BOUNDARY VALUE PROBLEMS IN SPACES OF DISTRIBUTIONS ON SMOOTH AND POLYGONAL DOMAINS

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Abstract. We study boundary value problems of the form $-\Delta u = f$ on Ω and $Bu = g$ on the boundary $\partial\Omega$, with either Dirichlet or Neumann boundary conditions, where Ω is a smooth bounded domain in $\mathbb{R}^n$ and the data f, g are *distributions*. This problem has to be first properly reformulated and, for practical applications, it is of crucial importance is to obtain the continuity of the solution u in terms of f and g. For $f = 0$, taking advantage of the fact that u is harmonic on Ω, we provide four formulations of this boundary value problem (one using non-tangential limits of harmonic functions, one using Green functions, one using the Dirichlet-to-Neumann map, and a variational one); we show that these four formulations are equivalent. We provide a similar analysis for $f \neq 0$ and discuss the roles of f and g, which turn to be somewhat interchangeable in the low regularity case. The weak formulation is more convenient for numerical approximation, whereas the non-tangential limits definition is closer to the intuition and easier to check in concrete situations. We extend the weak formulation to polygonal domains using weighted Sobolev spaces. We also point out some new phenomena for the "concentrated loads" at the vertices in the polygonal case.

Contents

Introduction

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$. Motivated in part by some problems in engineering [21], we want to study distributional boundary value problems of the form

$$-\Delta u = f \text{ on } \Omega, \quad Bu = g \text{ on } \partial\Omega, \tag{1}$$

where the data f, g are *distributions* in suitable Sobolev spaces and Bu denotes either the "restriction" of u to the boundary or the "normal derivative" $\partial_\nu u$ of u

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to the boundary (they have to be properly defined). Since u will not be a function, in general, but only a distribution with low regularity, one of the main issues is to make sense of the above boundary value problem.

The distributions f and g are also called *concentrated loads* or *concentrated couples* in the engineering literature if they are given by Dirac distributions (*i.e.,* point measures) or suitable derivatives of such distributions. A motivation for their study comes from problems in Structural Mechanical, where one studies, for instance, forces that are acting on a very small surface and are thus idealized with Dirac distributions. Then $-\Delta$ needs to be replaced with the Lamé (or elasticity) operator whose behavior is rather similar to that of Δ (both Δ and the Lamé operator are strongly elliptic). For this approach to have physical meaning, however, the continuous dependence of u (the solution of Equation (1)) on the data f and g is crucial, because the delta distribution appears in practice as a limit of a suitable sequence of functions. See [21] for a discussion of the continuity issue.

Problems similar to (1) (in spaces of distributions) were studied before, most notably by Lions and Magenes [15], Schechter [19], and Roitberg and his collaborators [18] (and the references therein). The case $f = 0$ was studied also by Seeley [20] and Strichartz [23]. In all these approaches to the distributional boundary value problem (1), one has to first appropriately reformulate it. This problem was studied from a numerical point of view–looking for interior approximations–in [3], where a weak formulation was used. This paper is a sequel of [3], and it helps, among other things, make a connection between the weak formulation of (1) from [3] and some strong formulations of (1) that are closer to the intuition.

The weak formulation is more convenient for numerical approximation, whereas the non-tangential limits definition is closer to the intuition and easier to check in concrete situations. The strong formulations are also closer in form to the case when the data f and g consists of smooth functions. These strong formulations of (1) also suggest a possible treatment of similar boundary value problems on a polygon. Knowing that the weak and strong formulations coincide will help us compare the approximate solutions to the exact solution.

In this paper, we treat the Neumann problem ($Bu = \partial_\nu u$) in detail, and we just mention some of the changes needed to treat the Dirichlet problem ($Bu = u$). If $f = 0$, the solution u of (1) is a harmonic function in any formulation of (1), and hence it is smooth on Ω. This allows us then to provide four approaches (or formulations) to Equation (1). A first formulation is the weak formulation from [3] using duality, which is similar in spirit to the approach of [18] and is reviewed in Section 1. A second approach is using the Neumann function (the Green function for the Dirichlet problem), taking advantage of the smoothness of the Neumann function on the boundary, which allows us to evaluate any distribution on the restriction of the Neumann function to the boundary. A third approach is to restrict u to surfaces $\partial\Omega_\epsilon$ near the boundary $\partial\Omega$, such that $\partial\Omega_\epsilon$ approaches the boundary if $\epsilon \to 0$. Then a result of Seeley guarantees that the restrictions $u|_{\partial\Omega_\epsilon}$ form a convergent family in a *negative order* Sobolev space $H^{-r}(\partial\Omega)$, $r > 0$. The limit will be then the definition of $u|_{\partial\Omega}$. The normal derivative $\partial_\nu u|_{\partial\Omega}$ can be defined similarly. Finally, the last approach is to use Green's representation formula for u in terms of $u|_{\partial\Omega}$ and $\partial_\nu u|_{\partial\Omega}$, which are related by the Dirichlet-to-Neumann map. This is done for the Neumann problem. The Dirichlet problem is completely similar, so we do not treat it in detail, but we nevertheless occasionally mention the changes needed to deal with this problem. The first and last approaches generalize to the case $f \neq 0$.

Let us now briefly describe the contents of the paper. In Section 1, we describe the four approaches mentioned above for the *homogeneous Neumann problem* and we show that they provide the same solution. Since the weak solution (or forumulation) is based on the "inf-sup" condition, we automatically obtain the continuous dependence of u on f and g (see for example [2]). In the second section, we provide a weak formulation of the *inhomogeneous Neumann problem*. The third section provides an alternative approach to the inhomogeneous problem using the method of layer potentials and an extension of our problem to $\mathbb{R}^n$. In section 3, we discuss the case of a half space (the case of a smooth, bounded domain is completely similar), using the second approach (that is, the one based on the Green or Neumann functions). We then point out some similarities and some differences that arise when we want to generalize this approach to an angle (the simplest non-smooth domain). In that case, the Neumann problem with data $\partial_\nu u = g = \delta_0$ ($\delta_0 = \delta$ at the vertex of the angle) can be solved similarly, in spite of the fact that $\partial_\nu u$ *is not defined* at the point 0, because the normal vector is not defined at this non-smooth point. On the other hand, the approach to the Dirichlet problem $u = g = \delta_0$ using Green functions gives $u = 0$ for an acute angle and $u = \infty$ for an obtuse angle. In the last section, we point out an approach to the weak formulation of the Dirichlet and Neumann problems on a curvilinear polygonal domain using weighted Sobolev spaces. The advantage of using weighted Sobolev spaces is that the full shift theorems for these domains are valid (Theorem 4.1) unlike the case of the usual Sobolev spaces. We can thus proceed exactly as in the case of a smooth domain, provided that we use weighted Sobolev spaces in place of the usual Sobolev spaces.

1. The homogeneous Neumann problem

The main result of this section is to establish the equivalence of four formulations of the homogeneous Neumann problem with distributional data. The first formulation is a variational formulation introduced in [3], the second formulation is in terms of the Neumann function, the third formulation is in terms of nontangential limits, and the last formulation is in terms of Green's formula and the Dirichlet-to-Neumann map. The order in which we introduce these formulations is that of increasing levels of abstractness. The last formulation is introduced mostly for technical rather than practical reasons, because it helps us establish the properties and, ultimately, the equivalence of the other three formulations. Yet another formulation is possible using the method of layer potentials used in Section 5. (See [13, 27] for a review of the method of layer potentials.)

1.1. **Sobolev spaces.** Let $\hat{f}(x) = \int_{\mathbb{R}^n} e^{-\imath x\cdot t} f(t)dt$ denote the Fourier transform on $\mathbb{R}^n$. In the following, by $H^s(\mathbb{R}^n)$, $s \geq 0$, we shall denote the usual Sobolev spaces on $\mathbb{R}^n$, namely

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n),\ \hat{f}(\xi)(1 + |\xi|^2)^{s/2} \in L^2(\mathbb{R}^n)\}.$$

Throughout this paper, $\Omega \subset \mathbb{R}^n$ will denote an open, connected, bounded subset with smooth boundary $\partial\Omega$. (That is, Ω is a smooth, bounded domain.) We shall denote by ∂_ν the directional derivative in the direction of the unit outer normal ν to $\partial\Omega$. (The only non-smooth domains considered in this paper are curvilinear polygonal domains in the plane, which will be denoted generically by $\mathbb{P}$.)

If $s \geq 0$, then $H^s(\Omega)$ is the space of restrictions of distributions $u \in H^s(\mathbb{R}^n)$ to Ω. We shall denote by $H_0^s(\Omega) \subset H^s(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H^s(\Omega)$. Our convention is that the *negative order Sobolev space* $H^{-s}(\Omega)$, $s > 0$, is the dual of $H^s(\Omega)$. Then $H^{-s}(\Omega)$ identifies with the distributions in $H^{-s}(\mathbb{R}^n)$ that vanish on $\mathcal{C}_c^\infty(\mathbb{R}^n \smallsetminus \overline{\Omega})$ (that is, distributions with support in $\overline{\Omega}$). If $u \in H^s(\Omega)$ and $v \in H^{-s}(\Omega)$, we shall denote by $\langle v, u\rangle = v(u)$, the value of the functional v on the function u.

The dual of $H_0^s(\Omega)$ will be denoted $H_0^{-s}(\Omega)$. There is a natural, continuous, surjective map (projection)

$$\Pi_0 : H^{-s}(\Omega) \to H_0^{-s}(\Omega) \tag{2}$$

that is dual to the inclusion $H_0^s(\Omega) \to H^s(\Omega)$. Any $u \in H_0^{-s}(\Omega)$ defines a linear map on $H_0^s(\Omega)$, and hence a distribution on Ω since $\mathcal{C}_c^\infty(\Omega) \subset H_0^s(\Omega)$. This is consistent with [18, 19]. The Sobolev spaces $H^s(\partial\Omega)$, $s > 0$, on the boundary can be defined by restriction, as the boundary values of the harmonic functions $u \in H^{s+1/2}(\Omega)$. For $s < 0$ we define the space $H^s(\partial\Omega)$ as the dual of $H^{-s}(\partial\Omega)$ (with pivot $L^2(\Omega)$).

We note, however, that in [6, 20, 23, 25], a different definition of the "negative order Sobolev spaces" $H^{-s}(\Omega)$ is used ($H^{-s}(\Omega) = H_0^{-s}(\Omega)$ in those papers).

Let $g \in H^s(\partial\Omega)$, with $s \in \mathbb{R}$ arbitrary. Then g is a distribution on $\partial\Omega$, *i.e.,* a continuous linear map $\mathcal{C}^\infty(\partial\Omega) \to \mathbb{C}$. We shall denote by $\langle g, \phi\rangle_{\partial\Omega} \in \mathbb{C}$ the value of g on the smooth function ϕ. If $\langle g, 1\rangle_{\partial\Omega} = 0$, then we shall say that g *has mean zero*. We shall endow $\partial\Omega$ with the induced surface measure dS.

We shall denote by $(u, v) := \int_\Omega u(x)\overline{v(x)}dx$ the L^2–inner product on Ω and by $(u, v)_{\partial\Omega} := \int_{\partial\Omega} u(x)\overline{v(x)}dS(x)$ the L^2–inner product on $\partial\Omega$. Both are linear in the first variable and conjugate linear in the second variable. Also, we shall denote by ∂_ν the derivative in the direction of the *outer* normal unit vector to the boundary of Ω.

1.2. **The Neumann problem.** We shall consider the *Neumann problem*

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega, \\ \ \ \partial_\nu u = g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

with $g \in H^s(\partial\Omega)$, $s \in \mathbb{R}$. If $s > 0$, it is known that this boundary value problem has a unique solution $u \in H^{s+3/2}(\Omega)$ satisfying $\langle u, 1\rangle := \int_\Omega u(x)dx = 0$ for any g such that $\langle g, 1\rangle_{\partial\Omega} := \int_{\partial\Omega} g(x)dS(x) = 0$ [6, 7, 25]. (The last two conditions will be referred to as the *vanishing mean conditions.*) Moreover, there is a constant $C_{\Omega,s}$ depending only on Ω and $s > 3/2$ such that, under the conditions above, we have

$$\|u\|_{H^{s+3/2}(\Omega)} \leq C_{\Omega,s}\|g\|_{H^s(\partial\Omega)}. \tag{4}$$

Any solution $u \in H^{3/2+s}(\Omega)$, $s > 0$, of Equation (3) (homogeneous Neumann problem), will be called a *classical solution* of (3).

The main goal of this section is to study several generalizations of the homogeneous Neumann problem (3) and of the above results to the case when $g \in H^{-s}(\partial\Omega)$ with $s > 3/2$. We shall show again that a solution $u \in H^{-s+3/2}(\Omega)$ exists and is unique (under the vanishing integrals conditions) and satisfies an analogue of the estimate of Equation (4). One of the main issues is to make sense of Equation (3) for $s < 0$. To this end, we provide four equivalent formulations of the homogeneous Neumann problem (3).

1.3. **The weak formulation.** Our first formulation of the homogeneous Neumann problem (3) is a *weak formulation*. This weak formulation of the homogeneous Neumann problem was introduced earlier in [3], using the "inf–sup" condition and an idea of Roitberg and his collaborators [18]. We now recall this weak formulation and some additional results from [3].

For our proof of the existence of the weak solution, we need the well posedness of the classical Neumann problem and then we proceed by duality. For this reason, we shall need to assume that $s > 3/2$ in order to prove the existence of the weak solution. Better estimates on the regularity of the solution of the homogeneous Neumann problem (3) will follow from the following sections, once we shall establish the equivalence of our four formulations of the homogeneous Neumann problem (Theorems 1.12 and 1.15).

Let us define $\tilde{H}^{3/2-s}(\Omega) := H^{3/2-s}(\Omega) \oplus H^{1-s}(\partial\Omega)$, where $s \in \mathbb{R}$. Intuitively, the second component u_1 of an element $u = (u_0, u_1)$ in $\tilde{H}^{3/2-s}(\Omega)$ should be thought of as some sort of trace of u at the boundary. We then define

$$\begin{aligned} &\tilde{B} : \tilde{H}^{3/2-s}(\Omega) \times H^{1/2+s}(\Omega) \to \mathbb{C} \quad \text{by} \\ &\tilde{B}(u, v) = -\langle u_0, \Delta v\rangle + \langle u_1, \partial_\nu v\rangle_{\partial\Omega}, \quad \text{where} \\ &\tilde{u} = (u_0, u_1) \in \tilde{H}^{3/2-s}(\Omega). \end{aligned} \tag{5}$$

With these preliminaries, we can now introduce weak solutions of Equation (3) for $g \in H^{-s}(\partial\Omega)$.

Definition 1.1. Let $g \in H^{-s}(\partial\Omega)$, $s \in \mathbb{R}$. We say that $u = (u_0, u_1) \in \tilde{H}^{3/2-s}(\Omega)$ satisfies Equation (3) in *weak sense* (or that *u is a weak solution* of the Equation (3)) if

$$\tilde{B}(u, v) = \langle g, v\rangle_{\partial\Omega}$$

for all $v \in \mathcal{C}^\infty(\overline{\Omega})$.

Remark 1.2. Similarly, the equation $\tilde{B}(u, v) = \langle f, v\rangle + \langle g, v\rangle_{\partial\Omega}$ provides a weak formulation of the inhomogeneous Neumann problem (1) (with $Bu = \partial_\nu u$). See Definition 2.1.

Let $X \subset \tilde{H}^{3/2-s}(\Omega) := H^{3/2-s}(\Omega) \oplus H^{1-s}(\partial\Omega)$ consist of the pairs $u = (u_0, u_1)$ satisfying $\langle u_0, 1\rangle = 0$. In fact, any linear condition defining a codimension one subspace $X \subset \tilde{H}^{3/2-s}(\Omega)$ that ensures $(1, 1) \notin X$ will work, in [3], the condition $\langle u_0, 1\rangle + \langle u_1, 1\rangle_{\partial\Omega} = 0$ was used. Also, let $Y \subset H^{1+k}(\Omega)$ consist of the functions V such that $\langle V, 1\rangle := \int_\Omega V(x)dx = 0$. Then the restriction of the form $\tilde{B}$ of Equation (5) to $X \times Y$ satisfies the "inf–sup" condition for $s > 3/2$ (see [2] for details on the "inf–sup" condition). This was proved in [3], Proposition 4.8, using the usual solvability results (*i.e.,* well-posedness) for the Neumann problem. We therefore obtain that the homogeneous Neumann problem has a weak solution, which we shall denote by $u = (u_W, u_1)$ in what follows. This leads to the following theorem from [3].

Theorem 1.3. *Let $s > 3/2$. Let $g \in H^{-s}(\partial\Omega)$ satisfy $\langle g, 1\rangle_{\partial\Omega} = 0$. Then there exists $u = (u_W, u_1) \in \tilde{H}^{3/2-s}(\Omega)$ satisfying the Equation (1) in weak sense. This solution is uniquely determined if $\langle u_W, 1\rangle = 0$ and then it satisfies*

$$\|u_W\|_{H^{3/2-s}(\Omega)} + \|u_1\|_{H^{1-s}(\partial\Omega)} \le C_{\Omega,s}\|g\|_{H^{-s}(\partial\Omega)},$$

for a constant that depends only on Ω and s.

We continue with some simple remarks from [3].

Remark 1.4. Let $u_0 \in H^r(\Omega)$, $r > 3/2$, and $u_1 := u_0|_{\partial\Omega} \in H^{r-1/2}(\partial\Omega)$ be the restriction of u_0 to the boundary. Then, for any $s > 3/2$, (u_0, u_1) is a weak solution of (1) if, and only if, it is a classical solution of (1). This gives the following.

Let $(u_W, u_1) \in \tilde{H}^{3/2-s}(\Omega)$ be a weak solution of the homogeneous Neumann problem (3) in the sense of the above definition with $s > 3/2$. If, in fact, $g \in H^r(\partial\Omega)$, $r > 0$, then it follows from the uniqueness of the weak solutions (up to a constant) that u_W is a classical solution of the homogeneous Neumann problem (3) and that u_1 is the restriction of u_W to the boundary.

Remark 1.5. If $u = (u_W, u_1)$ is a solution of Equation (3) in weak sense (Definition 1.1) then $\Delta u_0 = 0$ in distribution sense and u_1 is uniquely determined by u. In particular, $u_W \in \mathcal{C}^\infty(\Omega)$. Moreover, if $v = (v_W, v_1)$ is another solution of Equation (1), then $u_W - v_W = c$ and $u_1 - v_1 = c$, where c is a constant.

The last remark implies, in particular, that if (u_W, u_1) and (u_W, u_1') are two weak solutions of Equation (3), then $u_1' = u_1$. Therefore, it is justified to say that $u_W \in H^{3/2-s}(\Omega)$ is *a weak solution* of Equation (3) if there exists $u_1 \in H^{1/2-s}(\partial\Omega)$ such that $u = (u_W, u_1)$ is a weak solution of that equation.

Remark 1.6. For the Dirichlet problem we do not have to consider proper subspaces X and Y, see [4] for details. The Dirichlet problem in weighted Sobolev spaces on curvilinear polygonal domains is discussed in Section 4.

1.4. **Strong formulations of the homogeneous Neumann problem.** The other three formulations of the homogeneous Neumann problem (Equation (3)) for $g \in H^{-s}(\Omega)$ will be called *strong formulations* of the homogeneous Neumann problem. We now introduce these remaining three formulations of the homogeneous Neumann problem (using the Neumann function, non-tangential limits, and the Dirichlet-to-Neumann map, respectively), which will lead to solutions denoted u_N, u_L, and u_D, respectively. We shall show in Theorem 1.12 that $u_W = u_N = u_L = u_D$ as smooth functions on Ω, that is, as elements of $H_0^{3/2-s}(\Omega)$. In what follows, the most important definition of a strong solution will be the one using the Neumann function (respectively, using the Green function for the Dirichlet problem).

A common feature of the strong formulations of the homogeneous Neumann problem is that the resulting solutions u_N, u_L, and u_D are naturally defined only as elements of $H_0^{3/2-s}(\Omega)$ (recall that $H_0^r(\Omega)$ is the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H^r(\Omega)$ if $r \geq 0$ and $H_0^r(\Omega) := H_0^{-r}(\Omega)$ if $r < 0$). This is not a big loss of generality, as we shall show in Subsection 1.6 that any harmonic function in $H_0^{3/2-s}(\Omega)$, $s > 3/2$, has a canonical extensions to an element in $H^{3/2-s}(\Omega)$ and that the canonical extension of $u_N = u_L = u_D$ is u_W, the solution obtained from the weak formulation. Also, the strong formulation distinguish themselves from the weak formulation in that they are naturally defined for all $s \in \mathbb{R}$ (not only for $s > 3/2$, like the weak solution). Since the weak solution and the strong solutions turn out eventually to coincide, this provides us with additional regularity for the weak solution of the homogeneous Neumann problem (3).

1.4.1. *Definition of the solution u_N using the Neumann function.* Let $\Phi(x) = c_n|x|^{2-n}$ be the fundamental solution of Δ [6, 7, 25] ($\Phi(x) = c_2 \log|x|$ if $n = 2$), so that $\Delta(\Phi * \phi) = \phi$ for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, where c_n is a constant depending only on n and $*$ denotes the convolution product:

$$\Phi * \phi(x) = \int_{\mathbb{R}^n} \Phi(x-y)\phi(y)dy.$$

Note that we use a different sign convention than the one in [6, 8], where the constant c_n is negative ours. Let $N(x,y)$ be the *Neumann function* of Ω. That is,

$$N(x,y) := \Phi(x-y) - \phi_x(y), \quad \Delta_y \phi_x(y) = c \text{ and } \partial_{\nu_y} N(x,y) = 0. \tag{6}$$

The constant c above is chosen such that a solution to the Neumann problem $\partial_{\nu_y} N(x,y) = 0$ exists. Moreover, we chose the solution ϕ_x such that $\int_\Omega N(x,y)dy = 0$ for all x. This can be achieved by adding a constant to ϕ_x.

The solution u of (3) is then given by

$$u(x) = -\int_{\partial\Omega} N(x,y)g(y)dS(y). \tag{7}$$

(This follows from Green's representation formula, Equation (12), by substracting $\int_{\partial\Omega} \big(\partial_\nu \phi_x(y)u(y) - \phi_x(y)\partial_\nu u(y)\big)dS(y) = 0$. See [8] for details.)

We now observe that the function $N(x,y)$ is smooth as a function of $y \in \partial\Omega$ for any fixed $x \in \Omega$. Therefore the formula of Equation (7) still makes sense, provided that we replace integration over $\partial\Omega$ with the pairing between distributions and functions. We can therefore define

$$u_N(x) = -\langle g, N(x,\cdot)\rangle_{\partial\Omega}, \tag{8}$$

which defines a harmonic function $u_{0N} \in \mathcal{C}^\infty(\Omega)$.

Remark 1.7. For the homogeneous Dirichlet problem $\Delta u = 0$ on Ω and $u = g$ at the boundary, we just replace the Neumann function N with the Green function G. See [6, 7] or [8], for instance, for the definition of the Green function. Let $K(x,y) = \partial_{\nu_y} G(x,y)$. Then $u_G(x) = \langle g, K(x,\cdot)\rangle_{\partial\Omega}$. A detailed discussion of the Dirichlet problem will be included in [4].

1.4.2. *Definition of the solution u_L using non-tangential limits.* We now introduce a formulation of the homogeneous Neumann problem (3) with $g \in H^{-s}(\partial\Omega)$ using non-tangential limits of the solution u, taking advantage that u is a smooth function on Ω, since it is harmonic. Let Ω_ϵ be the set of points of Ω at distance $> \epsilon$ to $\partial\Omega$. Then, for ϵ small enough, Ω_ϵ is an open set with smooth boundary $\partial\Omega_\epsilon \simeq \partial\Omega$. The isomorphism can be chosen to associate to $x \in \partial\Omega_\epsilon$ the closest point on $\partial\Omega$, again, for ϵ small enough.

Let $H_0^r(\Omega)$ be the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H^r(\Omega)$ and $H_0^{-r}(\Omega) := H_0^r(\Omega)^*$, $r > 0$, as before. Let us define

$$\mathcal{K}(-r) = \{u \in H_0^{-r}(\Omega) = H_0^r(\Omega)^*,\ \Delta u = 0\}, \quad r > 0. \tag{9}$$

Here $\Delta u = 0$ is in distribution sense, that is, $\langle u, \Delta\phi\rangle = 0$ for all $\phi \in \mathcal{C}_c^\infty(\Omega)$. The following theorem is a particular case of the results in [20] (especially Theorem 7 on page 807). We shall still denote by ∂_ν the directional derivatives in the direction of the normal to the boundary to Ω_ϵ.

Theorem 1.8 (Seeley)**.** *For any $u \in \mathcal{K}(3/2-s) \subset H_0^{3/2-s}(\Omega)$, $s > 3/2$, the restrictions $g_t := \partial_\nu u|_{\partial\Omega_t}$ converge in $H^{-s}(\partial\Omega)$ to a distribution $Ru = g \in H^{-s}(\partial\Omega)$ as $t \to 0$. There exists a continuous linear map $\mathcal{P} : H^{-s}(\partial\Omega) \to \mathcal{K}(3/2-s)$ with image* *the distributions $u \in \mathcal{K}(3/2-s)$ with vanishing integral on Ω. We have $R\mathcal{P}g = g$ if, and only if, $\langle g, 1\rangle_{\partial\Omega} = 0$. Similarly, $\mathcal{P}Ru = u$ if, and only if, $\langle u, 1\rangle = 0$.*

For any $g \in H^{-s}(\partial\Omega)$ with $\langle g, 1\rangle_{\partial\Omega} = 0$, we shall then denote

$$u_L := \mathcal{P}(g) \in H_0^{3/2-s}(\Omega). \tag{10}$$

Remark 1.9. It is interesting to mention here that it was proved by Straube [22] that a harmonic function has distributional boundary values if, and only if, it grows no faster than some power of ρ^{-1}, where ρ is the distance to the (smooth) boundary.

1.4.3. *Definition of the solution u_D using the Dirichlet-to-Neumann map.* We now introduce the Dirichlet-to-Neumann map using the Cauchy space $\mathcal{C}(\Delta)$ of Δ. The Cauchy space $\mathcal{C}(\Delta)$ of Δ is defined as the set of limits at $t = 0$ of pairs (h_t, g_t), where h_t is the restriction of u to $\partial\Omega_t$ and g_t is the normal derivative of u on $\partial\Omega_t$. Then $\mathcal{C}(\Delta)$ consists of pairs of the form $(h, \mathcal{N}(h))$, with h a suitable distribution on $\partial\Omega$ and $\mathcal{N}$ the Dirichlet-to-Neumann map, which we is recalled in the next definition. In particular, we have that $g := Ru = 0$ only if u is a constant (the map R is as in Theorem 1.8). The continuity of $\mathcal{P}$ shows that

$$\|u_L\|_{H_0^{3/2-s}(\Omega)} \leq C_{\Omega,-s}\|g\|_{H^{-s}(\partial\Omega)} \tag{11}$$

if $\int_\Omega u(x)dx = 0$. Then, in Seeley's notation, $\mathcal{P}(g) = P(\mathcal{N}^{-1}g, g)$ if $\langle g, 1\rangle_{\partial\Omega} = 0$, with P a "multi-layer potential."

Let us now recall the definition of the Dirichlet-to-Neumann map used in the paragraph above and necessary for our last formulation of the homogeneous Neumann problem with distributional data.

Definition 1.10. The *Dirichlet-to-Neumann map* $\mathcal{N} : H^r(\partial\Omega) \to H^{r-1}(\partial\Omega)$, $r > 1$, is defined by

$$\mathcal{N}(h) = \partial_\nu u,$$

where $u \in H^{r+1/2}(\Omega)$ is the unique function satisfying $\Delta u = 0$ and $u = h$ on $\partial\Omega$.

It is known [24, 27] that $\mathcal{N}$ is an elliptic pseudodifferential operator of order 1, so it extends by continuity to a map $\mathcal{N} : H^r(\partial\Omega) \to H^{r-1}(\partial\Omega)$ for *all* $r \in \mathbb{R}$. Moreover, $\mathcal{N}(g) = 0$ if, and only if, u is constant and $\mathcal{N}$ has as range the space of distributions with mean zero. If we define $\mathcal{N}^{-1}$ to be 0 on constants and to be the inverse of $\mathcal{N}$ on distributions with mean zero, then $\mathcal{N}^{-1}$ will also be a pseudodifferential operator (of order -1 this time), so it also extends to a map $\mathcal{N}^{-1} : H^{r-1}(\partial\Omega) \to H^r(\partial\Omega)$ for any $r \in \mathbb{R}$.

In particular, if u is a classical solution of the homogeneous Neumann problem (3), the Dirichlet-to-Neumann map determines $u = \mathcal{N}^{-1}(g)$ at the boundary (with the usual assumptions that the means of u and g vanish). Green's representation formula [6, 9, 25] then allows us to recover u inside Ω:

$$u(x) = \int_{\partial\Omega} \big(\partial_\nu \Phi(x-y)u(y) - \Phi(x-y)\partial_\nu u(y)\big) dS(y), \tag{12}$$

where $\Phi(x) = c_n|x|^{2-n}$ is the fundamental solution of Δ ($\Phi(x) = c_2 \log|x|$, if $n = 2$).

Formula (12) extends to distributions (with the integral replaced by a pairing), which suggests to define for every $g \in H^{-s}(\partial\Omega)$,

$$u_D(x) = \langle g, \partial_\nu \Phi(x - \cdot)\rangle_{\partial\Omega} - \langle \mathcal{N}^{-1}(g), \Phi(x - \cdot)\rangle_{\partial\Omega}. \tag{13}$$

In this formula, $\Phi(x-\cdot)$ denotes the smooth function $y \to \Phi(x-y)$, where $y \in \partial\Omega$ and $x \in \Omega$. Similarly, $\partial_\nu \Phi(x-\cdot)$ denotes the normal derivative of $\Phi(x-y)$, regarded as a function of y alone (x is fixed) defined on the boundary. In particular, the normal derivative ∂_ν is in the y variable. This formula reduces to Equation (12) if g is a (distribution given by a) function.

Equation (13) defines $u_D \in \mathcal{C}^\infty(\Omega)$. The following proposition shows that, in fact, we obtain an element $u_D \in H^{3/2-s}(\Omega)$.

Proposition 1.11. *Let $s > 0$. Then for any $g \in H^{-s}(\partial\Omega)$ with zero mean, we have $u_D \in H_0^{3/2-s}(\Omega)$, $\Delta u_D = 0$ inside Ω, and*

$$\|u_D\|_{H^{3/2-s}(\Omega)} \leq C_{\Omega,s}\|g\|_{H^{-s}(\partial\Omega)},$$

for a constant $C_{\Omega,s}$ depending only on Ω and $s > 0$. A similar statement with $H_0^{3/2-s}(\Omega)$ replaced by $H^{3/2-s}(\Omega)$ holds true for $s \geq 0$.

Proof. Let $s > 0$ first. Also, let $g \in H^{-s}(\partial\Omega)$. Then

$$\mathcal{C}^\infty(\mathbb{R}^n) \ni \phi \to \int_{\partial\Omega} g(y)\phi(y)dS(y) \in \mathbb{C},$$

defines a distribution $g \otimes \delta_{\partial\Omega} \in H^{-s-1/2}(\mathbb{R}^n)$, as in [16], for example. Similarly, if $h = \mathcal{N}^{-1}(g) \in H^{1-s}(\partial\Omega)$, the linear map

$$\mathcal{C}^\infty(\mathbb{R}^n) \ni \phi \to \int_{\partial\Omega} \partial_\nu\phi(y)h(y)dS(y) \in \mathbb{C}$$

defines a distribution $h \otimes \delta'_{\partial\Omega} \in H^{-s-1/2}(\mathbb{R}^n)$. (See also Equation (22).) Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be equal to 1 in a neighborhood of $\overline{\Omega}$. We define $Qu(x) := \chi(x)[\Phi * u(x)]$. Then one can explicitly identify the distribution kernel of Q as $\chi(x)\Phi(x-y)$. The characterization of the kernels of pseudodifferential operators [26] then shows right away that Q is a pseudodifferential operator of order -2. Formula (12) then can be reformulated as

$$u = Q(g \otimes \delta_{\partial\Omega} + h \otimes \delta'_{\partial\Omega}). \tag{14}$$

Therefore $u \in H^{3/2-s}(\mathbb{R}^n)$ because Q defines a continuous linear map $H^r(\mathbb{R}^n) \to H^{r+2}(\mathbb{R}^n)$ for any $r \in \mathbb{R}$ (this can also be seen also directly using the Fourier transform, as in [3], Lemma 3.10, without making appeal to the fact that Q is a pseudodifferential of order -2). The continuous restriction map $H^{3/2-s}(\mathbb{R}^n) \to H_0^{3/2-s}(\Omega)$ then completes the proof in the case $s < 0$.

For $s > 0$, we see from the classical Green representation formula that u is the unique solution with mean zero of the homogeneous Neumann problem (3). The result then follows from the classical well posedness of the Neumann problem.

Finally, for $s = 0$, the result follows by interpolating between $-r$ and r for some small $r > 0$. □

1.5. **The equalities $\mathbf{u_W = u_N = u_L = u_D}$.** We now prove the main theorem of this section, stating that the four formulations of the homogeneous Neumann problem with distributional data coincide, and hence $u_W = u_N = u_L = u_D$, where u_W was introduced in Theorem 1.3, u_N was introduced in Equation (8), u_L was introduced in Equation (10), and u_D was introduced in Equation (13).

We are ready now to prove our main theorem. The interesting case is when $s > 3/2$, because otherwise it is a classical result. Recall that $\Pi_0 :: H^{-s}(\Omega) \to H_0^{-s}(\Omega)$ is the natural projection for $s > 0$.

Theorem 1.12. *Let $s \in \mathbb{R}$. Then for any $g \in H^{-s}(\partial\Omega)$, we have $u_N = u_L = u_D$ for any $x \in \Omega$. If $s > 3/2$, the we also have $u_N = u_L = u_D = \Pi_0(u_W)$. In particular, by denoting by u the common value, we have the estimate*

$$\begin{aligned} \|u\|_{H^{3/2-s}(\Omega)} &\le C_{\Omega,-s}\|g\|_{H^{-s}(\partial\Omega)} \quad \textit{if } s \le 3/2 \quad \textit{and} \\ \|u\|_{H_0^{3/2-s}(\Omega)} &\le C_{\Omega,-s}\|g\|_{H_0^{-s}(\partial\Omega)} \quad \textit{if } s > 3/2. \end{aligned} \tag{15}$$

Proof. For $g \in \mathcal{C}^\infty(\partial\Omega)$, it is known that $u_W(x) = u_N(x) = u_L(x) = u_D(x)$. For arbitrary $g \in \mathcal{C}^\infty(\partial\Omega)$, these relations follow by continuity. Indeed, the estimate of Theorem (1.12) was established for $u = u_W$ in Theorem 1.3. For $u = u_N$, u_L, or u_D, this estimate follows from Equation (8), from Theorem 1.8, and from Proposition 1.11. This shows that all these functions have the same restriction in the interior (which is an element of $H^{3/2-s}(\Omega)$. This is enough to show that $u_W = u_N = u_L = u_D$, since they are the canonical extensions of their restrictions to the interior, by Proposition 1.14 and by definition of u_N and u_L. □

The common function u of Theorem 1.12 will be the unique solution of the homogeneous distributional Neumann problem (3). For $s > 3/2$ we have $u \in H_0^{3/2-s}(\Omega)$, which is slightly inconvenient. However, we shall next extend th solution u in this case to an element in $u \in H^{3/2-s}(\Omega)$ that coincides with u_W.

1.6. **Canonical extensions.** We first define a canonical extension $\tilde{u} \in H^{3/2-s}(\Omega)$ of a harmonic function $u \in H_0^{-r}(\Omega)$. We use the ideas used to define u_D.

Proposition 1.13. *Let $u \in H_0^{3/2-s}(\Omega)$, $s > 3/2$, be such that $\Delta u = 0$ on Ω. Let χ_ϵ be the characteristic function of the set $\Omega_\epsilon := \{x \in \Omega, dist(x, \partial\Omega) > \epsilon\}$, as before. Then $\chi_\epsilon u \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \chi_\epsilon(x)u(x)\phi(x)dx$ converges as $\epsilon \to 0$ for any $\phi \in H^{s-3/2}(\mathbb{R}^n)$. The limit $\tilde{u} := \lim_{\epsilon\to 0} \chi_\epsilon u \in H^{3/2-s}(\Omega)$ maps to u in $H_0^{3/2-s}(\Omega)$ in the sense that $\Pi_0(\tilde{u}) = u$, and we have $\|\tilde{u}\|_{H^{3/2-s}(\Omega)} \le C\|u\|_{H_0^{3/2-s}(\Omega)}$.*

Proof. Let h_ϵ be the restriction of u to $\partial\Omega_\epsilon$ and $g_\epsilon = \partial_\nu u$ on $\partial\Omega_\epsilon$, as before. Then $(h_\epsilon, g_\epsilon) \to (h, g)$ in $H^{1-s}(\partial\Omega) \oplus H^{-s}(\partial\Omega)$, after we identify $\partial\Omega_\epsilon$ with $\partial\Omega$ using the closest point, by Seeley's results (Theorem 7 of [20]). Let Q be as in the proof of Proposition 1.11. Then $g_\epsilon \otimes \delta_{\partial\Omega_\epsilon} + h_\epsilon \otimes \delta'_{\partial\Omega_\epsilon} \to g \otimes \delta_{\partial\Omega} + h \otimes \delta'_{\partial\Omega}$ in $H_0^{-1/2-s}(\mathbb{R}^n)$ for $\epsilon \to 0$, and hence

$$\tilde{u} := Q(g \otimes \delta_{\partial\Omega} + h \otimes \delta'_{\partial\Omega}) = \lim_{\epsilon\to 0} Q(g_\epsilon \otimes \delta_{\partial\Omega_\epsilon} + h_\epsilon \otimes \delta'_{\partial\Omega_\epsilon}) = \lim_{\epsilon\to 0} \chi_\epsilon u, \tag{16}$$

by the continuity of Q.

For any $\phi \in \mathcal{C}_c^\infty(\Omega)$, we have $\langle \tilde{u}, \phi\rangle\langle \chi_\epsilon u, \phi\rangle = \langle u, \phi\rangle$ for ϵ small enough, so $\tilde{u}$ maps to u in $H_0^{3/2-s}(\Omega)$. Finally, it follows from [20] that $\|g_\epsilon\|_{H^{-s}(\partial\Omega)} \le C\|u\|_{H_0^{3/2-s}(\Omega)}$. This implies the last inequality. □

We are interested in the canonical extension $\tilde{u}$ because of the following result.

Proposition 1.14. *Let $u_{0W} := \Pi_0(u_W) \in \mathcal{C}^\infty(\Omega) \cap H_0^{3/2-s}(\Omega)$ be the canonical image of u_W. Then $u_W = \tilde{u}_{0W}$, the canonical extension of u_{0W}.*

Proof. If the data g is smooth enough, then u_W is a continuous function on $\overline{\Omega}$, and hence $u_W = \lim_{\epsilon\to 0} \chi_\epsilon u_W =: \tilde{u}_{0W}$ in $H^{3/2-s}(\Omega)$, the limit being taken in this space. For general g, the result follows by the continuity of the canonical extension and of the projection Π_0. □

In particular, Theorem 1.12 and Proposition 1.14 give the following result.

Theorem 1.15. *Let $s > 3/2$. Then $\tilde{u}_N = \tilde{u}_L = \tilde{u}_D = u_W$.*

2. The inhomogeneous Neumann problem

We now reformulate our problem (Equation (1)), based on the results in [18, 19, 20]. We shall consider in this section the *inhomogeneous Neumann problem*

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases} . \tag{17}$$

Green's formula,

$$\langle u, \Delta v\rangle = \langle \Delta u, v\rangle - \langle \partial_\nu u, v\rangle_{\partial\Omega} + \langle u, \partial_\nu v\rangle_{\partial\Omega}\,, \quad u, v \in \mathcal{C}^\infty(\overline{\Omega}), \tag{18}$$

tells us right away that, for $u, f \in \mathcal{C}^\infty(\overline{\Omega})$, $g \in \mathcal{C}^\infty(\partial\Omega)$, the inhomogeneous Neumann problem (17) is equivalent to

$$\langle u, \Delta v\rangle = -\langle f, v\rangle + \langle u, \partial_\nu v\rangle_{\partial\Omega} - \langle g, v\rangle_{\partial\Omega}\,, \quad \text{for all } v \in \mathcal{C}^\infty(\overline{\Omega}). \tag{19}$$

Roitberg proceeds then to define a solution of the Dirichlet problem (17) as a triple

$$(u_0, u_1, u_2) \in H^{3/2-s}(\Omega) \oplus H^{1-s}(\partial\Omega) \oplus H^{-s}(\partial\Omega),$$

such that $u_2 = g$ and

$$(u_0, \Delta v) = -(f, v) + \langle u_1, \partial_\nu v\rangle - \langle u_2, v\rangle\,, \quad \text{for all } v \in \mathcal{C}^\infty(\overline{\Omega}). \tag{20}$$

See [18, Section 0.2] for details.

Inspired by [19] and [20], we now reformulate the Equation (20) as follows. Let $f \in L^1(\Omega)$, then denote by $f\chi_\Omega$ the distribution defined by

$$\langle f\chi_\Omega, \phi\rangle = \int_\Omega f(x)\phi(x)dx\,, \quad \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n). \tag{21}$$

Also, for $g \in L^1(\partial\Omega)$ and any $\phi \in \mathcal{C}^\infty(\overline{\Omega})$, we shall denote as before

$$\begin{aligned} \langle g \otimes \delta_{\partial\Omega}, \phi\rangle &= \int_{\partial\Omega} g(x)\phi(x)dS(x) \\ \langle g \otimes \delta'_{\partial\Omega}, \phi\rangle &= \int_{\partial\Omega} g(x)\partial_\nu\phi(x)dS(x), \end{aligned} \tag{22}$$

(recall that dS is the measure on $\partial\Omega$ induced by the Riemannian metric and ∂_ν is the derivative in the direction of the unit outer normal). These definitions extend to f a distribution with support in Ω and to g a distribution on $\partial\Omega$.

Then Equation (20) is equivalent to

$$-\Delta(u_0\chi_\Omega) = f\chi_\Omega - u_1 \otimes \delta'_{\partial\Omega} + u_2 \otimes \delta_{\partial\Omega} \tag{23}$$

as *distributions* on $\mathbb{R}^n$. Let $h := f + g \otimes \delta_{\partial\Omega}$. Recall now the bilinear form $\tilde{B}(u, \phi) = -\langle u_0, \Delta\phi\rangle + \langle u_1, \phi\rangle_{\partial\Omega}$ introduced in Equation (5) where $u = (u_0, u_1)$. Then the Equation (23) is equivalent to

$$\tilde{B}(u, \phi) = \langle h, \phi\rangle, \qquad \text{where } u = (u_0, u_1) \text{ and } \phi \in \mathcal{C}^\infty(\overline{\Omega}). \tag{24}$$

We are ready now to formulate what we mean by the fact that u is a solution of the distributional, inhomogeneous Neumann problem, Equation (17), when f and g are distributions.

Definition 2.1. Let $h := f + g \otimes \delta_{\partial\Omega}$ and u be a distribution on $\mathbb{R}^n$ with support in $\overline{\Omega}$. We shall say that u is a weak solution of the inhomogeneous distributional Neumann problem, Equation (17) if u satisfies Equation (24) for all $\phi \in \mathcal{C}^\infty(\overline{\Omega})$.

The results of [3], especially Proposition 4.9 (see also Theorem 4.8 of that paper) give right away the following result.

Theorem 2.2. *Let $s > 3/2$. For any $h := f + g \otimes \delta_{\partial\Omega}$, where $f \in H^{-s-1/2}(\Omega)$ and $g \in H^{-s}(\partial\Omega)$, that satisfies $\langle h, 1\rangle := \langle f, 1\rangle + \langle g, 1\rangle_{\partial\Omega} = 0$, the inhomogeneous, distributional Neumann problem (17) has a weak solution $u = (u_0, u_1) \in \tilde{H}^{3/2-s}(\Omega) := H^{3/2-s}(\Omega) \oplus H^{1-s}(\partial\Omega)$. Any two solutions differ by a constant and, if we also impose $\langle u_0, 1\rangle = 0$, then the solution is unique and it satisfies*

$$\|u_0\|_{H^{3/2-s}(\Omega)} + \|u_1\|_{H^{1-s}(\Omega)} \leq C_{\Omega,s}\|h\|_{H^{-s-1/2}(\Omega)} \\ \leq C(\|f\|_{H^{-s-1/2}(\Omega)} + \|g\|_{H^{-s}(\partial\Omega)}).$$

Let us record the following simple consequence of our approach, which is probably the main difference between our approach and that of [18]. The following remark and the following example are relevant for the discussion of concentrated loads and couples [21].

Remark 2.3. The data f and g is *not* uniquelly determined by the solution $u = (u_0, u_1)$ of the inhomogeneous Neumann problem (17). Indeed, let p be a point on the boundary of Ω and δ_p the Dirac distribution (or measure) concentrated at p. Then the choices $(f, g) = (\delta_p, 0)$ and $(f, g) = (0, \delta_p)$, will lead to the same solution of the distributional, inhomogeneous Neumann problem. The reason is the solution $u = (u_0, u_1)$ depends only on h and $h = \delta_p$ in both cases. Let $N(x, y)$ be the Neumann function associated to Ω. Then the solution (u_0, u_1) is given by $u_0(x) = -N(x, p)$, $x \in \Omega$, and $u_1 = -N(x, p)$, $x \in \partial\Omega$. (So, in particular, u_1 is the "restriction at the boundary" of u_0.) Both functions are integrable, so there is no problem in defining u_0 and u_1 as distributions.

Let us include now a simple example related to the above remark.

Example 2.4. Let us consider now the inhomogeneous distributional Neumann problem with data $g = 0$ and $f = \delta_p$, $p \in \Omega$. Let us chose a sequence of points $p_n \in \Omega$, $p_n \to q$ as $n \to \infty$, and let us solve the problem $-\Delta v_n = \delta_{p_n}$, $\partial_\nu u = 0$ on $\partial\Omega$. Then $v_n \to u_0$, $v_n|_{\partial\Omega} \to u_1$ as $n \to \infty$, by the continuity of the weak solutions of the inhomogeneous distributional Neumann problem.

3. Examples

We now illustrate the results obtained in the previous sections and point out to some further developments.

3.1. **The half-space.** Let us consider $\Omega = \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, \infty)$. This is not a bounded domain and the Laplace operator $\Delta : H^2(\Omega) \to L^2(\Omega)$ is not invertible. Nevertheless, this is a domain for which we can easily write the Green function $G(x, y)$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ using the reflection principle [6], page 37, which leads to $G(x, y) = c_n\big(|x-y|^{2-n} - |\tilde{x}-y|^{2-n}\big)$, where $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$. Then $K(x, y) := \partial_\nu G(x, y)$ is

$$K(x, y) = \frac{c'_n x_n}{|x-y|^n}, \qquad x \in \Omega, y \in \partial\Omega,$$

where $c'_n = 2(n-2)c_n$. The boundary value problem $\Delta u = 0$, $u = \delta_p$ on the boundary $\partial\Omega$ then has the solution $u(x) = K(x,p)$. Let $p = 0$ and consider g_t =the restriction of $u(x) = c'_n x_n |x|^{-n}$ to $x_n = t$, $t > 0$. Then $\int_{x_n=t} g_t(x')dx' = 1$, $u \geq 0$, and $\int_{|x'|>\epsilon} u(x')dx' \to 0$ as $t \to 0$. This verifies that $g_t \to \delta_0$ in distribution sense.

If we replace the boundary value δ_p with $\partial^\alpha \delta_p$, then we obtain the solution $u_\alpha = (-1)^{|\alpha|}\partial^\alpha K(x,p)$.

Similarly, a solution of the homogeneous Neumann problem $\Delta u = 0$, $\partial_\nu u = \delta_p$ at the boundary, is obtained by replacing the function $K(x,y)$ with the Neumann function $N(x,y) = c_n\big(|x-y|^{2-n} + |\tilde{x}-y|^{2-n}\big)$.

3.2. **An angle.** Let us now look at the case of the angle $\mathbb{P} := \{(r,\theta), 0 < \theta < \alpha\}$, in polar coordinates $(r,\theta) \in [0,\infty) \times [0, 2\pi)$. Let $s = \pi/\alpha$. Then the map $z \to z^s$ maps $\mathbb{P}$ conformally onto the half-space $\mathbb{H} := \{\Im(z) > 0\}$, where $\Im(z) := b$ denotes the imaginary part of $z = a + bi \in \mathbb{C}$. Taking into account that the fundamental solution for the plane is $\Phi(z) = \frac{1}{2\pi}\log|z|$, the reflection principle gives $G_1(z,w) = \Phi(z-w) - \Phi(z-\overline{w})$ and $N_1(z,w) = \Phi(z-w) + \Phi(z-\overline{w})$ for the Green and Neumann functions of the half-plane $\mathbb{H}$. Therefore, the Green and Neumann functions of $\mathbb{P}$ are $G(z,w) = G_1(z^s, w^s)$ and $N(z^s, w^s)$, which gives the explicit formulas

$$G(z,w) = \frac{1}{2\pi}\log\left|\frac{z^s - w^s}{z^s - \overline{w}^s}\right| \text{ and } N(z,w) = \frac{1}{2\pi}\log|(z^s - w^s)(z^s - \overline{w}^s)|. \tag{25}$$

(We chose the branch of z^s that is obtained by removing the positive real axis.)

Let us consider now the Neumann problem $\Delta u_N = 0$ on $\mathbb{P}$ and $\partial_\nu u_N = \delta_w$, the Dirac distribution concentrated at the point w on the boundary $\partial\mathbb{P}$. Then the solution is given by $u_N(z) = N(z,w)$. If $w = 0$, this gives

$$u_N(z) = -\frac{1}{\alpha}\log|z|. \tag{26}$$

Let Ω_r be the intersection of $\mathbb{P}$ with the disc of radius r centered at the origin (*i.e.,* at 0). Then the "conservation property" $\int_{\partial\Omega_r} \partial_\nu u_N(z)dS(z)$ of this solution is immediately checked.

On the other hand, the solution of Dirichlet problem

$$\Delta u_D = 0 \text{ on } \mathbb{P} \text{ and } u_D = \delta_w \text{ on the boundary} \tag{27}$$

is given by $\partial_\nu K(z,w) = \pm r^{-1}\partial_\theta G(z,w)$, where $z = r(\cos\theta + i\sin\theta)$ and we have the sign $-$ for w real and $+$ otherwise (*i.e.,* for w on the upper half of the boundary). Explicitly, if $w \in \mathbb{R}_+$, then

$$K(z,w) = \frac{sw^{s-1}\Im(z^s)}{\pi|z^s - w^s|^2}. \tag{28}$$

Then, if we let $w \to 0$ ($w \in \mathbb{R}_+$), we obtain $K(z,0) = 0$ if $\alpha < \pi$ and $K(z,0) = \infty$ if $\alpha > \pi$. Thus the Dirichlet problem (27) seems not to be well posed for w at the angle (or corner), *unlike* the corresponding Neumann problem.

4. Polygonal domains

Let $\mathbb{P} \subset \mathbb{R}^2$ be a (curvilinear) polygonal domain. Then it is possible to extend to this case the definitions of the weak solutions. In view of the problems pointed out in the preceeding section, it is convenient weighted Sobolev spaces. Let us recall first the definitions of the weighted Sobolev spaces $\mathcal{K}^m_a(\mathbb{P})$ on $\mathbb{P}$ [14] and of the weighted Sobolev spaces $\mathcal{K}_a^m(\partial\mathbb{P})$ on the boundary $\partial\mathbb{P}$ [1]. See also [11, 10, 5] for a discussion of similar Sobolev spaces.

Let us denote by $\vartheta(x)$ the distance from a point $x \in \mathbb{P}$ to the closest vertex of $\mathbb{P}$. Then

$$\mathcal{K}_a^m(\mathbb{P}) := \{u : \mathbb{P} \to \mathbb{C},\ \vartheta^{|\alpha|-a}\partial^\alpha u \in L^2(\mathbb{P}),\quad |\alpha| \le m\}, \quad m \in \mathbb{Z}_+,\ a \in \mathbb{R}. \tag{29}$$

Similarly,

$$\mathcal{K}_a^m(\partial\mathbb{P}) := \{u : \partial\mathbb{P} \to \mathbb{C},\ \vartheta^{|\alpha|-a}\partial^\alpha u \in L^2(\partial\mathbb{P}),\quad |\alpha| \le m\}, \quad m \in \mathbb{Z}_+,\ a \in \mathbb{R}, \tag{30}$$

where only derivatives *tangent to the boundary* are to be used. More precisely, let us notice that we can identify each edge with $[0, 1]$. Then $\mathcal{K}_a^m(\partial\mathbb{P})$ consists of the functions $f : \partial\mathbb{P} \to \mathbb{C}$ that, on each edge, are such that $t^k(1-t)^k f^{(k)} \in L^2([0, 1])$, $0 \le k \le m$, after we identify that edge with $[0, 1]$. This last condition is equivalent to $[t(1-t)\partial_t]^k f \in L^2([0, 1])$, $0 \le k \le m$.

The spaces $\mathcal{K}_a^s(\partial\mathbb{P})$, $s \ge 0$, are defined by interpolating in s (for a fixed). The *negative order weighted Sobolev spaces* $\mathcal{K}_{-a}^{-s}(\partial\mathbb{P})$, $s \ge 0$, and $\mathcal{K}_{-a}^{-m}(\mathbb{P})$, $m \in \mathbb{Z}_+$, are defined by duality (with pivot L^2), for instance $\mathcal{K}_{-a}^{-m}(\mathbb{P}) \simeq \mathcal{K}_a^m(\mathbb{P})^*$ with the duality map extending the L^2–inner product. The following result are classical results of Kondratiev (with the additional identification of the Sobolev spaces at the boundary from [1]). Let us denote by $\alpha_j \in (0, 2\pi)$ the angles of the (curvilinear) polygonal domain $\mathbb{P}$.

Theorem 4.1. *Let $m \in \mathbb{Z}_+$, then the map*

$$\Delta_D : \mathcal{K}_{a+1}^{m+1}(\mathbb{P}) \to \mathcal{K}_{a-1}^{m-1}(\mathbb{P}) \oplus \mathcal{K}_{a+1/2}^{m+1/2}(\partial\mathbb{P}), \quad \Delta_D u = (\Delta u, u|_{\partial\mathbb{P}}),$$

is Fredholm for $a \ne k\pi/\alpha_j$, $k \in \mathbb{Z}$, $k \ne 0$. Similarly, the operator $\Delta_N : \mathcal{K}_{a+1}^{m+1}(\mathbb{P}) \to \mathcal{K}_{a-1}^{m-1}(\mathbb{P}) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial\mathbb{P})$, $\Delta_N u = (\Delta u, \partial_\nu u|_{\partial\mathbb{P}})$ is Fredholm for $a \ne k\pi/\alpha_j$, $k \in \mathbb{Z}$.

Note that $k \ne 0$ for the Dirichlet problem, while $k = 0$ is allowed for the Neumann problem. In fact, the map Δ_D turns out to be an isomorphism for $|a| < \pi/\alpha_{Max}$, where $\alpha_{Max} = \max \alpha_j$. On the other hand, Δ_N is *not* an isomorphism for $a = 0$ (it is not even Fredholm for this value of the weight a).

Theorem 4.1 can then be used to formulate a definition and prove the existence of weak solutions of the Dirichlet or Neumann problems with data in the spaces $\mathcal{K}_a^s$ along the lines of the ones in the preceeding sections. Let us outline this procedure (using the method from [4]) for the Dirichlet problem and $|a| < \pi/\alpha_{Max}$, for which we *do not* have to consider subspaces.

Let $\tilde{\mathcal{K}}_{a+1}^{1-s}(\mathbb{P}) := \mathcal{K}_{a+1}^{1-s}(\mathbb{P}) \oplus \mathcal{K}_{a-1/2}^{-1/2-s}(\partial\mathbb{P})$ and $D : \tilde{\mathcal{K}}_{a+1}^{1-s}(\mathbb{P}) \times \mathcal{K}_{a+1}^{1+s}(\mathbb{P}) \to \mathbb{C}$ be the bilinear functional defined by

$$-D(u, v) := \langle u_0, \Delta v\rangle + \langle u_1, v\rangle_{\partial\mathbb{P}}, \tag{31}$$

where $u = (u_0, u_1) \in \tilde{\mathcal{K}}_{a+1}^{1-s}(\mathbb{P})$.

Consider the inhomogeneous Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \mathbb{P}, \\ u = g & \text{on } \partial\mathbb{P}, \end{cases} \tag{32}$$

with $g \in \mathcal{K}_{a+1/2}^{1/2-s}(\partial\mathbb{P})$ and $f \in \mathcal{K}_{a-1}^{-1-s}(\mathbb{P})$, $s \in \mathbb{R}$, $|a| < \pi/\alpha_{Max}$.

Definition 4.2. Let $g \in \mathcal{K}^{1/2-s}_{a+1/2}(\partial\mathbb{P})$ and $f \in \mathcal{K}^{-1-s}_{a-1}(\mathbb{P})$, $s \in \mathbb{R}_+$. We say that $u = (u_0, u_1) \in \tilde{\mathcal{K}}^{1-s}_{a+1}(\mathbb{P})$ satisfies (32) in *weak sense*, or that u is a *weak solution* of the Dirichlet problem (32) if

$$D(u, v) = \langle f, v\rangle - \langle g, \partial_\nu v\rangle_{\partial\mathbb{P}}, \tag{33}$$

for all $v \in \mathcal{K}^{s+1}_{a+1}(\mathbb{P})$.

Then Theorem 4.1 and the fact that we get an isomorphism for $|a| < \pi/\alpha_{Max}$ for the Dirichlet problem and the "inf-sup" condition from [2] give the following result that is proved similarly to Proposition 4.8 in [3] or to the corresponding result in [4].

Theorem 4.3. *Let $g \in \mathcal{K}^{1/2-s}_{a+1/2}(\partial\mathbb{P})$ and $f \in \mathcal{K}^{-1-s}_{a-1}(\mathbb{P})$, $s \geq 0$, $|a| < \pi/\alpha_{Max}$. Then there exists a unique weak solution $u = (u_0, u_1) \in \tilde{\mathcal{K}}^{1-s}_{a+1}(\mathbb{P})$ for 32 and*

$$\|u_0\|_{\mathcal{K}^{1-s}_{a+1}(\mathbb{P})} + \|u_1\|_{\mathcal{K}^{-1/2-s}_{a-1/2}(\partial\mathbb{P})} \leq C_0\big(\|g\|_{\mathcal{K}^{1/2-s}_{a+1/2}(\partial\mathbb{P})} + \|f\|_{\mathcal{K}^{-1-s}_{a-1}(\mathbb{P})}\big),$$

for a constant $C_{\mathbb{P},s}$ that depends only on $\mathbb{P}$, s, and a.

5. The inhomogeneous Neumann problem revisited

In this section we carry an alternative discussion of the inhomogeneous Neumann problem using the methods from [12].

For technical reasons that will become clear in the following, we will replace $-\Delta$ with $-\Delta + V$, where V is a smooth function, following an idea from [17]. The advantage of this choice is that the Dirichlet problem on exterior domains for $-\Delta+V$, $V \geq \epsilon > 0$ outside a compact set, is well posed (*i.e.,* has a unique solution continuously depending on the data).

Definition 5.1. Let u be a distribution on $\mathbb{R}^n$ (or, more generally, on M) and $h := f + g \otimes \delta_{\partial\Omega}$. We say that u is a *distributional solution* of the (inhomogeneous distributional) Neumann problem (17), where f is a distribution supported on $\overline{\Omega}$ and g is a distribution on $\partial\Omega$, if u is supported on $\overline{\Omega}$ and there exists a distribution u_1 on $\partial\Omega$ such that

$$(-\Delta + V)u = h - u_1 \otimes \delta'_{\partial\Omega} := f + g \otimes \delta_{\partial\Omega} - u_1 \otimes \delta'_{\partial\Omega}\,, \tag{34}$$

as distributions on $\mathbb{R}^n$.

By "u supported on $\overline{\Omega}$" we mean $u(\phi) = 0$ for any $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n \smallsetminus \overline{\Omega})$, as usual. For $V = 0$, the above definition of the solution u is clearly equivalent to the definition of a weak solution (Definition 2.1). We now make some more comments.

Remark 5.2. (a) If f and g are smooth functions on $\overline{\Omega}$, respectively $\partial\Omega$, then they are determined by the knowledge of $h := f + g \otimes \delta_{\partial\Omega}$.
(b) The distribution u_2 turns out, in fact, to be determined by f, g, and the condition that the support of u be contained in $\overline{\Omega}$. (See Theorem 5.5 below.) By taking $u = u_0\chi_\Omega$, we $u_1 = g$ recover Roitberg's notion of solution of the Dirichlet problem when the data f and g are distributions.
(c) Our approach owes to [19] in that it has transformed the two equations of our boundary value problem in only one equation (Equation (34)) and the distributions f and g in one distribution, namely h. However, the domain of our operators are larger and our conditions stronger than those in [19].

We shall need the following technical lemmas.

Lemma 5.3. *Let $V \geq 0$ be a smooth function such that $V \geq 1$ outside a compact set. Then $V - \Delta : H^{r+1}(\mathbb{R}^n) \to H^{r-1}(\mathbb{R}^n)$ is an isomorphism for any $r \in \mathbb{R}$, and hence $(V - \Delta)^{-1}$ is a pseudodifferential operator of order -2.*

Proof. Let $u \in H^1(\mathbb{R}^n)$ be such that $(V - \Delta)u = 0$ in $H^{-1}(\mathbb{R}^n)$. The estimate

$$(\nabla u, \nabla u) + (Vu, u) = ((V - \Delta)u, u) = 0$$

shows that $V - \Delta : H^{r+1}(\mathbb{R}^n) \to H^{r-1}(\mathbb{R}^n)$ is injective for $r \geq 0$.

The result is well known if $V = 1$. Let us then write $V - \Delta = (1 - \Delta) - \phi$, with $\phi = 1 - V$ a function with compact support. Then $V - \Delta = (I - R)(1 - \Delta)$, with $R = \phi(1 - \Delta)^{-1}$, and hence R maps $H^{r-1}(\mathbb{R}^n)$ to $H_0^{r+1}(B)$, where B is a large ball. Since the inclusion $H_0^{r+1}(B) \to H^{r-1}(\mathbb{R}^n)$ is compact, by Rellich's lemma, we obtain that R is a compact map $H^{r-1}(\mathbb{R}^n) \to H^{r-1}(\mathbb{R}^n)$. We also have that $1 - R$ is injective, since $1 - \Delta$ and $V - \Delta$ are injective. By the Fredholm alternative, it follows that $1 - R$ is invertible. This shows that $V - \Delta : H^{r+1}(\mathbb{R}^n) \to H^{r-1}(\mathbb{R}^n)$ is an isomorphism for $r \geq 0$.

For $r \leq 0$, the result follows by taking duals. □

We shall denote by $\Omega_1 := \overline{\Omega}^c$ the open complement of (the closure of) Ω.

Lemma 5.4. *Let V be as in the previous lemma and $v \in H^{1/2-s}(\Omega_1)$, such that $(V - \Delta)v = 0$ on $\Omega_1 := \mathbb{R}^n \smallsetminus \overline{\Omega}$. Then there exists $u_1 \in H^{-s}(\partial\Omega) = H^{-s}(\partial\Omega_1)$, such that $v = (V - \Delta)^{-1}(u_1 \otimes \delta'_{\partial\Omega})$ on Ω_1.*

Proof. Let $w = (-\Delta + V)^{-1}(\phi \otimes \delta'_{\partial\Omega})$. Then, for ϕ smooth enough, the boundary values w_- of w at $\partial\Omega = \partial\Omega_1$ (approaching non-tangentially from the outside the domain Ω) are defined and of the form $w_- = (-\frac{1}{2} + K)\phi$, where K is a pseudodifferential operator of order -1, and hence compact. This is a standard result in the theory of layer potentials, see [7, 17, 26] for example.

If $w_- = 0$, it follows from the energy estimate

$$0 = ((-\Delta + V)w, w) = (\nabla w, \nabla w) + (Vw, w)$$

that $w = 0$. Consequently, $-\frac{1}{2} + K$ is injective. Since $-\frac{1}{2} + K$ is also Fredholm of index zero, if must be an isomorphism. The inverse $(-\frac{1}{2} + K)^{-1}$ is then a pseudodifferential operator of order zero which will act on any Sobolev space on $\partial\Omega$.

We can take then $u_1 = (-\frac{1}{2} + K)^{-1/2}v_-$, where $v_- \in H^{-s}(\partial\Omega)$ exist by Seeley's results (Theorem 7 in [20], used earlier for the Laplace operator, see Theorem 1.8). □

The main goal of this section is to prove the following theorem generalizing Theorem 1.12.

Theorem 5.5. *Let Ω be a smooth, bounded domain in $\mathbb{R}^n$ and $s > 1$. Assume $h := f + g \otimes \delta_{\partial\Omega} \in H^{-s}(\mathbb{R}^n)$. Then Equation (34) has a unique distributional solution $u \in H^{2-s}(\mathbb{R}^n)$ and there exists $C_{\Omega,s} > 0$, independent of h, such that*

$$\|u\|_{H^{2-s}(\Omega)} \leq C_{\Omega,s}\|h\|_{H^{-s}(\Omega)}.$$

Furthermore, $u_1 \in H^{3/2-s}(\partial\Omega)$ and $\|u_1\|_{H^{3/2-s}(\partial\Omega)} \leq C\|h\|_{H^{-s}(\Omega)}$, for some constant $C > 0$ independent of h.

Let $s > 1/2$, $g \in H^{1/2-s}(\partial\Omega)$, and $f \in H^{-s}(\mathbb{R}^n)$ be a distribution with support in $\overline{\Omega}$. Then $h := f + g \otimes \delta_{\partial\Omega} \in H^{-s}(\mathbb{R}^n)$ and has support in $\overline{\Omega}$, so it satisfies the assumptions of our theorem.

We are ready now to prove Theorem 5.5.

Proof. Choose $V \geq 0$ to be a smooth function pn $\mathbb{R}^n$ that is equal to 0 in a neighborhood of $\overline{\Omega}$ and is ≥ 1 outside some big ball.

Let $v = (-\Delta + V)^{-1}h \in H^{2-s}(\mathbb{R}^n)$, which is defined since the inverse map $(V - \Delta)^{-1} : H^{-s}(\mathbb{R}^n) \to H^{2-s}(\mathbb{R}^n)$ is defined and continuous by Lemma 5.3. Then $\|v\|_{H^{2-s}(\Omega)} \leq C\|h\|_{H^{-s}(\Omega)}$, for some constant $C > 0$. Moreover, $(-\Delta + V)v = 0$ on $\overline{\Omega}^c$. Let $u_1 \in H^{3/2-s}(\partial\Omega)$ such that $v = (-\Delta + V)^{-1}(u_1 \otimes \delta'_{\partial\Omega})$ on $\overline{\Omega}^c$. The existence of u_1 with this property is guaranteed by Lemma 5.4.

Then

$$u := v - (-\Delta + V)^{-1}(u_1 \otimes \delta'_{\partial\Omega})$$

has support in $\overline{\Omega}$ and satisfies Equation (34). The norm estimates follow from the continuity of all the maps involved. □

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