla\chi)$ without changing the energy.

Note that we assume an *a priori* L^∞ bound on $|u_\varepsilon|$, see Conjecture 1.6 below.

Theorem 1.3 (Compactness and Γ-convergence with external field)**.** *Assume that the external field satisfies* $h_{ex} = H|\log\varepsilon|$ *for some* $H > 0$*, and consider a sequence* $\{u_\varepsilon, A_\varepsilon\}$ *with*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex}) \le K|\log\varepsilon|^2$$

and $\|u_\varepsilon\|_{L^\infty} \le C$*. Set* $a_\varepsilon = \frac{1}{|\log\varepsilon|} A_\varepsilon$*. Then* $\{a_\varepsilon\}$ *is weakly precompact in* H^1*, and also the compactness assertions of the last theorem hold:* $v_\varepsilon = \frac{1}{|\log\varepsilon|} j(u_\varepsilon)$ *converges to* v *weakly in all* L^p*,* $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ *in* L^2*, and* $w_\varepsilon = \frac{J(u_\varepsilon)}{|\log\varepsilon|} \rightharpoonup w = \frac{1}{2}\operatorname{curl} v$*. Furthermore, the energy satisfies*

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|^2} G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex}) \\ &\qquad \ge G(v, a; H) = \frac{1}{2}\left(\int_U |v - a|^2 + \frac{\mu^2}{4}|\operatorname{curl} a - H|^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right). \end{aligned} \tag{1.17}$$

Conversely, for any $a \in H^1(U;\mathbb{R}^2)$ *and* $v \in L^2(U;\mathbb{R}^2)$ *such that* $w = \frac{1}{2}\operatorname{curl} v$ *is a Radon measure, there exists a sequence* $\{u_\varepsilon\}$ *in* $H^1(U;\mathbb{C})$ *with* $|u_\varepsilon| = 1$ *on* ∂U *and a sequence* $\{A_\varepsilon\} \in H^1(U;\mathbb{C})$ *in Coulomb gauge such that* $v_\varepsilon = \frac{1}{|\log\varepsilon|} j(u_\varepsilon) \rightharpoonup v$ *in* L^2*,* $w_\varepsilon = \frac{1}{|\log\varepsilon|} J(u_\varepsilon) \rightharpoonup w$ *in* $(C^{0,\beta})^*$*,* $a_\varepsilon = \frac{1}{|\log\varepsilon|} A_\varepsilon \rightharpoonup a$ *in* H^1*, and such that* (1.17) *holds with equality.*

This theorem follows from Propositions 7.1 and 7.2. As an application of the last theorem, we calculate the critical field h_{crit} for which vortices appear in nonzero minimizers of $G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex})$.

Corollary 1.4. *As $\varepsilon \to 0$, the critical field h_{crit} is given asymptotically by $H_1(\mu)\,|\log\varepsilon|$, where*

$$H_1(\mu) = \frac{2}{\mu^2 \max_U |z_\mu|}$$

and z_μ is the solution of

$$-\frac{\mu^2}{4}\Delta z_\mu + z_\mu + 1 = 0$$

with homogeneous Dirichlet boundary conditions. Concerning the dependence on μ, we have that $\mu^2 H_1(\mu) \to 2$ as $\mu \to 0$. Furthermore, $H_1(\mu)$ is decreasing in μ and converges to a limit $\overline{H}(U) > 0$ as $\mu \to \infty$. Finally, when $U \equiv B_R$, a ball of radius R, then

$$H_1(\mu, R) = \frac{2I_0\left(\frac{2R}{\mu}\right)}{\mu^2\left(I_0\left(\frac{2R}{\mu}\right) - 1\right)},$$

where I_0 is the modified Bessel function of zeroeth order.

The corollary follows from Theorem 1.3 using some analysis of the limit functional, see Proposition 8.1.

1.2. **Methodology with a comparison to Ginzburg-Landau.** We can compare (1.11) with the Ginzburg-Landau energy

$$G_{gl}(u, A; h_{ex}) := \frac{1}{2}\int_U |\nabla_A u|^2 + |\mathrm{curl}\, A - h_{ex}|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2 dx. \tag{1.18}$$

The Γ-limit of (1.18) has been the center of extensive research since the groundbreaking study of the corresponding functional without the gauge field

$$E_{gl}(u) := \frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2 dx \tag{1.19}$$

in the book of Bethuel-Brezis-Helein [3] and (1.19) is commonly referred to as the BBH energy. Here the authors offer a complete description of the small ε limit of energy minimizers to (1.19) by construction of comparison functions. Since energy about each vortex core is of size $\pi\,|\log\varepsilon|$, uniform energy bounds can be found by cutting out the vortices from the domain. Furthermore, the authors expand the energy (1.19) asymptotically to second order, which up to boundary effects, is a Coulomb potential. There has also been great success in higher dimensions including [28, 24, 4, 17, 1]. In higher dimensions the vorticity concentrates on $(n-2)$ dimensional, integer multiplicity, rectifiable currents. In the case of minimizing sequences the current is area minimizing, see [24].

Non minimizing sequences has also been the center of significant interest, see [29, 16, 23, 4, 31, 17, 30, 20, 21] among other places. The Γ-limit of the BBH energy was proven in Jerrard-Soner [17] in two dimensional domains. The higher dimensional Γ-limit was established jointly in Jerrard-Soner [17], who proved the compactness and energy lower bound, and Alberti-Baldo-Orlandi [1], who were able to construct the needed energy upper bound. The calculation of the first critical field H_1 for (1.18) can be found in [33, 32]. The Γ-limit full Ginzburg-Landau energy was proven in Jerrard-Soner [18], and the authors include a new derivation of H_1.

We briefly outline of rest of the paper. In Section 2 we provide several basic estimates on the Chern-Simons-Higgs energy. We make use of the Modica-Mortela method to prove strong convergence of $|u_\varepsilon| \to 1$ when $|u_\varepsilon| \geq \frac{1}{2}$ on the boundary. In Section 3 we prove the basic Jacobian estimate

$$\left| \int \phi J(u_\varepsilon) dx \right| \leq \pi d \, \|\phi\|_{L^\infty} + C\varepsilon^\gamma \, \|\phi\|_{C^{0,1}} \tag{1.20}$$

where $d \approx \left\lfloor \frac{1}{\pi} \int \frac{e_{csh}(u_\varepsilon)}{|\log \varepsilon|} dx \right\rfloor$, $\gamma \in (0,1)$, and C depends on $\int \frac{e_{csh}(u_\varepsilon)}{|\log \varepsilon|} dx$. The estimate, in the spirit of [17], follows from integration by parts and the co-area formula,

$$\begin{aligned} \int \phi J(u_\varepsilon) dx = \int_0^\infty \int_{\partial\Omega(t)} \nabla\phi \times j(u_\varepsilon) dl dt \approx \int_0^\infty \int_{\partial\Omega(t)} \tau \cdot \nabla\varphi \, dl dt \\ \approx \pi \int_0^\infty \deg(u_\varepsilon, \partial\Omega(t)) dt, \end{aligned}$$

where $u_\varepsilon = \rho e^{i\varphi}$ and $\Omega(t) = \{x \in U \text{ such that } \phi(x) > t\}$. Therefore, controlling $\int \phi J(u_\varepsilon) dx$ consists of finding estimates on the degree of u on level sets of ϕ. In order to establish (1.20) we can divide the level sets $t \in (0, \|\phi\|_{L^\infty})$ into high and low degree sets, and, not surprisingly, the lower degree level sets are rather easy to estimate. On the other hand the higher degree level sets are much more difficult to understand. In particular we expect

$$D_d = \{t \text{ such that } \deg(u_\varepsilon, \partial\Omega(t)) \geq d+1\},$$

where $d \approx \left\lfloor \frac{1}{\pi} \int_U \frac{e_{csh}(u_\varepsilon) dx}{|\log \varepsilon|} dx \right\rfloor$, should be of small measure otherwise there should be a violation of our energy bound. In [17, 18] this is accomplished via a covering lemma that relies on lower bounds on the Ginzburg-Landau energy. In the CSH case the Jerrard-Soner method fails; however, we provide a new approach to estimating the size of the set $|D_d|$, and we find

$$|D_d| \leq 8\varepsilon^{\alpha_\varepsilon} \left(1 + \exp\left(\frac{\eta_\varepsilon}{\pi(d+1)} \int_V e_{csh}(u_\varepsilon) dx \right) \right) \|\nabla\phi\|_{L^\infty}$$

where $0 \leq 1 - \alpha_\varepsilon \ll 1$ and $0 \leq \eta_\varepsilon - 1 \ll 1$, and where $V = \dot{\cup} B_{r_k}$ with $B_{r_k} \subseteq \operatorname{spt}(\phi)$ and $\sum r_k = \frac{|D_d|}{2\|\nabla\phi\|_{L^\infty}}$. This bound provides good control on

$|D_d|$ and allows us to establish (1.20) for both the $E_{csh}(u) = O(|\log\varepsilon|)$ and $E_{csh}(u) = O(|\log\varepsilon|^2)$ cases.

Sections 4–6 handle the proof of compactness and Γ-convergence of the CSH energy for energy of size $O(|\log\varepsilon|)$ and $O(|\log\varepsilon|^2)$. Our arguments are similar to the approach found in [17, 18]. Section 4 establishes the compactness of the Jacobian in a weak Banach space $(C^{0,\beta})^*$ for energies of size $O(|\log\varepsilon|)$ and $O(|\log\varepsilon|^2)$. Here we can lift the restriction on the domain being two-dimensional, and we can show that the limiting Jacobian is an $(n-2)$-dimensional, integer multiplicity rectifiable current, see Theorems 4.1 and 4.3. We make use of estimate (1.20) and methods developed in [17] to establish this result.

Section 5 provides the lim inf condition on the CSH energy, and this lower bound follows almost directly from estimate (1.20). Section 6 completes the Γ-limit proof by constructing the upper bounds in both the $O(|\log\varepsilon|)$ and $O(|\log\varepsilon|^2)$ cases. Here we make use of constructions of [19, 31]. Section 7 then establishes the Γ-limit in the presence of the magnetic field potential and the external magnetic field. Here we rely on the assumption that $\|u_\varepsilon\|_{L^\infty} \leq C$, independent of ε, see Conjecture 1.6 below.

Finally, in Section 8 we study the limiting energy of the full CSH energy functional in the $O(|\log\varepsilon|^2)$ case, as computed in Section 7. The critical field calculations are similar in spirit to the critical field calculation for Ginzburg-Landau energy (1.18).

1.3. **Open Problems.** When $\mu = \mu(\varepsilon)$ varies with ε, then there are many classes of Γ-limits. Such limits are studied in the forthcoming [22]. We end the introduction with several open questions that naturally arise from the analysis.

Conjecture 1.5. *Theorems 1.1 and 1.2 hold without any assumption on $|u_\varepsilon|$ on the boundary ∂U.*

A major difference between the Ginzburg-Landau functional and the Chern-Simons-Higgs functional is that the trivial solution $u \equiv 0$ is minimizing in the absence of boundary conditions and an external field. However, if we choose boundary conditions that ensure $|u_\varepsilon| \to 1$ on the boundary, the problems become very similar again, since any sequence $\{u_\varepsilon\}$ with an energy bound $E_{csh}(u_\varepsilon) \leq C\varepsilon^{-\gamma}$, $\gamma < 1$, must converge to a limit u with either $|u| = 1$ or $|u| = 0$ a.e. in U. This follows from the theory of scalar phase transitions, e.g. the result of [26], which shows that $\mathrm{Per}_U(|u| = 0) \leq \liminf_{\varepsilon\to 0} \varepsilon E_{csh}(u_\varepsilon) = 0$.

When $\lim_{\varepsilon\to 0} |u_\varepsilon| = 0$ then Theorems 1.1-1.2 follow easily; however, if $\lim_{\varepsilon\to 0} |u_\varepsilon| = 1$, we still need to establish a rate of convergence of $|u_\varepsilon| \to 1$ in order to use the Jacobian bound found in Proposition 3.2.

Conjecture 1.6. *Theorem 1.3 holds without assuming that $\|u_\varepsilon\|_{L^\infty}$ is uniformly bounded for sequences $\{u_\varepsilon, A_\varepsilon\}$ of critical points of* (1.11).

The uniform L^∞ bound on $|u_\varepsilon|$ is used to establish the compactness of the full energy $G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex})$. If the bound is absent, the set on which $\|u_\varepsilon\|_{L^\infty}$ is unbounded can allow for the failure of a bound on $\|\operatorname{curl} A - h_{ex}\|_{L^2}$ even when $\left\| \frac{|\operatorname{curl} A - h_{ex}|}{|u_\varepsilon|} \right\|_{L^2}$ is bounded. However, for critical points of the energy, there may be enough regularity to establish this conjecture.

Finally, the dynamics of vortices in the full CSH equations (1.7)-(1.10) can also be considered. In this case it is possible to generate more refined Jacobian estimates in terms of the CSH energy that establishes the rate of Γ-convergence, see [20, 21]. Such estimates provide sufficient control to establish the dynamics of topological vortices. Rigorous results that establish the vortex motion law are found in [15, 23] for the nonlinear wave equation and [34, 9] for the Maxwell-Higgs equations.

2. Basic Energy Bounds

Let $u = \rho e^{i\varphi} : U \to \mathbb{C}$ and $A : U \to \mathbb{R}^2$ then we define two CSH energy densities

$$g_{csh}(u, A; h_{ex}) = \frac{1}{2}\left[|\nabla_A u|^2 + \frac{\mu^2}{4}\frac{|\operatorname{curl} A - h_{ex}|^2}{|u|^2} + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 \right]$$

$$e_{csh}(u) = \frac{1}{2}\left[|\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 \right]$$

We note

$$g_{csh}(u, A; h_{ex}) \geq \frac{1}{2}\left[|\nabla \rho|^2 + \frac{1}{\varepsilon^2}\rho^2\left(1 - \rho^2\right)^2 \right] = e_{csh}(|u|);$$

therefore, $\int_U g_{csh}(u, A; h_{ex})dx \geq \int_U e_{csh}(|u|)dx$ provides a simple lower bound that will be exploited throughout. We have the following useful energy bounds.

Lemma 2.1. *Suppose $|U| \leq \int_U e_{csh}(|u|)dx$ and $\varepsilon \leq 1$ then*

$$\|\rho\|^2_{H^1(U)} \leq 5 \int_U e_{csh}(|u|)dx \tag{2.1}$$

$$\|j_A(u)\|_{L^\alpha(U)} \leq C_\alpha \int_U g_{csh}(u, A; h_{ex})dx \tag{2.2}$$

$$\|h - h_{ex}\|_{L^\alpha(U)} \leq \frac{C_\alpha}{\mu} \int_U g_{csh}(u, A; h_{ex})dx \tag{2.3}$$

for all $1 \leq \alpha < 2$ and $C_\alpha \to \infty$ as $\alpha \to 2$.

Proof. 1. By Young's inequality

$$\rho^6 + \rho^2 = \rho^6 - 2\rho^4 + \rho^2 + 2\rho^4 \leq \rho^2\left(1 - \rho^2\right)^2 + \frac{2\rho^6}{3} + \frac{2^3}{3}$$

so $\frac{\rho^6}{3} + \rho^2 \leq \rho^2 \left(1-\rho^2\right)^2 + \frac{2^3}{3}$ which implies

$$\begin{aligned}\int_U |\nabla \rho|^2 + \rho^2 dx &\leq \int_U |\nabla \rho|^2 + \rho^2 \left(1-\rho^2\right)^2 dx + \frac{8}{3}|U| \\ &\leq 2\int_U e_{csh}(|u|)dx + \frac{8}{3}|U|.\end{aligned}$$

By the assumptions on the energy, this shows (2.1). Furthermore, this implies by Sobolev embedding

$$\|\rho\|_{L^p(U)} \leq C_p \|\rho\|_{H^1(U)} \leq C_p \sqrt{\int_U e_{csh}(|u|)dx} \tag{2.4}$$

for any $p < \infty$.

2. Since $|\nabla_A u|^2 = \left|\frac{j_A(u)}{\rho}\right|^2 + |\nabla \rho|^2 \geq \left|\frac{j_A(u)}{\rho}\right|^2$,

$$\begin{aligned}\|j_A(u)\|_{L^\alpha(U)} = \left\|\frac{j_A(u)}{\rho}\rho\right\|_{L^\alpha(U)} &\leq \left\|\frac{j_A(u)}{\rho}\right\|_{L^2(U)} \|\rho\|_{L^{\frac{2\alpha}{2-\alpha}}(U)} \\ &\leq C_\alpha \sqrt{\int_U g_{csh}(u,A;h_{ex})dx}\sqrt{\int_U e_{csh}(|u|)dx} \\ &\leq C_\alpha \int_U g_{csh}(u,A;h_{ex})dx\end{aligned}$$

follows from (2.4).

3. Finally, since $\int_U \frac{\mu^2}{8}\frac{|h-h_{ex}|^2}{\rho^2}dx \leq \int_U g_{csh}(u,A;h_{ex})dx$ then

$$\begin{aligned}\|h-h_{ex}\|_{L^\alpha(U)} = \left\|\frac{h-h_{ex}}{\rho}\rho\right\|_{L^\alpha(U)} &\leq \left\|\frac{h-h_{ex}}{\rho}\right\|_{L^2(U)} \|\rho\|_{L^{\frac{2\alpha}{2-\alpha}}(U)} \\ &\leq \frac{C_\alpha}{\mu} \sqrt{\int_U g_{csh}(u,A;h_{ex})dx}\sqrt{\int_U e_{csh}(|u|)dx} \\ &\leq \frac{C_\alpha}{\mu} \int_U g_{csh}(u,A;h_{ex})dx\end{aligned}$$

follows from (2.4). □

Remark 2.2. If $\int_U e_{csh}(|u|)dx < |U|$ then replace (2.1) by $\|\rho\|_{H^1} \leq C$, (2.2) by $\|j_A(u)\|_{L^\alpha} \leq C_\alpha \sqrt{\int_U g_{csh} dx}$, and (2.3) by $\|h-h_{ex}\|_{L^\alpha} \leq \frac{C_\alpha}{\mu}\sqrt{\int_U g_{csh} dx}$.

We have an important covering argument for $\{|u| < 1/2\}$ that exploits the Modica-Mortola trick [25, 26], used with great success by Sandier for complex Ginzburg-Landau energies [29].

Lemma 2.3. *Suppose $\rho \geq \frac{1}{2}$ on ∂U, then we have*

$$\left\{x \in U : |u| < \frac{1}{2}\right\} \subset \cup B_{r_j}$$

with

$$\sum r_j \leq 4\varepsilon \int_U e_{csh}(|u|)dx.$$

Proof. For any open set $A \subseteq U$, we define

$$\mathcal{H}^1_\infty(A) = \inf\left\{\sum 2r_j : A \subset \sum B_{r_j}(y_j)\right\}$$

then $\mathcal{H}^1_\infty(A) \leq \mathcal{H}^1(\partial A \cap U)$, as noted in [29].

Note that Cauchy-Schwarz implies

$$\frac{|\nabla\rho|^2}{2} + \frac{1}{2\varepsilon^2}\rho^2\left(1-\rho^2\right)^2 \geq \frac{1}{\varepsilon}\rho\left|1-\rho^2\right||\nabla\rho|.$$

So for $\alpha \in (0,1)$,

$$\begin{aligned}\int_U e_{csh}(|u|)dx &\geq \frac{1}{\varepsilon}\int_U \rho\left|1-\rho^2\right||\nabla\rho|\,dx \\ &= \frac{1}{\varepsilon}\int_{t=0}^{\infty} t\left|1-t^2\right|\mathcal{H}^1\left(\rho^{-1}(t)\right)dt \\ &\geq \frac{1}{\varepsilon}\int_{t=\alpha}^{1} t\left(1-t^2\right)\mathcal{H}^1\left(\rho^{-1}(t)\right)dt\end{aligned}$$

From the bound above and the fact that

$$\alpha \mapsto \mathcal{H}^1_\infty\left(\{x : \rho(x) \leq \alpha\}\right)$$

is an increasing function, we have

$$\begin{aligned}\int_U e_{csh}(|u|)dx &\geq \frac{1}{\varepsilon}\int_{t=\alpha}^{1} t\left(1-t^2\right)\mathcal{H}^1\left(\rho^{-1}(t)\right)dt \\ &\geq \frac{1}{\varepsilon}\int_{t=\alpha}^{1} t\left(1-t^2\right)\mathcal{H}^1_\infty\left(\{x : \rho < t\}\right)dt \\ &\geq \frac{1}{\varepsilon}\mathcal{H}^1_\infty\left(\{x : \rho < \alpha\}\right)\int_{t=\alpha}^{1} t\left(1-t^2\right)dt,\end{aligned}$$

where we need the assumption $\rho \geq \frac{1}{2}$ on ∂U between the first and second lines. Choosing $\alpha = \frac{1}{2}$ and noting that $\int_{t=\frac{1}{2}}^{1} t\left(1-t^2\right)dt = \frac{9}{64}$, then $\mathcal{H}^1_\infty(\{x : \rho(x) \leq \frac{1}{2}\}) \leq 8\varepsilon\int_U e_{csh}(|u|)dx$. □

In particular, this implies control on the rate of convergence of $\rho \to 1$, which will be used in the proof of compactness of the Jacobian. We recall that a nontopological vortex is a region where $|u| < \frac{1}{2}$ and has a phase with trivial winding number about this region.

Corollary 2.4. *Suppose $|u| \geq \frac{1}{2}$ on ∂U. Then the sum of the radii of balls covering the non topological vortices is $O\left(\varepsilon \int_U e_{csh}(|u|)dx\right)$. Furthermore, we have bounds on the rate of convergence of $\rho \to 1$. In particular*

$$\left\|1-\rho^2\right\|_{L^2} \leq 8\varepsilon \int_U e_{csh}(|u|)dx, \quad \left\|1-\rho^2\right\|_{L^3} \leq 4\left(\varepsilon \int_U e_{csh}(|u|)dx\right)^{\frac{2}{3}} \tag{2.5}$$

so long as $\int_U e_{csh}(|u|)dx \geq |U|$. If $\int_U e_{csh}(|u|)dx < |U|$ then $\|1-\rho^2\|_{L^2} \leq \varepsilon 8\,|U|$ and $\|1-\rho^2\|_{L^3} \leq \varepsilon^{\frac{2}{3}} 4\,|U|$.

Proof. From

$$\begin{aligned}\int_U \left(1-\rho^2\right)^2 dx &= \int_{\{\rho\geq\frac{1}{2}\}} \left(1-\rho^2\right)^2 dx + \int_{\{\rho\leq\frac{1}{2}\}} \left(1-\rho^2\right)^2 dx \\ &\leq 4\int_{\{\rho\geq\frac{1}{2}\}} \rho^2\left(1-\rho^2\right)^2 dx + \left|\left\{\rho \leq \frac{1}{2}\right\}\right| \\ &\leq 8\varepsilon^2 \int_U e_{csh}(|u|) + \pi\left(4\varepsilon\int_U e_{csh}(|u|)\right)^2,\end{aligned}$$

where in the last line we use $\left(\sum r_j\right)^2 \geq \sum r_j^2$ if $r_j > 0$.

Next $\left(1-\rho^2\right)^3 = \left(1-\rho^2\right)^2 - \rho^2\left(1-\rho^2\right)^2$, so

$$\begin{aligned}\int_U \left(1-\rho^2\right)^3 dx &= \int_U \left(1-\rho^2\right)^2 dx - \int_U \rho^2\left(1-\rho^2\right)^2 dx \\ &\leq (8+16\pi)\left(\varepsilon\int_U e_{csh}(|u|)dx\right)^2 + 2\varepsilon^2\int_U e_{csh}(|u|)dx.\end{aligned}$$

□

3. Basic Jacobian Bound

In this section we show a relationship between the Jacobian

$$J(u) = \det \nabla u = \frac{1}{2}\operatorname{curl} j(u)$$

and the energy density $e_{csh}(u)$. Set $\phi \in C_c^{0,1}(U)$ a Lipschitz function vanishing on ∂U. We define $\Omega(t) = \{x \in U \text{ such that } \phi(x) > t\}$ then $\partial\Omega(t)$ is a level set ϕ. Let

$$\operatorname{Reg}(\phi) := \left\{\begin{array}{l} t \in [0, \|\phi\|_{L^\infty}] \text{ such that } \partial\Omega(t) = \phi^{-1}(t), \\ \partial\Omega(t) \text{ rectifiable, and } \mathcal{H}^1(\partial\Omega(t)) < \infty \end{array}\right\}.$$

By the co-area formula $|\mathrm{Reg}(\phi)| = \|\phi\|_{L^\infty}$ and $t \in \mathrm{Reg}(\phi)$ implies $\partial\Omega(t)$ is a union of finite Jordan curves, $\Gamma_i(t)$. We set, as in [17],

$$\Gamma(t) := \cup\{\text{components of } \partial\Omega(t) \text{ such that } \min_{x\in\Gamma_i(t)} |u| > \frac{1}{2}\}$$
$$\gamma(t) := \cup\{\text{components of } \partial\Omega(t) \text{ such that } \min_{x\in\Gamma_i(t)} |u| \le \frac{1}{2}\}.$$

We set $d \in \mathbb{Z}^+$ and define

$$D^d := \{t \in \mathrm{Reg}(\phi) : |\deg(u;\Gamma(t))| \ge d+1 \text{ or } \mathcal{H}^1(\gamma(t)) \ge \varepsilon\}$$
$$A := \mathrm{Reg}(\phi)\backslash D^d.$$

Furthermore, define

$$D_\gamma := \{t \in \mathrm{Reg}(\phi) : \mathcal{H}^1(\gamma(t)) \ge \varepsilon\}$$
$$D_d := \{t \in \mathrm{Reg}(\phi) : \Gamma(t) \text{ is nonempty and } |\deg(u;\Gamma(t))| \ge d+1\}$$

so that

$$D_\gamma \cap D_d = D^d.$$

We will choose d in a special way in Section 4. In Subsection 3.1 we offer a bound on $|D_d|$ in terms of the excess energy. Let us define

$$E_\phi(u) = \int_{\mathrm{spt}(\phi)} e_{csh}(u)dx$$

for short. The main results of this section are

Proposition 3.1. *Suppose $U \subset \mathbb{R}^2$ and $u \in H^1(U;\mathbb{C})$ then*

$$\begin{aligned}\left|\int_U \phi J(u)dx\right| \le \left(\pi d + \varepsilon^{\frac{1}{2}}\right)\|\phi\|_{L^\infty} \\ + \varepsilon^{\frac{1}{3}}\|\nabla\phi\|_{L^\infty}\left[2E_\phi^2(u) + 3E_\phi(u) + \frac{|\mathrm{spt}(\phi)|}{4}\right] \\ + \frac{|D_d|}{4}E_\phi(u)\end{aligned} \tag{3.1}$$

for any $\varepsilon \le 1$.

We defer the proof of Proposition 3.1. In order to use (3.1), we need to estimate $|D_d|$. This is controlled by the following result.

Proposition 3.2. *Suppose $|u| \ge 1 - \frac{1}{|\log\varepsilon|}$ on ∂U and let $\varepsilon \in (0, e^{-2})$. Define*

$$\varepsilon^{\alpha_\varepsilon} = \varepsilon\,|\log\varepsilon|^2 \int_U e_{csh}(u)dx \qquad \eta_\varepsilon = \frac{1}{1-\frac{2}{|\log\varepsilon|}} > 1,$$

then

$$|D_d| \le 8\varepsilon^{\alpha_\varepsilon}\|\nabla\phi\|_{L^\infty}\left(1 + \exp\left(\frac{\eta_\varepsilon}{\pi d_\star}\int_V e_{csh}(u)dx\right)\right) \tag{3.2}$$

where $d_\star = d+1$. *Here* $V \equiv \dot\cup_j B_{r_j}$ *is the union of disjoint balls* B_{r_j} *with* $B_{r_j} \subseteq \mathrm{spt}(\phi) \subseteq U$ *and* $\sum_j r_j = \frac{|D_d|}{2\|\nabla\phi\|_{L^\infty}}$.

Remark 3.3. Estimate (3.2) implies a weaker estimate

$$|D_d| \leq 8\varepsilon^{\alpha_\varepsilon} \|\nabla\phi\|_{L^\infty} \left(1 + \exp\left(\frac{\eta_\varepsilon}{\pi d_\star} E_\phi(u)\right)\right) \tag{3.3}$$

which is sufficient for the $E_1(u) \approx |\log\varepsilon|$ case. In this case we choose $d+1 \geq \frac{E_\phi(u)}{\pi|\log\varepsilon|}$, which provides a good bound on D_d. For the large energy $E_\phi(u) \approx |\log\varepsilon|^2$, we need the refined estimate (3.2).

The proof of Proposition 3.2 is deferred until Subsection 3.1. Although the proof of Proposition 3.2 employs some ideas of [17], it is fundamentally different. This is because the argument of [17] relies on point-wise lower bounds of $\int_{\partial B_s} e_{gl}(u)dl > 0$ on each radius ∂B_s. In the CSH case the point-wise lower bound of $\int_{\partial B_s} e_{csh}(u)dl$ is **zero** for each radius s, even when $\deg(u, \partial B_s)) \neq 0$. To overcome this difficulty, we use a strongly modified version of the method originating in [29], see also [31]. In fact our method can be used for energies of the form

$$\int |\nabla u|^2 + \frac{1}{\varepsilon^2} W(|u|^2)dx,$$

where $W(s) = s^p(1-s)^q$.

Proof of Proposition 3.1. The proof follows from a decomposition. In particular

$$\begin{aligned}\int_U \phi J(u)dx &= \frac{1}{2}\int_A \int_{\Gamma(t)} j(u)\cdot \mathbf{t} d\mathcal{H}^1 dt \\ &\quad + \frac{1}{2}\int_A \int_{\gamma(t)} j(u)\cdot \mathbf{t} d\mathcal{H}^1 dt + \frac{1}{2}\int_D \int_{\partial\Omega_t} j(u)\cdot \mathbf{t} d\mathcal{H}^1 dt \\ &= I_{A,\Gamma} + I_{A,\gamma} + I_D.\end{aligned}$$

From Lemma 3.5, Lemma 3.6, and Lemma 3.8 below we have

$$\begin{aligned}\left|\int_U \phi J(u)dx\right| &\leq \pi d \|\phi\|_{L^\infty} + 2\varepsilon \|\nabla\phi\|_{L^\infty} E_\phi(u) \\ &\quad + \frac{1}{2}\varepsilon^{\frac{1}{2}} \|\nabla\phi\|_{L^\infty} E_\phi(u) + \varepsilon^{\frac{1}{2}} \|\phi\|_{L^\infty} \\ &\quad + \varepsilon^{\frac{1}{3}} \|\nabla\phi\|_{L^\infty} \left[2E_\phi^2(u) + \frac{1}{2}E_\phi(u) + \frac{|\mathrm{spt}(\phi)|}{4}\right] \\ &\quad + \frac{1}{4}|D_d| E_\phi(u).\end{aligned}$$

The bound follows. $\square$

In order to finish the proof of Proposition 3.1, we need to establish Lemma 3.5, Lemma 3.6, and Lemma 3.8. We start with a basic estimate of Jerrard-Soner [17].

Lemma 3.4 (Jerrard-Soner)**.** *For any set S,*

$$\int_S \left| \int_{\partial\Omega(t)} j(u) \cdot \mathbf{t} d\mathcal{H}^1 \right| dt \leq \frac{|S|}{2} \int_{\mathrm{spt}(\phi)} e_{csh}(u) dx \tag{3.4}$$

and for any nonnegative function f,

$$\int_0^T \int_{\partial\Omega(t)} f(x) d\mathcal{H}^1 dt \leq \|\nabla\phi\|_{L^\infty} \int_{\mathrm{spt}(\phi)} f(x) dx \tag{3.5}$$

Proof. For any $t \in \mathrm{Reg}(\phi)$, Stokes' Theorem implies

$$\int_{\partial\Omega(t)} j(u) \cdot \mathbf{t} d\mathcal{H}^1 = \frac{1}{2} \int_{\Omega(t)} J(u) dx$$

Since $|J(u)| = \frac{1}{2} \operatorname{curl} j(u) \leq \frac{1}{2} |\nabla u|^2 \leq e_{csh}(u)$, then we get the first identity. On the other hand from the co-area formula,

$$\begin{aligned} \int_0^T \int_{\partial\Omega(t)} f d\mathcal{H}^1 dt &= \int_{\mathrm{spt}(\phi)} f\, |\nabla\phi|\, dx \\ &\leq \|\nabla\phi\|_{L^\infty} \int_{\mathrm{spt}(\phi)} f dx. \end{aligned}$$

□

We bound these three terms via the follow lemmas.

Lemma 3.5. *We have*

$$\left| \int_A \int_{\partial\Omega(t)} j(u) \cdot \mathbf{t} \mathcal{H}^1 dt \right| \leq 2\pi d_\lambda \|\phi\|_{L^\infty} + 4\varepsilon^{\frac{1}{2}} \|\nabla\phi\|_{L^\infty} E_\phi(u)$$

Proof. For $t \in A$, $|u| \geq \frac{1}{2}$. We set $v = \frac{u}{|u|}$ and so $j(v) = \frac{j(u)}{|u|^2}$. Therefore,

$$\int_{\Gamma(t)} j(v) \cdot \mathbf{t} d\mathcal{H}^1 = 2\pi \deg\left(u; \Gamma(t)\right)$$

Therefore,

$$\int_{\Gamma(t)} j(u) \cdot \mathbf{t} d\mathcal{H}^1 = 2\pi \deg\left(u; \Gamma(t)\right) + \int_{\Gamma(t)} j(u) \frac{|u|^2 - 1}{|u|^2} \cdot \mathbf{t} d\mathcal{H}^1$$

This implies (since $j(u) \leq |u|\,|\nabla u|$) then

$$\begin{aligned}\int_A \left| \int_{\Gamma(t)} j(u) \cdot \mathbf{t} d\mathcal{H}^1 - 2\pi \deg\left(u; \Gamma(t)\right) \right| dt &\leq \int_A \int_{\Gamma(t)} |\nabla u| \left| \frac{|u|\,\left||u|^2 - 1\right|}{|u|^2} \right| d\mathcal{H}^1 \\ &\leq 4\varepsilon \int_A \int_{\Gamma(t)} e_{csh}(u) d\mathcal{H}^1 \\ &\leq 4\varepsilon \left\|\nabla \phi\right\|_{L^\infty} E_\phi(u)\end{aligned}$$

The bound follows from noting $|A| \leq \|\phi\|_{L^\infty}$. □

Lemma 3.6. *Let $\gamma(t)$ such that $\mathcal{H}^1(\gamma(t)) \leq \varepsilon$ with $\varepsilon \leq 1$ then*

$$\left| \int_A \int_{\gamma(t)} j(u) \cdot \mathbf{t}\, d\mathcal{H}^1 dt \right| \leq \varepsilon^{\frac{1}{2}} \left\|\nabla \phi\right\|_{L^\infty} E_\phi + \varepsilon^{\frac{1}{2}} \left\|\phi\right\|_{L^\infty}.$$

Proof. 1. We claim that any $x \geq 0$ satisfies $x \leq \frac{x(x-1)^2}{b} + \left(\frac{b}{4} + 1\right)$ for any $b > 0$. In particular if $x \geq 0$ then by Young's Inequality,

$$x \leq \frac{2}{3} x^{\frac{3}{2}} + \frac{1}{3} = \frac{2}{3}\sqrt{x}\,(x-1) + \frac{2}{3}\sqrt{x} + \frac{1}{3} \leq \frac{2}{3}\sqrt{x}\,(x-1) + \frac{x}{3} + \frac{2}{3}$$

so $\frac{2}{3}x \leq \frac{2}{3}\sqrt{x}\,(x-1) + \frac{2}{3}$. Applying Cauchy-Schwarz,

$$x \leq \sqrt{x}\,(x-1) + 1 \leq \frac{x\,(x-1)^2}{b} + \left(\frac{b}{4} + 1\right),$$

and so yields the claim.

2. Therefore, since $|j(u)| \leq |u|\,|\nabla u|$ then

$$\begin{aligned}|j(u)| \leq |\nabla u|\,|u| \leq \frac{a}{2}|\nabla u|^2 + \frac{1}{2a}|u|^2 &= \frac{a}{2}\left(|\nabla u|^2 + \frac{1}{a^2}|u|^2\right) \\ &\leq \frac{a}{2}\left(|\nabla u|^2 + \frac{|u|^2\left(1 - |u|^2\right)^2}{a^2 b}\right) + \frac{1}{2a}\left(1 + \frac{b}{4}\right)\end{aligned}$$

Set $a = \varepsilon^\alpha$ and $b = \varepsilon^{2(1-\alpha)}$ then $|j(u)| \leq \varepsilon^\alpha e_{csh}(|u|) + \varepsilon^{-\alpha}$ so long as $\varepsilon \leq 1$. Since $|A| \leq T = \|\phi\|_{L^\infty}$ and $\mathcal{H}^1(\gamma(t)) < \varepsilon$ for every $t \in A$ then from Lemma 3.4

$$\begin{aligned}|I_{A,\gamma}| &\leq \int_A \int_{\gamma(t)} \varepsilon^\alpha e_{csh}(u) d\mathcal{H}^1 dt + \int_A \int_{\gamma(t)} \varepsilon^{-\alpha} d\mathcal{H}^1 dt \\ &\leq \varepsilon^\alpha \left\|\nabla\phi\right\|_{L^\infty} E_\phi(u) + \varepsilon^{-\alpha} \int_A \mathcal{H}^1(\gamma(t)) dt \\ &\leq \varepsilon^\alpha \left\|\nabla\phi\right\|_{L^\infty} E_\phi(u) + \varepsilon^{1-\alpha} \int_A dt \\ &\leq \varepsilon^\alpha \left\|\nabla\phi\right\|_{L^\infty} E_\phi(u) + \varepsilon^{1-\alpha} \left\|\phi\right\|_{L^\infty}.\end{aligned}$$

Setting $\alpha = \frac{1}{2}$ finishes the proof.

□

In order to bound the D^d terms, we prove a lower bound on contours.

Lemma 3.7. *Suppose $\mathcal{H}^1(\gamma(t)) \geq \varepsilon$ and $\|u\|_{L^\infty(\gamma(t))} \geq \varepsilon^\alpha$ then*

$$\int_{\gamma(t)} e_{csh}(u) d\mathcal{H}^1 \geq \frac{1}{8}\frac{1}{\varepsilon^{1-2\alpha}}.$$

Proof. The proof is similar to, but weaker than, a result in [17]. Fix a connected component $\Gamma_i(t)$ of $\gamma(t)$ and set $\rho := |u|$ and

$$\beta_i := \int_{\Gamma_i(t)} \frac{1}{2} |\nabla \rho|^2 \, d\mathcal{H}^1$$

By the definition of $\gamma(t)$ there is a point $x_\star \in \Gamma_i(t)$ such that $\varepsilon^\alpha \leq x_\star \leq \frac{1}{2}$. Parametrizing $\Gamma_i(t)$ by arclength with

$$\Gamma_i(t) = \{x(s) | s \in [0, G_i]\} \qquad G_i := \mathcal{H}^1(\Gamma_i(t))$$

with $x_\star = x(0) = x(G_i)$. Then since $|\dot{x}(s)| = 1$,

$$\begin{aligned} \rho(x(s)) &= \rho(x(0)) + \int_0^s \nabla\rho(x(r)) \cdot \dot{x}(r) dr \\ &\leq \frac{1}{2} + \sqrt{s}\sqrt{\int_0^s |\nabla\rho(x(r))|^2 \, dr} \leq \frac{1}{2} + \sqrt{s\beta_i} \leq \frac{3}{4} \end{aligned}$$

and $\rho(x(s)) \geq \varepsilon^\alpha - \sqrt{s\beta_i} \geq \frac{\varepsilon^\alpha}{2}$, so long as $s \leq \sigma_i := \min\{G_i, \frac{1}{16\beta_i}, \frac{\varepsilon^{2\alpha}}{4\beta_i}\}$. This implies for $x \in [0, \sigma_i]$

$$\frac{1}{4}\rho^2(x(s))\left(1 - \rho^2(x(s))\right) \geq \frac{1}{4}\left(\frac{\varepsilon^\alpha}{4}\right)^2 \left(1 - \left(\frac{3}{4}\right)^2\right) = \varepsilon^{2\alpha}\frac{7^2}{4^4}$$

Therefore,

$$\int_{\Gamma_i(t)} e_{csh}(u) d\mathcal{H}^1 \geq \beta_i + \int_{\Gamma_i(t)} \frac{1}{4\varepsilon^2}\rho^2\left(1-\rho^2\right) d\mathcal{H}^1 \geq \beta_i + \frac{\sigma_i}{\varepsilon^2}\frac{\varepsilon^{2\alpha}7^2}{4^4}$$

and minimizing over β_i we find (for $\varepsilon^{2\alpha} \leq \frac{1}{4}$),

$$\begin{aligned} \beta_i + \frac{\sigma_i \varepsilon^{2\alpha-2}7^2}{4^4} &= \beta_i + \frac{\varepsilon^{2\alpha-2}7^2}{4^4}\min\{G_i, \frac{\varepsilon^{2\alpha}}{4\beta_i}\} \\ &\geq \min\left\{\frac{\varepsilon^{2\alpha-2}7^2 G_i}{4^4}, \frac{\varepsilon^{2\alpha-1}7}{2^5}\right\} \\ &= \frac{\varepsilon^{2\alpha}}{\varepsilon}\frac{7}{2^5}\min\{\frac{G_i 7}{\varepsilon 2^3}, 1\} \end{aligned}$$

Summing over components, we find

$$\begin{aligned}\int_{\gamma(t)} e_{csh}(u)d\mathcal{H}^1 &= \sum_{\Gamma_i(t) \text{ components of } \gamma(t)} \int_{\Gamma_i(t)} e_{csh}(u)d\mathcal{H}^1 \\ &\geq \frac{\varepsilon^{2\alpha}}{\varepsilon}\frac{7}{2^5}\min\{\frac{\sum G_i 7}{\varepsilon 2^3}, 1\} \geq \frac{\varepsilon^{2\alpha}}{\varepsilon}\frac{7}{2^5}\min\{\frac{\varepsilon 7}{\varepsilon 2^3}, 1\} \\ &\geq \frac{\varepsilon^{2\alpha}}{\varepsilon}\frac{7^2}{2^8} \geq \varepsilon^{2\alpha-1}\frac{48}{2^8} = \varepsilon^{2\alpha-1}\frac{3(2^4)}{2^8} \geq \varepsilon^{2\alpha-1}\frac{2^5}{2^8},\end{aligned}$$

which finishes the lower bound. □

Recall

$$\begin{aligned}D_\gamma &:= \{t \in \mathrm{Reg}(\phi) : \mathcal{H}^1(\gamma(t)) \geq \varepsilon\} \\ D_d &:= \{t \in \mathrm{Reg}(\phi) : \Gamma(t) \text{ is nonempty and } |\deg(u; \Gamma(t))| \geq d+1\}\end{aligned}$$

Lemma 3.8.

$$\left|\int_D \int_{\gamma(t)} j(u)\cdot \mathbf{t} d\mathcal{H}^1 dt\right| \leq \varepsilon^{\frac{1}{3}} \|\nabla\phi\|_{L^\infty}\left[4E_\phi^2(u) + E_\phi(u) + \frac{|\mathrm{spt}(\phi)|}{2}\right] + \frac{1}{2}|D_d| E_\phi(u)$$

Proof. We first consider D_γ bounds.

1. We subdivide D_γ:

$$\begin{aligned}D_\gamma^1 &:= \{t \in \mathrm{Reg}(\phi) : \mathcal{H}^1(\gamma(t)) \geq \varepsilon \text{ and } \|u(\gamma(t))\|_{L^\infty} \geq \varepsilon^\alpha\} \\ D_\gamma^2 &:= \{t \in \mathrm{Reg}(\phi) : \mathcal{H}^1(\gamma(t)) \geq \varepsilon \text{ and } \|u(\gamma(t))\|_{L^\infty} < \varepsilon^\alpha\}.\end{aligned}$$

and we consider a bound on D_γ^1 first. Since

$$\int_{D_\gamma}\int_{\partial\Omega(t)} e_{csh}(u)d\mathcal{H}^1 dt \geq \int_{D_\alpha^1} \frac{\varepsilon^{2\alpha-1}}{8}dt = \frac{|D_\alpha^1|}{\varepsilon^{1-2\alpha}8}$$

then

$$\left|D_\gamma^1\right| = 8\varepsilon^{1-2\alpha}\int_{D_\gamma^1}\int_{\partial\Omega(t)} e_{csh}(u)d\mathcal{H}^1 dt \leq C\varepsilon^{1-2\alpha}\|\nabla\phi\|_{L^\infty} E_\phi(u)$$

Therefore,

$$\left|\int_{D_\gamma^1}\int_{\gamma(t)} j(u)\cdot\mathbf{t} d\mathcal{H}^1 dt\right| \leq \frac{\left|D_\gamma^1\right|}{2}\int_{\mathrm{spt}(\phi)} e_{csh}(u)dx \leq 4\varepsilon^{1-2\alpha}\|\nabla\phi\|_{L^\infty} E_\phi^2(u)$$

This implies

$$\int_{D_\gamma^1}\int_{\gamma(t)} j(u)\cdot\mathbf{t} d\mathcal{H}^1 dt \leq 4\varepsilon^{1-2\alpha}\|\nabla\phi\|_{L^\infty} E_\phi^2(u). \tag{3.6}$$

2. Next for D_γ^2 we have $|j(u)| \leq |u|^{\frac{1}{2}} |\nabla u| |u|^{\frac{1}{2}} \leq |u| \frac{|\nabla u|^2}{2} + \frac{|u|}{2}$ then

$$\int_{D_\gamma^2} \int_{\gamma(t)} |j(u)| \, d\mathcal{H}^1 dt \leq \int_{D_\gamma^2} \int_{\gamma(t)} |u| \frac{|\nabla u|^2}{2} + \frac{|u|}{2} d\mathcal{H}^1 dt$$
$$\leq \varepsilon^\alpha \|\nabla\phi\|_{L^\infty} \int_{\operatorname{spt}(\phi)} \left(e_{csh}(u) + \frac{1}{2} \right) dx$$

or

$$\int_{D_\gamma^2} \int_{\gamma(t)} j(u) \cdot \mathbf{t} \, d\mathcal{H}^1 dt \leq \varepsilon^\alpha \|\nabla\phi\|_{L^\infty} \left[E_\phi(u) + \frac{|\operatorname{spt}(\phi)|}{2} \right]. \tag{3.7}$$

We choose $\alpha = \frac{1}{3}$ and the bound on D_γ follows.

3. Finally for D_d we have

$$\left| \int_{D_d} \int_{\Gamma(t)} j(u) \cdot \mathbf{t} d\mathcal{H}^1 dt \right| \leq \frac{|D_d|}{2} \int_{\operatorname{spt}(\phi)} e_{csh}(u) dx \leq \frac{1}{2} |D_d| \, E_\phi(u).$$

In the following subsection we show that D_d must have small measure. □

3.1. **Control on D_d.** In order to prove Proposition 3.2 we show that $|D_d|$ is controlled by the energy and the degree d. We first define $\omega_t = \{x \in U : |u(x)| \leq t\}$ to be the level set of $|u|$. Here we make use of the boundary condition on $|u|$.

Lemma 3.9. *Suppose $|u| \geq 1 - \frac{1}{|\log\varepsilon|}$ on ∂U and set $t^\star \in \left(0, 1 - \frac{1}{|\log\varepsilon|}\right]$ then*

$$\mathcal{H}^1_\infty(\omega_{t^\star}) \leq 4\varepsilon \, |\log\varepsilon|^2 \, E_1(u) \tag{3.8}$$

for $\varepsilon \leq \frac{1}{8}$.

Proof. Repeating the argument from Lemma 2.3,

$$\int_U e_{csh}(u) dx \geq \frac{1}{2\varepsilon} \int_0^1 t \left(1 - t^2\right) \mathcal{H}^1 \left(\rho^{-1}(t)\right) dt$$
$$\geq \frac{\mathcal{H}^1 \left(\rho^{-1}\left(t^\star\right)\right)}{2\varepsilon} \int_{t_\star}^1 t - t^3 dt$$
$$\geq \frac{\mathcal{H}^1 \left(\rho^{-1}\left(t^\star\right)\right)}{2\varepsilon} \int_{1-\frac{1}{|\log\varepsilon|}}^1 t - t^3 dt$$
$$= \frac{\mathcal{H}^1 \left(\rho^{-1}\left(t^\star\right)\right)}{2\varepsilon} \left(\frac{t^2}{2} - \frac{t^4}{4} \right) \Bigg|_{1-\frac{1}{|\log\varepsilon|}}^1$$

We evaluate the integral and

$$\left(\frac{t^2}{2}-\frac{t^4}{4}\right)\Big|_{1-\frac{1}{|\log\varepsilon|}}^{1} = \frac{1}{|\log\varepsilon|^2}-\frac{1}{|\log\varepsilon|^3}+\frac{1}{4\,|\log\varepsilon|^4}$$
$$\geq \frac{1}{|\log\varepsilon|^2}-\frac{1}{|\log\varepsilon|^3} = \frac{1}{|\log\varepsilon|^2}\left(1-\frac{1}{|\log\varepsilon|}\right)$$

Therefore,

$$\mathcal{H}^1_\infty(\omega_{t^\star}) \leq 2\varepsilon\frac{|\log\varepsilon|^2}{1-\frac{1}{|\log\varepsilon|}}\int_U e_{csh}(u)dx \leq 4\varepsilon\,|\log\varepsilon|^2\int_U e_{csh}(u)dx$$

so long as $|\log\varepsilon| \geq 2$. □

In this part we use the ideas of Sandier in [29] to compute lower bounds on the $\mathbb{S}^1$-valued map. The following covering lemma follows from an iteration process for $\mathbb{S}^1$-valued maps and is a slight modification of the method introduced by Sandier [29].

Lemma 3.10. *Suppose ω is a compact subset of $V \subset\subset U$ and let $r(\omega) = \sum r_j(0)$ be the minimum sum of radii of balls $B_{r_j(0)}$ covering ω, i.e.*

$$r(\omega) = \inf\left\{\sum r_j(0) \text{ such that } \omega \subset \cup B_{r_j(0)}\right\} = \mathcal{H}^1_\infty(\omega)$$

Let $v : U\backslash\omega \mapsto \mathbb{S}^1$ then for each $s \geq 0$ there exists a collection of balls $\mathcal{K}_s = \{B_{r_k(s)}\}$ such that

(1) *$r_k(s)$ is an increasing function of s for each k.*

(2) *For any subset of balls $\{B_{r_{k_j}(s)}\} \subseteq \{B_{r_k(s)}\}$ in $\mathcal{K}_s$*

$$\frac{1}{2}\int_{\cup_{k_j}B_{r_{k_j}(s)}\backslash\omega}|\nabla v|^2dx \geq \pi\left(\sum_{k_j}\left|d_{k_j}\right|\right)\log\frac{\sum_{k_j} r_{k_j}(s)}{r(\omega)}$$

where $d_{k_j} = \deg(v, \partial B_{r_{k_j}(s)})$.

Proof. 1. We start with a family of disjoint balls $\{B_k(0)\}$ with $\omega = \cup B_k(0)$ and set $d_j = \deg(v, \partial B_k)$. For a later time $t > 0$ we define a new family of balls $\{B_i(t)\}$ with radii $r_i(t)$ and degrees $d_i(t)$. We also define a seed size $\varepsilon_i(t)$ of $B_i(t)$. We set $\varepsilon_i(0) = r_i(0)$. Finally we define an expansion function

$$\alpha(t) = \log\frac{r_j(t)}{\varepsilon_k(0)}$$

identical for *all* k. We now grow the balls $r_k(t)$. We have two cases:

(1) If $\overline{B_i(t)} \cap \overline{B_j(t)} = \varnothing$ for all $i \neq j$ then **Expand**
Leave the $\varepsilon_i(s)$ constant as $r_i(s)$ increases continuously such that each $\alpha(s) = \log \frac{r_i(s)}{\varepsilon_i(s)}$ is the same. This implies $\alpha(s)$ increases continuously.

(2) If $\overline{B_i(t)} \cap \overline{B_j(t)} \neq \varnothing$ some $i \neq j$ then **Merge**
Include both balls in the smallest ball that contains both balls. If the closure hits another ball, continue merging until we have a ball disjoint from all other balls in the family. This ball contains $B_{i_1}(t), \ldots, B_{i_k}(t)$ and has radius $r(t) \leq r_{i_1}(t) + \cdots + r_{i_k}(t)$ and degree $d(t) = d_{i_1} + \cdots + d_{i_k}(t)$. The only issue is the redefine the seed size. In particular we want $\alpha(t) = \log \frac{r(t)}{\varepsilon(t)}$ for the $r(t)$ defined above. Therefore, we take $\varepsilon(t) = r(t)e^{-\alpha(t)}$.

This process can be continued indefinitely.

2. We collect some facts about the expansion and merging process.

(1) $\omega \subset \cup B_i(t)$ for all $t \geq 0$. This is trivial.

(2) For any subset $\{k_j\} \subset \{k\}$
$$\alpha(t) = \log \frac{r_i(t)}{\varepsilon_i(t)} = \log \frac{\sum_j r_{k_j}(t)}{\sum_j \varepsilon_{k_j}(t)}.$$
This follows from the simple observation that if $\frac{a}{c} = \frac{b}{d}$ then $\frac{a+b}{c+d} = \frac{a}{c} = \frac{b}{d}$.

(3) The upper bound
$$\varepsilon_i(t) \leq \sum_{j \text{ such that } B_j(0) \subset B_i(t)} r_j(0).$$
This fact holds through expansion, so we need only check that it holds through merging. Suppose at time t we merge $B_{i_1}(t), \ldots, B_{i_k}(t)$ into B with radius r and seed size ε. Then $\log \frac{r}{\varepsilon} = \alpha(t) = \log \frac{r_j(t)}{\varepsilon_j(t)} = \log \frac{\sum_k r_{i_k}(t)}{\sum_k \varepsilon_{i_k}(t)}$. Therefore,
$$\frac{r}{\varepsilon} = \frac{\sum_{i_k} r_{i_k}(t)}{\sum_k \varepsilon_{i_k}(t)}$$
or
$$\varepsilon = \varepsilon_{i_k}(t) \frac{r}{\sum_{i_k} r_{i_k}(t)} \leq \varepsilon_{i_k}(t)$$
since $r \leq \sum_{i_k} r_{i_k}(t)$.

which concludes the needed facts on the growth process.

3. For any annulus $\frac{1}{2}\int_{B_r \setminus B_\varepsilon} |\nabla v|^2 dx \geq \pi d^2 \log \frac{r}{\varepsilon} \geq \pi |d| \log \frac{r}{\varepsilon}$ for our $\mathbb{S}^1$-valued function v. Following our growth strategy for the $r(t)$'s and $\varepsilon(t)$'s we have

bounds

$$\frac{1}{2}\int_{\cup B_{r_j}(t)\setminus B_\varepsilon(t)} |\nabla v|^2 dx \geq \pi \sum_j |d_j| \log \frac{r_j(t)}{\varepsilon_j(t)} = \pi \left(\sum_j |d_j|\right) \log \frac{r_j(t)}{\varepsilon_j(t)}$$

$$= \pi \left(\sum_j |d_j|\right) \log \frac{\sum_j r_j(t)}{\sum_j \varepsilon_j(t)}$$

Since $\sum_j \varepsilon_j(t) \leq \sum_j \varepsilon_j(0) \leq |\omega| = r(\omega)$ then we get the required lower bound. $\square$

Definition 3.11. Recall the different types of domains:

$$\omega_t = \{x \in U \text{ such that } |u| \leq t\}$$

and

$$\Omega(t) = \{x \in U \text{ such that } \phi(x) < t\}.$$

Note $\Gamma(t) = \partial\Omega$.

We can now turn to the

Proof of Proposition 3.2. We will choose our set V in Step 6. We recall an energy bound of [29], see also [31].

1. Let $\Theta(t) = \int_{V\setminus\omega_t} |\nabla\varphi|^2\, dx$ where $u = \rho e^{i\varphi}$. Then following [29]

$$\int_V e_{csh}(u)dx = \frac{1}{2}\int_0^\infty \left[\int_{\partial\omega_t\cap V} |\nabla\rho| + \frac{t^2(1-t^2)^2}{\varepsilon^2\,|\nabla\rho|} dl - t^2\Theta'(t)\right] dt$$

by the co-area formula. Cauchy-Schwarz and integration by parts yields

$$\int_V e_{csh}(u)dx \geq \int_0^1 \left[\int_{\partial\omega_t\cap V} \frac{|\nabla\rho|}{2} + \frac{t^2(1-t^2)^2}{2\varepsilon^2|\nabla\rho|} dl + \int_{V\setminus\omega_t} |\nabla\varphi|^2 dx\right] dt$$

or

$$\int_V e_{csh}(u)dx \geq \int_0^1 a(t) + 2tb(t)dt$$

with

$$a(t) = \int_{\partial\omega_t\cap V} \frac{|\nabla\rho|}{2} + \frac{t^2(1-t^2)^2}{2\varepsilon^2|\nabla\rho|} dl$$

$$b(t) = \frac{1}{2}\int_{V\setminus\omega_t} |\nabla\varphi|^2 dx.$$

Therefore,

$$\begin{aligned}\int_0^1 2tb(t)dt &\geq \int_{\frac{1}{|\log\varepsilon|}}^{1-\frac{1}{|\log\varepsilon|}} t\left(\int_{V\setminus\omega_t}|\nabla\varphi|^2\,dx\right)dt\\ &\geq \left(\inf_{t\in\left[\frac{1}{|\log\varepsilon|},1-\frac{1}{|\log\varepsilon|}\right]}\int_{V\setminus\omega_t}|\nabla\varphi|^2\,dx\right)\int_{\frac{1}{|\log\varepsilon|}}^{1-\frac{1}{|\log\varepsilon|}} tdt\\ &= \frac{1}{2}\left(1-\frac{2}{|\log\varepsilon|}\right)\int_{V\setminus\omega_{\overline{t}}}|\nabla\varphi|^2\,dx\end{aligned}$$

for $\overline{t} = 1-\frac{1}{|\log\varepsilon|}$ since $\int_{V\setminus\omega_t}|\nabla\varphi|^2\,dx$ is a decreasing function in t. This yields a lower bound

$$\frac{1}{1-\frac{2}{|\log\varepsilon|}}\int_V e_{csh}(u)dx \geq \frac{1}{2}\int_{V\setminus\omega_{\overline{t}}}|\nabla\varphi|^2\,dx \tag{3.9}$$

with $\overline{t} = 1-\frac{1}{|\log\varepsilon|}$.

2. To simplify further discussion, we redefine $U \equiv \{x\in U$ such that $\phi(x) > 0\}$. As an artifact of the proof, we use a subdomain $U^\varepsilon \subset U$ which is roughly within a distance $\varepsilon^{\alpha_\varepsilon}$ from the boundary of U. We determine the boundary via foliation of the domain by the level sets of our test function ϕ. We claim there are $t\in(0,5\varepsilon^{\alpha_\varepsilon}\|\nabla\phi\|_{L^\infty})$ such that $\partial\Omega(t)\cap\omega_{\overline{t}}=\varnothing$, and where

$$4\varepsilon^{\alpha_\varepsilon} = 4\varepsilon\,|\log\varepsilon|^2\,E_1(u) \geq \mathcal{H}^1_\infty(\omega_{\overline{t}})$$

is the bound from Lemma 3.9. To prove this let

$$G = \{t\in(0,5\varepsilon^{\alpha_\varepsilon}\|\nabla\phi\|_{L^\infty})\cap\operatorname{Reg}(\phi)\text{ such that }\partial\Omega(t)\cap\omega_{\overline{t}}\neq\varnothing\}$$

Then $G\subset\cup\phi(B_{r_k(0)})$, since $\omega_{\overline{t}}\subset\cup B_{r_k(0)}$, a set of disjoint balls, and

$$\begin{aligned}|G| &\leq 2\,\|\nabla\phi\|_{L^\infty}\sum r_k(0)\\ &\leq 4\,\|\nabla\phi\|_{L^\infty}\,\varepsilon\,|\log\varepsilon|^2\,E_1(u)\\ &= 4\,\|\nabla\phi\|_{L^\infty}\,\varepsilon^{\alpha_\varepsilon}\end{aligned} \tag{3.10}$$

Assume that there are no such $t\in(0,5\varepsilon^{\alpha_\varepsilon}\|\nabla\phi\|_{L^\infty})$, then $|G| = 5\varepsilon^{\alpha_\varepsilon}\|\nabla\phi\|_{L^\infty}$ which contradicts (3.10). Therefore, there exists a set of $t\in(0,5\varepsilon^{\alpha_\varepsilon}\|\nabla\phi\|_{L^\infty})\cap\operatorname{Reg}(\phi)$ of positive measure with $\Gamma(t)\cap\cup B_{r_k(0)}=\varnothing$. Choose one such t_ε.

Note that $\Gamma(t_\varepsilon)$ is a finite collection of disjoint, closed Jordan curves, $\{\Gamma_j\}$, so their interiors are well-defined. For each Γ_j let I_j be its interior, i.e. $\partial I_j = \Gamma_j$.

We define $U^\varepsilon = \cup I_j$, which satisfies the following properties:

$$\{x \in U \text{ such that } \phi(x) \geq t_\varepsilon\} \subseteq U^\varepsilon, \tag{3.11}$$

$$\partial U^\varepsilon \cap \omega_{\bar{t}} = \varnothing, \tag{3.12}$$

$$\operatorname{dist}(\partial U^\varepsilon, U) \leq \varepsilon^{\alpha_\varepsilon}. \tag{3.13}$$

The last fact follows from $\phi \in C_c^1(U)$; ϕ vanishes before the boundary. For a $y \in U$ and $x \in U^\varepsilon$ the mean value theorem implies

$$|x-y| \left\|\nabla\phi\right\|_{L^\infty} \geq |\phi(x) - \phi(y)| = \phi(x) \geq t_\varepsilon = 2\varepsilon^{\alpha_\varepsilon} \left\|\nabla\phi\right\|_{L^\infty}$$

by the definition of t_ε. We define

$$D_d^\varepsilon = D_d \cap \{t \geq t_\varepsilon\}$$

and bound $|D_d^\varepsilon|$ in the following steps.

3. Unleash the expand and merge process of Lemma 3.10 for initial domain $\omega_{\bar{t}}$, which we can think of as a union of disjoint balls $\cup B_{r_j(0)}$. For each s we have from Lemma 3.10 a set of balls $\mathcal{K}_s = \{B_{r_k(s)}\}$ with $\omega_{\bar{t}} \subset \cup B_{r_k(s)}$. We define

$$\mathcal{K}_s^{int} = \{B_{r_k(s)} \in \mathcal{K}_s \text{ such that } B_{r_k(s)} \subset U^\varepsilon\}$$

where the balls of $\mathcal{K}_s^{int}$ are wholly included in our subdomain U^ε.

We claim we can continue the expansion process until some time s_σ with

$$\sum_{B_{r_j(s_\sigma)} \in \mathcal{K}_{s_\sigma}^{int}} r_j(s_\sigma) = \sigma = \frac{|D_d^\varepsilon|}{2\left\|\nabla\phi\right\|_{L^\infty}}. \tag{3.14}$$

Let

$$\Upsilon(s) = \sum_{B_{r_k(s)} \in \mathcal{K}_s^{int}} r_k(s).$$

From the expansion process defined in Lemma 3.10, each $r_k(s)$ with $B_{r_k(s)} \in \mathcal{K}_s$ is continuously increasing until a merging happens. Furthermore, if a merge of $B_{k_1}, \dots, B_{k_l}$ happens at s, then the new ball's radius $r(s) \leq \sum_j r_{k_j}(s)$. Finally, by definition and the nesting of balls in $\mathcal{K}_s$, balls can only leave $\mathcal{K}_s^{int}$ and not enter at a later time. These three facts imply that

$$\Upsilon(s) \text{ is } \begin{cases} \text{lower semi-continuous} \\ \text{increasing on continuous intervals} \\ \text{nonincreasing on discontinous points.} \end{cases} \tag{3.15}$$

Set

$$\sigma^{max} = \limsup_{s>0} \Upsilon(s)$$

to be the largest covering of components of $\omega_{\bar{t}}$ that lie inside U^ε and

$$s_\star = \limsup_{s>0} \{s \text{ such that } \Upsilon(s) \neq \varnothing\}$$

to be the last time which contains a vortex ball inside U^ε. Note $\sigma^{max} \le \frac{\operatorname{diam}(U)}{2}$ implies $s_\star < \infty$.

We prove (3.14) by contradiction. Suppose the claim (3.14) is false, then $\Upsilon(s) < \sigma$ for all $s \ge 0$. We now prove a contradiction by showing that if $\Upsilon(s) < \sigma$ for all $s \in [0, s_\star]$, then we can find $s_\delta > s_\star$ such that $\Upsilon(s) > 0$ for all $s \in [0, s_\delta]$, which contradicts the definition of $s_\star$. Let a and b satisfy $0 \le a \le b$, then we define

$$\widetilde{\mathcal{K}}_b^a = \left\{ B_{r_l(b)} \in \mathcal{K}_b \text{ such that there exists } B_{r_k(a)} \in \mathcal{K}_a^{int} \text{ with } B_{r_k(a)} \subset B_{r_l(b)} \right\}$$

and

$$C_b^a = \left\{ t \in (t_\varepsilon, \|\phi\|_{L^\infty}) \cap \operatorname{Reg}(\phi) \text{ such that } \Gamma(t) \cap \left[\bigcup_{B_{r_k(b)} \in \widetilde{\mathcal{K}}_b^a} B_{r_k(b)} \right] \ne \varnothing \right\}.$$

This definition lets us grow balls from a previous time without worrying about balls leaving $\mathcal{K}_a^{int}$. Since the merge and expand process of Lemma 3.10 yields continuously increasing radii of balls except on a countable number of of jumps we have two cases. Either we can find an $s_\delta > s_\star$ with $s_\delta - s_\star \ll 1$ and

$$\sum_{B_{r_l(s_\delta)} \in \widetilde{\mathcal{K}}_{s_\delta}^{s_\star}} r_l(s_\delta) < \sigma \tag{3.16}$$

or there is a jump at the exit time $s_\star$. The case where there is a jump at the exit time $s_\star$ is handled in Step 4 below. We suppose (3.16) holds.

For any $0 \le a \le b$,

$$C_b^a \subseteq \bigcup_{B_{r_k(b)} \in \widetilde{K}_b^a} \phi(B_{r_k(b)})$$

so by (3.16)

$$\left| C_{s_\star}^{s_\delta} \right| \le 2 \|\nabla \phi\|_{L^\infty} \sum_{B_{r_l(s_\delta)} \in \widetilde{\mathcal{K}}_{s_\delta}^{s_\star}} r_l(s_\delta) < 2 \|\nabla \phi\|_{L^\infty} \sigma = |D_d^\varepsilon|$$

This implies $|D_d^\varepsilon| > \left| C_{s_\delta}^{s_\star} \right|$, so there exists a $t_0 \in D_d^\varepsilon \backslash C_{s_\delta}^{s_\star}$. In particular $\Gamma(t_0) \subset U^\varepsilon \backslash \omega_{\overline{t}}$ is a Jordan curve with

$$\deg(u, \Gamma(t_0)) \ge d + 1 \ne 0. \tag{3.17}$$

However, by (3.12) and

$$\omega_{\overline{t}} \cap U^\varepsilon \subset \bigcup_{B_{r_k(s_\delta)} \in \widetilde{\mathcal{K}}_{s_\delta}^{s_\star}} B_{r_k(s_\delta)},$$

we see that $|u| \geq \bar{t} = 1 - \frac{1}{|\log \varepsilon|} > 0$, and as a consequence $\frac{u}{|u|}$ is a well-defined vector field in

$$U^\varepsilon \backslash \bigcup_{B_{r_k(s_\delta)} \in \mathcal{K}_{s_\delta}^{s_\star}} \overline{B_{r_k(s_\delta)}}.$$

Since $s_\star$ is the last time that there can exist balls inside U^ε and since $B_{r_k(s_\star)} \subset B_{r_l(s_\delta)}$ for some $B_{r_l(s_\delta)} \in \widetilde{K}_{s_\delta}^{s_\star}$ then $B_{r_l(s_\delta)} \cap \partial U^\varepsilon \neq \varnothing$ for each $B_{r_l(s_\delta)} \in \widetilde{K}_{s_\delta}^{s_\star}$. However, Lemma 3.12 below implies

$$U^\varepsilon \backslash \bigcup_{B_{r_k(s_\delta)} \in \widetilde{\mathcal{K}}_{s_\delta}^{s_\star}} \overline{B_{r_k(s_\delta)}}$$

must be a disjoint union of simply connected sets. Therefore, given any Jordan curve

$$\gamma \subset U^\varepsilon \backslash \bigcup_{B_{r_k(s_\star)} \in \widetilde{\mathcal{K}}_{s_\delta}^{s_\star}} \overline{B_{r_k(s_\star)}},$$

the Brouwer Fixed Point Theorem implies $\deg(\frac{u}{|u|}, \gamma) = 0$ since each component is simply connected. Therefore,

$$\deg(u, \Gamma(t_0)) = 0,$$

which in turn contradicts (3.17). Therefore, there exists a ball $B_{r_l(s_\delta)} \in \widetilde{K}_{s_\delta}^{s_\star}$ such that $B_{r_l(s_\delta)} \subset U_\varepsilon$, and this in turn implies $\mathcal{K}_{s_\delta}^{int} \neq \varnothing$. Hence $\Upsilon(s_\delta) \geq r_l(s_\delta) > 0$, which contradicts the definition of $s_\star$.

This contradiction implies $\Upsilon(s) < \sigma$ for all $s \geq 0$ is false; hence, there exists $s \in [0, s_\star]$ such that $\Upsilon(s) \geq \sigma$. By the continuity properties (3.15) of $\Upsilon(s)$ and the definitions of $\mathcal{K}_s^{int}$ and $s_\star$, there exists a set of at least one time(s) $\{s_\sigma^j\} \in [0, s_\star]$ such that

$$\Upsilon(s_\sigma^j) = \sum_{B_{r_k(s_\sigma^j)} \in \mathcal{K}_{s_\sigma^j}^{int}} r_k(s_\sigma^j) = \sigma.$$

We choose $s_\sigma = \min_j \{s_\sigma^j\}$ to be the first time $s \in [0, s_\star]$ that achieves (3.14). By (3.15) this s_σ is a point of full continuity, a fact which is used in Step 5.

4. If in Step 3 there is a jump $\sum_{B_{r_l(s)} \in \widetilde{\mathcal{K}}_s^{s_\star}} r_l(s)$ at $s = s_\star$ then we restart the *entire* process by redefining U_ε with a different level set of ϕ. In particular since (a) there is a set of t's of positive measure such that $\partial\Omega(t) \cap \omega_{\bar{t}} = \varnothing$ with $0 \leq t \leq 5\varepsilon^{\alpha_\varepsilon} \|\nabla\phi\|_{L^\infty}$ and (b) the number of jumps of $\sum_{B_{r_l(s)} \in \widetilde{\mathcal{K}}_s^{s_\star}} r_l(s)$ is at most countably infinite and **independent** of ϕ, then we can choose a new t_ε so that both $\partial\Omega(t_\varepsilon) \cap \omega_{\bar{t}} = \varnothing$ and (3.16) hold. We then proceed with the rest of the argument in Step 3.

5. We now estimate the degree of the balls in $\mathcal{K}_{s_\sigma}^{int}$. Since s_σ is a point of full continuity of $\Upsilon(s)$, there exists a slightly earlier time $s_- < s_\sigma$ with $s_\sigma - s_- \ll 1$ such that for all $s_- \leq s \leq s_\sigma$ no jumps in $\Upsilon(s)$ occur and the balls in the set $\mathcal{K}_s^{int}$ are identical except for their radii. In particular:

$$\Upsilon(s_-) < \sigma \tag{3.18}$$

and

$$\sum_{B_{r_k(s_-)} \in \mathcal{K}_{s_-}^{int}} \left|\deg(u, \partial B_{r_k(s_-)})\right| = \sum_{B_{r_k(s_\sigma)} \in \mathcal{K}_{s_\sigma}^{int}} \left|\deg(u, \partial B_{r_k(s_\sigma)})\right|. \tag{3.19}$$

This follows from (3.15), the definition of s_σ, and the conservation of degree in the expand and merge process of Lemma 3.10.

Let

$$C_a = \left\{ t \in (t_\varepsilon, \|\phi\|_{L^\infty}) \cap \operatorname{Reg}(\phi) \text{ such that } \Gamma(t) \cap \left[\bigcup_{B_{r_k(a)} \in \mathcal{K}_a^{int}} B_{r_k(a)} \right] \neq \varnothing \right\}.$$

then by (3.18) and the argument in Step 3 we have

$$\left|C_{s_-}\right| < |D_d^\varepsilon|,$$

and hence there is $t_0 \in D_d^\varepsilon \backslash C_{s_-}$. So

$$\begin{aligned} d_\star \leq |\deg(u, \Gamma(t_0))| &= \left| \sum_{B_{r_k(s_-)} \in \Omega(t_0) \cap U^\varepsilon} \deg(u, \partial B_{r_k(s_-)}) \right| \\ &\leq \sum_{B_{r_k(s_-)} \in \mathcal{K}_{s_-}^{int}} \left|\deg(u, \partial B_{r_k(s_-)})\right| \end{aligned}$$

by the definition of D_d^ε. Thus by (3.19) we have

$$d_\star \leq \sum_{B_{r_k(s_\sigma)} \in \mathcal{K}_{s_\sigma}^{int}} \left|\deg(u, \partial B_{r_k(s_\sigma)})\right|. \tag{3.20}$$

6. Returning to our Step 1 estimate,

$$
\begin{aligned}
&\frac{1}{1-\frac{2}{|\log\varepsilon|}}\int_V e_{csh}(u)dx \\
&\quad\geq \frac{1}{2}\int_{V\setminus\omega_{\bar{t}}} |\nabla\varphi|^2\,dx && \text{by (3.9)}\\
&\quad\geq \frac{1}{2}\int_{\cup_{B_{r_k(s_\sigma)}\in\mathcal{K}^{int}_{s_\sigma}} B_{r_k(s_\sigma)}\setminus\omega_{\bar{t}}} |\nabla\varphi|^2\,dx \\
&\quad\geq \pi\left(\sum_{B_{r_k(s_\sigma)}\in\mathcal{K}^{int}_{s_\sigma}} |d_k|\right)\log\frac{\sum_{B_{r_k(s_\sigma)}\in\mathcal{K}^{int}_{s_\sigma}} r_k(s_\sigma)}{r(\omega_{\bar{t}})} && \text{by Lemma 3.10}\\
&\quad\geq \pi d_\star \log\frac{\sigma}{r(\omega_{\bar{t}})} && \text{by (3.14) and (3.20)}
\end{aligned}
$$

so long as $V \supseteq B_{r_k(s_\sigma)} \in \mathcal{K}^{int}_{s_\sigma}$. We can now define

$$V \equiv \bigcup_{B_{r_k(s_\sigma)}\in\mathcal{K}^{int}_{s_\sigma}} B_{r_k(s_\sigma)} \tag{3.21}$$

with $\sum r_k(s_\sigma) = \sigma$. We see V is composed of a union of disjoint balls $B_{r_k(\sigma)} \subseteq \mathrm{spt}(\phi)$ such that $\sum r_k(s_\sigma) = \sigma$.

7. Recall $d_\star = d+1$ and $\bar{t} = 1 - \frac{1}{|\log\varepsilon|}$ then by Lemma 3.9

$$r(\omega_{\bar{t}}) \leq 4\varepsilon\,|\log\varepsilon|^2\,E_1(u) = 4\varepsilon^{\alpha_\varepsilon}$$

Set $\eta_\varepsilon = \frac{1}{1-\frac{2}{|\log\varepsilon|}} > 1$ then by Step 6 above

$$\pi d_\star \log\frac{\sigma}{4\varepsilon^{\alpha_\varepsilon}} \leq \eta_\varepsilon \int_V e_{csh}(u)dx$$

and so

$$\sigma \leq 4\varepsilon^{\alpha_\varepsilon}\exp\left(\frac{\eta_\varepsilon}{\pi d_\star}\int_V e_{csh}(u)dx\right).$$

Since $\sigma = \frac{|D^\varepsilon_d|}{2\|\nabla\phi\|_{L^\infty}}$ and $|D_d| \leq 5\varepsilon^{\alpha_\varepsilon}\|\nabla\phi\|_{L^\infty} + |D^\varepsilon_d|$, the bound follows. This finishes the proof of Proposition 3.2. □

We finish this section with a lemma about the connectedness of two dimensional sets used in Step 3 of the proof of Proposition 3.2.

Lemma 3.12. *Let $A \subset \mathbb{R}^2$ be a simply connected, open, bounded set, and let $\{B_k\}$ be a countable collection of balls such that for each k,*

$$\partial A \cap B_k \neq \varnothing.$$

Then $A\setminus\cup\overline{B_k}$ is a union of disjoint, simply connected sets.

Proof. Since A is simply connected, then $A \setminus \cup \overline{B_k} = \dot{\cup} C_j$, a disjoint union of connected open sets. We claim that each C_j is simply connected.

Suppose not, then there is a C_j that is not simply connected. Since C_j is open, connected, and non-simply connected, we can find a Jordan curve $\gamma \subset C_j$ which is not contractible to a point. Since γ is a Jordan curve, it has a well-defined interior $\mathrm{Int}(\gamma)$. Furthermore, since γ is not contractible to a point, there exists $x_0 \in \mathrm{Int}(\gamma)$ with $x_0 \notin C_j$.

But $\gamma \subset A$, since $C_j \subset A$. Since γ is a closed Jordan curve in A, then it is contractible to a point. Hence, $x_0 \in A$. Finally, we see $x_0 \in A$ but $x_0 \notin C_j \subset A \setminus \cup \overline{B_k}$. Therefore, $x_0 \in B_n$ for some B_n in the collection of balls $\{B_k\}$. Since $B_n \cap C_j = \varnothing$ and $x_0 \in B_n$ then

$$B_n \subset \mathrm{Int}(\gamma) \subset A.$$

Thus $B_n \cap \partial A = \varnothing$, which contradicts our hypothesis. □

4. Compactness of the Jacobian $J(u_\varepsilon)$ via $E_{csh}(u_\varepsilon)$ bounds

Given bounds on $J(u)$ from Propositions 3.1-3.2, we can establish compactness results of the spirit of those found in [1, 17, 18, 30]. In particular the we show that sequences $\{J(u_\varepsilon)\}$ are pre-compact in the weak norm $(C_c^{0,\beta})^*$. Here we can lift restriction on $u_\varepsilon \in H^1(\mathbb{R}^2;\mathbb{C})$ and let $u_\varepsilon \in H^1(\mathbb{R}^m;\mathbb{C})$ for $m \geq 2$.

Theorem 4.1. *Let $U \subset \mathbb{R}^m$, and suppose that $u^\varepsilon \in H^1(U;\mathbb{C})$ is a collection of smooth functions such that*

$$\sup_{\varepsilon \in (0,1]} \int_U \frac{e_{csh}(u_\varepsilon)}{|\log \varepsilon|} dx = K_1^U < \infty. \tag{4.1}$$

Suppose U is a simply connected, bounded domain with smooth boundary, and suppose $|u_\varepsilon| \geq 1 - \frac{1}{|\log \varepsilon|}$ on ∂U. Then there exists a subsequence $\varepsilon_n \to 0$ and a Radon measure $\overline{J}$ such that

1. *$J(u^\varepsilon)$ converges to a limit $\overline{J}$ in the $\left(C^{0,\beta}\right)^*$ norm for every $\beta > 0$.*
2. *$\frac{1}{\pi}\overline{J}$ is $m-2$ dimensional integer multiplicity rectifiable; and*
3. *If $\overline{\nu}$ is the weak limit of a subsequence of $\{\nu_\varepsilon\} \equiv \frac{e_{csh}(u_\varepsilon)}{|\log \varepsilon|}$, then $\left|\overline{J}\right| \ll \overline{\nu}$ and $\frac{d\left|\overline{J}\right|}{d\overline{\nu}} \leq 1$. In particular, $\left|\overline{J}\right|(U) \leq K_1^U$.*

Remark 4.2. When $m = 2$ then Theorem 4.1 implies $J(u_\varepsilon) \rightharpoonup \frac{1}{\pi}\overline{J} = \sum_{j=1}^d d_j \delta_{a_j}$ in $(C_c^{0,\beta})^*$ for all $\beta \in (0,1]$ with $d_j \in \mathbb{Z}$ and $\left|\overline{J}\right| = \pi \sum_j |d_j| \leq K_1^U$. In other words the limiting Jacobian will condense down to a finite number of delta functions with total mass bounded by ν_ε.

As a consequence we have

Theorem 4.3. *Suppose*

$$K_2^U = \sup_{0<\varepsilon<\varepsilon_0} \int_{\mathrm{spt}(\phi)} \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|^2} dx < \infty \tag{4.2}$$

Set $v_\varepsilon = \frac{j(u_\varepsilon)}{|\log\varepsilon|}$ *and* $w_\varepsilon = \frac{J(u_\varepsilon)}{|\log\varepsilon|} = \frac{1}{2}\operatorname{curl} v_\varepsilon$, *then* w_ε *is precompact in* $(C^{0,\beta})^*$ *for any* $\beta > 0$. *Furthermore,* $\|v_\varepsilon\|_{L^p} \le C$ *for* $1 \le p < \frac{\gamma+2}{\gamma+1}$ *and all* $\varepsilon > 0$. *Finally,* $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ *weakly in* L^2 .

The proof of Theorem 4.1 follows from the arguments of [17].

Sketch of the proof of Theorem 4.1. We now sketch the proof of Theorem 4.1, using several technical results of [17]. Since the $m \ge 3$ case follows from a 2-dimensional slicing argument, we first consider the two dimensional case $u_\varepsilon \in H^1(\mathbb{R}^2;\mathbb{C})$.

1. Given the assumptions in Proposition 3.2 and supposing (4.1) holds, then we have the following Jacobian bound: Fix $\lambda = (1,2]$ and $d_\lambda = \left\lfloor \frac{\lambda}{\pi}\int_U \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|} dx \right\rfloor$, then

$$\left|\int \phi J(u_\varepsilon) dx\right| \le \pi d_\lambda \|\phi\|_{L^\infty} + C\varepsilon^{\frac{\lambda-1}{2\lambda}} \|\phi\|_{C^{0,1}} . \tag{4.3}$$

for all $\varepsilon \in \left(0, e^{-\frac{8}{\lambda-1}}\right)$, where $C = C(\lambda, K_1^U, \mathrm{spt}(\phi))$. To establish (4.3) we set $d = d_\lambda$ in Proposition 3.1, then $d_\lambda^\star = d_\lambda + 1 = \left\lfloor \frac{\lambda}{\pi}\frac{E_\phi(u_\varepsilon)}{|\log\varepsilon|}\right\rfloor + 1 \ge \frac{\lambda}{\pi}\frac{E_\phi(u_\varepsilon)}{|\log\varepsilon|}$, which implies $1 + \exp\left(\frac{\eta_\varepsilon}{\pi d_\lambda^\star} E_\phi(u_\varepsilon)\right) \le 2\varepsilon^{-\frac{\eta_\varepsilon}{\lambda}}$ where $\eta_\varepsilon = \frac{1}{1-\frac{2}{|\log\varepsilon|}}$. Therefore, from Proposition 3.2 and Remark 3.3 we have the inequality

$$\begin{aligned} \frac{|D_{d_\lambda}|}{4} E_\phi(u_\varepsilon) &\le \varepsilon^{\alpha_\varepsilon} |\log\varepsilon| \frac{K_1^U}{2}\left(1 + \exp\left(\frac{\eta_\varepsilon}{\pi d_\lambda^\star} E_\phi(u_\varepsilon)\right)\right) \|\nabla\phi\|_{L^\infty} \\ &\le \frac{\left(K_1^U\right)^2}{2} \varepsilon |\log\varepsilon|^4 \left(1 + \exp\left(\frac{\eta_\varepsilon}{\pi d_\lambda^\star} E_\phi(u_\varepsilon)\right)\right) \|\nabla\phi\|_{L^\infty} \\ &\le C\varepsilon^{\frac{\lambda-1}{2\lambda}} \|\nabla\phi\|_{L^\infty} , \end{aligned} \tag{4.4}$$

so long as there is a constant $C \ge \varepsilon^{1-\frac{\eta_\varepsilon}{\lambda}-\frac{\lambda-1}{2\lambda}} |\log\varepsilon|^4 = \varepsilon^{\frac{\lambda+1-2\eta_\varepsilon}{2\lambda}} |\log\varepsilon|^4$.

Setting $\lambda - 1 = \delta > 0$ then $\delta - \frac{4}{|\log\varepsilon|} \le \lambda + 1 - 2\eta_\varepsilon \le \delta$ so long as $\varepsilon \le e^{-4}$. Therefore, for $\varepsilon \le e^{-\frac{8}{\delta}}$

$$\frac{\delta}{4\lambda} \le \frac{\lambda+1-2\eta_\varepsilon}{2\lambda} \le \frac{\delta}{2\lambda},$$

which means $\varepsilon^{\frac{\lambda+1-2\eta_\varepsilon}{2\lambda}} |\log\varepsilon|^4 \le \varepsilon^{\frac{\delta}{4\lambda}} |\log\varepsilon|^4$. By a simple calculus calculation $\varepsilon^a |\log\varepsilon|^b \le (\frac{b}{ae})^b$, so

$$\varepsilon^{\frac{\delta}{4\lambda}} |\log\varepsilon|^4 \le \left(\frac{16\lambda}{\delta e}\right)^4 ,$$

which completes (4.4). The other terms of (3.1) are also controlled by $C\varepsilon^{\frac{\lambda-1}{2\lambda}}$ for $\lambda \in (1,2]$. This completes (4.3).

2. We claim that $J(u_\varepsilon)$ is strongly precompact in $(C^{0,\beta})^*$ for all $\beta > 0$. By (4.3) and Proposition 3.2 of [17] we can decompose $J(u_\varepsilon) = J_0^\varepsilon + J_1^\varepsilon$, both signed measures, with $\|J_0^\varepsilon\|_{(C^0)^*} \leq C$ and $\|J_1^\varepsilon\|_{(C^1)^*} \leq C\varepsilon^\alpha$ where $\alpha = \frac{\lambda-1}{2\lambda}$. By Lemma 3.4 $(C^0)^*$ is compactly embedded in $(C^{0,\beta})^*$; therefore, $\{J_0^\varepsilon\}$ is precompact in $(C^{0,\beta})^*$.

Next we show $\{J_1^\varepsilon\}$ is precompact in $(C^{0,\beta})^*$. Since $\|J(u_\varepsilon)\|_{(C^0)^*} = \sup \left|\int \phi J(u_\varepsilon)\right| \leq \|J(u_\varepsilon)\|_{L^1} \leq C\|\nabla u\|_{L^2}^2$, then $\|J_1^\varepsilon\|_{(C^0)^*} \leq \|J(u_\varepsilon)\|_{(C^0)^*} + \|J_0^\varepsilon\|_{(C^0)^*} \leq C|\log\varepsilon| + C$. By Lemma 3.3 of [17] we can interpolate the weak norms

$$\|J_1^\varepsilon\|_{(C^{0,\beta})^*} \leq C\|J_1^\varepsilon\|_{(C^0)^*}\|J_1^\varepsilon\|_{(C^{0,1})^*} \leq C\varepsilon^\alpha|\log\varepsilon|,$$

so $\{J_1^\varepsilon\}$ is also precompact in $(C^{0,\beta})^*$. This finishes the claim.

3. We now establish the limit of these measures $\{J(u_\varepsilon)\}$, and the argument in this step is essentially the same as in the proof of Theorem 3.1 in [17]. Set $\nu_\varepsilon = \frac{e_{csh}(u_\varepsilon)}{\pi|\log\varepsilon|}$ then $\nu_\varepsilon \rightharpoonup \nu$, Radon measure on U. We claim that $\overline{J}$ is supported only on x_0 such that $\lim_{r\to 0}\nu_\varepsilon(B_r(x_0)\cap U) \geq \pi$. If there is a subsequence ε_n with $\int_{B_{r_0}(x_0)\cap U} d\nu_\varepsilon \leq \alpha < \pi$ for all ε_n sufficiently small then, using Proposition 3.1 with $\lambda = \frac{\alpha+\pi}{\alpha}$ yields

$$\int \phi\overline{J}dx = \lim_{n\to\infty}\int \phi J(u_{\varepsilon_n})dx = 0$$

since $d_\lambda = 0$ for such ϕ. Thus, $x_0 \notin \operatorname{spt}(J)$. Since $\nu_\varepsilon(U) \leq K_1^U < \infty$, there exists only finitely many points a_j such that $\lim_{r\to 0}\nu_\varepsilon(B_r(a_j)\cap U) \geq \pi$. Thus, $\overline{J} = \pi\sum c_i\delta_{a_j}$ for the limiting Jacobian measure, $\overline{J}$.

We show that c_j's are integers. Take $a_1 \subset U$ with $\operatorname{dist}(a_1, \partial U) > r_1$. We choose Lipschitz test function $\phi = (r_1 - |x-a_1|)^+$ then $d_\lambda = \left\lfloor \frac{\lambda}{\pi}\nu_\varepsilon(B_{r_1}(a_j))\right\rfloor \leq \frac{1}{\pi}\lim_{r\to 0}\nu_\varepsilon(B_r(a_j))$. As in [17] we set

$$A_n = \left\{ \begin{array}{l} t \in (0, \|\phi\|_{L^\infty}) \cap \operatorname{Reg}(\phi) \text{ such that } \Gamma(t) \neq \varnothing \\ \text{ and } |\deg(u_{\varepsilon_n}, \partial\Omega(t))| \leq d_\lambda \end{array} \right\}$$

so that $A_n = A \cap \{t : \Gamma(t) \neq \varnothing\}$, where A is as in Proposition 3.1 and where the n refers to an ε_n and u_{ε_n}. From (4.3) we can show that $|A_n| \geq \|\phi\|_{L^\infty} - \varepsilon^\alpha C$. If $t \in A_n$ then there is a component $\Gamma(t)$ nonempty. This implies $\phi^{-1}(t) = \partial B_{r_1-t}(a_1)$ satisfies $\min_{\partial B_{r_1-t}(a_1)}|u_{\varepsilon_n}| \geq \frac{1}{2}$ and $|\deg(u_{\varepsilon_n}, \partial B_{r_1-t}(a_1))| \leq d_\lambda \leq \frac{1}{\pi}\lim_{r\to 0}\nu_\varepsilon(B_r(a_j))$.

We claim that there is an integer $d_n \leq d_\lambda$ such that

$$S_n^{d_n} = \{r \in [0, r_1] \text{ such that } \min_{\partial B_r(a_i)}|u_{\varepsilon_n}| \geq \frac{1}{2}, \deg(u_{\varepsilon_n}, \partial B_r(a_1)) = d_n\}$$

has measure at least $k_1 = \frac{r_1}{3d_\lambda}$. Further, we define $\Sigma_n \subset S_n^{d_n}$ with $|\Sigma_n| = k_0$. Defining $\psi_n(x) = f_n(|x - a_1|)$ where $f_n(r) = |[r, r_1] \cap \Sigma_n|$. Then for some r, $\deg(u, \psi^{-1}(t)) = \deg(u, \partial B_r(a_1)) = d_n$ for a.e. $t \in (0, k_0)$. It is easy to see from Proposition 3.1 that

$$\int \psi_n J(u_{\varepsilon_n}) dx = \pi d_n k_1 + C\varepsilon^\alpha$$

and

$$0 = \lim_{n\to\infty} \left| \int \psi_n J(u_{\varepsilon_n}) dx - \pi c_1 \psi_n(a_1) \right| = \lim_{n\to\infty} \left| \int \psi_n J(u_{\varepsilon_n}) dx - \pi c_1 k_1 \right|.$$

We see that $d_n = c_1$, an integer, with $d_n \le d_\lambda$. This completes the compactness argument for $u_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$.

4. We now turn to higher dimensional domains. If $u : \mathbb{R}^n \to \mathbb{C}$ then the supercurrent is the one-form

$$j(u) = (iu, du) = \sum_{i=1}^{n} (iu, \partial_{x_i} u)\, dx^i$$

whereas the Jacobian $J(u)$ is a two-form

$$J(u) = \det\left(u_{x_i}, u_{x_j}\right) = \frac{1}{2} dj(u).$$

Component-wise we can write

$$J(u) = \sum_{i<j} J^{ij}(u) dx^i \wedge dx^j = \frac{1}{2} \sum_{i,j} J^{ij}(u) dx^i \wedge dx^j$$

where

$$J^{ij}(u) = \left(i\partial_{x_i} u, \partial_{x_j} u\right) = \det(u_{x_i}, u_{x_j}).$$

The Jacobian can be viewed as a co-dimension 2 current acting on $(n-2)$-forms via

$$J(u)(\phi) = \frac{1}{2} \int_U J(u) \wedge \phi dx$$

for $\phi \in \Lambda^{n-2}(\mathbb{R}^n)$. Here $\Lambda^k(\mathbb{R}^n)$ is the space of smooth k-forms on $\mathbb{R}^n$ and $\Lambda_0^k(U)$ are those forms with compact support in $U \subset \mathbb{R}^n$.

The proof that $\frac{1}{\pi}\overline{J}$ is an integer-multiplicity rectifiable current was shown in the Ginzburg-Landau case by Jerrard-Soner [17] via a two-dimensional slicing argument, see also [1]. Later Sandier-Serfaty in [30] prove that Jacobians with variable metrics have a similar compactness property. In both cases the authors make use of the fact that an $(n-2)$-dimensional current is integer-multiplicity rectifiable if and only if almost every projected two-dimensional slice is an integer-multiplicity rectifiable 0-current, a fact proved in [36, 19] and later [2]. Therefore, we need only identify what happens on each two-dimensional slice of $J(u_\varepsilon)$, as we have done in Step 1 – Step 3. Since the rest of the proof is pretty much identical to arguments in [17, 30], we point the reader to [17] for a precise treatment.

$\square$

We now consider the large E_{csh} compactness, Theorem 4.3.

Proof of Theorem 4.3. Set $v_\varepsilon = \frac{j(u_\varepsilon)}{|\log\varepsilon|}$, $w_\varepsilon = \frac{J(u_\varepsilon)}{|\log\varepsilon|} = \frac{1}{2}\operatorname{curl} v_\varepsilon$, and $\int_U e_{csh}(u_\varepsilon)dx \leq g_\varepsilon = \varepsilon^{-\gamma}$.

1. From Lemma 3.2 and the energy bound (4.2) we have

$$|D_d| \leq 8\varepsilon^{\alpha_\alpha} \|\nabla\phi\|_{L^\infty} \left(1 + \exp\left(\frac{\eta_\varepsilon}{\pi d_\star}\int_V e_{csh}(u)dx\right)\right)$$

where

$$\eta_\varepsilon = \frac{1}{1 - \frac{2}{|\log\varepsilon|}}$$

$$\varepsilon^{\alpha_\varepsilon} = \varepsilon\,|\log\varepsilon|^2 \int_U e_{csh}(u)dx \leq K_2^U \varepsilon\,|\log\varepsilon|^4$$

$$V \equiv \dot{\bigcup_j} B_{r_j} \qquad \sum r_j = \frac{|D_d|}{2\,\|\nabla\phi\|_{L^\infty}}.$$

We fix $\lambda \in (1,2]$ and define

$$d_\lambda = \left\lfloor \frac{\lambda}{\pi}\int_V \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|}dx \right\rfloor.$$

Setting $d_\lambda^\star = d_\lambda + 1$ then

$$\frac{|\log\varepsilon|}{\lambda} \geq \frac{1}{\pi d_\lambda^\star}\int_V e_{csh}(u_\varepsilon)dx;$$

which implies

$$|D_{d_\lambda}| \leq 16 K_2^U \varepsilon\,|\log\varepsilon|^4 \|\nabla\phi\|_{L^\infty} \varepsilon^{\frac{\eta_\varepsilon}{\lambda}}.$$

Returning to the bound in (3.1), we see

$$\begin{aligned}\frac{1}{2}|D_{d_\lambda}|\,E_\phi(u_\varepsilon) &\leq 8\left(K_2^U\right)^2 \varepsilon^{1-\frac{\eta_\varepsilon}{\lambda}} |\log\varepsilon|^6 \|\nabla\phi\|_{L^\infty} \\ &\leq C\varepsilon^{\frac{\lambda-1}{2\lambda}} \|\nabla\phi\|_{L^\infty}\end{aligned}$$

by following the argument of Step 1 of the proof Theorem 4.1. This implies the bound

$$\left|\int \phi w_\varepsilon dx\right| \leq \pi d_\lambda \|\phi\|_{L^\infty} + C\varepsilon^{\frac{\lambda-1}{2\lambda}} \|\phi\|_{C^{0,1}}. \tag{4.5}$$

As in the proof of Theorem 4.1, once we have estimate (4.5), the compactness of w_ε in $(C^{0,\beta})^*$ for all $\beta \in (0,1)$ follows.

2. We claim $\|v_\varepsilon\|_{L^p} \le C(K_1^U)$ for all $1 \le p < \frac{2+\gamma}{1+\gamma}$ for $0 \le \gamma < 2$. This follows from

$$\int |v_\varepsilon|^p \, dx \le \left(\int \frac{|\nabla u_\varepsilon|^2}{|\log \varepsilon|^2} dx\right)^{\frac{p}{2}} \left(\int |u_\varepsilon|^{\frac{2p}{2-p}}\right)^{\frac{2-p}{2p}} \le \left(K_2^U\right)^{\frac{p}{2}} \|u_\varepsilon\|_{L^{\frac{2p}{2-p}}}.$$

From Corollary 2.4 we have $\|1-\rho_\varepsilon\|_{L^6} \le C\varepsilon^{\frac{1}{3}} g_\varepsilon^{\frac{1}{3}}$ since $|1-\rho_\varepsilon|^2 \le |1-\rho^2|$, and $|\nabla \rho_\varepsilon| \le |\nabla u_\varepsilon|$ then $\|\rho_\varepsilon\|_{L^2} \le C g_\varepsilon^{\frac{1}{2}}$. By Sobolev estimates

$$\|1-\rho_\varepsilon\|_{L^r} \le \|1-\rho_\varepsilon\|_{L^6}^{\theta} \|\nabla u_\varepsilon\|_{L^q}^{1-\theta} \le C\left(\varepsilon^{\frac{1-\gamma}{3}}\right)^{\theta} \left(\varepsilon^{-\frac{\gamma}{2}}\right)^{1-\theta} = C\varepsilon^{\frac{\theta}{6}(2+\gamma)-\frac{\gamma}{2}}$$

for $\frac{1}{r} = \frac{\theta}{6} + \left(\frac{1}{q} - \frac{1}{2}\right)(1-\theta)$ and $p < 2$. Therefore, letting q get arbitrarily close to 2 implies $\theta > \frac{3\gamma}{2+\gamma}$ and hence $r < \frac{2(\gamma+2)}{\gamma}$. Returning to the original bound, we get a uniform L^p bound on v_ε so long as $1 \le p < \frac{\gamma+2}{\gamma+1}$. Therefore, if $v_\varepsilon \rightharpoonup v$ in L^p_{loc} then $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$.

Since $E_1(u_\varepsilon) \le g_\varepsilon \le \varepsilon^{-2}$ and $|\rho_\varepsilon| \ge \frac{1}{2}$ on the boundary then $\rho_\varepsilon \to 1$ strongly in L^s for all $1 \le s \le 6$ by Corollary 2.4. Therefore, if $\frac{v_\varepsilon}{\rho_\varepsilon} \rightharpoonup v$ weakly in L^2 then $v_\varepsilon = \frac{v_\varepsilon}{\rho_\varepsilon}\rho_\varepsilon$ converges weakly to v in L^p_{loc} for $1 \le p_0$ where $p_0 = \frac{\gamma+2}{\gamma+1}$. Finally, if $v_\varepsilon \rightharpoonup v$ then by strong convergence of $\rho_\varepsilon \to 1$, we can show that $\frac{v_\varepsilon}{\rho_\varepsilon} \rightharpoonup v$. Since $\frac{v_\varepsilon}{\rho_\varepsilon}$ precompact in L^2 implies $\frac{v_\varepsilon}{\rho_\varepsilon} \rightharpoonup v$ weakly in L^2.

□

5. Lower bounds for $E_{csh}(u_\varepsilon)$

In this section we show the lower bound parts of Theorem 1.1 and 1.2.

Proposition 5.1. *For every sequence $\{u_\varepsilon\}$ such that*

$$\sup_{0<\varepsilon<\varepsilon_0} \int_U \frac{e_{csh}(u_\varepsilon)}{|\log \varepsilon|} dx = K_1^U < \infty$$

converging to u in $W^{1,1}(U;\mathbb{C})$,

$$\liminf_{\varepsilon \to 0} \int_U \frac{e_{csh}(u_\varepsilon)}{|\log \varepsilon|} dx \ge \pi \left\|\overline{J}\right\|_{\mathcal{M}}$$

if $\left\|\overline{J}\right\|_{\mathcal{M}} < +\infty$.

Proof. This follows from (3.1) and our choice of $d_\lambda = \left\lfloor \frac{\lambda}{\pi} \frac{E_1(u_\varepsilon)}{|\log\varepsilon|} \right\rfloor$:

$$\begin{aligned}\pi\left|\int_U \phi \overline{J} dx\right| &= \pi \lim_{n\to\infty}\left|\int_U \phi J(u_{\varepsilon_n}) dx\right| \\ &\leq \lambda \|\phi\|_{L^\infty} \liminf_{n\to\infty}\int_U \frac{e_{csh}(u_{\varepsilon_n})}{|\log\varepsilon_n|}dx\end{aligned}$$

for every $\lambda > 1$. Letting $\lambda \to 1$ yields the bound. □

For $E_{csh}(u_\varepsilon) = O(|\log\varepsilon|^2)$ we have the following lower bound.

Proposition 5.2. *Let sequence $\{u_\varepsilon\}$ satisfy*

$$\sup_{0<\varepsilon<\varepsilon_0}\int_U \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|^2}dx = K_2^U < \infty.$$

Set $v_\varepsilon = \frac{j(u_\varepsilon)}{|\log\varepsilon|}$ and $w_\varepsilon = \frac{J(u_\varepsilon)}{|\log\varepsilon|} = \frac{1}{2}\operatorname{curl} v_\varepsilon$. Suppose $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ in L^2 and $v_\varepsilon \rightharpoonup v$ in L^p_{loc} for $1 \leq p < 2$. Then $w = \frac{1}{2}\operatorname{curl} v$ is a measure and

$$\liminf_{\varepsilon\to0}\int \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|^2}dx \geq \frac{1}{2}\left[\|v\|_{L^2}^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right] \tag{5.1}$$

Proof. We use essentially the proof of Theorem 6.1 of [18], but we include it for the sake of completeness.

1. From Proposition 3.2 there is a set $V = \dot{\cup} B_{r_k}$ with $B_{r_k} \subsetneq \operatorname{spt}(\phi)$ where the Jacobian concentrates. We define

$$\chi_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \cup B_{r_j} \\ 0 & \text{otherwise} \end{cases},$$

then for any h we have $\left\|h\frac{v_\varepsilon}{|u_\varepsilon|}\chi_\varepsilon\right\|_{L^2}^2 \leq \int |h|^2\chi_\varepsilon \int \left|\frac{v_\varepsilon}{|u_\varepsilon|}\right|^2$. By the dominated convergence theorem, the first integral converges to zero and the second is bounded by the assumptions. Thus $\frac{v_\varepsilon}{|u_\varepsilon|}\chi_\varepsilon \rightharpoonup 0$ in L^2 and $\frac{v_\varepsilon}{|u_\varepsilon|}(1-\chi_\varepsilon) \rightharpoonup v$ in L^2. Therefore,

$$\begin{aligned}\liminf_{\varepsilon\to0}\int_{U\setminus\cup B_{r_j}} \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|^2}dx &\geq \liminf_{\varepsilon\to0}\int_{U\setminus\cup B_{r_j}} \frac{1}{2}\left|\frac{\nabla u_\varepsilon}{|\log\varepsilon|}\right|^2 dx \\ &\geq \liminf_{\varepsilon\to0}\int_{U\setminus\cup B_{r_j}} \frac{1}{2}\left|\frac{v_\varepsilon}{|u_\varepsilon|}\right|^2 dx \\ &\geq \liminf_{\varepsilon\to0}\int_U \frac{1}{2}\left|\frac{v_\varepsilon}{|u_\varepsilon|}(1-\chi_\varepsilon)\right|^2 dx \\ &\geq \frac{1}{2}\|v\|_{L^2}\end{aligned}$$

2. We have $w_\varepsilon = \frac{1}{2}\operatorname{curl} v_\varepsilon \rightharpoonup w$ in the sense of distribution with $w = \frac{1}{2}\operatorname{curl} v$. Let $\psi \in C_c^\infty(U)$ then

$$\begin{aligned}
\sup_{\|\psi\|_{L^\infty}\le 1}\left|\int_U \psi \operatorname{curl} v dx\right| &= \sup_{\|\psi\|_{L^\infty}\le 1}\left|\lim_{\varepsilon\to 0}\int_U \psi \frac{J(u_\varepsilon)}{|\log\varepsilon|}dx\right| \\
&\le \sup_{\|\psi\|_{L^\infty}\le 1} \lambda \|\psi\|_{L^\infty} \liminf_{\varepsilon\to 0}\int_{U\cap\cup B_{r_j}} \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|^2}dx \\
&= \lambda \liminf_{\varepsilon\to 0}\int_{U\cap\cup B_{r_j}} \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|^2}dx.
\end{aligned}$$

Adding this bound to Step 1 yields

$$\frac{1}{\lambda}\liminf_{\varepsilon\to 0}\int_U \frac{e_{csh}(u_\varepsilon)}{|\log\varepsilon|^2}dx \ge \frac{1}{2}\|\operatorname{curl} v\|_{\mathcal{M}} + \frac{1}{2}\|v\|_{L^2}.$$

We take $\lambda \to 1$ which finishes (5.1). □

6. Upper bounds for $E_{csh}(u_\varepsilon)$

In this section, we establish the upper bounds corresponding to the lower bounds of the last section, thus finishing the proofs of Theorems 1.1 and 1.2 The constructions we use are heavily based on those of Jerrard-Soner [18].

Proposition 6.1. *For every $v \in L^2(U;\mathbb{R}^2)$ such that $w = \frac{1}{2}\operatorname{curl} v$ is a Radon measure, there exists a sequence $\{u_\varepsilon\}$ in $H^1(U;\mathbb{C})$ such that*

$$v_\varepsilon := \frac{1}{|\log\varepsilon|} j(u_\varepsilon) \rightharpoonup v \quad \text{in } L^2(U;\mathbb{R}^2) \tag{6.1}$$

$$w_\varepsilon := \frac{1}{|\log\varepsilon|} J(u_\varepsilon) \rightharpoonup w \quad \text{in } (C^{0,\alpha})^* \text{ for every } \alpha > 0 \tag{6.2}$$

Furthermore, the energy of the sequence satisfies

$$\limsup_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|^2}\int_U e_{csh}(u) \le \frac{1}{2}\left(\|v\|_{L^2}^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right). \tag{6.3}$$

The proof follows from the following proposition by a standard approximation argument, since $C^\infty(U)$ is "dense in energy" in the limit spaces. Note that by Theorem 4.3 the weak convergence of v_ε in L^p for $1 \le p < 2$ holds if and only if $\frac{v_\varepsilon}{|u_\varepsilon|}$ converges weakly in L^2, and their respective weak limits are the same.

Proposition 6.2. *Let $v \in C^\infty(U;\mathbb{R}^2)$. Let $\{d_\varepsilon\}$ be an increasing sequence with $d_\varepsilon \to \infty$ and $\varepsilon d_\varepsilon \to 0$. Then there exists a sequence of functions $\{u_\varepsilon\} \in$*

$H^1(U;\mathbb{C})$ *with the following properties:*

$$|u_\varepsilon| \le 1 \text{ in } U, \quad |u_\varepsilon| = 1 \text{ on } \partial U \tag{6.4}$$

$$v_\varepsilon := \frac{j(u_\varepsilon)}{d_\varepsilon} \rightharpoonup v \quad \text{in } L^p \text{ for all } p < 2 \tag{6.5}$$

$$w_\varepsilon := \frac{Ju_\varepsilon}{d_\varepsilon} \rightharpoonup \frac{1}{2}\operatorname{curl} v =: w \quad \text{in } W^{-1,p} \text{ for all } p < 2 \tag{6.6}$$

and the energy satisfies

$$E_\varepsilon(u_\varepsilon) \le \frac{d_\varepsilon^2}{2}\|v\|_{L^2(U)}^2 + d_\varepsilon |\log\varepsilon| \, \|w\|_{L^1(U)} + o(d_\varepsilon^2) \tag{6.7}$$

as $\varepsilon \to 0$.

Proof. We check that in the proof of Proposition 7.1 of [18], there holds indeed $|u_\varepsilon| \le 1$ and $|u_\varepsilon| = 1$ on ∂U To be precise, we need to check this for Lemma 7.2 from there. The construction shows that $|u_\varepsilon| = \rho_\varepsilon$ satisfies $\|(\rho_\varepsilon)^p - 1\|_{L^q(U)} \to 0$ for all $p, q \in [1,\infty)$.

In fact, ρ_ε is a function that is 1 outside $B_\varepsilon(a_i^\varepsilon)$, $i = 1, \dots, N_\varepsilon$, for some points a_i^ε that satisfy $|a_i^\varepsilon - a_j^\varepsilon| \ge c d_\varepsilon^{-\frac{1}{2}}$ and $dist(a_i^\varepsilon, \partial U) \ge c d_\varepsilon^{-\frac{1}{2}}$, and ρ_ε satisfies $|\rho_\varepsilon| \le 1$ everywhere.

Since $|u_\varepsilon| \le 1$, we have

$$\frac{1}{\varepsilon^2}\int |u_\varepsilon|^2(1 - |u_\varepsilon|^2)^2 \le CN_\varepsilon \le Cd_\varepsilon \|w\|_M = o(d_\varepsilon^2)$$

as in [18]. The statements on the convergence of the other terms in the energy are independent of the difference between our functional and the Ginzburg-Landau functional, we sketch the argument here for the convenience of the reader. The proof in [18] relies on a Hodge-type decomposition of $L^2(U;\mathbb{R}^2)$ into vector fields that are curls of functions that are zero on the boundary, gradients, and harmonic vector fields. Gradients and harmonic vector fields are easily approximated as in Lemma 7.4 of [18], and the combination is done as in the proof of Proposition 7.1. For the vector fields that are curls with zero boundary conditions, the measure w is first approximated by a sum w_ε of point masses at appropriately chosen points a_i^ε as above. In the next step, a vector field $\hat{v}_\varepsilon$ is defined as

$$\hat{v}_\varepsilon = -2\operatorname{curl}\Delta_D^{-1} w_\varepsilon,$$

where Δ_D denotes the Laplacian with zero Dirichlet boundary data. Setting $\hat{u}_\varepsilon = e^{i\varphi_\varepsilon}$, where $\nabla\varphi_\varepsilon = d_\varepsilon \hat{v}_\varepsilon$, the functions $\hat{u}_\varepsilon$ then satisfy $\frac{j(\hat{u}_\varepsilon)}{d_\varepsilon} = v_\varepsilon$ and $\frac{J(u_\varepsilon)}{d_\varepsilon} = w_\varepsilon$. Using the cutoff function ρ_ε above, and defining $u_\varepsilon = \rho_\varepsilon \hat{u}_\varepsilon$, one obtains a sequence with the desired properties. ☐

In the scaling $\int_U e_{csh}(u) \approx C|\log\varepsilon|$, we have the following upper bound result:

Proposition 6.3. *Let $J = \pi \sum_{i=1}^N d_i \delta_{a_i}$ for $a_i \in U$. Then there exists a sequence $\{u_\varepsilon\}$ with $\frac{1}{|\log \varepsilon|} J(u_\varepsilon) \rightharpoonup J$ and*

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_U e_{csh}(u) = \|J\|_{\mathcal{M}(U)} . \tag{6.8}$$

Proof. This can be shown by a construction similar to the one above, but simpler. (For the case of $|d_i| = 1$, it is already contained in [3]): Choose points $b^\varepsilon_{i,j} \to a_i$ with $|b^\varepsilon_{i,j} - b^\varepsilon_{i,k}| \geq \varepsilon |\log \varepsilon|$ for $1 \leq j < k \leq |d_i|$. Construct u_ε as a function that satisfies $|u_\varepsilon| = 1$ in $U \setminus \bigcup B_\varepsilon(b^\varepsilon_{i,j})$ and satisfies $\deg(u_\varepsilon, \partial B_\varepsilon(b^\varepsilon_{i,j}) = \operatorname{sgn}(d_i)$ for all i, j, k. Similar to the construction in [3], this can be done with an energy bounded by $|J| |\log \varepsilon| + C \log |\log \varepsilon|$, or even by $|J| |\log \varepsilon| + C$ if all d_i are ± 1. □

7. Extension to the full CSH energy

In this section we discuss the generalization of the results on $E_\varepsilon(u)$ to $G_\varepsilon(u, A; h_{ex})$, for the scaling $G_\varepsilon \leq C |\log \varepsilon|^2$ and $h_{ex} = H |\log \varepsilon|$, proving Theorem 1.3.

We use

$$G_\varepsilon(u, A; h_{ex}) = \frac{1}{2} \int |\nabla_A u|^2 + \frac{\mu^2}{4} \frac{|\operatorname{curl} A - h_{ex}|^2}{|u|^2} + \frac{1}{\varepsilon^2} |u|^2 (1 - |u|^2)^2.$$

The functional has a gauge invariance: it stays invariant under the gauge transformation $(u, A) \to (ue^{i\chi}, A + \nabla \chi)$: $G_\varepsilon(ue^{i\chi}, A + \nabla\chi; h_{ex}) = G_\varepsilon(u, A; h_{ex})$ for any $\chi \in H^1(U)$.

The appropriate gauge-invariant quantities are the supercurrent $j_{A_\varepsilon} = j(u_\varepsilon) - |u_\varepsilon|^2 A_\varepsilon$ and the magnetic field $\operatorname{curl} A_\varepsilon$. For simplicity, we will fix the gauge invariance by choosing the Coulomb gauge: We can choose A such that $\nabla \cdot A = 0$, $A \cdot \nu = 0$ and $\int A = 0$. (Otherwise, let ξ be a solution of $\Delta \xi = -\nabla \cdot A$, $\nu \cdot \nabla \xi = -A \cdot \nu$. Then $\tilde{A} = A + \nabla \xi - \int_U (A + \nabla \xi)$ satisfies these assumptions). We use $(\tilde{u}, \tilde{A})$ with $\tilde{u} = ue^{i\xi}$ instead of (u, A).

We also make the physically reasonable assumption that $\|u_\varepsilon\|_{L^\infty} \leq C$ for some constant C.

Proposition 7.1. *Assume $G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex}) \leq K |\log \varepsilon|^2$ and $\|u_\varepsilon\|_{L^\infty} \leq C$. Set $a_\varepsilon = \frac{1}{|\log \varepsilon|} A_\varepsilon$, where A_ε is in Coulomb gauge. Then a_ε is weakly compact in H^1, and the compactness assertions of Proposition 5.2 hold: $v_\varepsilon = \frac{1}{|\log \varepsilon|} j(u_\varepsilon)$ converges to v weakly in all L^p, $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ in L^2, and $w_\varepsilon = \frac{J(u_\varepsilon)}{|\log \varepsilon|} \rightharpoonup w = \frac{1}{2} \operatorname{curl} v$.*

Furthermore, the energy satisfies

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|^2} G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex}) \geq G(v,a;H) = \frac{1}{2}\left(\int_U |v-a|^2 + \frac{\mu^2}{4}|\operatorname{curl} a - H|^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right). \tag{7.1}$$

Proof. By the L^∞ bound on u_ε, it follows that $\frac{1}{|u_\varepsilon|^2} \geq c$, hence $\|\operatorname{curl} a_\varepsilon - H\|_{L^2}^2 \leq \frac{8cK}{\mu^2}$. This together with the Coulomb gauge implies a H^1 bound on a_ε, so for a subsequence $a_\varepsilon \rightharpoonup a$ in H^1.

Decomposing

$$G_\varepsilon(u,A;h_{ex}) = E_\varepsilon(u_\varepsilon) - \int j(u)\cdot A + \frac{1}{2}\int |A|^2|u|^2 + \frac{\mu^2}{4|u|^2}|\operatorname{curl} A - h_{ex}|^2, \tag{7.2}$$

we see that all we need to deal with is $\int j(u)\cdot A$.

Adapting again the proof of [18], we have that

$$\begin{aligned}
|A_\varepsilon \cdot j(u_\varepsilon)| &\leq \frac{1}{4}\frac{|j(u_\varepsilon)|^2}{|u_\varepsilon|^2} + |u_\varepsilon|^2|A_\varepsilon|^2 \\
&\leq \frac{1}{4}|\nabla u_\varepsilon|^2 + \left(|u_\varepsilon|\left||u_\varepsilon|^2 - 1\right| + 1\right)|A_\varepsilon|^2 \\
&\leq \frac{1}{4}|\nabla u_\varepsilon|^2 + |A_\varepsilon|^2 + \frac{1}{4\varepsilon^2}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2 + 2\varepsilon^2|A_\varepsilon|^4 \\
&\leq \frac{1}{2}E_\varepsilon(u_\varepsilon) + |A_\varepsilon|^2 + 2\varepsilon^2|A_\varepsilon|^4.
\end{aligned}$$

By the uniform H^1 bound on a_ε, it follows via (7.2) and Sobolev embedding that

$$\frac{1}{2}E_\varepsilon(u_\varepsilon) \leq K|\log\varepsilon|^2 + C|\log\varepsilon|^2 + C\varepsilon^2|\log\varepsilon|^4 \leq C|\log\varepsilon|^2. \tag{7.3}$$

This shows that the compactness and lower bound results of Proposition 5.2 are applicable for u_ε, and we obtain that v_ε is compact as claimed.

We now continue as on page 555 of [18] and decompose

$$G_\varepsilon(u_\varepsilon, A_\varepsilon; h_{ex}) = G_\varepsilon^1(u_\varepsilon, A_\varepsilon) + G_\varepsilon^2(u_\varepsilon, A_\varepsilon) + G_\varepsilon^3(u_\varepsilon, A_\varepsilon) + G_\varepsilon^4(u_\varepsilon, A_\varepsilon), \tag{7.4}$$

where

$$G_\varepsilon^1(u_\varepsilon, A_\varepsilon) = E_\varepsilon(u_\varepsilon) \tag{7.5}$$

$$G_\varepsilon^2(u_\varepsilon, A_\varepsilon) = \frac{|\log\varepsilon|^2}{2}\int_U |a_\varepsilon|^2 + \frac{\mu^2\,|\log\varepsilon|^2}{8}\int_U |\operatorname{curl} a_\varepsilon - H|^2 \tag{7.6}$$

$$G_\varepsilon^3(u_\varepsilon, A_\varepsilon) = \frac{|\log\varepsilon|^2}{2}\int_U \left(|u_\varepsilon|^2 - 1\right)|a_\varepsilon|^2 \tag{7.7}$$

$$G_\varepsilon^4(u_\varepsilon, A_\varepsilon) = -\,|\log\varepsilon|^2\int_U a_\varepsilon\cdot v_\varepsilon. \tag{7.8}$$

For G_ε^1, we use the lower bound of Proposition 5.2 and obtain

$$\liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|^2}G_\varepsilon^1(u_\varepsilon, A_\varepsilon) \geq \frac{1}{2}\left(\|v\|_{L^2}^2 + \|\operatorname{curl} v\|_{\mathcal{M}}\right). \tag{7.9}$$

By the weak H^1 convergence and lower semicontinuity, we find that

$$\liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|^2}G_\varepsilon^2(u_\varepsilon, A_\varepsilon) \geq \frac{\mu^2}{8}\|\operatorname{curl} a - H\|_{L^2}^2 + \frac{1}{2}\|a\|_{L^2}^2 \tag{7.10}$$

For the third term, we need the analysis of Section 2 that shows that $\|1-|u_\varepsilon|^2\|_{L^2}^2 \leq C\varepsilon^2\,|\log\varepsilon|^4$, hence

$$|G_\varepsilon^3(u_\varepsilon, A_\varepsilon)| \leq \left(\int_U (|u_\varepsilon|^2-1)^2\right)^{1/2}\left(\int_U |A_\varepsilon|^4\right)^{1/2} \leq C\varepsilon\,|\log\varepsilon|^4 \to 0. \tag{7.11}$$

The convergence

$$\frac{1}{|\log\varepsilon|^2}G_\varepsilon^4(u_\varepsilon, A_\varepsilon) \to -\int_U a\cdot v$$

follows from the weak convergence of v_ε in L^p with $p<2$ and the strong convergence $a_\varepsilon \to a$ in L^q for all q that is implied by the weak H^1 convergence and Sobolev embedding.

Summing up the terms, we obtain the lower bound as claimed. □

Proposition 7.2. *Given $a \in H^1(U;\mathbb{R}^2)$ and $v \in L^2(U;\mathbb{R}^2)$ such that $w = \frac{1}{2}\operatorname{curl} v$ is a Radon measure, there exists a sequence $\{A_\varepsilon\} \in H^1(U;\mathbb{R}^2)$ and $u_\varepsilon \in H^1(U;\mathbb{C})$ such that $v_\varepsilon = \frac{1}{|\log\varepsilon|}j(u_\varepsilon)$ satisfies $v_\varepsilon \rightharpoonup v$ in L^p_{loc} with $p<2$, $\frac{v_\varepsilon}{|u_\varepsilon|} \rightharpoonup v$ in L^2. Also, $\frac{1}{|\log\varepsilon|}A_\varepsilon \rightharpoonup a$ in H^1.*

Proof. We choose the sequence $\{u_\varepsilon\}$ constructed in Proposition 6.2. To find A_ε, we set $A_\varepsilon = |\log\varepsilon|\,a_\varepsilon$, where a_ε is defined as the Coulomb gauge solution of

$$\operatorname{curl} a_\varepsilon = H + |u_\varepsilon|(\operatorname{curl} a - H) \tag{7.12}$$

We thus have

$$\operatorname{curl} a_\varepsilon - \operatorname{curl} a = (H - \operatorname{curl} a)(1-|u_\varepsilon|), \tag{7.13}$$

and since $|u_\varepsilon| \leq 1$, it follows that $(\operatorname{curl} a_\varepsilon)$ is bounded in L^2. The weak limit must be $\operatorname{curl} a$, since

$$\int |\operatorname{curl} a_\varepsilon - \operatorname{curl} a| \leq \left(\int (1-|u_\varepsilon|)^2\right)^{1/2} \left(\int |H - \operatorname{curl} a|^2\right)^{1/2} \to 0 \tag{7.14}$$

Elliptic regularity shows that $(a_\varepsilon - a)$ is bounded in H^1, and we obtain $a_\varepsilon \rightharpoonup a$ in H^1.

To see the convergence of the energy, we need to check the convergence of the terms G^1_ε to G^4_ε as above. The convergence of G^1_ε was dealt with in the proof of Proposition 7.1. We observe that

$$\frac{\operatorname{curl} A_\varepsilon - h_{ex}}{|u_\varepsilon|^2} = \frac{|u_\varepsilon|^2}{|u_\varepsilon|^2} |\log \varepsilon|^2 \, |\operatorname{curl} a - H|^2, \tag{7.15}$$

and $a_\varepsilon \to a$ strongly in L^2 by the compactness of the embedding $H^1 \subset L^2$, so G^2_ε converges as claimed. □

The terms G^3_ε and G^4_ε converge by the same arguments as used in the lower bound proof.

8. Application: Critical h_{ex}

In this section we analyze the limit functional

$$G(v, a; H) = \frac{1}{2} \|v - a\|^2_{L^2} + \frac{\mu^2}{8} \|\operatorname{curl} a - H\|^2_{L^2} + \frac{1}{2} \|\operatorname{curl} v\|_{\mathcal{M}} \tag{8.1}$$

We look for the critical field H_1 such that for $H < H_1$, minimizers of (8.1) will satisfy $\operatorname{curl} v = 0$, and for $H > H_1$, $\operatorname{curl} v \neq 0$. The critical field in the original scaling will then be $h_{ext} \approx H_1 |\log \varepsilon|$. Similar results for the Ginzburg-Landau functional are due to Sandier-Serfaty [31], [32] where also the relation to a free boundary problem is derived; while this can be done here, we omit this for the sake of brevity. We will follow the argument in [18] but keep the dependence on the extra parameter μ. We show

Proposition 8.1. *The critical field $H_1 = H_1(\mu)$ can be calculated by $H_1(\mu) = \frac{2}{\mu^2 \max_U |z_\mu|}$, where z_μ is the solution of $-\frac{\mu^2}{4}\Delta z_\mu + z + \mu + 1 = 0$ in U with homogeneous Dirichlet boundary data. The function $\mu \mapsto H_1(\mu)$ has the following properties:*

1. *$H_1(\mu)$ is a decreasing function*
2. *$\mu^2 H_1(\mu) \to 2$ as $\mu \to 0$*
3. *$H_1(\mu) \to \overline{H}(U)$ as $\mu \to \infty$, where $\overline{H}(U)$ is given by $\overline{H}(U) = \frac{1}{2 \sup_U w}$, and w is the solution of $\Delta w = -1$ in U, $w = 0$ on ∂U.*

We will show this by comparing G with the functional

$$F(a; H) = \frac{1}{2} \|a\|^2_{L^2} + \frac{\mu^2}{8} \|\operatorname{curl} a - H\|^2_{L^2}. \tag{8.2}$$

The minimizer a_* of F satisfies the Euler-Lagrange equations

$$a_* + \frac{\mu^2}{4}\operatorname{curl}(\operatorname{curl} a_* - H) = 0 \text{in } U, \qquad \operatorname{curl} a_* - H = 0 \text{on } \partial U. \tag{8.3}$$

Taking the curl of this equation, we obtain

$$\operatorname{curl} a_* - H - \frac{\mu^2}{4}\Delta(\operatorname{curl} a_* - H) + H = 0, \tag{8.4}$$

so setting $Hz_\mu = \operatorname{curl} a_* - H$ we see that z_μ is the solution of

$$-\frac{\mu^2}{4}\Delta z_\mu + z_\mu + 1 = 0, \quad z_\mu = 0 \text{ on } \partial U. \tag{8.5}$$

We now decompose the energy of (8.1) by setting $a = a_* + b$. We obtain

$$\begin{aligned} G(v, a; H) = \frac{1}{2}\int_U |a_*|^2 + |b|^2 + 2a_* \cdot b + \frac{\mu^2}{8}\int_U |\operatorname{curl} a_* - H|^2 + |\operatorname{curl} b|^2 \\ + 2(\operatorname{curl} a_* - H)\operatorname{curl} b - \int_U (v \cdot a_* + v \cdot b) + \frac{1}{2}\int_U |v|^2 + \frac{1}{2}\|\operatorname{curl} v\|_{\mathcal{M}} \end{aligned} \tag{8.6}$$

Integrating by parts and using (8.3) and (8.4), we see that

$$\begin{aligned} \int_U v \cdot a_* &= -\frac{\mu^2}{4}\int_U v \cdot \operatorname{curl}(\operatorname{curl} a_* - H) \\ &= -\frac{\mu^2}{4}H\int_U v \cdot \operatorname{curl} z_\mu = \frac{\mu^2}{4}H\int_U z_\mu \operatorname{curl} v \end{aligned} \tag{8.7}$$

and

$$\int_U (a_* + \frac{\mu^2}{4}\operatorname{curl}(\operatorname{curl} a_* - H)) \cdot b = 0 \tag{8.8}$$

We can thus rewrite

$$\begin{aligned} G(v, a; H) = \int_U \frac{\mu^2}{8}|\operatorname{curl} b|^2 + \frac{1}{2}|v - b|^2 + F(a_*) + \frac{1}{2}\|\operatorname{curl} v\|_{\mathcal{M}} - \frac{\mu^2}{4}H\int \operatorname{curl} v z_\mu \\ \geq F(a_*) + \|\operatorname{curl} v\|_{\mathcal{M}}\left(\frac{1}{2} - \frac{\mu^2}{4}H\max_U |z|\right) \end{aligned} \tag{8.9}$$

It follows that the minimizer of G satisfies $\operatorname{curl} v = 0$ if and only if

$$\frac{1}{2} - \frac{\mu^2}{4}H\max_U |z_\mu| < 0,$$

that is if and only if

$$H < H_1(\mu) = \frac{2}{\mu^2 \max_U |z_\mu|}.$$

The dependence of z_μ (and hence H_1) on μ can be calculated explicitly in the case of $U = B_R(0)$: The solution of (8.5) is the given by

$$z_\mu^R(r) = \frac{I_0(\frac{2r}{\mu})}{I_0(\frac{2R}{\mu})} - 1,$$

where I_0 is the modified Bessel function of zeroeth order. It follows that

$$\max |z_\mu^R| = 1 - \frac{1}{I_0(\frac{2R}{\mu})}.$$

Since $I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}$ for $x \to \infty$ and $I_0(x) = 1 + \frac{1}{4}x^2 + O(x^4)$ as $x \to 0$, it follows that $\max |z_\mu^R| \to 1$ as $\mu \to 0$ and $\mu^2 \max |z_\mu^R| \to R^2$ as $\mu \to \infty$. Finally we see that

$$H_1(\mu) = \frac{2I_0\left(\frac{2R}{\mu}\right)}{\mu^2\left(I_0\left(\frac{2R}{\mu}\right) - 1\right)}.$$

The general behavior is similar:

Proposition 8.2. *The solution z_μ of* (8.5) *has the following properties:*

(1) *$-\mu^2 z_\mu(x)$ is monotonically increasing in μ for every $x \in U$.*
(2) *$\sup_U |z_\mu| \to 1$ as $\mu \to 0$*
(3) *$\frac{\mu^2}{4} \sup_U |z_\mu| \to A(U)$ as $\mu \to \infty$, where $A(U) = \sup_U y$ and y is the solution of $\Delta y = -1$ in U, $y = 0$ on ∂U.*

Proof. We set $y_\mu = -\frac{\mu^2}{4} z_\mu$. Then y_μ solves $\Delta y_\mu - \frac{4}{\mu^2} y_\mu + 1 = 0$. Differentiating, we obtain that $w_\mu = \frac{\partial}{\partial\mu} y_\mu$ solves

$$\Delta w_\mu - \frac{4}{\mu^2} w_\mu = -\frac{8}{\mu^3} y_\mu \le 0, \tag{8.10}$$

and so by the maximum principle, $w_\mu \ge 0$, which proves the first claim.

The second claim follows since $z_\mu \to -1$ in every $U' \subset\subset U$, for the third we observe that y_μ as defined above converges to a solution of $\Delta y = -1$, and $A(U) = \sup_U y$. □

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