Constructing Non-Abelian Vortices with Arbitrary Gauge Groups

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Talk based on the work: arXiv:0802.1020.
In collaboration with:
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Outline

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Motivations

- Solitons play crucial role in the dynamics of gauge theories:
  - vortices are relevant in some mechanisms of confinement: (QCD?).
  - could be relevant in the understanding of n-A dualities.
- SUSY theories:
  - semiclassical analysis $\Rightarrow$ quantum regime.
  - Vortices provide an interesting mapping between $4D$ and $2D$ dynamics.
- Cosmological solitons: cosmic string.

- Mostly studied for $U(N)$ gauge theories, but...
  - How physics depends on the gauge group?
  - GUT theories?

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The Model

Non-Abelian $(2+1)d$ $G = G' \times U(1)$ gauge theory with $N_{\rm F}$ "fundamental" flavour.

$$\mathcal{L}_{Gauge} = -\frac{1}{4e^2} F^0_{\mu\nu} F^{0\mu\nu} - \frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + (\mathcal{D}_\mu H_A)^\dagger \mathcal{D}^\mu H_A - \frac{e^2}{2} \left| H_A^\dagger t^0 H_A - \frac{v^2}{\sqrt{2N}} \right|^2 - \frac{g^2}{2} |H_A^\dagger t^a H_A|^2,$$

H : $N_{\rm C} \times N_{\rm F}$ matrix of squark fields;

- (Truncated) bosonic sector of $\mathcal{N} = 2$ SUSY theory;
- The FI term v triggers a spontaneous breaking of $U(1)$ (rich structure of the vacuum: Higgs/Higgs-Coulomb phases)
- $SU(N_{\rm F})$ flavour symmetry: $H \to H\, U_F$

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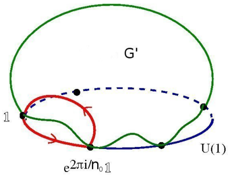
Non-Abelian Vortices

Consider the fully Higgsed, color-flavor locked vacuum:

$$\langle H \rangle \sim \mathbf{1}_N.$$

Non-Abelian BPS vortices supported by $\pi_1[G' \times U(1)] = \mathbb{Z}$.

$$G = (G' \times U(1))/\mathbb{Z}_{n_0}$$
$$\Downarrow$$
$$T_{G' \times U(1)} = \frac{1}{n_0} T_{U(1)}$$



- Vacuum color-flavor symmetry $G'_{\mathrm{C+F}}$
  - broken by a vortex solutions: Orientational moduli: $G'_{\mathrm{C+F}}/H_{\mathrm{C+F}}$
- The vacuum has a big degeneracy:
  - usually we find: $\pi_1(\mathcal{M}_{Vac.\ man.}) = 0$: Semilocal (size) moduli

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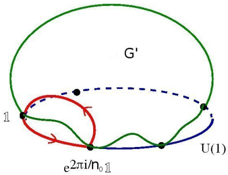

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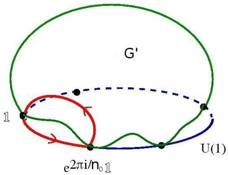

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Interlude: "Semilocal" vortex

Extended Abelian Higgs model

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + |\mathcal{D}_\mu H_1|^2 + |\mathcal{D}_\mu H_2|^2 - \frac{e^2}{2} \left(|H_1|^2 + |H_2|^2 - v^2 \right)^2$$

$$\pi_1(U(1) = S^1) = \mathbb{Z}$$

The absence of topological stability causes the appearance of a size modulus:

$$H_{vortex}(r, \theta) = f(r, a)(e^{i\theta}, a/r)$$

- $a = 0$, "local": $f(r, 0) = f_{ano}(r) \sim 1 - \alpha exp(-ve^2 r)$
- $a \neq 0$, "semilocal": $f(r, a) \sim 1 - \frac{1}{2}|a|^2/r^2$
  - $H(r, \theta) \sim \frac{1}{\sqrt{r^2 + |a|^2}}(e^{i\theta} r, a), \quad |a| \gg 1/ve^2$
  - Semilocal vortices interpolate between "ano" vortices and $\mathbb{C}P^1$ lumps (M. Hindmarsh, Phys. Rev. Lett. 68, 1263 - 1266)

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Strong coupling limit.
$e, g \to \infty$. NLσM

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Maxwell terms disappear, gauge fields become auxiliary.

The fields are forced to lie in the vacuum

- To eliminate all gauge fields and have a well defined NLσM we must consider a fully Higgsed vacuum.

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Result of the strong coupling limit

$$\mathcal{N} = 2, \quad (2+1)d, \quad \text{NL}\sigma\text{M} \quad \text{with} \quad \mathcal{M}_{Target} = \mathcal{M}_{Higgs}$$

The sigma model inherits the natural Kahler metric of the Higgs branch of the original theory.

Lumps

Solitons in $(2+1)d, \; NL\sigma M$ supported by $\pi_2(\mathcal{M}_{Target})$

- Scale invariance (classical) $\Rightarrow$ degenerate set of lumps with arbitrary size.
- Singular, zero-size, lumps $\Rightarrow$ small lump singularities (similar to small instantons singularities)

From vortices to lumps

$g, e \to \infty$

$$G' \times U(1) \text{ Gauge theory} \longrightarrow NL\sigma M \text{ on } \mathcal{M}_{Higgs}$$
$$\text{Vortices} \longrightarrow \text{Lumps}$$

- "BPS" correspondence: $T_{vortex} = M_{lump}$
- Toplogical correspondence: $\pi_2(\mathcal{M}_{Higgs}) = \pi_2(\mathcal{M}_{Vac.\,man.}/G) = \pi_1(G)$
  - $\pi_1(\mathcal{M}_{Vac.\,man.}) = 1 \quad \Leftrightarrow \quad$ Existence of semilocal vortices

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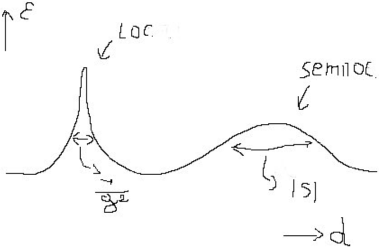

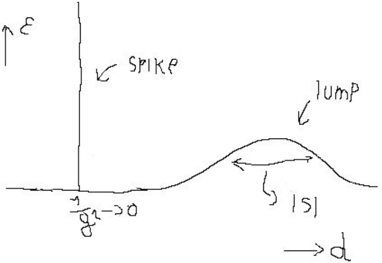

Constructing the Vortices

The strategy:From lumps to vortices

Step 1: Lump construction

- Construct the lump solutions in the related NLσM and their moduli space.
  - Lump construction is completely algebrical;
  - Contains (small-lumps) singularities.

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Step 2: Deforming the Lump

- Vortices as "deformed" lumps.
  - Algebraic equations deformed to differential equations (BPS equations);
  - Small lump singularities resolved to regular "local" vortices.

Step 1: Lump construction

The BPS equations for lumps

$$\Rightarrow \mathcal{D}_{\bar{z}} H \equiv \frac{1}{2}(\mathcal{D}_1 + i\mathcal{D}_2) H = 0,$$

$$\left| H^\dagger t^0 H - \frac{v^2}{\sqrt{2N}} \right|^2 = |H^\dagger t^a H|^2 = 0.$$

- $\mathcal{D}_{\bar{z}} H = 0$ implies "covariant holomorphicity":

$$H_{lump}(z, \bar{z}) = S^{-1}_{lump}(z, \bar{z}) H_0(z), \quad S \in G^{\mathbb{C}}$$

  - The "Moduli Matrix" $H_0(z)$ encodes the moduli space parameters
  - Residual $G^{\mathbb{C}}$ invariance: $V(z) \in G'^{\mathbb{C}}$

- Tha second line contains algebraic equations for S_{lump}.

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The Generalized Rational Map Construction

- Recall that lump configurations identify a mapping:
  - $\mathbb{C}P^1 \mapsto \mathcal{M}_{moduli\ space\ vacua} \cdots$
- ...and the following description of the moduli space of vacua (SUSY)
  - $\mathcal{M}_{moduli\ space\ vacua} = \{I_G^i,$ Holomorphic G invariants of the fileds $H\}$

(Luty, Taylor: Phys.Rev.D53:3399-3405,1996)

Holomorphic mapping

$$I_G^i(H)(z,\bar{z}) = I_G^i(S_{lump}^{-1} H_0)(z,\bar{z}) = I_G^i(H_0)(z)$$
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The Generalized Rational Map Construction: boundary conditions

Boundary conditions

$$I_G^i(H_0)\Big|_{|z|\to\infty} = I_G^i(H_{\rm vev}) \equiv I^i_{\rm vev}$$

$$G = G' \times U(1) \quad \Rightarrow \quad I_G^{(i,j)}(H_0) = I^i_{G'}(H_0(z))/I^j_{G'}(H_0(z))$$

G' invariants with the same $U(1)$ charge.

Definition of the lump number: k

$$I^i_{G'}(H_0) = I^i_{\rm vev} z^{\nu n_i} + \mathcal{O}(z^{\nu n_i - 1}) \equiv I^i_{\rm vev} z^{k n_i/n_0} + \mathcal{O}(z^{k n_i/n_0 - 1})$$

$$\nu = k/n_0, \quad n_0 \equiv GCD\{n_i\}$$

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Step 2: "Deforming" the lump

The BPS equations for vortices

The algebraic (vacuum) equations for the lumps...

$$\begin{aligned} \left(H^\dagger t^0 H - \frac{v^2}{\sqrt{2N}} \right) &= 0 \\ (H^\dagger t^a H) &= 0 \\ \mathcal{D}_{\bar{z}} H &= 0 \end{aligned}$$

- The "covariant holomorphicity" condition is untouched, the same construction hold for vortices.
  - $H_{lump}(z, \bar{z}) = S^{-1}_{lump}(z, \bar{z}) H_0(z) \Rightarrow$
- Finite gauge coupling deformations regularize the small lump singularities.

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-\frac{\sqrt{2N}}{e^2}F^0_{12} + \left(H^\dagger t^0 H - \frac{v^2}{\sqrt{2N}}\right) &= 0 \\
-\frac{4}{g^2}F^a_{12}t^a + (H^\dagger t^a H) &= 0 \\
\mathcal{D}_{\bar{z}} H &= 0
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- The "covariant holomorphicity" condition is untouched, the same construction hold for vortices.
  - $H_{lump}(z, \bar{z}) = S^{-1}_{lump}(z, \bar{z}) H_0(z) \Rightarrow H_{vort}(z, \bar{z}) = S^{-1}_{vort}(z, \bar{z}) H_0(z)$
- Finite gauge coupling deformations regularize the small lump singularities.

Step 2: "Deforming" the lump

The BPS equations for vortices

The algebraic (vacuum) equations for the lumps...
get deformed to the BPS equations for the vortices.

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A well known example: $U(N)$

The moduli space of vortices

1. List all the $G' = SU(N)$ invariants:
   - $I_{SU}(H_0) = \det H_0$ with charge N $\Rightarrow$ $n_0 = 1/N$
2. Impose the boundary conditions constraints:
   - $\det H_0(z) = z^{\nu N} + \mathcal{O}(z^{\nu N-1}) = z^k + \mathcal{O}(z^{k-1})$
   - $U(1)$ winding of the fundamental vortex is: $\nu = k/N = 1/N$
3. Write down the moduli space for non-Abelian $U(N)$ vortices:

$$\mathcal{M}_{U(N);k} = \{H_0(z) \mid \det H_0(z) = z^k + \mathcal{O}(z^{k-1}) \mid H_0(z) \sim V(z)H_0(z)\}$$

$$\dim_{\mathbb{C}}(\mathcal{M}_{U(N),k}) = kN$$

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The moduli space of the fundamental vortex

The following choice fixes any residual $V(z)$ equivalence:

$$H_0 = \begin{pmatrix} z - z_0 & 0 \\ \mathbf{b} & \mathbf{1}_{N-1} \end{pmatrix} \quad + \quad \text{permutations}$$

- Position of the vortex: parameterized by z_0.
- Internal moduli space of the vortex, $\mathbb{C}P^{N-1}$: parameterized by $\mathbf{b}$

The $U(N)$ vortex is an embedding of an Abelian vortex

$$H = U(N)_{C+F} \begin{pmatrix} H_{ANO} & 0 \\ 0 & \mathbf{1}_{N-1} \end{pmatrix} U(N)^{\dagger}_{C+F}$$

- All internal zero modes are N-G modes generated by breaking of the global symmetries of the vacuum by the vortex:

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Orthogonal and Symplectic groups

The moduli space of vortices

1. $SO/USp(2M)$ are embedded into $SU(2M)$ by:

$$U^T J U = J, \quad J = \begin{pmatrix} \mathbf{0}_M & \mathbf{1}_M \\ \pm\mathbf{1}_M & \mathbf{0}_M \end{pmatrix}$$

2. There are $N(N \pm 1)/2$ invariants with charge 2 and a charge $2M$ one:

$$\{I_{SO,USp} = H_0^T J H_0, \ \det H_0\} \qquad (n_0 = 1/2)$$

3. We get the defining constraints for the moduli matrix:

$$H_0^T(z) J H_0(z) = z^k J + \mathcal{O}(z^{k-1})$$

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Orthogonal and Symplectic groups

The moduli space of the fundamental vortex

The $V(z)$-fixed moduli matrix is the following:

$$H_0 = \begin{pmatrix} z\mathbf{1}_M - \mathbf{A} & \mathbf{C}_{S/A} \\ \mathbf{B}_{A/S} & \mathbf{1}_M \end{pmatrix} \quad + \quad \text{permutations}$$

$$T_{SO/USp(2M)} = 2\pi v^2 \frac{k}{n_0} = \pi v^2 = M\, T_{U(2M)}$$

- New properties:
  - The vortex generally is not an embedding of a $U(N)$-vortex
  - Moduli space very rich, only partially generated by global symmetries $\Rightarrow$ Vortex basically semilocal
  - It can split into M "fractional vortices".
- The rich structure of the moduli space is a consequence of the big degeneracy of the vacuum (much bigger than $U(N)$ case)

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Local Vortices

Reduce the moduli space to the subspace of local vortices:

- Enhancing of normalizable modes:
  - Local vortices has the maximum number of normalizable modes.
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Local Vortices

"Degenerate" rationl map construction for local vortices

- Local vortices corresponds to degenerations of the rational map construction:

$$I^i_{G'}(H_{0,\text{local vortex in } z=z_0}) = (z - z_0)^{n_i/n_0}\, I^i_{\text{vev}}(z)$$

- Only local vortices: complete degeneration.

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$$H_{0,\mathrm{local}} = \begin{pmatrix} (z - z_0)\mathbf{1}_M & 0 \\ \mathbf{B}_{A/S} & \mathbf{1}_M \end{pmatrix}$$

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$$H_0 = G'(N)_{C+F} \begin{pmatrix} (z - a)\mathbf{1}_M & 0 \\ 0 & \mathbf{1}_M \end{pmatrix} G'(N)^{\dagger}_{C+F}$$

$$\mathcal{M}_{k=1} \simeq \mathbb{C} \times G'_{\mathrm{C+F}}/H_{\mathrm{C+F}} = \mathbb{C} \times [SO, USp(2M)]/U(M),$$

Isotropic Grassmanians

Exceptional Groups

E_6:
There is a rank-3 symmetric tensor: Γ_{ijk}. The conditions on the moduli matrix is

$$\Gamma_{i_1 i_2 i_3} (H_0)^{i_1}{}_{j_1} (H_0)^{i_2}{}_{j_2} (H_0)^{i_3}{}_{j_3} \sim \Gamma_{j_1 j_2 j_3} z^k,$$

and the $U(1)$ winding number is quantized as $\nu = k/3$.
E_7:
There are 2 invariant tensors: d_{ijkl} and f_{ij} respectively of rank 4 and 2. The moduli matrix is constrained:

$$d_{i_1 i_2 i_3 i_4} (H_0)^{i_1}{}_{j_1} (H_0)^{i_2}{}_{j_2} (H_0)^{i_3}{}_{j_3} (H_0)^{i_4}{}_{j_4} \sim d_{j_1 j_2 j_3 j_4} z^{2k}$$
$$f_{i_1 i_2} (H_0)^{i_1}{}_{j_1} (H_0)^{i_2}{}_{j_2} \sim f_{j_1 j_2} z^k,$$

and the vortices are quantized in half integers: $\nu = k/2$.

Conclusion and further developments

Conlcusions

- We have given a general prescription to construct 1/2 BPS vortex solutions in a very wide class of non-Abelian gauge theories.
- It can be easily generalized to any $\mathcal{N} = 2$ gauge theory (arbitrary gauge groups, matter content, representations...).

Further developments

- Work out the prescription in concrete models;
- extend the formalism to other kind of solitons (monopoles, domain walls) and 1/4-1/8 BPS composite-solitons (monopoles confined by vortices, vortex-domain wall junctions);
- Non trivial checks to n-A dualities which relate different gauge groups ($SO \Leftrightarrow USp$)

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