Statistical QCD with non-positive measure

Kim Splittorff

with: Jac Verbaarschot

James Osborne

Gernot Akemann

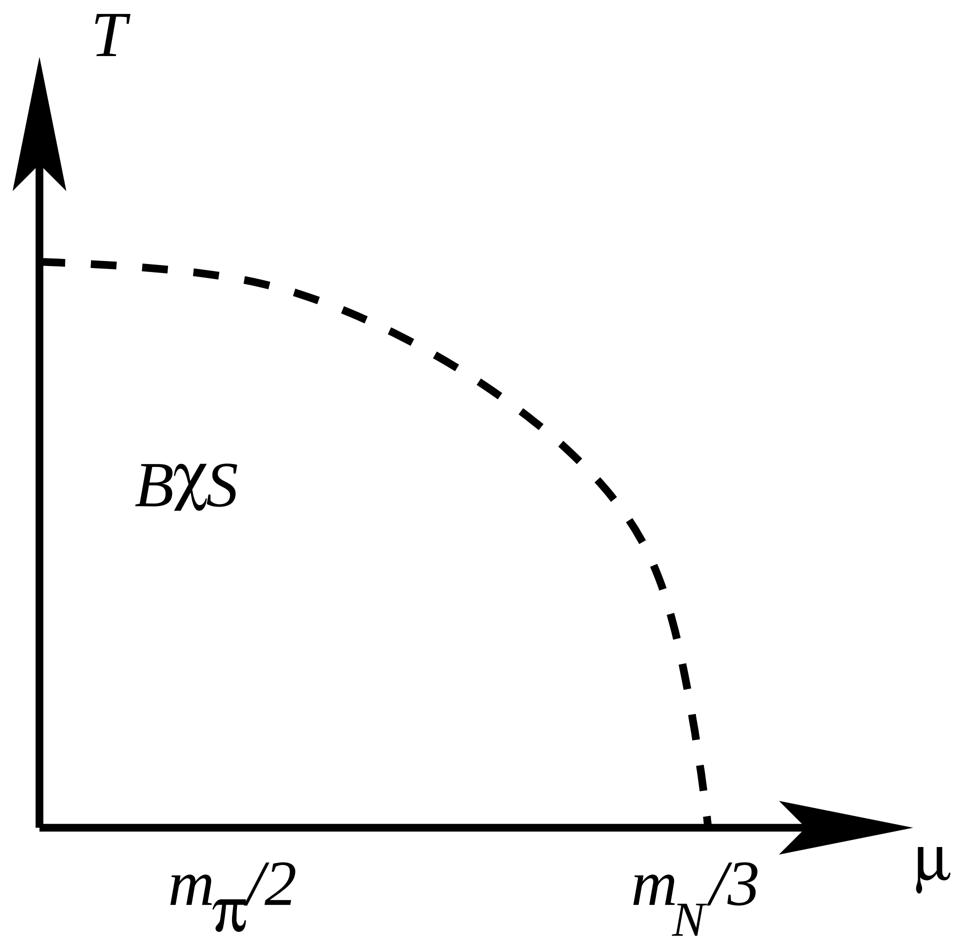
Niels Bohr Institute

What QCD at non zero chemical potential

Problem The Sign problem

Here Chiral condensate, Dirac spectrum and Sign Problem

The Big Picture



de Forcrand Philipsen JHEP 0701 (2007) 077

McLerran Pisarski NPA 796 (2007) 83

The sign problem

$$Z_{N_f=2} = \int dA \, \det{}^2(D + \mu\gamma_0 + m) \, e^{-S_{\rm YM}}$$

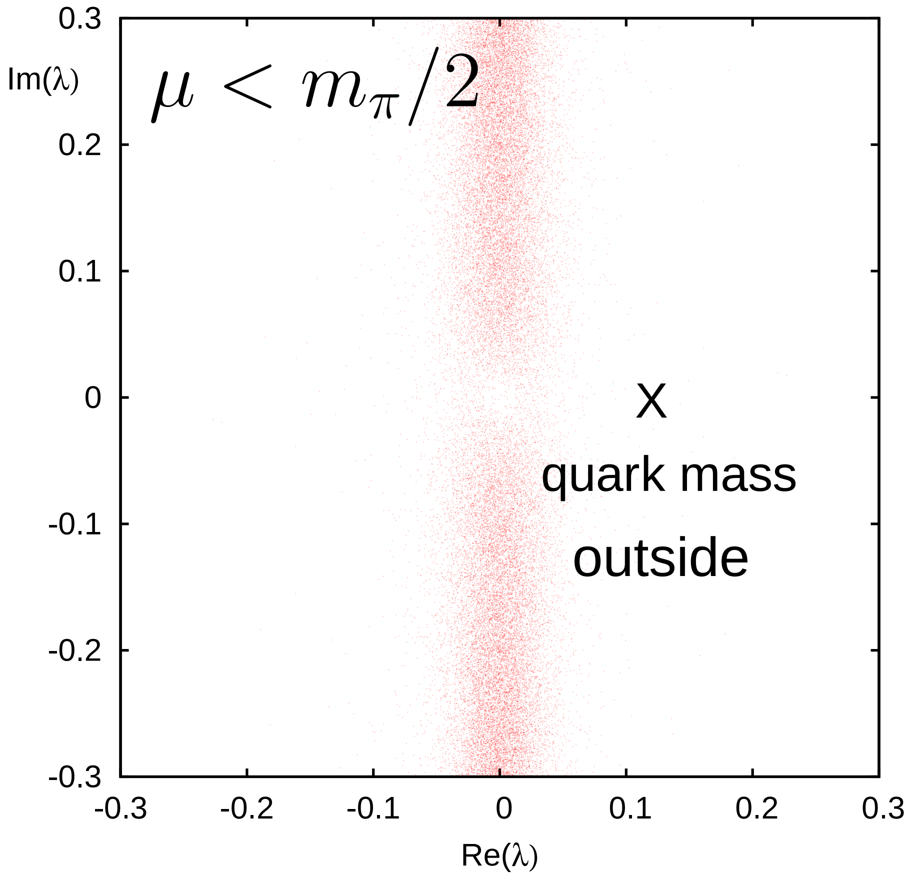
Anti Hermitian

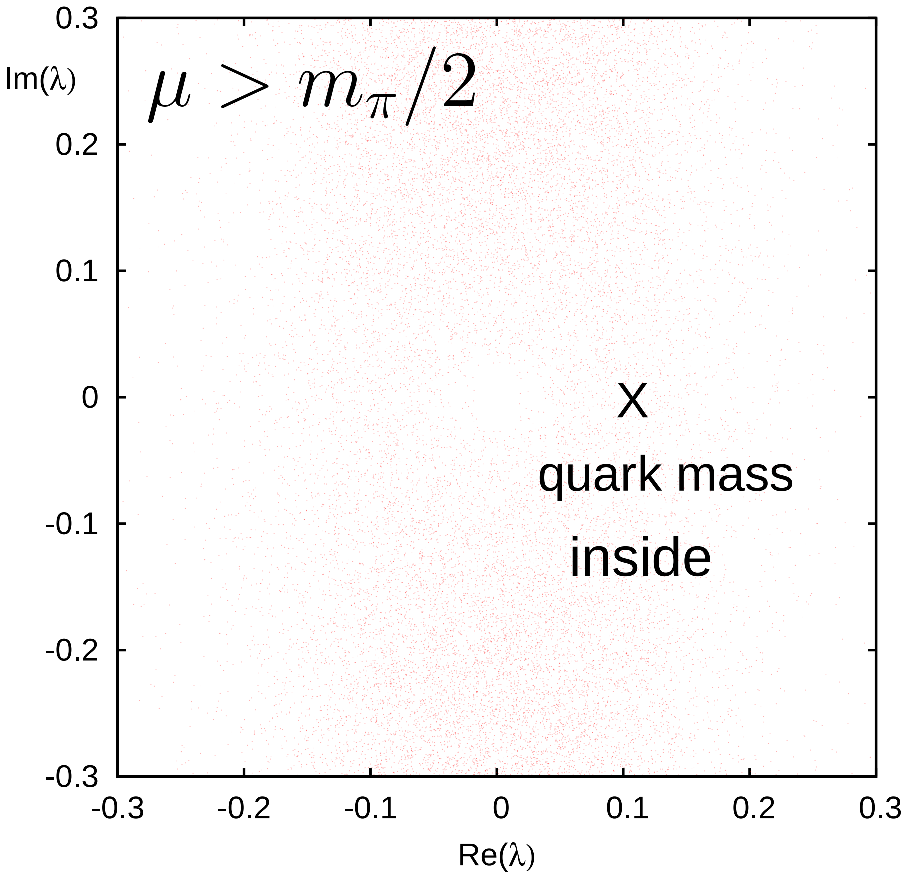
Hermitian

$$\det{}^2(D + \mu\gamma_0 + m) = |\det(D + \mu\gamma_0 + m)|^2 e^{2i\theta}$$

The measure is not real and positive

Quark mass & the eigenvalue distribution



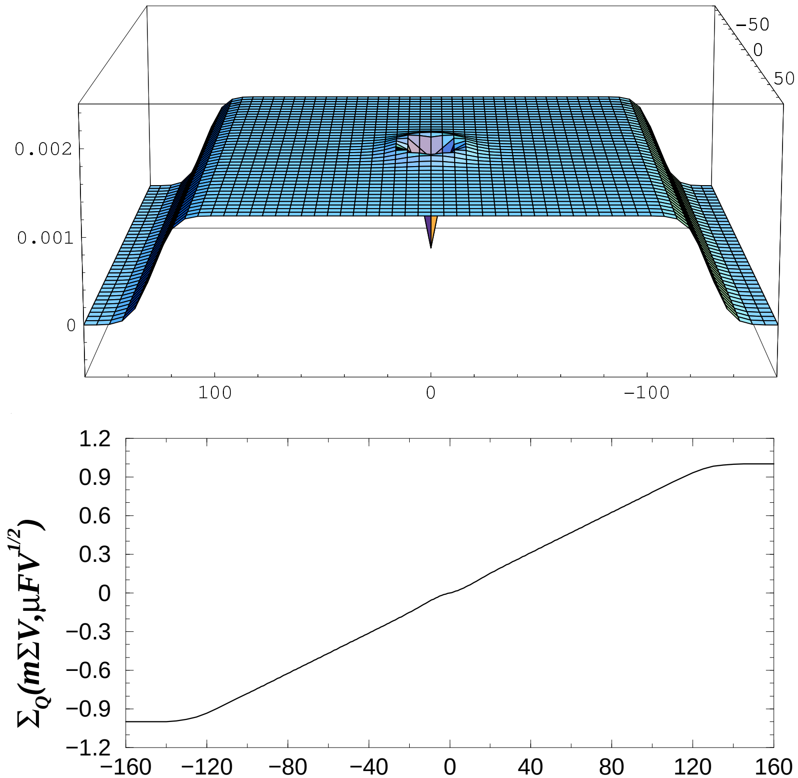


$$(D + \mu\gamma_0)\psi_k = z_k\psi_k \qquad \det(D + \mu\gamma_0 + m) = \prod_k z_k + m$$

Bloch Wettig Lattice 2006 Gibbs PRINT-86-0389

Davies Klepfish PLB 256 (1991) 68 Lombardo Kogut Sinclair PRD 54 (1996) 2303

The quenched chiral condensate

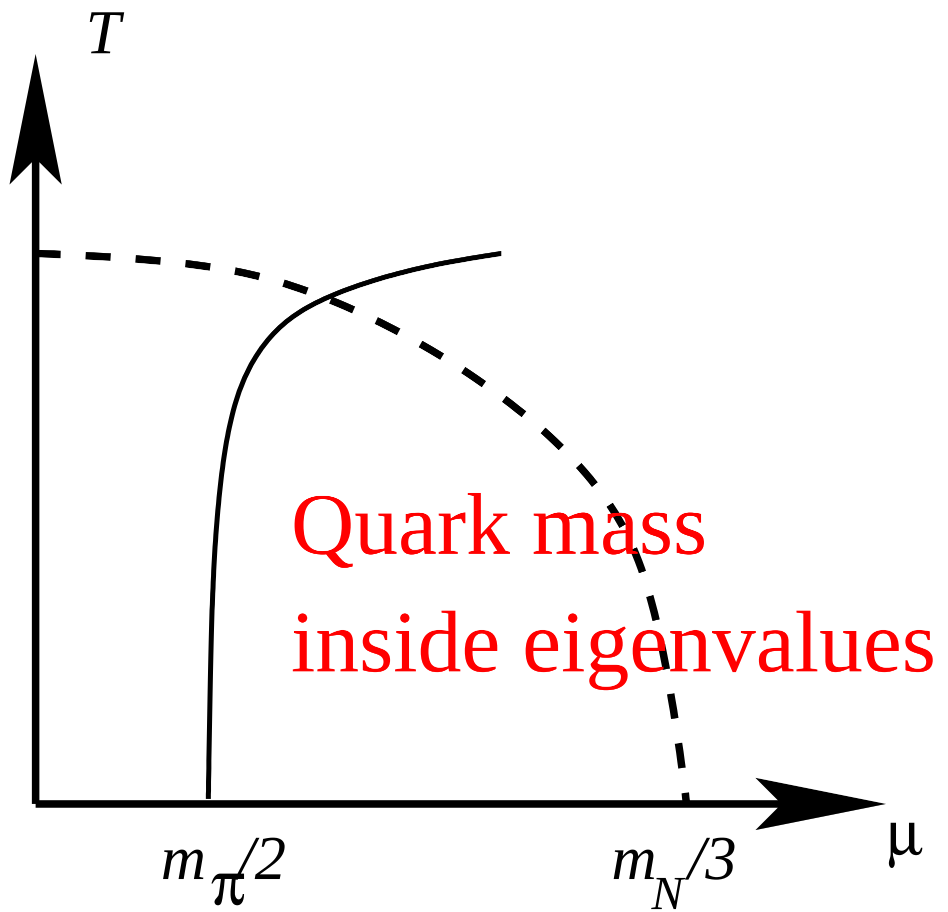


Electrostatic analogy:

Eigenvalues = charges, quark mass = test charge

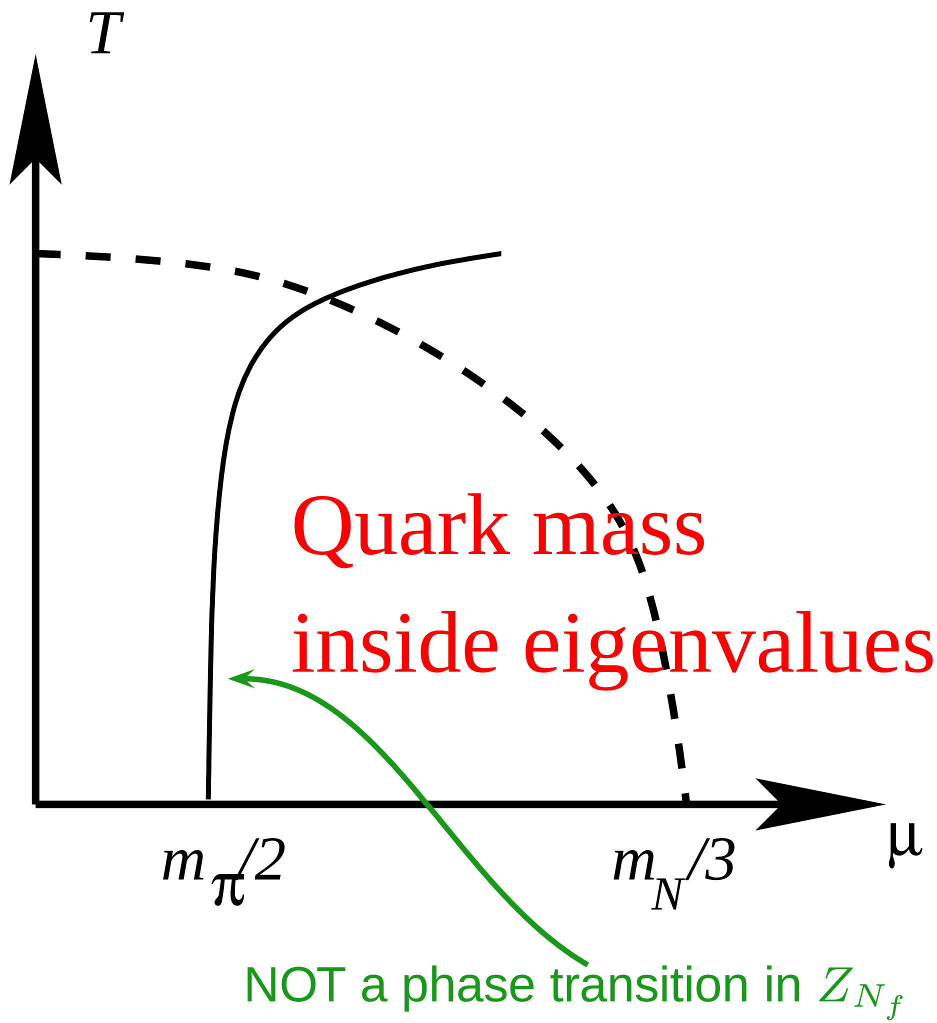
Barbour et al. NPB 275 (1986) 296

The big picture



Chiral condensate is independent of μ at $T = 0$ and $\mu < m_N/3$

The big picture



Chiral condensate is independent of μ at $T = 0$ and $\mu < m_N/3$

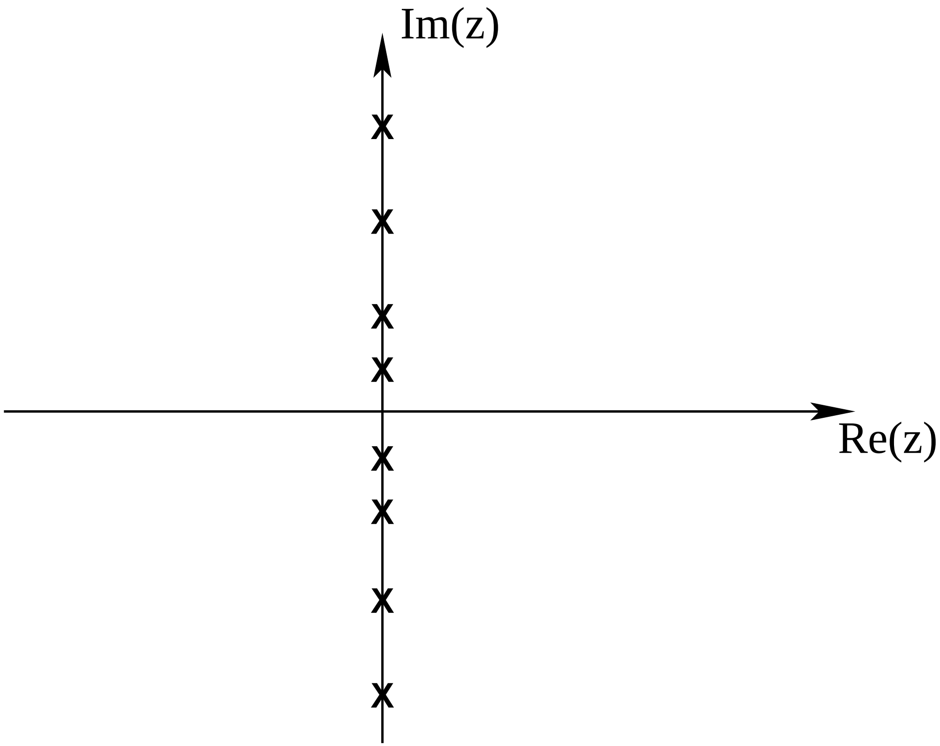
The Silver Blaze Problem

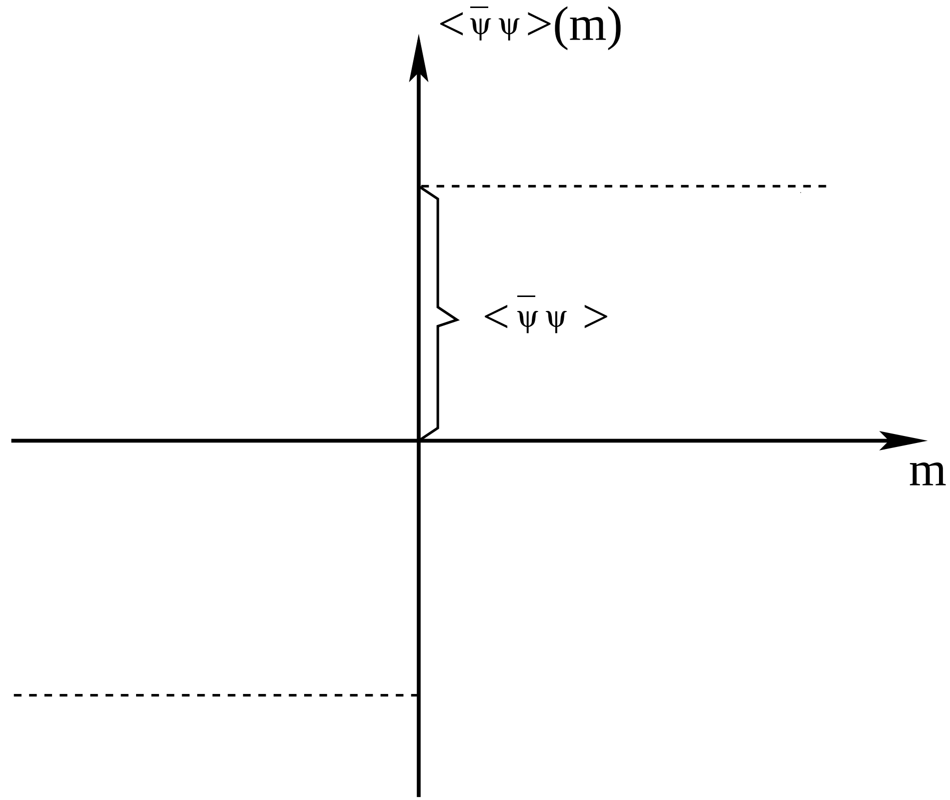
Sir Arthur Conan Doyle *The Memoirs of Sherlock Holmes: Silver Blaze*

Thomas D . Cohen PRL (2003) 222001

$\mu = 0$

Banks Casher

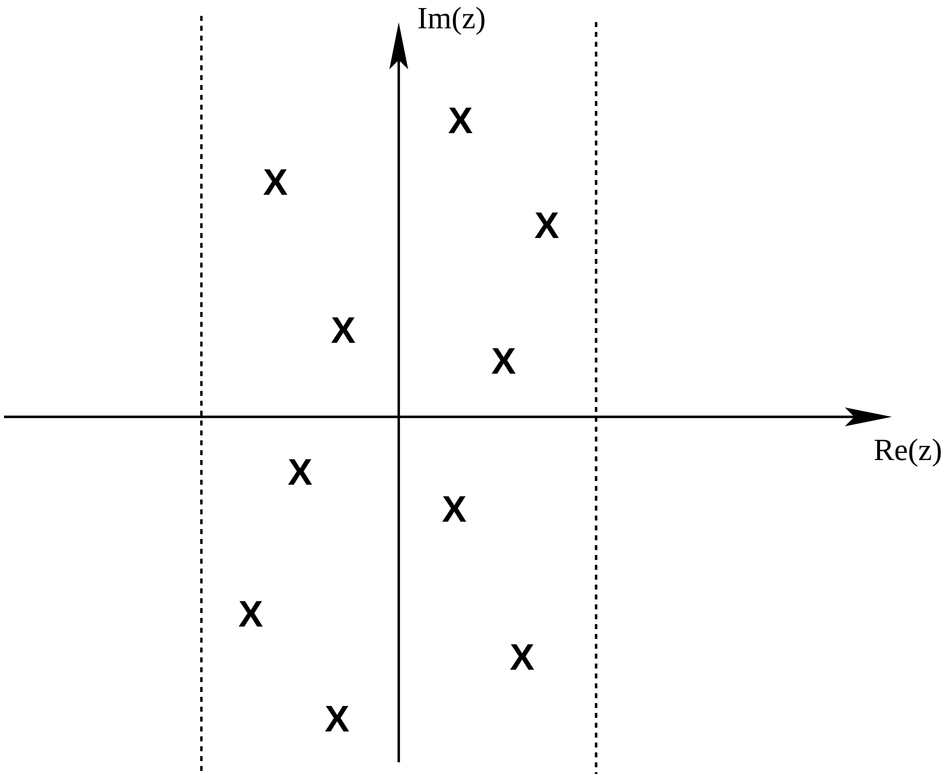


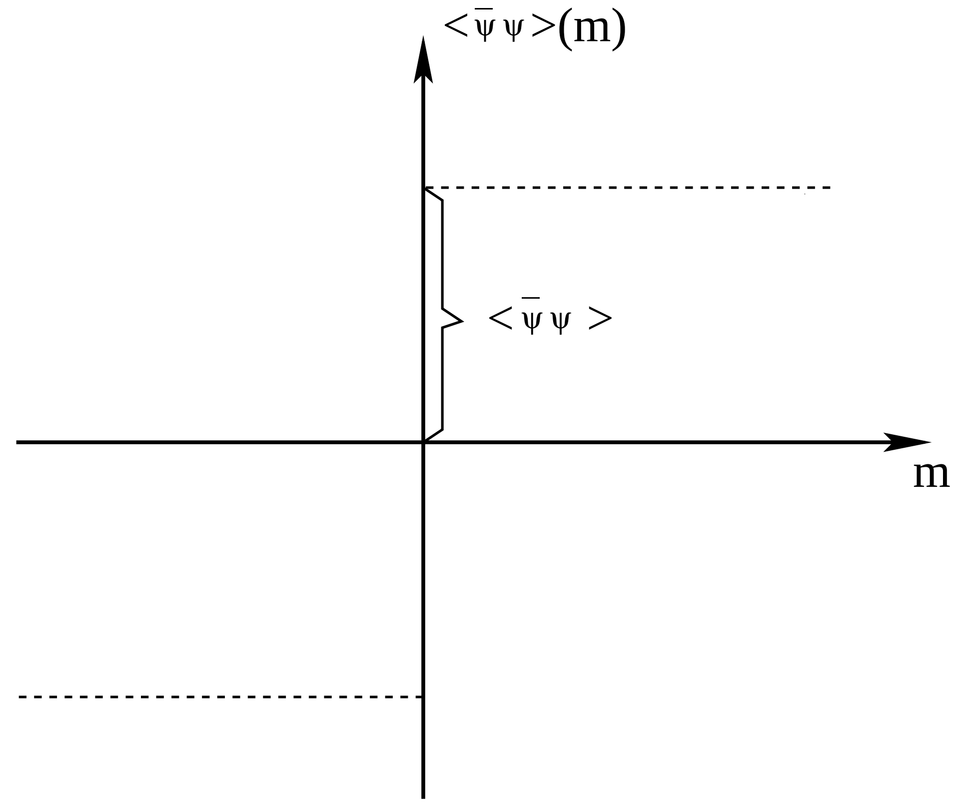


$$\boxed{\langle\bar{\psi}\psi\rangle = \frac{\pi}{V}\rho(0)}$$

Banks Casher NPB 169 (1980) 103

$\mu \neq 0$ *The silver blaze problem*





Eigenvalues move into the complex plane
the discontinuity of the chiral condensate remains

Barbour et al. NPB 275 (1986) 296
Gibbs PLB 182 (1986) 369

Cohen PRL 91 (2003) 222001

We need — Microscopic-regime of QCD

The eigenvalue density $z\langle\bar{\psi}\psi\rangle \ll \frac{1}{\sqrt{V}}$

SBχS — The basic assumption

Chiral limit $m\langle\bar{\psi}\psi\rangle \ll \frac{1}{\sqrt{V}}$

Small chemical potential $\mu^2 F_\pi^2 \ll \frac{1}{\sqrt{V}}$

Notice $\mu \sim m_\pi$

Gasser, Leutwyler, PLB 184 (1987) 83, PLB 188 (1987) 477

Neuberger, PRL 60 (1988) 889

Leutwyler, Smilga, PRD 46 (1992) 5607

Shuryak, Verbaarschot, NPA 560 (1993) 306

Stephanov PRL 76 (1996) 4472

Akemann PRL 89 (2002) 072002, J.Phys. A36 (2003) 3363

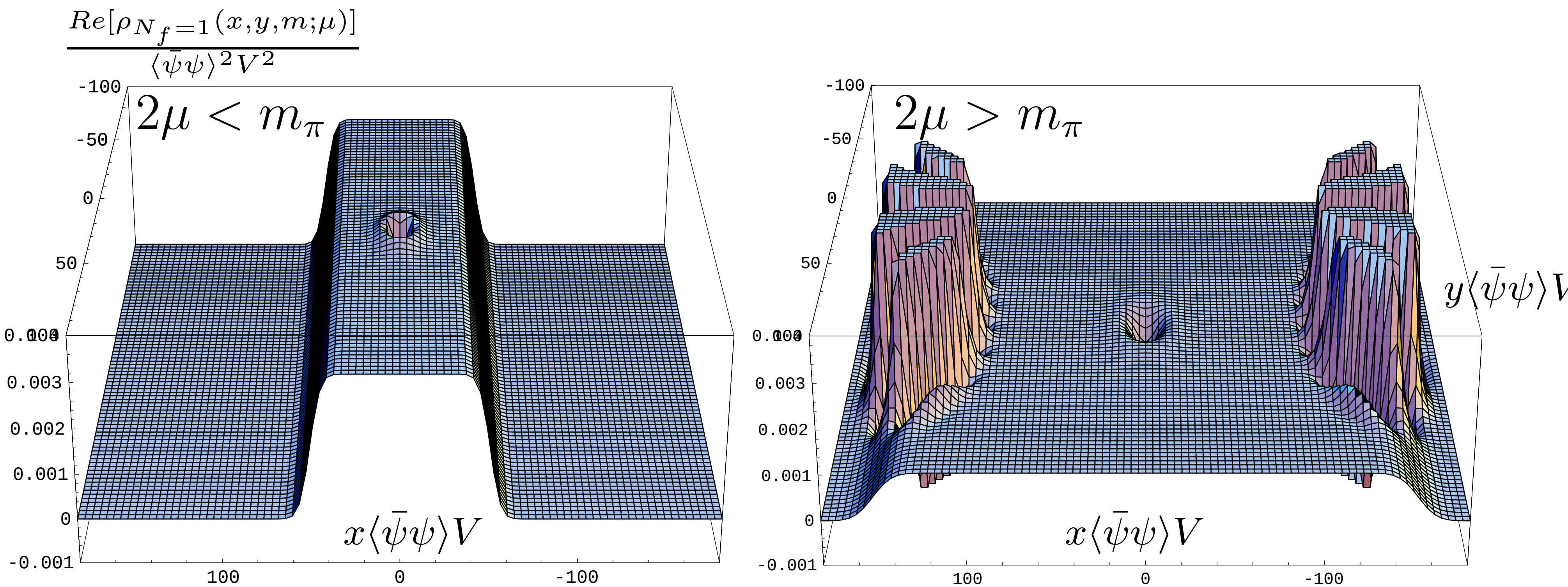
Splittorff, Verbaarschot, NPB 683 (2004) 467

Osborn PRL 93 (2004) 222001

Akemann Osborn Splittorff Verbaarschot NPB 712 (2005) 287

The unquenched eigenvalue density

$m\langle\bar{\psi}\psi\rangle V = 100$ increasing $2\mu^2 F_\pi^2 V$



For $2\mu > m_\pi$ the density is complex and oscillates

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The chiral condensate from the eigenvalue density

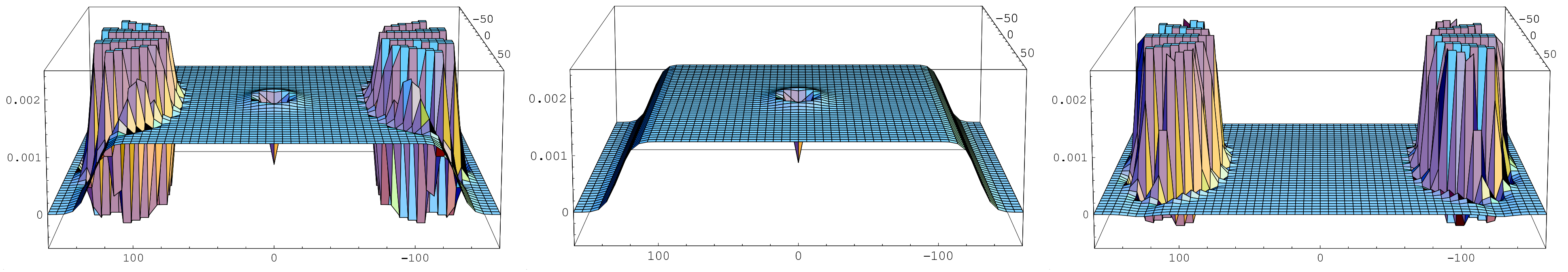
$$\begin{aligned}\langle\bar\psi\psi\rangle(m) &= \frac{1}{V}\partial_m \log Z(m;\mu)\\ &= \frac{1}{V}\int dxdy\,\rho(x,y)\frac{1}{x+iy+m}\end{aligned}$$

$\boxed{\text{The oscillations of the density are responsible for chiral symmetry breaking}}$

Osborn Splittorff Verbaarschot PRL 94 (2005) 202001

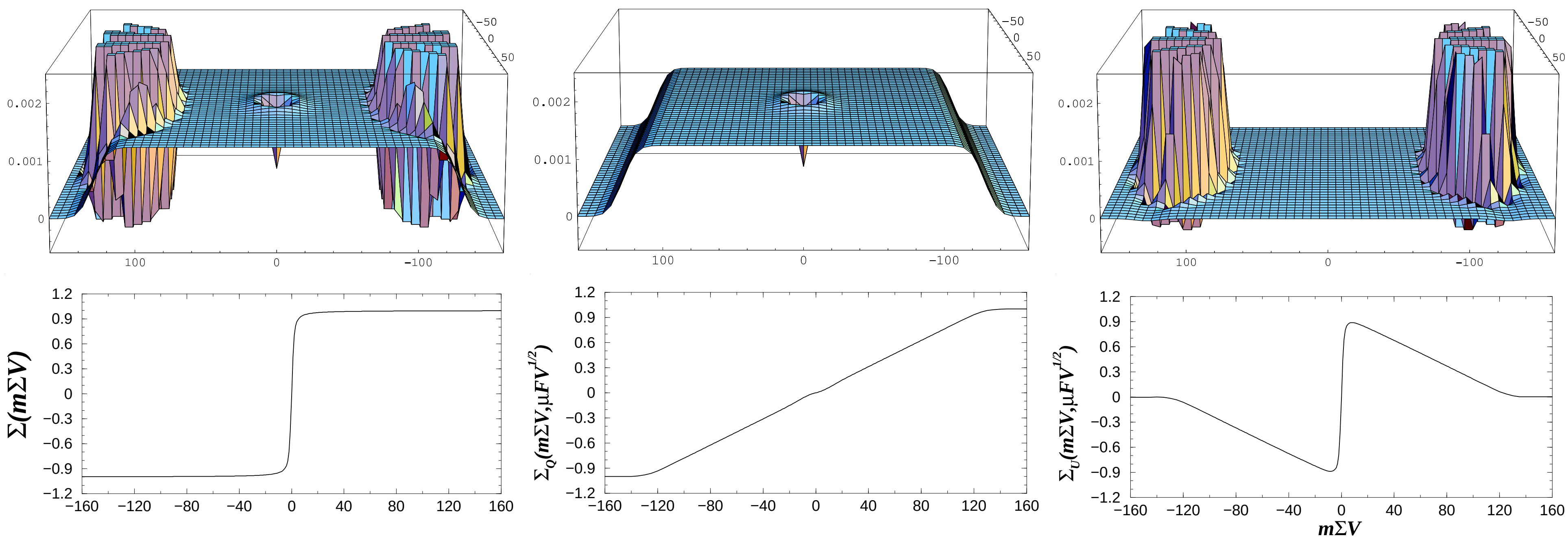
Ravagli Verbaarschot arXiv:0704.1111

The unquenched eigenvalue density



Structure: $\rho_{N_f=1} = \rho_Q + \rho_U$

The unquenched chiral condensate



Structure: $\Sigma_{N_f=1}(m) = \Sigma_Q(m) + \Sigma_U(m)$

Banks-Casher $\mu = 0$

Accumulation of eigenvalues on the y-axis is responsible for chiral symmetry breaking

OSV $\mu \neq 0$

The oscillations of the eigenvalue density are responsible for chiral symmetry breaking

Observation: The complex oscillations of the spectral correlation functions take part on the microscopic scale: period $\sim 1/V$ amplitude $\sim \exp(V)$

Fourier transform of eigenvalue density

$$\rho_{N_f}(x,y) = \left\langle \sum_k \delta(x-x_k)\delta(y-y_k) \right\rangle_{N_f} \qquad \tilde{\rho}_{N_f}(x,t) \equiv \int_{-\infty}^{\infty} dy e^{-iyt} \rho_{N_f}(x,y)$$

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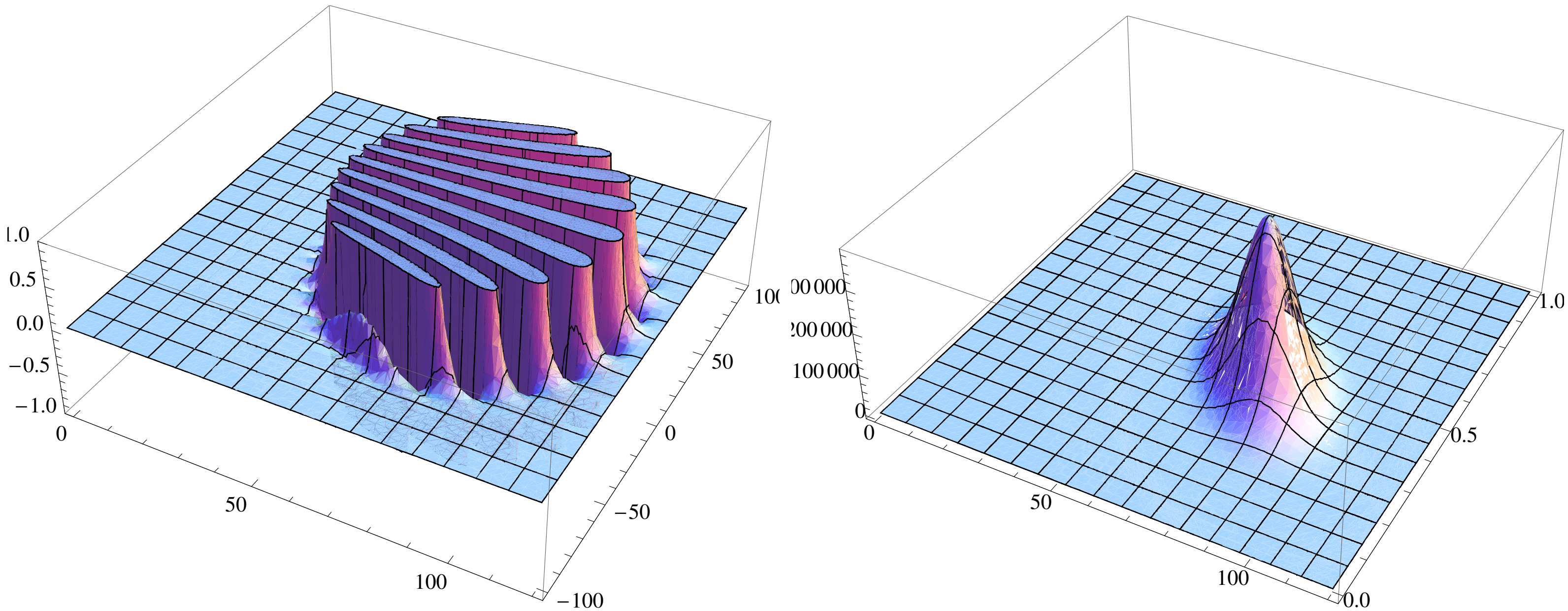
$$\Sigma_{N_f}(m) = \int dx dy \, \frac{\rho_{N_f}(x,y)}{x + iy - m} = \int dx dt \tilde{\rho}(x,t) e^{t(m-x)} \theta(x - m)$$

Notice: the integrand is positive if $\tilde{\rho}(x,t) : \mathbb{R}^2 \to \mathbb{R}_+$.

Oscillating part - Asymptotic limit

$$\rho_U(x,y) = \frac{1}{4\pi\mu^2} e^{V(m-x-iy)(m+3x-8\mu^2-iy)/(8\mu^2)}$$

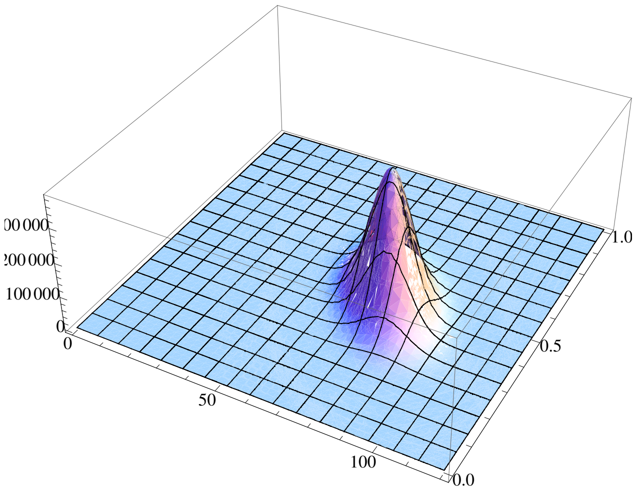


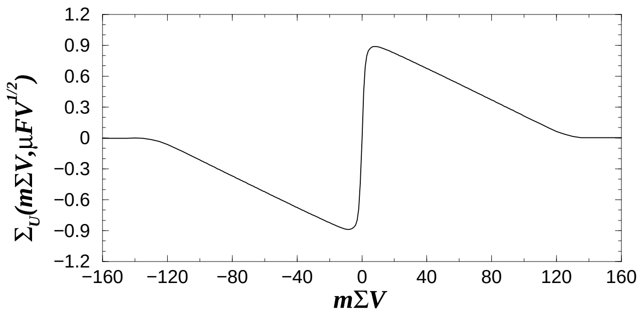
$$\tilde{\rho}_U(x,t) = \frac{1}{\sqrt{2\pi\mu^2}} e^{V(-2\mu^2(t-1)^2 - x^2/2\mu^2 - mt - (t-2)x)}$$

Note $\tilde{\rho}(x,t) : \mathbb{R}^2 \to \mathbb{R}_+$!

Asymptotic limit part 2

$\tilde{\rho}_U(x,t)$





$$\Sigma(m) = \frac{1}{\sqrt{2\pi\mu^2}}\int_{-2\mu^2}^{2\mu^2} dx \int_{-\infty}^{\infty} dt e^{-2\mu^2(t-1)^2 - x^2/2\mu^2 - 2(t-1)x}\theta(x-m)$$

As expected ($0 < m < 2\mu^2$)

$$\Sigma(m) = \frac{1}{2\mu^2}(2\mu^2 - m)$$

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Exact microscopic result

Exact microscopic partition function and condensate

$$Z_{N_f=1}(m) = I_\nu(m) \qquad \Sigma_{N_f=1}(m) = \frac{I'_\nu(m)}{I_\nu(m)}$$

$$\Sigma_{N_f}(m) = \int dxdy \, \frac{\rho_{N_f}(x,y)}{x+iy-m}$$

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Exact microscopic partition function and condensate

$$Z_{N_f=1}(m) = I_\nu(m) \qquad \Sigma_{N_f=1}(m) = \frac{I'_\nu(m)}{I_\nu(m)}$$

$$\Sigma_{N_f}(m) = \int dx dy \, \frac{\rho_{N_f}(x,y)}{x+iy-m}$$

Exact microscopic eigenvalue density

$$\begin{aligned} \rho^{(\nu)}_{N_f=1}(\hat{z},\hat{z}^*) \quad &= \quad \frac{|\hat{z}|^2}{2\pi\hat{\mu}^2} K_\nu\left(\frac{|\hat{z}|^2}{4\hat{\mu}^2}\right) \mathrm{e}^{-\frac{\hat{z}^2+\hat{z}^{*\,2}}{8\hat{\mu}^2}} \\ &\times \left(\int_0^1 dt\; t\; \mathrm{e}^{-2\hat{\mu}^2 t^2} I_\nu(\hat{z}t) I_\nu(\hat{z}^* t) - \frac{I_\nu(\hat{z})}{I_\nu(\hat{m})} \int_0^1 dt\; t\; \mathrm{e}^{-2\hat{\mu}^2 t^2} I_\nu(\hat{m}t) I_\nu(\hat{z}^* t)\right) \end{aligned}$$

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Chiral Random Matrix Theory

The partition function

$$\mathcal{Z}_N^{N_f}(m;\mu) \equiv \int d\Phi d\Psi \, e^{-N \mathrm{Tr} \Phi^\dagger \Phi} e^{-N \mathrm{Tr} \Psi^\dagger \Psi} \det{}^{N_f}(\, \mathcal{D}(\mu) + m \,)$$

The *Dirac operator* is given by

$$\mathcal{D}(\mu) = \begin{pmatrix} 0 & i\Phi + \mu\Psi \\ i\Phi^\dagger + \mu\Psi^\dagger & 0 \end{pmatrix}$$

Φ and Ψ are complex $(N+\nu) \times N$ matrices

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Chiral Random Matrix Theory

The partition function - eigenvalue representation

$$\mathcal{Z}_N^{N_f}(m;\mu) = \int \prod_{k=1}^{N} d^2 z_k \left|\Delta_N(\{z_l^2\})\right|^2 |z_k|^{2\nu+2}$$
$$\times K_\nu\left(\frac{N(1+\mu^2)}{2\mu^2}|z_k|^2\right) e^{-\frac{N(1-\mu^2)}{4\mu^2}(z_k^2+{z_k^*}^2)}(m^2 - z_k^2)^{N_f}$$

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The complex orthogonal polynomial method

$$\mathcal{Z}_N^{N_f=1}(m;\mu) = m^\nu p_N(m;\mu)$$

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The complex orthogonal polynomial method

$$\mathcal{Z}_N^{N_f=1}(m;\mu) \quad = \quad m^\nu p_N(m;\mu)$$

$$p_k(z;\mu) = \left(\frac{1-\mu^2}{N}\right)^k k! L_k^\nu\left(-\frac{Nz^2}{1-\mu^2}\right)$$

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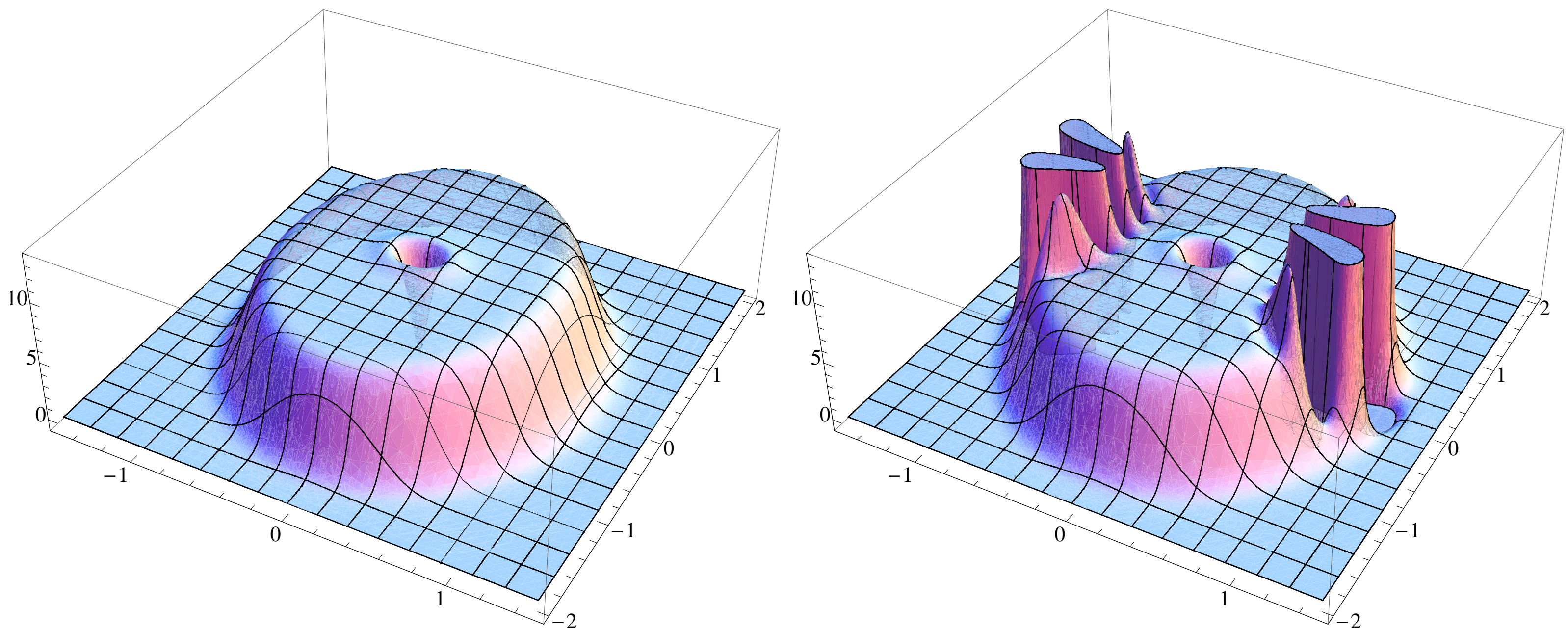
OSV relation at finite N

$$\mathcal{Z}_N^{N_f=1}(m) \quad = \quad m^\nu p_N(m) \qquad\qquad \Sigma_N^{N_f=1}(m) = \frac{dp_N(m)/dm}{p_N(m)} + \frac{\nu}{m}$$

The eigenvalue density at $N = 20$



Osborn Splittorff Verbaarschot arXiv:0805.1303

OSV relation at finite N

$$\mathcal{Z}_N^{N_f=1}(m) \quad = \quad z^\nu p_N(m) \qquad\qquad \Sigma_N^{N_f=1}(m) = \frac{dp_N(m)/dm}{p_N(m)} + \frac{\nu}{m}$$

The eigenvalue density at finite N from the orthogonal polynomials

$$\rho_N^{N_f=1} = 2w(z, z^*; \mu) \sum_{k=0}^{N-1} \frac{p_k(z^*)(p_k(z) - p_N(z)p_k(m)/p_N(m))}{r_k}$$

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The integral

$$\Sigma_N(m) \quad = \quad \int dxdy \frac{\rho_N^{N_f=1}(x,y)}{x+iy+m} = \frac{dp_N(m)/dm}{p_N(m)} + \frac{\nu}{m}$$

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$$\Sigma_N(m) \quad = \quad \frac{1}{V}\int dxdy\frac{\rho_N^{N_f=1}(x,y)}{x+iy+m} = \frac{dp_N(m)/dm}{p_N(m)} + \frac{\nu}{m}$$

can be done *using the orthogonality of the polynomials*

$$\int_{\mathbb{C}} d^2z\ w(z,z^*;\mu)\ p_k(z;\mu)\ p_l(z;\mu)^* \ \ = \ \delta_{kl}\ r_k^\nu$$

Osborn Splittorff Verbaarschot arXiv:0805.1303

Conclusions

Chiral symmetry breaking linked to oscillations at the microscopic scale

Shows the numerical difficulties in dealing with the sign problem

Complete proof from finite N chiral Random Matrix Theory

At finite N cancellations are due to orthogonality of the polynomials

Conclusions

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Complete proof from finite N chiral Random Matrix Theory

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Conjecture: The complex oscillations always have a microscopic period

Additional slides

Why can we compute the spectral density ?

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The replica way of writing the eigenvalue density

$$\rho^{N_f}(z, z^*, m; \mu) = \lim_{n\to 0} \frac{1}{\pi n} \partial_{z^*} \partial_z \log \mathcal{Z}_{N_f,n}(m, z, z^*; \mu)$$

generating functionals for the eigenvalue density

$$\mathcal{Z}_{N_f,n}(m, z, z^*; \mu) = \int \mathrm{d}A \det(D_\eta \gamma_\eta + \mu\gamma_0 + m)^{N_f} |\det(D_\eta\gamma_\eta + \mu\gamma_0 + z)|^{2n} \, \mathrm{e}^{-S_{\mathrm{YM}}(A)}$$

Stephanov PRL **76** (1996) 4472

Central observation

The eigenvalue z *and its complex conjugate* z^* *appears as the mass of two conjugate fermions.*

very small eigenvalues $\leftrightarrow$ **very** light quarks

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The eigenvalue z and its complex conjugate z^* appears as the mass of two conjugate fermions.

very small eigenvalues $\leftrightarrow$ **very** light quarks

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$\Rightarrow$ Zero mode of the pions dominates

$$Z_{N_f,n} = \int_{U(N_f+2n)} dU e^{-\frac{V}{4}F_\pi^2\mu^2 \mathrm{Tr}[U,B][U^{-1},B] + \frac{1}{2}m\langle\bar{\psi}\psi\rangle V \mathrm{Tr}(U+U^{-1})}$$

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Stephanov PRL **76** (1996) 4472

The Replica Limit of the Toda Lattice Equation

$$\partial_z \partial_{z^*} \log Z_{N_f,n} = 4zz^* n \frac{Z_{N_f,n+1} Z_{N_f,n-1}}{[Z_{N_f,n}]^2}$$

Take $n \to 0$ in this equation

$$\rho_{N_f}(z, z^*, m; \mu) = 4zz^* \frac{Z_{N_f,n=1}(m, z, z^*; \mu) Z_{N_f,n=-1}(m|z, z^*; \mu)}{[Z_{N_f}(m; \mu)]^2}$$

Problems

Verbaarschot, Zirnbauer, J. Phys. A **18**, 1093 (1985)

Kamenev Mézard J.Phys.A **32** 4373 (1999); PRB **60** 3944 (1999)

Yurkevich, Lerner, PRB **60**, 3955 (1999)

M.R. Zirnbauer, cond-mat/9903338

Solution

Kanzieper, PRL **89**, 250201 (2002)

Splittorff, Verbaarschot, PRL **90**, 041601 (2003)