THE MASTER SPACE OF N=1 GAUGE THEORIES

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Butti, Forcella, Zaffaroni	hepth/0611229
Forcella, Hanany, Zaffaroni	hepth/0701236
Butti,Forcella,Hanany,Vegh, Zaffaroni,	arXiv 0705.2771
Forcella,Hanany,He,Zaffaroni,	arXiv 0801.1585
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FIELD THEORY MOTIVATIONS:

- Study of the moduli space (vacua and spectrum)
- Counting problems in supersymmetric gauge theories

D-BRANE MOTIVATIONS:

- N=1 supersymmetric theories appear on the worldvolume of D-branes - computation in gravitational duals
- Statistical properties of the BPS states and relation to black holes entropies

INTRINSIC MOTIVATIONS:

- Interplay with math:
  (Combinatorics, Rep. Theory and Quivers)

Moduli space of N=1 Theories:

In supersymmetric gauge theories with chiral matter superfields X, gauge group G and a superpotential W(X)

$$\boxed{M = F // G_c}$$

F term solutions $\qquad \dfrac{\partial W(X)}{\partial X} = 0$

D term solutions
modulo gauge transformations $\qquad G \longrightarrow G_c$
== Complexified Gauge Group

parametrized by field VEVS $\longrightarrow$ flat directions

$\simeq$

parametrized by gauge invariant BPS chiral operators $\longrightarrow$ holomorphic functions on moduli space

EXAMPLE: Classical SQCD

$$(N_c, N_f)$$

$i = 1, \ldots, N_f$
$\alpha = 1, \ldots, N_c$

$$Q_i^{\alpha}, \overline{Q_{\alpha i}}$$

$$G = SU(N_c)$$

$W = 0$ no F-terms

Gauge invariant chiral operators (mesons and baryons):

$$Q_i^{\alpha} \overline{Q_{\alpha j}} \qquad B_{i_1 \ldots i_{N_f - N_c}} = \varepsilon_{i_1 \ldots i_{N_f}} Q_{i_1}^{\alpha_1} \ldots Q_{i_{N_c}}^{\alpha_{N_c}} \varepsilon_{\alpha_1 \ldots \varepsilon_{N_c}} \qquad \overline{B}_{i_1 \ldots i_{N_f - N_c}} = \varepsilon_{i_1 \ldots i_{N_f}} \overline{Q}_{i_1}^{\alpha_1} \ldots \overline{Q}_{i_{N_c}}^{\alpha_{N_c}} \varepsilon_{\alpha_1 \ldots \varepsilon_{N_c}}$$

For $3N_c / 2 < N_f < 3N_c$ IR fixed point

For $N_f \geq N_c + 1$ no quantum corrections to the moduli space

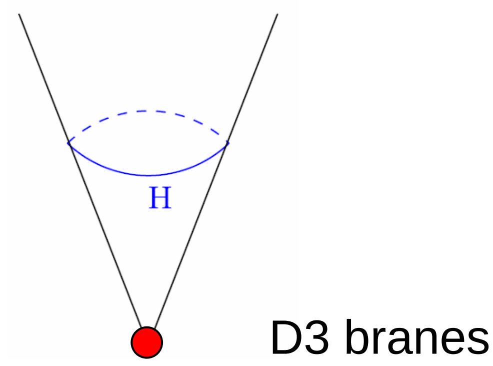
$$\mathrm{M} = \mathbb{C}^{2N_c N_f} // G$$

properties studied:
[Pouliot].[Romelsberger]
[Hanany-He-Mekarreya-Vejjala]

N=1 QUIVER GAUGE THEORIES

N=1 gauge theories live on D branes probing conical Calabi-Yau singularities:

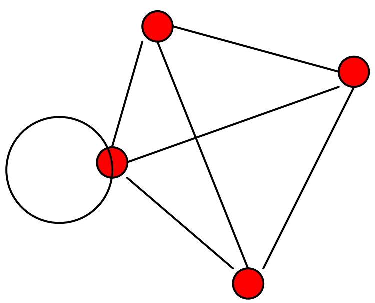
$$\mathbb{R}^{1,3} \times X \equiv C(H)$$



•Physical branes: 4d conformal gauge theories

•Fractional branes: non conformal theories
(confinement, cascades, susy breaking)

•All theory of Quiver Type:



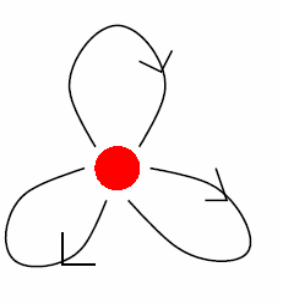
gauge groups $G = \prod_i^g U(N_i)$

bifundamental or adjoints fields X

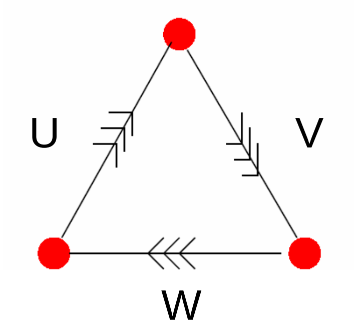
superpotential W(X)

Orbifolds of N=4 SYM:

$\mathbb{C}^3$



$\mathbb{C}^3 / Z_3$

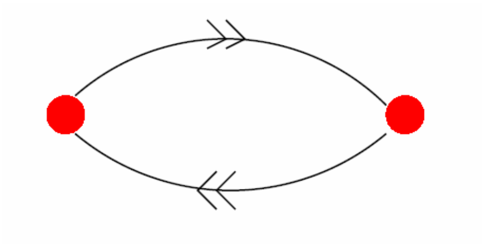


$$W = \Phi_1[\Phi_2, \Phi_3]$$

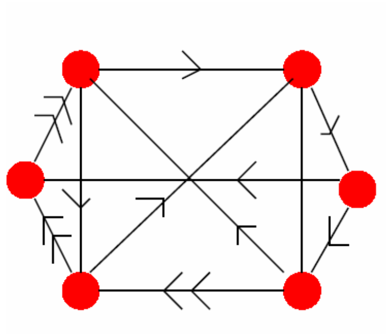
$$W = \varepsilon_{ijk} U_i V_j W_k$$

Toric Calabi-Yaus – generalizations of abelian orbifolds:

$\mathrm{C}\,(\mathrm{T}^{1,1})$



$C(L^{152})$



$$W = \varepsilon_{ij} \varepsilon_{pq} A_i B_p A_j B_q$$

$$W = \ldots$$

Correspondence CALABI-YAU and N=1 QUIVER gauge theories:

- classification for toric CY (dimers/tiling)

[Franco,Kennaway,Hanany,Vegh,Wecht]

- few metrics known ($H = S^5, T^{1,1}, Y^{p,q}, L^{p,q,r}$) but many interesting questions solved without knowledge of the metric.

[Gauntlett,Martelli,Sparks,Waldram]
[Cvetic,Lu,Page,Pope]

- As a prediction from AdS/CFT (near horizon $AdS_5 \times H$) an infinite class of 4d SUPERCONFORMAL THEORIES: $N_i \equiv N$

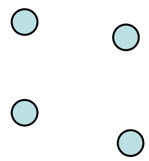
Many tests:[klebanov-witten]
[Benvenuti,Franco,Hanany,Martelli,Sparks]
[Butti,Zaffaroni] …

COMMON LORE:

$\boxed{\text{QFT theory moduli space} \quad == \quad \text{brane moduli space}}$

For N=4 SYM: $\Phi_i \quad \mathrm{N} \times \mathrm{N}$ matrices

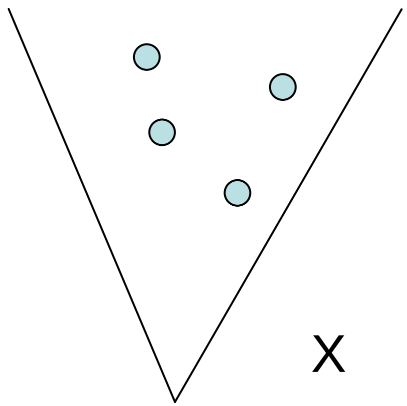
$$[\Phi_i, \Phi_j] = 0 \ \rightarrow \ \Phi_i \ \text{ diagonal}$$



$\mathbb{C}^3$

For a generic CY X:

Mesonic moduli space $\subset M$



$Sym(X^N)$

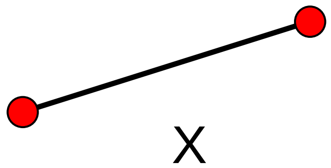
Baryonic directions

FI terms; blowing up modes

Baryonic directions:

In the IR theory: all abelian gauge factors decouple

$U(N) \longrightarrow SU(N)$ U(1) appears as global baryonic symmetries

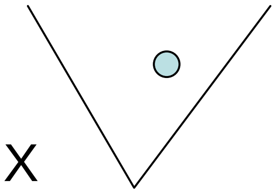


di-baryons: det X

Baryonic flat directions: $\det X \neq 0$

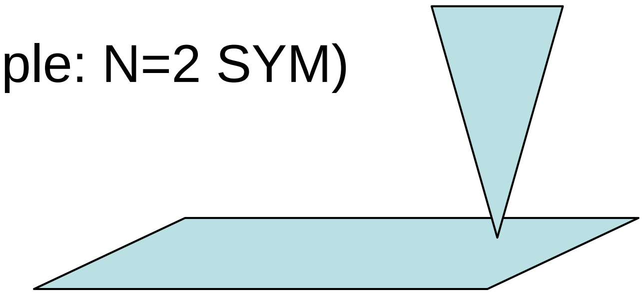
It turns out that:

the N=1 moduli space (MASTER SPACE)



X

- Contains the mesonic moduli space X; dim =3
- It is itself a Calabi-Yau manifold of dim = g+2
- Generically reducible (example: N=2 SYM)



the N>1 moduli space

- Determined by the master space
- dim = 3 N + g -1

In theories with symmetries, information encoded in a single function, known as Hilbert Series

$$H(t) = \sum n(k) t^k$$

generating functions for holomorphic functions/ chiral operators

t = chemical potential for global symmetries
n(k) = number of hol. functions of charge k
= number of BPS operators of charge k

Density of states with charge
Structure of moduli space: dimensions, generators, relations

Math=degree
Phys= # d.o.f

$$H(t) = \frac{P(t)}{(1-t)^d} \xrightarrow[t \longrightarrow 1]{} \frac{c}{(1-t)^d}$$

dimension

N=1 MASTER SPACE $\mathcal{F}^\flat$ = solution of F-terms

(no gauge groups in the IR)

N=4 SYM

Trivial F-terms:

$$[\Phi_i, \Phi_j] = 0$$

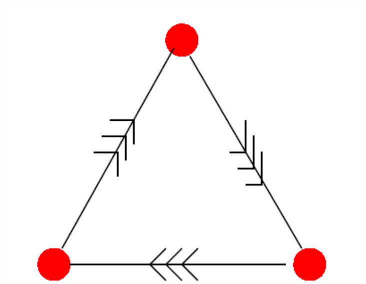
$$\Phi_i \rightarrow q_i$$

N=1 moduli space

$$\mathbb{C}^3$$

$$H(q) = \frac{1}{(1-q_1)(1-q_2)(1-q_3)}$$

$\mathbb{C}^3 / \mathbb{Z}_3$



$$W = \varepsilon_{ijk} u_i v_j w_k$$

Non trivial F terms:

$$u_i v_j = u_j v_i$$

$$u_i w_j = u_j w_i$$

$$w_i v_j = w_j v_i$$

$$\mathbb{C}^6 / \{1, 1, 1, -1, -1, 1\}$$

$$H(t) = \frac{1 + 4t + t^2}{(1-t)^5}$$

$\mathcal{X}$	$\dim(\mathcal{F}^\flat)$	${}^{\mathrm{Irr}}\mathcal{F}^\flat$	$H(t;\ {}^{\mathrm{Irr}}\mathcal{F}^\flat)$
$\mathbb{C}^3$	3	$\mathbb{C}^3$	$(1-t)^{-3}$
Conifold	4	$\mathbb{C}^4$	$(1-t)^{-4}$
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$	4	Conifold $\times\ \mathbb{C}$	$\frac{1+t}{(1-t)^4}$
$\mathbb{C}^3/\mathbb{Z}_3$	5	$\mathbb{C}^6/\{1,1,1,-1,-1,1\}$	$\frac{1+4t+t^2}{(1-t)^5}$
dP_1	6	$-$	$\frac{1+4t+7t^2+4t^3+t^4}{(1-t)^6(1+t)^2}$

Algebraic variety dW=0

Symplectic quotient ${}^{\mathrm{Irr}}\mathcal{F}^\flat = \mathbb{C}^c//(\mathbb{C}^*)^{c-g-2}$

Palindromic property $H(t)=t^w H(1/t)$

${}^{\mathrm{Irr}}\mathcal{F}^\flat$ is itself a toric CALABI-YAU

N>1: Computing gauge invariants

The problem of finding (classical) gauge invariants goes back to the nineteenth century (invariant theory).

$N \times M$ matrices X_{ij}

$$R[X_{ij}] = \mathbb{C}[X_{ij}] / \{\partial W(X_{ij}) = 0\}$$

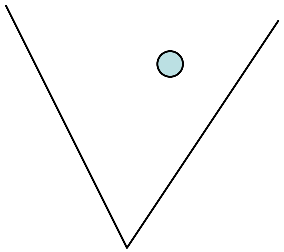
$$R^{INV} = R[X_{ij}] // G$$

- General methods due to Hilbert: free resolutions, syzygies…
- Now algorithmical (Groebner basis)
- Explicit formulae when W=0 (Molien)
- With computers and computer algebra programs really computable (but for small values of N and M)
- Still very hard to get general formulae for generic N,M

For quiver theories: $\mathcal{F}^\flat$ is the key to N>1

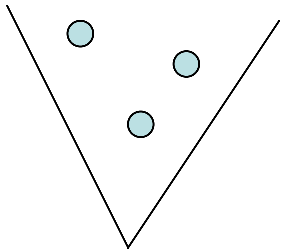
Warming up: N=4 SYM

N=1



$$\mathcal{F}^\flat = \mathbb{C}^3$$

N>1



$$\mathcal{F}^\flat = Sym((\mathbb{C}^3)^N)$$

$$\Phi_1^{n_1}\Phi_2^{n_2}\Phi_3^{n_3} \longrightarrow t_1^{n_1}t_2^{n_2}t_3^{n_3}$$

$$\mathrm{H(t)}=\frac{1}{(1-t_1)(1-t_2)(1-t_3)}$$

$$PE[\nu H(t)] = \sum \nu^N g_N(t)$$

$$\equiv Exp(\sum_{k=1} H(t^k)\upsilon^k / k)$$

pletystic exponential
counts symmetric products

General Conjecture: given the expansion of the N=1 generating function in sectors with definite baryonic charge

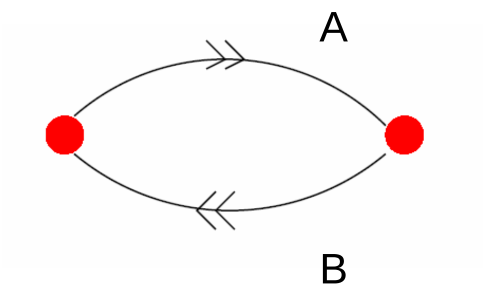
$$g_1(\{t_i\};CY) = \sum_{B=-\infty}^{\infty} b^B g_{1,B}(\{t_i\};CY)$$

the partition function at finite N is given by

$$\sum_{N=0}^{\infty} \nu^N g_N(\{t_i\};CY) = \sum_{B=-\infty}^{\infty} b^B PE[\nu g_{1,B}(\{t_i\};CY)]$$

based on geometrical quantization of wrapped D3 branes in AdS dual.

Full partition function for the conifold



$$(A_1, A_2, B_1, B_2) \longrightarrow (t_1, t_1, t_2, t_2)$$

baryonic charge $(1,1,-1,-1)$

N=1 Trivial F-terms:

$$A_i\, B_p\, A_j = A_j\, B_p\, A_i$$

N=1 moduli space

$$\mathbb{C}^4$$

$$g_1(t) = \frac{1}{(1-t_1)^2(1-t_2)^2} = \sum_B (g_{1,B}(t) \equiv \sum (n+1)(n+1+B) t_1^n t_2^{n+B})$$

Finite N:

$$\sum g_N(t) \nu^N = \sum_B PE[\nu g_{1,B}(t)]$$

Checked against explicit computation for N=2,3.

Some specific examples: conifold N=2

$$g_2(t_1, t_2; \mathcal{C}) = \frac{1 + t_1 t_2 + t_1^2 t_2^2 - 3t_1^4 t_2^2 - 3t_1^2 t_2^4 + t_1^5 t_2^3 + t_1^3 t_2^5 - 3t_1^3 t_2^3 + 4t_1^4 t_2^4}{(1 - t_1^2)^3 (1 - t_1 t_2)^3 (1 - t_2^2)^3}$$

generators:

$$\begin{aligned}
3t_1^2 &\quad\longrightarrow\quad \epsilon\epsilon A_1 A_1 \ , \ \epsilon\epsilon A_1 A_2 \ , \ \epsilon\epsilon A_2 A_2 \\
3t_2^2 &\quad\longrightarrow\quad \epsilon\epsilon B_1 B_1 \ , \ \epsilon\epsilon B_1 B_2 \ , \ \epsilon\epsilon B_2 B_2 \\
4t_1 t_2 &\quad\longrightarrow\quad \mathrm{tr}(A_1 B_1) \ , \ \mathrm{tr}(A_1 B_2) \ , \ \mathrm{tr}(A_2 B_1) \ , \ \mathrm{tr}(A_2 B_2)
\end{aligned}$$

Some specific examples: conifold N=3

$$g_3(t_1, t_2; \mathcal{C}) = \frac{F(t_1, t_2)}{(1-t_1^3)^4(1-t_1t_2)^3(1-t_1^2t_2^2)^3(1-t_2^3)^4}$$

(F degree 24 polinomial)

$$PE^{-1}[g_3(t_1, t_2; \mathcal{C})] = 4t_1^3 + 2t_1^4t_2 + 2t_1t_2^4 + 4t_2^3 + 4t_1t_2 + 9t_1^2t_2^2 + \ldots$$

generators:

$$\begin{array}{rcl}
4t_1^3 & \rightarrow & \epsilon\epsilon A_iA_jA_k \; : \; 4 \text{ operators} \\
2t_1^4t_2 & \rightarrow & \ldots : \; 2 \text{ non factorizable baryons} \quad \epsilon\epsilon(A_lB_mA_i)(A_j)(A_k) \\
2t_1t_2^4 & \rightarrow & \ldots : \; 2 \text{ non factorizable baryons} \\
4t_2^3 & \rightarrow & \epsilon\epsilon B_iB_jB_k \; : \; 4 \text{ operators} \\
4t_1t_2 & \rightarrow & \mathrm{tr}(A_iB_j) \; : \; 4 \text{ operators} \\
9t_1^2t_2^2 & \rightarrow & \mathrm{tr}(A_iB_jA_kB_l) \; : \; 9 \text{ operators}
\end{array}$$

SEIBERG INVARIANCE

The master space partition function is the same for Seiberg dual theories

- Extension of Seiberg duality to N=1

HIDDEN SYMMETRIES

$\mathcal{X}$	$\dim(\mathcal{F}^\flat)$	${}^{\mathrm{Irr}}\mathcal{F}^\flat$	$H(t;\ {}^{\mathrm{Irr}}\mathcal{F}^\flat)$	Global Symmetry
$\mathbb{C}^3$	3	$\mathbb{C}^3$	$(1-t)^{-3}$	$U(3)$
Conifold	4	$\mathbb{C}^4$	$(1-t)^{-4}$	$U(1)_R \times SU(4)_H$
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$	4	Conifold $\times\,\mathbb{C}$	$\frac{1+t}{(1-t)^4}$	$U(1)_R \times SU(2)_R \times U(1)_B \times SU(2)_H$
$\mathbb{C}^3/\mathbb{Z}_3$	5	$\mathbb{C}^6/\{1,1,1,-1,-1,1\}$	$\frac{1+4t+t^2}{(1-t)^5}$	$U(1)_R \times SU(3)_M \times SU(3)_H$
dP_1	6	$-$	$\frac{1+4t+7t^2+4t^3+t^4}{(1-t)^6(1+t)^2}$	$U(1)_R \times SU(2)_M \times U(1)^3 \times SU(2)_H$

DUAL INTERPRETATION

BPS states can be obtained by quantizing wrapped D3 branes in the AdS dual: $g_1(t)$ counts classical configuration of supersymmetric D3

Comments and Conclusions

Counting problems in N=1 gauge theories are continuous source of interesting results:

General understanding of the structure of BPS partition functions for selected class of theories

Non trivial statistical properties

Discretized Kahler moduli and similarity with Nekrasov/ Topological string partition functions

Other interesting questions: superconformal index
¼ BPS partition function (black holes)
non conformal models

APPENDIX

Pletystic Exponential counts symmetrized products of elements P in a set S with generating function

$$g_1(q) = \sum_{n \in S} q^n$$

Introduce a new parameter: υ

$$\boxed{g(q,\upsilon) = \frac{1}{\prod_{n \in S}(1-\upsilon q^n)} = \sum_{N=1} \upsilon^N g_N(q)}$$

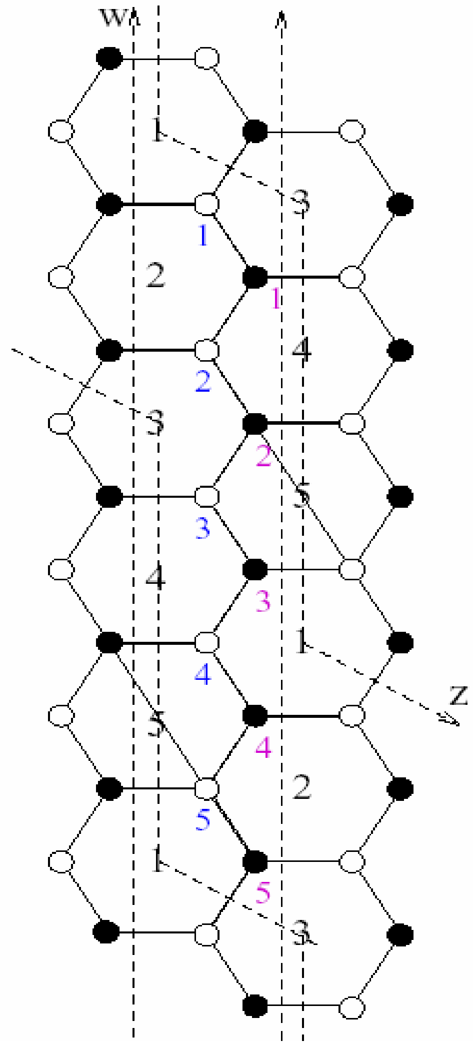
$$\Downarrow$$

$$Exp(\sum_{n \in S} \log(1-\upsilon q^n)) = Exp(\sum_{k=1}^{\infty} \sum_{n \in S} \upsilon^k q^{kn}/k) = Exp(\sum_{k=1}^{\infty} g_1(q^k)\upsilon^k/k)$$

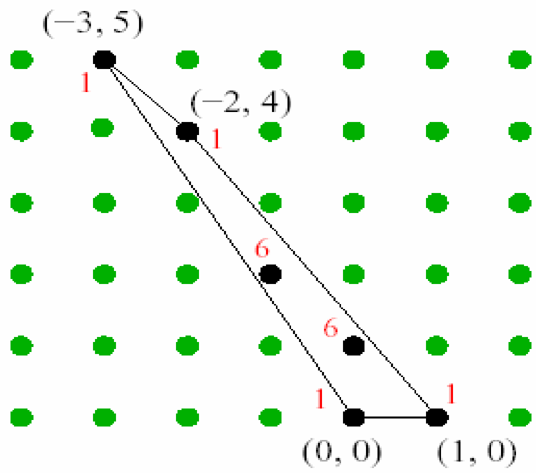
Connection to dimers:

Okounkov,Nekrasov,Vafa – Franco,Kennaway,Hanany,Vegh,Wecht

L^{152}

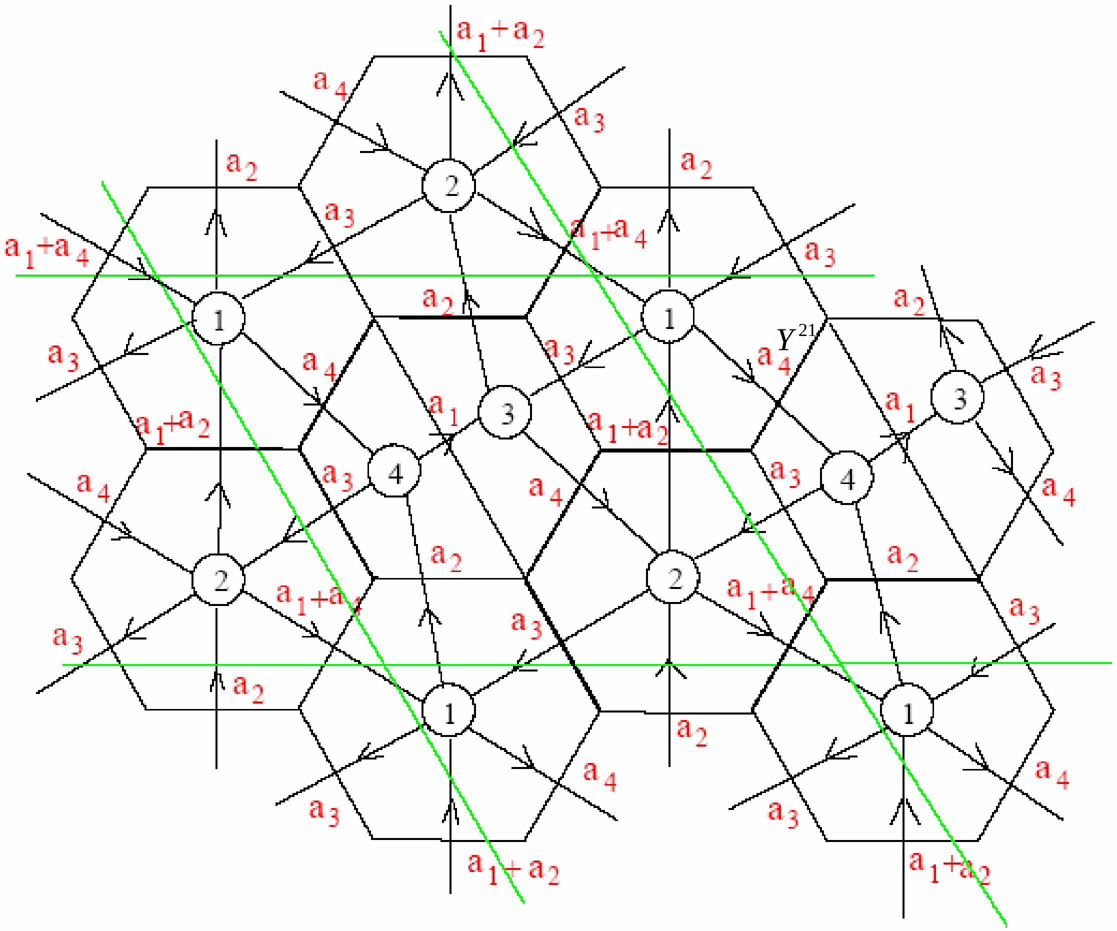


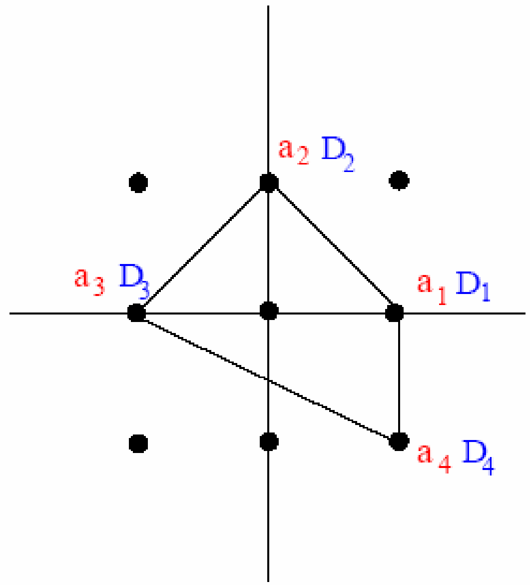
$$K = \left(\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 0 & 0 & w & z \\ 2 & 1 & 1 & 0 & 0 & w \\ 3 & wz^{-1} & 1 & 1 & 0 & 0 \\ 4 & 0 & wz^{-1} & 1 & 1 & 0 \\ 5 & 0 & -wz^{-1} & wz^{-1} & 1 & 1 \end{array}\right)$$



Dimers, combinatorics and charges:

Hanany-Witten construction for local CY





delPezzo 1 = Y^{21}

Connection to a-maximization:

Central charge of the CFT determined by combinatorial data:

$$\boxed{a = \frac{9}{32}\,\mathrm{Tr}\,R^3 = \sum_{i,j,k} |< \mathrm{V_i}, \mathrm{V_j}, \mathrm{V_k} >|\, a_i a_j a_k}$$

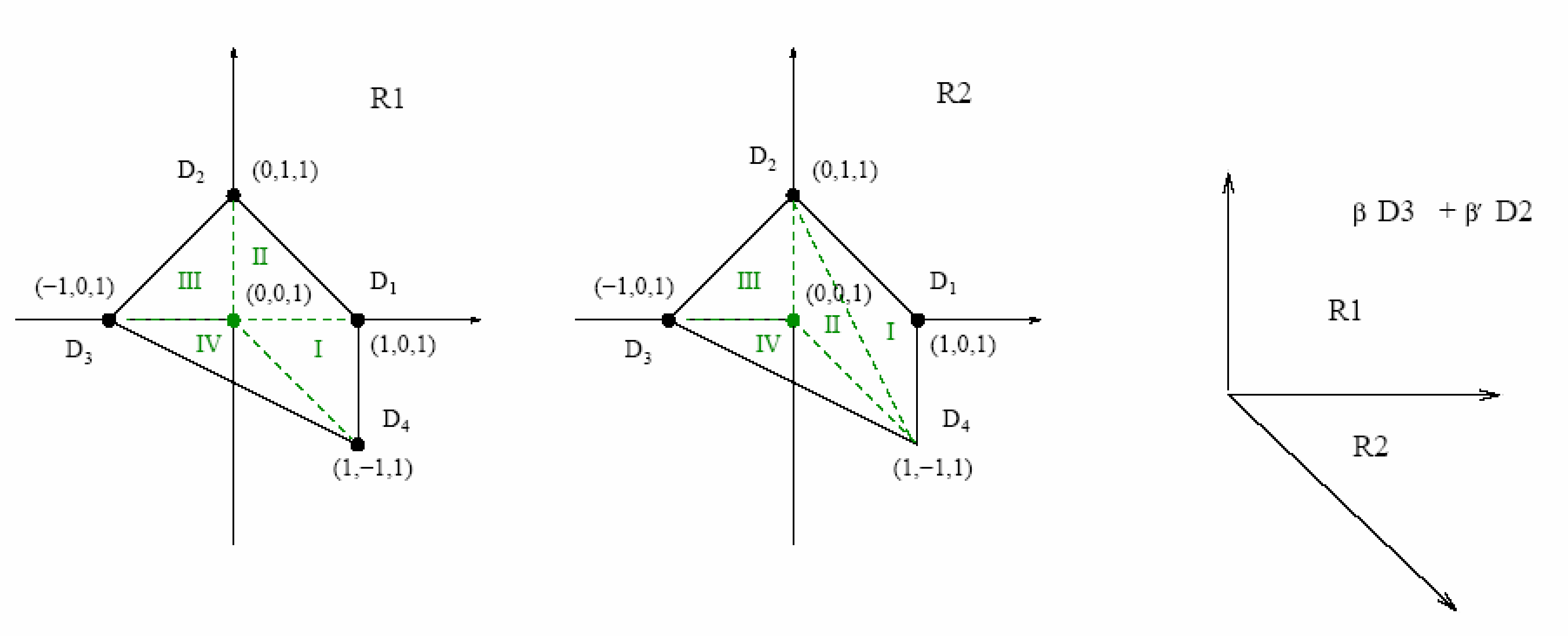
$$\sum_{i=1}^{d} a_i = 2$$

Butti,Zaffaroni
Benvenuti,Pando-Zayas,Tachikawa
Lee,Rey

Thanks to a-maximization (Intriligator,Wecht) the exact R-charge of the CFT is obtained by maximizing a

In general toric CY B is replaced by a sum over the Kahler cone

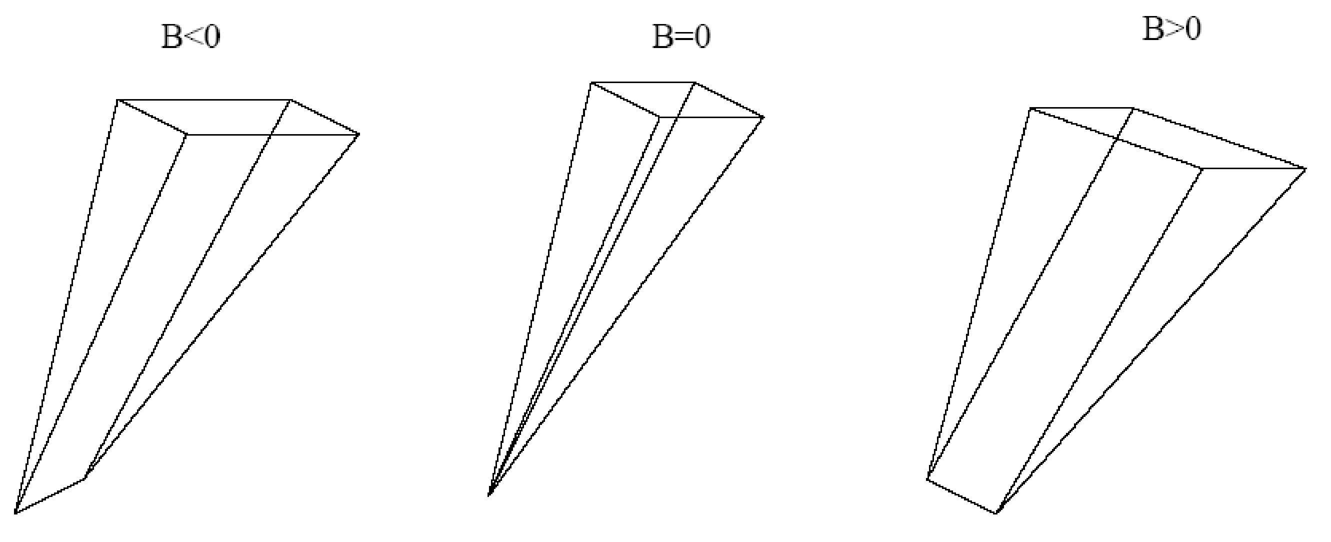
$$g_1(t;CY) = \sum_{\beta \in GKZ} g_{1,\beta}(t)$$



Y21 example

DIGRESSION: geometrical interpretation

$g_{1,B}(q)$
counts integer points
in convex polytopes



Relation with
kahler modulus: