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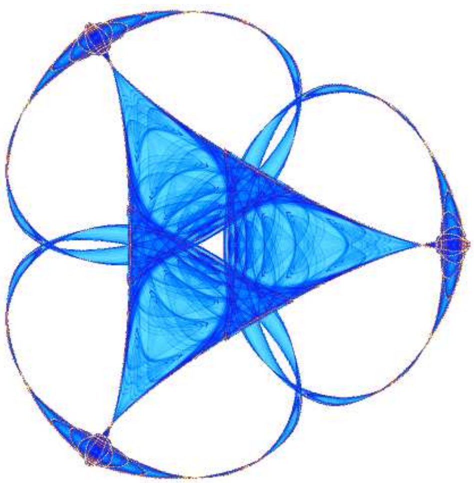

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The Cardinality of Sets of k-Independent Vectors over Finite Fields

S. B. Damelin[*], G. Michalski and Gary L. Mullen

Abstract

A set of vectors is k-independent if all its subsets with no more than k elements are linearly independent. We obtain a result concerning the maximal possible cardinality $Ind_q(n, k)$ of a k-independent set of vectors in the n-dimensional vector space F_q^n over the finite field F_q of order q. Namely, we give a necessary and sufficient condition for $Ind_q(n, k) = n + 1$.

1 Introduction

For q a prime power, let F_q denote the finite field of order q, and let F_q^n denote the n-dimensional vector space of all n-tuples over F_q. For $k \geq 1$ an integer, we say that a set of vectors $A \subseteq F_q^n$ is **k-independent** if all its subsets with at most k elements are linearly independent. We are interested in the maximal possible cardinality, $Ind_q(n, k)$, of a k-independent subset of F_q^n. Clearly $(k + 1)$-independence implies k-independence, and since any set of $n + 1$ vectors in a vector space of dimension n must be linearly dependent, we can assume that $1 \leq k \leq n$. We clearly have

$$q^n - 1 = Ind_q(n, 1) \geq Ind_q(n, 2) \geq \cdots \geq Ind_q(n, n) \geq n + 1.$$

It is not hard to see that the last inequality in the sequence above holds because we can use the n unit vectors and the all ones vector of length n

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over F_q to form an n-independent set containing $n+1$ vectors. On the other hand, the first inequality above becomes an equality when $q = 2$, for over F_2, 2-independence is equivalent to 1-independence. Actually, a general formula for $Ind_q(n, 2)$ is not hard to find. Recalling that any two vectors are linearly independent over a field if and only if they are not scalar multiples of each other, we have

Observation 1. *Let q be a prime power, and $n \geq 1$ an integer. Then*

$$Ind_q(n, 2) = \frac{q^n - 1}{q - 1}. \tag{1.1}$$

Using the n unit vectors and the all ones vector of length n over F_q, it is easy to see that $Ind_q(n, k) \geq n + 1$ for any k with $1 \leq k \leq n$.

In [1], the authors investigated formulae for $Ind_2(n, k)$ in two extreme cases: they first noted the well known formula for the case when $k = 3$. More importantly, they considered the case when $k \geq 2n/3$. The results from [1] are stated in the theorem below, where m and n are positive integers.

Theorem 1. *The following formulae hold:*

(a)
$$Ind_2(n, 3) = 2^{n-1}, \text{ for } n \geq 3. \tag{1.2}$$

(b)
$$Ind_2(n, n-m) = n + 1, \text{ for } n \geq 3m + 2, \; m \geq 0. \tag{1.3}$$

(c)
$$Ind_2(n, n-m) = n + 2, \text{ for } n = 3m + i, \; i = 0, 1, \; m \geq 2. \tag{1.4}$$

In this paper we generalize the result stated in part (b) of Theorem 1. We present a simple condition on q, n and k which is both necessary and sufficient for $Ind_q(n, k) = n + 1$ to hold.

Our main result is the following:

Theorem 2. *Let q be a prime power, and let k and n be integers with $2 \leq k \leq n$. Then $Ind_q(n, k) = n + 1$ if and only if*

$$k \geq \frac{q}{q+1}(n+1)$$

In particular, in the case $q = 2$, Theorem 2 says that the inequality in Theorem 1 (b) is not only sufficient, but also necessary.

We note that our current result (Theorem 2) is most effective when the prime power q is small. For example when $q = 2$ and n is large, we are able to determine the value of $Ind_2(n, k)$ for many values of k, in particular for all k in the range $(2n + 2)/3 \leq k \leq n$.

2 k-Extensions and k-Closures

Clearly, when calculating $Ind_q(n, k)$, we can restrict our attention to **maximal** k-independent sets; i.e. those k-independent sets that don't have proper supersets that are still k-independent.

Observation 2. *Every maximal k-independent set contains a basis of F_q^n over F_q.*

Proof. For $X \subseteq F_q^n$, we use $span\,(X)$ to denote the linear subspace generated by X. If $A \subseteq F_q^n$ is maximal k-independent then every element of F_q^n is a linear combination of (less than k) elements of A; i.e. $span\,(A) = F_q^n$. Consider a maximal linearly independent $B \subseteq A$. If follows (by maximality of B) that $A \subseteq span\,(B)$, and therefore $F_q^n = span\,(A) \subseteq span\,(B)$; i.e. B is a basis of F_q^n. □

Since k-independence is preserved by automorphisms of F_q^n, in the light of Observation 2, while studying $Ind_q(n, k)$ we can restrict our attention even further, namely to the supersets of the standard basis, which we denote by B. We shall say that a set $W \subseteq F_q^n$ is a k**-extension** (of B) if W is disjoint from B, and $W \cup \mathsf{B}$ is k-independent; if $W \cup \mathsf{B}$ is *maximal* k-independent then W will be called a k**-closure** (of B). Let $Cls_q(n, k)$ denote the maximal possible cardinality of a k-closure in F_q^n. The above remarks imply that

$$Ind_q(n, k) = n + Cls_q(n, k). \tag{2.1}$$

Theorem 2 determines exactly for which q, n and k, singletons are the only possible nonempty k-extensions of the standard basis B, and therefore, the only possible k-closures in F_q^n; i.e. for which q, n and k we have $Cls_q(n, k) = 1$.

3 The Proof of Theorem 2

Throughout this section q is a prime power, and n and k are integers with $2 \le k \le n$. We will need the following notation.

For $\mathbf{a} = (a_1, \ldots, a_n) \in F_q^n$, $\mathrm{supp}(\mathbf{a})$ will denote the *support* of the vector $\mathbf{a}$, i.e. the set $\{i : a_i \neq 0, i = 1, \ldots, n\}$; the cardinality of this set will be denoted by $\|\mathbf{a}\|$. ($\|\ \| : F_q^n \to R^+$ satisfies the usual norm conditions, where the absolute value is replaced by the trivial valuation on F_q. In particular, $\|\alpha\mathbf{a}\| = \|\mathbf{a}\|$, for every $\alpha \in F_q^*$.)

We use $|X|$ to denote the cardinality of the set X; and if x is a real number, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the largest integer not larger than x, and the smallest integer not smaller than x, respectively. We begin with a characterization of k-extensions in terms of $\|\ \|$.

Lemma 1. *Suppose that $W \neq \emptyset$ is disjoint from the standard basis B. Then W is a k-extension if and only if for every nonempty $U \subseteq W$ and $\{\alpha_\mathbf{u} : \mathbf{u} \in U\} \subseteq F_q^*$, we have*

$$\left\| \sum_{\mathbf{u} \in U} \alpha_\mathbf{u} \mathbf{u} \right\| > k - |U|.$$

Proof. Suppose first that W is a k-extension, and let

$$\mathbf{w} = \sum_{\mathbf{u} \in U} \alpha_\mathbf{u} \mathbf{u}$$

be as above. By expanding $\mathbf{w}$ in the standard basis we get

$$\mathbf{w} = \sum_{\mathbf{u} \in U} \alpha_\mathbf{u} \mathbf{u} = \sum_{\mathbf{v} \in C} \beta_\mathbf{v} \mathbf{v}$$

for some $C \subseteq \mathsf{B}$, with $|C| = \|\mathbf{w}\|$, and $\beta_\mathbf{v} \in F_q^*$, for $\mathbf{v} \in C$. It follows that $U \cup C$ is a linearly dependent subset of the k-independent set $W \cup \mathsf{B}$, and therefore its cardinality $|U \cup C| = |U| + \|\mathbf{w}\|$ must be greater than k; i.e. $||\mathbf{w}|| > k - |U|$, as required.

Next, suppose that W is not a k-extension, (i.e. $W \cup \mathsf{B}$ is not k-independent). Then for some $U \subseteq W, C \subseteq \mathsf{B}$ with $|U| + |C| \le k$, and some $\alpha_\mathbf{u}, \beta_\mathbf{v} \in F_q^*$, for $\mathbf{u} \in U$ and $\mathbf{v} \in C$, we have

$$\sum_{\mathbf{u} \in U} \alpha_\mathbf{u} \mathbf{u} + \sum_{\mathbf{v} \in C} \beta_\mathbf{v} \mathbf{v} = \mathbf{0}.$$

In particular,

$$\left\|\sum_{\mathbf{u}\in U}\alpha_{\mathbf{u}}\mathbf{u}\right\| = \left\|-\sum_{\mathbf{v}\in C}\beta_{\mathbf{v}}\mathbf{v}\right\| = |C| \leq k-|U|. \qquad \square$$

Lemma 1 will be used in the proof of Theorem 2 through the following corollary.

Corollary 1.

(a) *If W is a k-extension then $\|\mathbf{a}\| \geq k$, for every $\mathbf{a} \in W$.*

(b) *A singleton $\{\mathbf{a}\} \subseteq F_q^n - B$ is a k-extension if and only if $\|\mathbf{a}\| \geq k$.*

(c) *Suppose $\mathbf{a}, \mathbf{b} \in F_q^n - B$ are distinct. Then $\{\mathbf{a}, \mathbf{b}\}$ is a k-extension if and only if $\|\mathbf{a}\|, \|\mathbf{b}\| \geq k$ and $\|\alpha\mathbf{a}+\beta\mathbf{b}\| \geq k-1$, for all $\alpha, \beta \in F_q^*$.*

(d) *Suppose $W \subseteq F_q^n - B$ consists of vectors with pairwise disjoint supports. Then W is a k-extension if and only if $\|\mathbf{a}\| \geq k$, for every $\mathbf{a} \in W$.*

Proof. The proofs of parts (a), (b), and (c) are straightforward from Lemma 1. In proving part (d) we use the fact that if U consists of vectors with pairwise disjoint supports then for every $\{\alpha_{\mathbf{u}} : \mathbf{u} \in U\} \subseteq F_q^*$

$$\left\|\sum_{\mathbf{u}\in U}\alpha_{\mathbf{u}}\mathbf{u}\right\| = \sum_{\mathbf{u}\in U}\|\alpha_{\mathbf{u}}\mathbf{u}\| = \sum_{\mathbf{u}\in U}\|\mathbf{u}\|. \qquad \square$$

One consequence of Corollary 1 is a slight improvement on the lower bound on $Ind_q(n,k)$ given in the introduction ($Ind_q(n,k) \geq n+1$, for $2 \leq k \leq n$).

Observation 3. $Ind_q(n,k) \geq n + \lfloor n/k \rfloor$.

Proof. Let $m = \lfloor n/k \rfloor$. Partition the set $\{1,\ldots,km\}$ into k-element subsets $A_1,\ldots,A_m$. For each $i=1,\ldots,m$, let $\mathbf{a}_i$ be any vector with $\mathrm{supp}(\mathbf{a}_i) = A_i$. The set $\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}$ is a k-extension by Corollary 1(d). $\square$

Next, in connection with Corollary 1(d), we are going to take a closer look at $\|\alpha\mathbf{a}+\beta\mathbf{b}\|$, for $\mathbf{a},\mathbf{b}\in F_q^n$ and $\alpha,\beta\in F_q^*$. Let A and B denote the support of $\mathbf{a}$ and $\mathbf{b}$, respectively, and let $\mathbf{a}=(a_1,\ldots,a_n)$ and $\mathbf{b}=(b_1,\ldots,b_n)$. For any $\xi\in F_q^*$ we define $X_\xi := \{i\in A\cap B : a_i/b_i=\xi\}$. It is not hard to see that the support of $\alpha\mathbf{a}+\beta\mathbf{b}$ equals $A\cup B - X_{-\beta/\alpha}$. In particular,

$$\|\alpha\mathbf{a}+\beta\mathbf{b}\| = |A\cup B| - \left|X_{-\beta/\alpha}\right|.$$

So if $\ell(\mathbf{a},\mathbf{b})=\max_{\xi\in F_q^*}|X_\xi|$ then we have

$$\min_{\alpha,\beta\in F_q^*}\|\alpha\mathbf{a}+\beta\mathbf{b}\| = |A\cup B| - \ell(\mathbf{a},\mathbf{b}). \tag{3.1}$$

Note also that

$$|A\cap B| \leq (q-1)\ell(\mathbf{a},\mathbf{b}). \tag{3.2}$$

Indeed, $|A\cap B| = \left|\bigcup_{\xi\in F_q^*} X_\xi\right| = \sum_{\xi\in F_q^*}|X_\xi| \leq (q-1)\max_{\xi\in F_q^*}|X_\xi|$.

Lemma 2. *Suppose $\mathbf{a},\mathbf{b}\in F_q^n$ are distinct, and let A and B denote the support of $\mathbf{a}$ and $\mathbf{b}$, respectively.*

(a) *If $\|\mathbf{a}\|,\|\mathbf{b}\|\geq k$ then $\{\mathbf{a},\mathbf{b}\}$ is a k-extension iff*

$$\ell(\mathbf{a},\mathbf{b}) \leq |A\cup B| - k + 1. \tag{3.3}$$

(b) *If $\{\mathbf{a},\mathbf{b}\}$ is a k-extension then*

$$2k-n \leq |A\cap B| \leq (q-1)(n-k+1).$$

Proof. (a) Follows from Corollary 1(c), since by (3.1) the inequality (3.3) is equivalent to $\min_{\alpha,\beta\in F_q^*}\|\alpha\mathbf{a}+\beta\mathbf{b}\|\geq k-1$.

(b) By Corollary 1(a), we have $|A|,|B|\geq k$, and so the first inequality follows because

$$|A|+|B|-|A\cap B| = |A\cup B| \leq n.$$

The second inequality follows from (3.2) and part (a) of this lemma:

$$|A\cap B| \leq (q-1)\ell(\mathbf{a},\mathbf{b}) \leq (q-1)(|A\cup B|-k+1). \qquad \square$$

Corollary 2. *If $Ind_q(n,k) \geq n+2$, then $q-1 \geq \frac{2k-n}{n-k+1}$.*

Proof. Indeed, if $Ind_q(n,k) \geq n+2$, then there are $\mathbf{a}$ and $\mathbf{b}$ such that $\{\mathbf{a},\mathbf{b}\}$ is a two-element k-extension (cf. 2.1). But then by Lemma 2(b),

$$q-1 \geq \frac{|\mathrm{supp}(\mathbf{a}) \cap \mathrm{supp}(\mathbf{b})|}{n-k+1} \geq \frac{2k-n}{n-k+1}. \qquad \square$$

In the proof of our last lemma we shall need a basic combinatorics observation. Suppose X and Y are finite sets with $Y \neq \emptyset$. By a **partition** of X by Y we understand any family $\{X_y : y \in Y\}$ of subsets of X such that the union of the family equals X, and its members are pairwise disjoint, with some of them possibly empty. We define the **index** of the partition $\{X_y : y \in Y\}$ to be the maximal of the cardinalities $|X_y|$, for $y \in Y$.

Observation 4. *For any finite sets X, Y, with $Y \neq \emptyset$, there is a partition of X by Y with the index not greater than $\lceil |X|/|Y| \rceil$.*

Lemma 3. *Suppose that r and s are positive integers with $r, s \leq n \leq r+s$. Then there exist distinct $\mathbf{a}, \mathbf{b} \in F_q^n$ such that*

(a) $\|a\| = r,\ \|b\| = s$,

(b) $\ell(\mathbf{a},\mathbf{b}) \leq \left\lceil \frac{r+s-n}{q-1} \right\rceil$,

(c) $|\mathrm{supp}(\mathbf{a}) \cup \mathrm{supp}(\mathbf{b})| = n$.

Proof. Let $\mathbf{a} = (1,\dots,1,0,\dots,0)$, with $\|\mathbf{a}\| = r$. Let X be the $(r+s-n)$-element set $\{n-s+1,\dots,r\}$. Let $\left\{X_\beta : \beta \in F_q^*\right\}$ be a partition of X by F_q^* with the index not greater than $\lceil (r+s-n)/(q-1) \rceil$ (cf. Observation 4). We define $\mathbf{b} = (b_1,\dots,b_n)$ where

$$b_i = \begin{cases} 0 & \text{if } \ i \leq n-s \\ \beta & \text{if } \ i \in X_\beta \\ 1 & \text{if } \ i > r. \end{cases}$$

It is clear that $\|\mathbf{b}\| = s$. Also,

$$\ell(\mathbf{a},\mathbf{b}) = max_{\beta \in F_q^*} |X_\beta| \leq \lceil (r+s-n)/(q-1) \rceil .$$

To see that condition (c) holds note that $\mathrm{supp}(\mathbf{a}) \cap \mathrm{supp}(\mathbf{b}) = X$, and so $|\mathrm{supp}(\mathbf{a}) \cup \mathrm{supp}(\mathbf{b})| = \|\mathbf{a}\| + \|\mathbf{b}\| - |X| = r+s-(r+s-n) = n$. $\square$

Proof of Theorem 2: Note that the condition $k \geq \frac{q}{q+1}(n+1)$ is equivalent to

$$q < \frac{2k-n}{n-k+1} + 1. \tag{3.4}$$

If (3.4) holds then Corollary 2 implies that $Ind_q(n,k) \leq n+1$; i.e. $Ind_q(n,k) = n+1$ (see the remark preceding Observation 3).

Now suppose that (3.4) does not hold. Note that this implies

$$\left\lceil \frac{r+s-n}{q-1} \right\rceil \leq n-k+1. \tag{3.5}$$

We will show that $Ind_q(n,k) \geq n+2$. By Observation 3 this is true if $2k \leq n$. Suppose then that $2k > n$. By Lemma 3 applied to $r = s = k$, there exist distinct $\mathbf{a}, \mathbf{b} \in F_q^n$ such that $\|\mathbf{a}\| = \|\mathbf{b}\| = k$, $\ell(\mathbf{a}, \mathbf{b}) \leq \lceil (r+s-n)/(q-1) \rceil$, and $|\mathrm{supp}(\mathbf{a}) \cup \mathrm{supp}(\mathbf{b})| = n$. To complete the proof it is enough to show that $\{\mathbf{a}, \mathbf{b}\}$ is a k-extension. The latter follows from Lemma 2(a) since by (3.5) and the properties of $\mathbf{a}$ and $\mathbf{b}$ above

$$\ell(\mathbf{a}, \mathbf{b}) \leq n-k+1 = |\mathrm{supp}(\mathbf{a}) \cup \mathrm{supp}(\mathbf{b})| - k + 1. \qquad \square$$

4 An application to sets of k-orthogonal hypercubes

In [1, Section 3], numerous applications of Theorem 1 were given related to the construction of hypercubes and orthogonal arrays, pseudo (t, m, s)-nets and linear codes. We refer the reader to the paper [1] and the references cited therein for a comprehensive account of these applications.

We now present an application of our results to the construction of sets of orthogonal hypercubes. By a *hypercube of dimension n and order b* is meant a $b \times \cdots \times b$ array consisting of b^n cells, based upon b distinct symbols arranged so that each of the b symbols appears the same number of times, namely $b^n/b = b^{n-1}$ times. For $2 \leq k \leq n$, a set of k such hypercubes is said to be *k-orthogonal* if upon superpositioning of the k hypercubes, each of the b^k distinct ordered k-tuples appears the same number of times, i.e. $b^n/b^k = b^{n-k}$ times. Finally a set of $r \geq k$ such hypercubes is said to be

k-*orthogonal* if any subset of k hypercubes is k-orthogonal. When $n = k = 2$ these ideas reduce to the usual notion of mutually orthogonal latin squares of order b.

Given a set of k-independent vectors of length n over F_q, we can build sets of k-orthogonal hypercubes of order q and dimension n. Let $a_1x_1 + \cdots + a_nx_n$ denote a vector of length n in a k-orthogonal set. One can then construct a hypercube of order q and dimension n by placing the field element $a_1b_1 + \cdots + a_nb_n$ in the cell of the hypercube labelled by $(b_1, \ldots, b_n)$, where each $b_i \in F_q$. Since each coefficient vector $(a_1, \ldots, a_n)$ has at least one nonzero entry, it is clear that the array represented by the vector is indeed a hypercube of dimension n and order q.

Moreover, given k such vectors from a k-independent set, the corresponding set of k hypercubes will be k-orthogonal. This follows from the fact that the k vectors are k-independent over F_q, and hence the $k \times n$ matrix obtained from the coefficients of the k vectors will have rank k. Hence each element of F_q^k will be picked up exactly q^{n-k} times, so the k hypercubes are indeed k-orthogonal. This construction thus yields $Ind_q(n, k)$, k-orthogonal hypercubes of dimension n and order q.

We now raise a question regarding hypercubes of prime power orders. Let q, n, and k be such that they satisfy Theorem 2 so that $Ind_q(n, k) = n + 1$. Then as above, we can construct $n + 1$ hypercubes, each of dimension n and order q, which are k-orthogonal.

Question: If q is a prime power and the values of q, n and k satisfy Theorem 2 so that $Ind_q(n, k) = n + 1$, is it possible to have more than $n + 1$ hypercubes of order q and dimension n which are k-orthogonal?

We close by referring the reader to [2] and [3] for discussions of latin squares and hypercubes.

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