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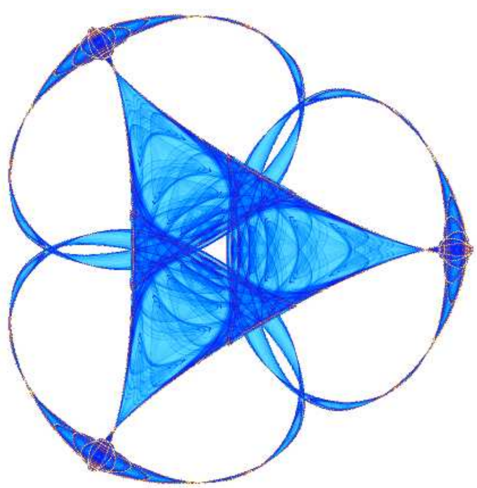

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A SHARP NATURAL CHARACTERIZATION OF A^1

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Abstract. We use the natural maximal operator to give a simple characterization of A^1 that yields sharp bounds on the A^1 constant of a weight in terms of its A^∞ constant and the BLO norm of its logarithm. Along the way we reveal further the symmetry between the Hardy-Littlewood maximal operator M and the Cruz-Uribe–Neugebauer minimal operator m (in particular, showing m bounded from BMO to BUO), clarifying the reciprocity of their behavior on the reverse Hölder and A^p structures of A^∞.

1. Introduction

In 1995 Cruz-Uribe and Neugebauer [4] introduced the minimal operator, $mf(x) = \inf_{Q\ni x} \frac{1}{|Q|}\int_Q |f|$, to fill a role with respect to the reverse Hölder class structure of the A^∞ weights paralleling that played by the Hardy-Littlewood maximal operator M with respect to the coexistent A^p structure. In analogy with A^1, they defined the limiting class $RH^\infty = \{w \mid \exists\, c > 0 \text{ s.t. } cmw(x) \geq w(x) \text{ a.e.}\}$ and showed that, just as in the A^p situation, $RH^\infty \subset \bigcap_s RH^s$, and $w \in RH^\infty$ if and only if $w \in RH^s$ for all $s > 1$ with $RH^s(w)$ independent of s. They also revealed a symmetry between the behavior of the operators on their respective classes: that, just as M maps A^∞ into A^1, $m(RH^1) \subset RH^\infty$ (where $RH^1 = A^\infty$). Further, they proved that $w \in RH^\infty$ if and only if $w^s \in RH^\infty$ for any $s > 0$, used this fact to obtain the characterization $RH^\infty = e^{BUO}$, and demonstrated the existence of a near-reciprocity between the limiting classes, namely, that $A^1 = \frac{1}{RH^\infty \cap A^2}$. In other words, they began to establish the notion that m and the RH^s classes in some sense mirror M and the A^p classes.

In this paper we use the natural extremal operators introduced in [9] to strengthen and clarify this parallel by first in section 2 making precise the equivalence of the behavior of the Hardy-Littlewood maximal operator and the Cruz-Uribe–Neugebauer minimal operator on A^∞ and BMO; in particular, we give the new result that m is bounded from BMO to BUO (in analogy with the boundedness of M from BMO to BLO). In section 3, by permitting fractional reverse Hölder classes, we make clear the reciprocal nature of the A^p and RH^s structures themselves, using a result of Strömberg and Wheeden [12] to extend the observation that $A^{1+\frac{1}{s}} = \frac{1}{RH^s \cap A^2}$, $s > 0$ to its full range. In section 4, we finish by using the natural maximal operator to give our best result: a new, sharp characterization of A^1, bounding the A^1 constant of a weight in terms of its A^∞ constant and the BLO norm of its logarithm; further, the same method with slight modifications then yields a proof of the characterization of RH^∞ that explains the asymmetry between the limiting classes. As application we give simple, new proofs of various known properties of A^1.

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2. Natural Extremal Operators and BMO

To study the behavior of the Hardy-Littlewood maximal operator on BMO and A^∞, we previously saw (in [9]) that the key was to consider the corresponding "natural maximal operator" $M^\natural$, defined by

$$M^\natural f(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q f.$$

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Exactly the same procedure can be followed for the Cruz-Uribe–Neugebauer minimal operator, yielding analogous results. For the sake of completeness we list those results here, demonstrating the more important of them.

Lemma 1 (Commutation)**.** *For all $w \in A^\infty$, $0 \le [\log m^\natural - m^\natural \log]w(x) \le \log A^\infty(w)$.*

Proof. By the reverse Jensen inequality for A^∞ and Jensen's inequality,

$$\frac{1}{|Q|}\int_Q w \le A^\infty(w)e^{\frac{1}{|Q|}\int_Q \log w} \le A^\infty(w)\frac{1}{|Q|}\int_Q w.$$

Take the infimum over all cubes Q containing x and then take the logarithm. $\square$

Lemma 2 (Characterization of BUO)**.** *$\phi \in BUO \iff m^\natural\phi(x) \ge \phi(x) - C$ a.e., in which case $||\phi||_{BUO}$ equals the infimum of such C.*

(Recall that BLO denotes the functions of bounded lower oscillation, i.e. functions ϕ such that over all cubes Q, $\frac{1}{|Q|}\int_Q \phi - \inf_Q \phi \le C$; we denote by $||\phi||_{BLO}$ the infimum of such C. BUO, the functions of bounded *upper* oscillation, is defined similarly.)

Theorem 1. *$m^\natural$ maps BMO boundedly into BUO.*

Proof. Let $\phi \in BMO$. As a consequence of the John-Nirenberg Inequality, $\phi(x) = \frac{||\phi||_*}{c_n}\log w(x)$ for some $w \in A^2$, with $A^2(w) \le e$ and $c_n = \frac{1}{2^{n+2}e}$, where n denotes the dimension [6, p. 166]. Thus, by the commutation lemma,

$$m^\natural\phi(x) = \frac{||\phi||_*}{c_n}[\log mw(x) + b(x)]$$

for some $||b||_\infty \le 1$. Consider the $\log mw$ term: as mentioned in the introduction, m maps RH^1 into $RH^\infty = e^{BUO}$; so it is in BUO. Further, its norm depends on $RH^\infty(mw)$, which depends ultimately on the A^p class and A^p constant of w, which depend on nothing. Thus $||m^\natural\phi||_{BUO} \le C_n||\phi||_*$. $\square$

Alternatively, we could have simply observed that since $M^\natural(f)(x) = -m^\natural(-f)(x)$, the above is equivalent to the previously shown [9] boundedness of $M^\natural$ from BMO into BLO. In either case, we get the following new result, paralleling the boundedness of M from BMO to BLO.

Corollary. *m maps BMO boundedly into BUO.*

Proof. $m\phi = m^\natural(|\phi|) \in BUO$, and $||m\phi||_{BUO} = ||m^\natural(|\phi|)||_{BUO} \le 2C_n||\phi||_*$. $\square$

We complete our list of statements with the following (assuming the weak result that $M(A^\infty) \subset A^\infty$ with $A^\infty(Mw)$ dependent only on $A^p(w)$ and p).

Theorem 2 (Equivalence)**.** *The boundedness of $m^\natural : BMO \to BUO$ implies $m(RH^1) \subset RH^\infty$ with $RH^\infty(mw)$ dependent on $RH^s(w)$ and s.*

Note that if we compile the logical implications and equivalences for both M and m, we have the following diagram.

$$\begin{array}{ccc}
M : A^\infty \to A^1 & & m : RH^1 \to RH^\infty \\
\Updownarrow & & \Updownarrow \\
M^\natural : BMO \to BLO & \iff & m^\natural : BMO \to BUO \\
\Downarrow & & \Downarrow \\
M : BMO \to BLO & & m : BMO \to BUO
\end{array}$$

In other words, the behavior of $M^\natural$ and $m^\natural$ on BMO is in fact logically equivalent to that of M and m on A^∞ (RH^1), respectively; thus, although the two structures on A^∞ are not precisely reciprocal, the respective behavior of the extremal operators on them is exactly equivalent.

3. Near-Reciprocity of Structures on A^∞

At this point, it is natural to ask to what extent the A^p and RH^s structures are reciprocal. For completeness, we note that if we allow *fractional* reverse Hölder classes for $0 < s < 1$, defined by $w = e^\phi \in RH^s$ if $\sup_Q(\frac{1}{|Q|}\int_Q e^{s[\phi(x)-\phi_Q]}dx)^{1/s} = RH^s(w) < \infty$, it is not difficult, using the observation of Strömberg and Wheeden that a weight w is in RH^s if and only if $w^s \in A^\infty$ [12], to show the following extension of the observation that $A^{1+\frac{1}{s}} = \frac{1}{RH^s \cap A^2}, s > 0$ (which, since $M(f)(x) = \frac{1}{m(\frac{1}{f})(x)}$ a.e., implies that $M : A^\infty \to A^1$ is morally equivalent to $m : RH^1 \to RH^\infty$, a rough, partial version of the equivalence given in section 2).

Theorem 3 (Structure Near-Reciprocity). $(1 \le p < \infty,\ 0 < s \le \infty)$

1. *If $w \in A^p$, then $\frac{1}{w} \in RH^s$ for $s = \frac{1}{p-1}$. Precisely, $RH^s(\frac{1}{w}) \le A^p(w)$.*
2. *If $\frac{1}{w} \in RH^s \cap \frac{1}{A^\infty}$ then $w \in A^p$ for $p = 1 + \frac{1}{s}$: for $0 < s \le 1$, $A^p(w) \le A^\infty(w)RH^s(\frac{1}{w})$; for $1 < s \le \infty$, $A^p(w) \le A^\infty(\frac{1}{w})A^\infty(w)RH^s(\frac{1}{w})$.*

The above is not sufficiently symmetric to ensure a set of precise equivalences as one might desire, but it does imply the following about the behavior of the extremal operators on the fractional reverse Hölder classes: roughly, that m and M transform and preserve respectively the fractional reverse Hölder classes as they do the standard ones.

Corollary (M and m on fractional RH^s). *Let $0 < s < 1$. If $w \in RH^s \cap \frac{1}{A^\infty}$, then $mw \in RH^\infty$ and $Mw \in RH^s$, where $RH^\infty(mw)$ and $RH^s(Mw)$ depend on only s, $RH^s(w)$, and $A^\infty(\frac{1}{w})$.*

Proof. By the theorem, $w \in RH^s$ and $\frac{1}{w} \in A^\infty$ imply that $\frac{1}{w} \in A^{1+\frac{1}{s}}$. Since M maps A^p into A^1, $M(\frac{1}{w}) \in A^1$, and thus $mw = \frac{1}{M(\frac{1}{w})} \in RH^\infty$. Further, $RH^\infty(mw) = RH^\infty(\frac{1}{M(\frac{1}{w})}) \le A^1(M(\frac{1}{w}))$. Now, $A^1(M(\frac{1}{w}))$ depends on only s and $A^{1+\frac{1}{s}}(\frac{1}{w})$; by the theorem, the latter depends on only s, $RH^s(w)$, and $A^\infty(\frac{1}{w})$.

Similarly, since m preserves A^p classes, $m(\frac{1}{w}) \in A^{1+\frac{1}{s}}$, i.e. $\frac{1}{Mw} \in A^{1+\frac{1}{s}}$. Using the theorem once more implies $Mw \in RH^s$; the dependence of $RH^s(Mw)$ on only s, $RH^s(w)$, and $A^\infty(\frac{1}{w})$ follows as before. □

4. Asymmetric Sharp Characterizations of A^1 and RH^∞

A variation of the technique used to prove the equivalence theorem in section 2 yields our best result, a satisfying and further, sharp, characterization of A^1, refining the result of Coifman-Rochberg ([3], cited in [5]) that $w \in A^2 \cap e^{BLO} \iff w \in A^1$. One direction is well known [6, p. 157].

Theorem 4 (Characterization of A^1). *$w \in A^1 \iff w \in A^\infty \cap e^{BLO}$. Precisely: ($e^{||\log w||_{BLO}} \le$) $A^1(w) \le A^\infty(w)e^{||\log w||_{BLO}}$.*

Proof. $\Leftarrow$) Say $w \in e^{BLO} \cap A^\infty$; $\log w \in BLO$. By the characterization of BLO [9],

$$M^\natural \log w(x) \le \log w(x) + ||\log w||_{BLO} \text{ a.e..}$$

Further, since $w \in A^\infty$, the commutation lemma for $M^\natural$ implies

$$\log M^\natural w(x) - \log A^\infty(w) \le M^\natural \log w(x) \text{ a.e..}$$

Combining the two yields

$$\log M^\natural w(x) \le \log A^\infty(w) + \log w(x)) + ||\log w||_{BLO} \text{ a.e.;}$$

i.e.,

$$Mw(x) \le [A^\infty(w)e^{||\log w||_{BLO}}]w(x) \text{ a.e..}$$

Considering the case of constant weights shows the bound to be sharp. □

In the same manner, we also obtain an alternate demonstration of the characterization $RH^\infty = e^{BUO}$. We include the proof since it is instructive to see how and why the asymmetry between the limiting classes arises. First recall that $RH^\infty(w) = \inf\{c \mid cmw(x) \ge w(x) \text{ a.e.}\}$.

Theorem 5 (Characterization of RH^∞). *$w \in RH^\infty \iff w \in e^{BUO} \cap A^\infty (= e^{BUO})$. Precisely: $RH^\infty(w) \le e^{||\log w||_{BUO}} \le A^\infty(w)RH^\infty(w)$.*

Proof. $\Leftarrow$) Say $w \in e^{BUO}$; $\log w \in BUO$. By the $m^\natural$ characterization of BUO,

$$m^\natural \log w(x) \geq \log w(x) - ||\log w||_{BUO} \text{ a.e..}$$

Now, although we do not have $w \in A^\infty$, we only need the part of the commutation lemma based on Jensen's inequality, i.e.,

$$\log m^\natural w(x) \geq m^\natural \log w(x) \text{ a.e..}$$

Thus

$$\log m^\natural w(x) \geq \log w(x) - ||\log w||_{BUO} \text{ a.e.,}$$

i.e.,

$$mw(x) \geq e^{-||\log w||_{BUO}} w(x) \text{ a.e..}$$

as desired.

$\Rightarrow$) Say $w \in A^\infty \cap RH^\infty$. By the reverse Jensen inequality and the definition of RH^∞, we see

$$A^\infty(w) e^{\frac{1}{Q}\int_Q \log w} \geq \frac{1}{|Q|}\int_Q w \geq \frac{1}{RH^\infty(w)} w(x),$$

i.e., $\log A^\infty(w) + \log RH^\infty(w) \geq \log w(x) - \frac{1}{Q}\int_Q \log w$ for all $x \in Q$. Taking the supremum over $x \in Q$ shows the BUO norm bounded by $\log A^\infty(w) + \log RH^\infty(w)$. As in the previous theorem, the bounds are sharp. $\square$

Thus the characterization of A^1 clarifies the exact nature of the near-reciprocity between A^1 and RH^∞. Moreover, it makes transparent various known properties of A^1, for example, the following (cited in [5], proved in [8] by Johnson and Neugebauer):

Proposition. *If $w \in A^\infty$ and $w^s \in A^1$ for any $s > 0$ then $w \in A^1$.*

Proof. $w^s \in A^1$ implies $\log w^s = s \log w \in BLO$; since BLO is closed under multiplication by positive scalars, $\log w \in BLO$ also. $\square$

Proposition. *If $w, w^{-1} \in A^1$ then $w \approx 1$.*

Proof. $w, w^{-1} \in A^1$ implies $\log w$ and $-\log w$ are in BLO; thus $\log w \in L^\infty$. $\square$

For a deeper example, consider the following result of Cruz-Uribe and Neugebauer [4], which becomes a simple consequence of the well-known duality $w \in A^p \iff w^{-\frac{1}{p-1}} \in A^{p'}$.

Theorem (Cruz-Uribe–Neugebauer)**.** $(p > 1)$. $w \in A^1 \iff w^{1-p} \in RH^\infty \cap A^p$.

Proof. $\Rightarrow$) $w \in A^1 \implies w \in A^{p'} \implies w^{1-p} \in A^p$; further, since $A^1 \subset e^{BLO}$, $w^{1-p} \in e^{BUO} = RH^\infty$; so $w^{1-p} \in RH^\infty \cap A^p$.

$\Leftarrow$) $w^{1-p} \in A^p \implies w^{(1-p)(1-p')} = w \in A^{p'}$, and $w^{1-p} \in RH^\infty = e^{BUO} \implies w^{(1-p)(1-p')} \in e^{BLO}$; thus $w \in A^\infty \cap e^{BLO} = A^1$. $\square$

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