SURFACE AND BULK GROWTH UNIFIED

By

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IMA Preprint Series # 2045

(May 2005)

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May 20, 2005

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Notice

This paper is due to appear in the book *Mechanics of Material Forces* (Proceedings of the homonymous EuroMech Colloquium held in Kaiserslautern, Germany, May 21-24, 2003) edited by P. Steinmann and G.A. Maugin, to be published in the Springer Series "Advances in Mechanics and Mathematics." The manuscript was first submitted to the editors on July 30, 2003, and finally accepted in its present form on July 21, 2004.

> *Solid surfaces can have their physical area changed in two ways, either by creating or destroying surface without changing surface structure and properties per unit area, or by an elastic strain ... along the surface keeping the number of surface lattice sites constant ... .*
>
> —J.W. Cahn, 1980

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Abstract

I have been puzzled for a long time by the unnatural divide between the theory of bulk growth—strikingly underdeveloped—and that for surface growth—much better developed, along apparently independent lines.

Recent advances in growth mechanics (DiCarlo and Quiligotti, 2002) make it now possible to subsume growth phenomena of both kinds under one and the same format, where surface growth is obtained as an infinitely intense bulk growth confined in a layer of vanishingly small thickness.

This has allowed me to recover the results collected in Gurtin (2000) from the standpoint of DiCarlo and Quiligotti (2002). In particular, I am able to construe Gurtin's technique of referential control volumes that evolve in time as a special application of the principle of virtual power.

1 Introduction

Any continuum theory of growth—be it modelled as spread in bulk or concentrated on surfaces—hinges on two key issues: in kinematics, extra degrees of freedom have to be introduced, in order to distinguish growth from deformation; in dynamics, new balance laws have to be provided, apt to govern the evolution of such degrees of freedom.

Both issues are much subtler for bulk than for surface growth, however, demanding a sharper veering from the customary route in continuum mechanics. In my mind, here lies the basic difficulty which hindered surface and bulk growth theories from progressing on a par, within a common frame.

The theory of bulk growth which I (still) prefer is the one set forth by DiCarlo and Quiligotti (2002).[1] It is the only one I am aware of where the evolution law for bulk growth is obtained as a constitutively augmented balance. In order to make this paper reasonably self-contained, Sect. 2 offers a résumé of that theory, adapted to the present discussion.

As a paradigm for surface growth theories, I take the tract by Gurtin (2000): it is a convenient reference book, where the role of an independent configurational balance is duly stressed—from a point of view, I may add, which in several respects differs from my own.

[1]This is not at all obvious, since I can change my mind. But in this respect I did not. My coauthor did (Quiligotti, 2002), on the (not minor) issue of invariance requirements, convinced by an argument intimated by Green and Naghdi (1971) and spelled out in Casey and Naghdi (1980, 1981), Casey (1987), and elsewhere. For reasons I expound elsewhere (DiCarlo, in preparation), I find that old argument faulty and misleading.

The quote from Cahn (1980) I put in the prologue is a paraphrase of Gibb's discussion of multiphase equilibria. I took it from the introductory chapter of Gurtin (2000), where it is completed by the following comment from the author: "The creation of surface involves configurational forces, while stretching the surface involves standard forces." Despite the fact that Gibbs, Cahn and Gurtin had only surface phenomena in mind (namely, the evolution of phase interfaces), it is perfectly legitimate—and much to the point—to rephrase the whole statement in terms of bulk phenomena, as follows (italicized words are my own):

> Solid *bodies* can have their physical *volume* changed in two ways, either by creating or destroying *bulk* without changing *bulk* structure and properties per unit *volume*, or by an elastic strain *within the body* keeping the number of *bulk* lattice sites constant. The creation of *bulk* involves configurational *couples*, while stretching the *body* involves standard forces.

By *couple* I mean a tensor quantity—not necessarily skew—having the physical dimensions of length$\times$force. Why configurational couples are germane to bulk growth is explained in Sect. 2.2; explaining how they match with configurational forces, proper to surface growth, is one of the main aims of this paper.

2 A continuum theory of bulk growth

The growing bodies considered here are standard Cauchy continua: the only kinematic descriptor ascribed to their points is place in ordinary physical space. In order to distinguish growth from deformation, *two* evolving configurations are associated with each body element: its *current* configuration, describing how it is actually placed in space, and its *relaxed* configuration, describing how it "would like" to be placed. The field of relaxed configurations need not be (and usually is not) compatible, not even locally.

This is a good old kinematic idea, primarily introduced to distinguish between elastic and viscoplastic strains by Kröner (1960) and Lee (1969), and much later imported into growth modelling by Rodriguez et al. (1994) (see also Taber (1995)). The original contribution by DiCarlo and Quiligotti (2002) is in dynamics. As summarized in the following, we obtain the evolution law for bulk growth as a constitutively augmented *new* balance, the balance of configurational (or remodelling) couples, *independent* of the standard force balance.

2.1 Kinematics

We regard a body as a smooth manifold $\mathcal{B}$ (with boundary $\partial\mathcal{B}$), and call **placement** any smooth embedding

$$p : \mathcal{B} \to \mathcal{E} \tag{1}$$

of the body into the Euclidean place manifold $\mathcal{E}$, whose translation space will be denoted by $\mathrm{V}\mathcal{E}$. Tangent vectors on the body manifold itself are called **line elements**. The set of all line elements attached to a single body-point $b \in \mathcal{B}$ is called the **body element** at b, and denoted $\mathrm{T}_b\mathcal{B}$ (the *tangent space* to $\mathcal{B}$ at b). The union of all body elements is denoted $\mathrm{T}\mathcal{B}$ (the *tangent bundle* of $\mathcal{B}$).

The **body gradient** ∇p of a placement p is a tensor field on $\mathcal{B}$, whose value at any given point b, denoted by $\nabla p|_b$, maps linearly the body element $\mathrm{T}_b\mathcal{B}$ onto $\mathrm{V}\mathcal{E}$. We call **stance** any tensor field of this kind, *be it a gradient or not*. Therefore, a stance is any smooth mapping

$$\mathrm{P} : \mathrm{T}\mathcal{B} \to \mathrm{V}\mathcal{E}\,, \tag{2}$$

such that the restriction $\mathrm{P}|\mathrm{T}_b\mathcal{B}$ is a linear embedding, for all $b \in \mathcal{B}$. If a stance happens to be the gradient of a placement, we say that it is **induced** by that placement: all placement induces a stance, but a general stance is not induced by any placement, not even locally. We describe growth by the time evolution of the **relaxed stance** $\mathbb{P}$, while motion is described as the time evolution of the **actual placement** p.

The **complete motion** of a growing body is a family of pairs $(p, \mathbb{P})$ smoothly parametrized by the *time line* $\mathcal{T}$ (identified with the real line), and the velocity **realized** along that motion at the time $\tau \in \mathcal{T}$ is the pair of fields (a superposed dot denoting time differentiation):

$$(\,\dot{p}(\tau), \dot{\mathbb{P}}(\tau)\mathbb{P}(\tau)^{-1}) : \mathcal{B} \to \mathrm{V}\mathcal{E} \times (\mathrm{V}\mathcal{E} \otimes \mathrm{V}\mathcal{E})\,. \tag{3}$$

The linear space of **test velocities** $\mathfrak{T}$, comprising all smooth fields

$$(\mathrm{v}, \mathbb{V}\,) : \mathcal{B} \to \mathrm{V}\mathcal{E} \times (\mathrm{V}\mathcal{E} \otimes \mathrm{V}\mathcal{E})\,, \tag{4}$$

will play a central role in the next subsection. The **visible velocity** of body-points (with physical dimensions length/time) is given by the vector field v, while the *tensor* field $\mathbb{V}$ gives the **growth velocity** of the corresponding body elements (with physical dimensions 1/time).

2.2 Dynamics

To us, a **force** is primarily a continuous linear real-valued *functional* on the space of test velocities, whose *value* we call the **working** expended by that force.[2] We assume that the total working expended on any test velocity $(\mathrm{v}, \mathbb{V}) \in \mathfrak{T}$ admits the following integral representation:

$$\int_{\mathcal{B}} -(\mathrm{s}\cdot\mathrm{v} + \mathbb{C}\cdot\mathbb{V} + \mathbb{S}\cdot\mathrm{Dv}) + \int_{\mathcal{B}} (\mathrm{b}\cdot\mathrm{v} + \mathbb{B}\cdot\mathbb{V}) + \int_{\partial\mathcal{B}} \mathrm{t}_{\partial\mathcal{B}}\cdot\mathrm{v}\,, \tag{5}$$

where the integrals are taken with respect to the *relaxed* volume and surface area of body elements, and D denotes the *relaxed* gradient:

$$\mathrm{Dv} := (\nabla\mathrm{v})\mathbb{P}^{-1}\,. \tag{6}$$

Because of the compound structure of test velocities (4), the force functional splits additively into a **brute force**, dual to v, and a **remodelling force**, dual to $\mathbb{V}$. Another important splitting is between the **inner** working, given by the first bulk integral in (5), and the **outer** working, given by the remaining sum. The **brute self-force** per unit volume s, the **outer brute bulk-force** per unit volume b, and the **brute boundary-force** per unit area $\mathrm{t}_{\partial\mathcal{B}}$ take values in $\mathcal{VE}$; the **remodelling self-couple** per unit volume $\mathbb{C}$, the **brute Piola stress** $\mathbb{S}$ (also a specific couple!), and the **outer remodelling couple** per unit volume $\mathbb{B}$ take values in $\mathcal{VE}\otimes\mathcal{VE}$.

All balance laws are systematically provided by the **principle of null working**: the total working expended on any test velocity should be zero, i.e., the total force should be the null functional. Skipping the balance of brute forces, which is standard, I present here only the **balance of remodelling couples**:

$$-\mathbb{C} + \mathbb{B} = 0 \quad \text{on } \mathcal{B}\,. \tag{7}$$

2.3 Constitutive theory

Our treatment of constitutive issues rests on two pillars (altogether independent of balance): the *principle of material indifference* to change in observer, and the

[2] The physical dimensions of the working are God given: energy/time. Since the prototypal space of test velocities is $\mathcal{VE}$, with physical dimensions length/time, forces get typically identified with elements of $\mathcal{VE}$ itself, with physical dimensions energy/length (= "force" *by definition!*). As velocities have *not* dimensions length/time, in general (think of growth velocity $\mathbb{V}$), a general force has *not* dimensions "force". Another force related issue is that the space $\mathfrak{T}$ of test velocities should be endowed with the structure of a *topological* vector space, so as to make meaningful the physically essential requirement that force functionals be *continuous*. I leave gaps like this to be filled in by the mathematically conscious reader.

dissipation principle. Both of them deliver strict selection rules on the constitutive prescription for the *inner* force. Such *a priori* restrictions do not apply to the outer force, which is regarded as an adjustable control on the process. This inner/outer dichotomy does not pertain to the physics of interactions, but to the limitations of the model: what appears as an outer interaction within a given theory may always be accounted for—in principle—as an inner interaction within a broader and more cumbersome theory. In an all-embracing model there would be no outer interactions at all. In our model of growth mechanics, the *outer* remodelling couple $\mathbb{B}$ has a determinative role whenever growth—be it surface or bulk—is powerfully controlled by non-mechanical phenomena, such as biochemical reactions in living tissues.

I will only flash the outcome of the first principle, as applied by DiCarlo and Quiligotti (2002), while summarizing with some more detail the machinery of the second one. Material indifference rules out non-trivial values of the brute self-force s and non-symmetric values of the *Cauchy stress* $\mathrm{T} := |\det \mathrm{F}|^{-1}\, \mathbb{S}\, \mathrm{F}^{\top}$, where the **warp**

$$\mathrm{F} := \mathrm{D}p = (\nabla p)\, \mathbb{P}^{-1}. \tag{8}$$

measures how the actual stance, i.e., the body gradient of the actual placement, differs from the relaxed stance. If we further assume that the response of the body element at b filters off from $(p, \mathbb{P})$ all information other than $p|_b$, $\nabla p|_b$, and $\mathbb{P}|_b$, we obtain from the same principle that

$$\mathbb{S}(b,\tau) = \mathrm{R}(b,\tau)\, \check{\mathbb{S}}_b(\mathrm{U}|_b, \mathbb{P}|_b, \tau)\,, \qquad \mathbb{C}(b,\tau) = \check{\mathbb{C}}_b(\mathrm{U}|_b, \mathbb{P}|_b, \tau)\,, \tag{9}$$

the *rotation* R and the *stretch* U being, respectively, the orthogonal and the right positive-symmetric factor of the warp (8): $\mathrm{F} = \mathrm{R}\,\mathrm{U}$.

To have a notion of dissipation, an additional energetic descriptor is needed. We postulate the existence of an additive real-valued *free energy* $\Psi(\mathcal{P}) = \int_{\mathcal{P}} \psi$, measuring the inner energy available to body-parts. Since we take integrals over body-parts with respect to the relaxed volume, ψ represents the free energy per unit *relaxed* volume. We call *power expended* along a process at time τ the *opposite* of the working expended by the inner force due to the process on the velocity *realized* at time τ. Hence, the power expended measures the working done by an *outer* force *balanced* with the *constitutively determined* inner force. The **dissipation principle** we enforce requires that the *power dissipated*—defined as the difference between the power expended along a process and the time derivative of the free energy—should be non-negative, for all body-parts, at all times.[3] This localizes into:

$$\dot{\psi} + \psi\, \mathrm{I}\cdot\mathbb{V} \leq \mathbb{S}\cdot \mathrm{Dv} \,+\, \mathbb{C}\cdot\mathbb{V}\,, \tag{10}$$

[3]Due account is to be taken of the fact that the relaxed-volume form, let's say μ, *evolves* in time, as dictated by the growth velocity $\mathbb{V} = \dot{\mathbb{P}}\mathbb{P}^{-1}$: $\dot{\Psi}(\mathcal{P}) = \big(\int_{\mathcal{P}} \psi\, \mu\big)^{\cdot} = \int_{\mathcal{P}} (\psi\, \mu)^{\cdot} = \int_{\mathcal{P}} (\dot{\psi}\, \mu + \psi\, \dot{\mu}) = \int_{\mathcal{P}} (\dot{\psi} + \psi\, \mathrm{I}\cdot\mathbb{V})\, \mu$ (I denotes the identity on $\mathcal{V}\mathcal{E}$; $\mathrm{I}\cdot\mathbb{V}$ is the trace of $\mathbb{V}$).

it being intended that v and $\mathbb{V}$ are given by (3), $\mathbb{S}$ and $\mathbb{C}$ by (9), and ψ has to be related to the process by an extra constitutive mapping. Our **main constitutive assumption** selects a rather special, but very interesting constitutive class, beautifully accounted for by Epstein (1999). We posit that, at each body-point, the present value of the free energy per unit relaxed volume depends only on the *present* value of the *warp* at that point:

$$\psi(b,\tau) = \check{\psi}_b\,(\mathrm{F}(b,\tau))\,. \tag{11}$$

The requirement that (10) be satisfied along all processes is fulfilled if and only if for each b (which will be dropped from now on) the responses $\check{\mathbb{S}}$ and $\check{\mathbb{C}}$ satisfy (∂ denotes differentiation):

$$\check{\mathbb{S}} = \partial\check{\psi} + \overset{+}{\mathbb{S}}\,, \qquad \check{\mathbb{C}} = \mathbb{E} + \overset{+}{\mathbb{C}}\,, \tag{12}$$

with

$$\mathbb{E} := \check{\psi}\,\mathrm{I} - \mathrm{F}^{\top}\check{\mathbb{S}} \tag{13}$$

the **Eshelby coupling** between brute mechanics and remodelling, and the *extra-energetic responses* $\overset{+}{\mathbb{S}}\,, \overset{+}{\mathbb{C}}$ restricted by the **reduced dissipation inequality**

$$\overset{+}{\mathbb{S}}\cdot\dot{\mathrm{F}} + \overset{+}{\mathbb{C}}\cdot\mathbb{V} \geq 0\,, \tag{14}$$

to be abided by in the same sense as (10). As is seen, the Eshelbian coupling is mandatory (within the constitutive class we are considering) and independent of any special assumption on $\check{\mathbb{S}}$. Additional couplings—through the outer remodelling couple $\mathbb{B}$, in particular—are *not* ruled out.

It should be stressed that $\mathbb{B}$, while trivial in most—if not all—applications to "dead" engineering materials, plays a major role even in the simplest biomechanical applications, where it describes the mechanical feedback from the biochemical control system: think of so-called *stress-dependent growth laws* (Taber, 1995). Therefore, the idea—emphasized by Epstein and Maugin (2000)—that the Eshelby coupling is *the* "driving force of irreversible growth" is untenable for any smartly controlled material. The Eshelby coupling *by itself* does not drive any smart growth; rather, it drives a dull—though nontrivial—visco-plastic flow, as shown by DiCarlo, Nardinocchi and Teresi (forthcoming).

3 Piecewise compatible bulk growth

Let $\mathfrak{C}_{\mathrm{P}}$ be a collection of open disjoint *patches*, whose closures cover the body manifold $\mathcal{B}$, and $\mathfrak{C}_{\mathrm{F}}$ the corresponding collection of *interfaces*. In order to cover also the body boundary $\partial\mathcal{B}$ with (closures of) interfaces in $\mathfrak{C}_{\mathtt{F}}$, it is convenient to augment $\mathfrak{C}_{\mathtt{P}}$ with an idle patch $\mathcal{P}_{ext}$, the *exterior* of $\mathcal{B}$.

Now, enforce a *local compatibility constraint* on the relaxed stance $\mathbb{P}$, requiring that on each patch $\mathcal{P} \in \mathfrak{C}_{\mathtt{P}}$ there is a local placement

$$\boldsymbol{p}_{\mathcal{P}} : \mathcal{P} \to \mathcal{E}\,, \tag{15}$$

smooth up to $\partial\mathcal{P}$, such that

$$\mathbb{P}|\mathcal{P} = \nabla \boldsymbol{p}_{\mathcal{P}}\,. \tag{16}$$

Parallelly, restrict the testing of the remodelling force to *piecewise compatible* growth velocities, i.e., on tensor fields $\mathbb{V}$ that, on each patch $\mathcal{P} \in \mathfrak{C}_{\mathtt{P}}$, are the relaxed gradient of a vector field $\mathbf{w}_{\mathcal{P}}$, smooth up to $\partial\mathcal{P}$:

$$\mathbb{V}|\mathcal{P} = \mathrm{D}\mathbf{w}_{\mathcal{P}} = (\nabla \mathbf{w}_{\mathcal{P}})\,\mathbb{P}^{-1}. \tag{17}$$

Then, denoting by $\mathbb{A} := \mathbb{B} - \mathbb{C}$ the total remodelling couple per unit relaxed volume ($\mathbb{A} = 0$ in $\mathcal{P}_{ext}$), the working expended on $\mathbb{V}$ is given by

$$\begin{aligned}\int_{\mathcal{B}} \mathbb{A}\cdot\mathbb{V} &= \sum_{\mathcal{P}\in\mathfrak{C}_{\mathtt{P}}} \int_{\mathcal{P}} \mathbb{A}\cdot\mathbb{V} = \sum_{\mathcal{P}\in\mathfrak{C}_{\mathtt{P}}} \int_{\mathcal{P}} \mathbb{A}\cdot(\mathrm{D}\mathbf{w}_{\mathcal{P}}) \\ &= \sum_{\mathcal{P}\in\mathfrak{C}_{\mathtt{P}}} \left(\int_{\partial\mathcal{P}} (\mathbb{A}\,\mathbf{n}_{\partial\mathcal{P}})\cdot\mathbf{w}_{\mathcal{P}} - \int_{\mathcal{P}} (\mathrm{Div}\,\mathbb{A})\cdot\mathbf{w}_{\mathcal{P}} \right),\end{aligned} \tag{18}$$

with $\mathbf{n}_{\partial\mathcal{P}}$ the *outward* unit normal to the relaxed shape of $\partial\mathcal{P}$. Hence, the principle of null working yields the following balances: for each $\mathcal{P} \in \mathfrak{C}_{\mathtt{P}}$,

$$\mathrm{Div}\,(\mathbb{E} + \overset{+}{\mathbb{C}} - \mathbb{B}) = 0 \quad \text{on } \mathcal{P}\,, \tag{19a}$$

$$(\mathbb{E} + \overset{+}{\mathbb{C}} - \mathbb{B})\,\mathbf{n}_{\partial\mathcal{P}} = 0 \quad \text{on } \partial\mathcal{P}\,. \tag{19b}$$

This set compares with, but is weaker than, (7) plus (9_2) and (12_2).

Eq. (19a) matches with the *configurational force balance* (5–10) on page 37 of Gurtin (2000), provided that his *configurational stress* $\boldsymbol{C}$ is identified with $\mathbb{E}$—as implied by the correspondence between (13) and the *Eshelby relation* (6–9) on page 43—, and his *configurational body forces*, *internal* $\boldsymbol{g}$ and *external* $\boldsymbol{e}$, are related to my inner and outer remodelling couples by

$$\boldsymbol{g} \leftarrow \mathrm{Div}\,\overset{+}{\mathbb{C}}\,, \qquad \boldsymbol{e} \leftarrow -\mathrm{Div}\,\mathbb{B} \tag{20}$$

Here and in the following, *assignment statements* such as

$$q_{\,surf} \leftarrow T\, q_{\,bulk} \tag{21}$$

should be read as meaning that the constitutive assignment for quantity $q_{\,surf}$ in surface-growth theory is obtained as the T-image of the constitutive assignment for quantity $q_{\,bulk}$ in bulk-growth theory.

Analogously, (19b) agrees (modulo an obvious change in notation) with the *interfacial force balance* (7.7_2) given by Cermelli and Gurtin (1994, Part B: theory of *incoherent* interfaces *without* interfacial structure), provided that their *interfacial configurational forces*, *internal* $\mathbf{e}_{\partial\mathcal{P}}$ and *external* $\boldsymbol{f}_{\partial\mathcal{P}}$, are assigned according to

$$\mathbf{e}_{\partial\mathcal{P}} \leftarrow \overset{+}{\mathbb{C}}\,\mathbf{n}_{\partial\mathcal{P}}\,, \qquad \boldsymbol{f}_{\partial\mathcal{P}} \leftarrow -\mathbb{B}\,\mathbf{n}_{\partial\mathcal{P}} \tag{22}$$

Imitating Gurtin (2000), I will now concentrate on coherent interfaces.

4 Coherent interfaces

Interface coherency is obtained by enforcing the further constraint that the local relaxed placement in any two adjoining patches $\mathcal{P}_+$, $\mathcal{P}_-$ be *continuous* across the interface $\mathcal{S} := \partial\mathcal{P}_+ \cap \partial\mathcal{P}_-$: the jump across $\mathcal{S}$, $[\![\,\boldsymbol{p}\,]\!]_{\mathcal{S}} := \boldsymbol{p}^+_{\mathcal{S}} - \boldsymbol{p}^-_{\mathcal{S}}$ (the limit from $\mathcal{P}_+$ minus that from $\mathcal{P}_-$), should vanish. If test velocities are analogously restricted by the condition $[\![\,\mathbf{w}\,]\!]_{\mathcal{S}} = 0$ on all $\mathcal{S} \in \mathfrak{C}_{\mathrm{F}}$, then the integrals over patch boundaries in (18) yield

$$\sum_{\mathcal{P}\in\mathfrak{C}_{\mathrm{P}}} \int_{\partial\mathcal{P}} (\,\mathbb{A}\,\mathbf{n}_{\partial\mathcal{P}}) \cdot \mathbf{w}_{\mathcal{P}} = -\sum_{\mathcal{S}\in\mathfrak{C}_{\mathrm{F}}} \int_{\mathcal{S}} (\,[\![\,\mathbb{A}\,]\!]_{\mathcal{S}} \mathbf{m}_{\mathcal{S}}) \cdot \mathbf{w}_{\mathcal{S}}\,, \tag{23}$$

where $\mathbf{w}_{\mathcal{S}} := \mathbf{w}^+_{\mathcal{S}} = \mathbf{w}^-_{\mathcal{S}}$, $\mathbf{m}_{\mathcal{S}} := \mathbf{n}_{\partial\mathcal{P}_-} = -\mathbf{n}_{\partial\mathcal{P}_+}$ (pay attention to signs and draw a sketch!). Hence, on each $\mathcal{S} \in \mathfrak{C}_{\mathrm{F}}$,

$$[\![\,\mathbb{E} + \overset{+}{\mathbb{C}} - \mathbb{B}\,]\!]_{\mathcal{S}}\,\mathbf{m}_{\mathcal{S}} = 0\,, \tag{24}$$

in accord with the *configurational force balance at the interface* (11–8b) on page 68 of (Gurtin, 2000), provided his *interface configurational forces*, *internal* $\boldsymbol{g}^{\mathcal{S}}$ and *external* $\boldsymbol{e}^{\mathcal{S}}$, be given by

$$\boldsymbol{g}^{\mathcal{S}} \leftarrow [\![\,\overset{+}{\mathbb{C}}\,]\!]_{\mathcal{S}}\mathbf{m}_{\mathcal{S}}\,, \qquad \boldsymbol{e}^{\mathcal{S}} \leftarrow -[\![\,\mathbb{B}\,]\!]_{\mathcal{S}}\mathbf{m}_{\mathcal{S}}\,. \tag{25}$$

To obtain a genuine surface-growth theory, the growth process should be further specialized, confining the *realized* growth velocity $\mathbb{V} = \dot{\mathbb{P}}\mathbb{P}^{-1}$ (cf. (3)) in thin layers around interfaces. For $\varepsilon > 0$ small enough, let

$$\mathcal{L}_\varepsilon := \{\,\boldsymbol{p}_{\mathcal{S}}(b) + \zeta\,\mathbf{m}_{\mathcal{S}}(b) \,|\, b \in \mathcal{S}\,, -\varepsilon < \zeta < \varepsilon\,\} \tag{26}$$

be the ε-thickening of the relaxed shape of an interface $\mathcal{S}$, and $\mathcal{L}_\varepsilon^+$, $\mathcal{L}_\varepsilon^-$ its upper ($\zeta > 0$) and lower ($\zeta < 0$) halves, respectively. Assume then (my definition of choice for tensor multiplication implying that $(\mathbf{m}\otimes\mathbf{w})\,\mathbf{r} = (\mathbf{m}\cdot\mathbf{r})\,\mathbf{w}$):

$$\mathbb{V} = \pm\varepsilon^{-1}\,\mathbf{m}_{\mathcal{S}}\otimes\mathbf{w}_{\mathcal{S}} \quad \text{on } \mathcal{L}_\varepsilon^\mp\,, \quad \mathbb{V} = 0 \quad \text{elsewhere.} \tag{27}$$

Notice that the minus sign prevails in $\mathcal{L}_\varepsilon^+$, and vice versa: the tensor field (27) is, to within $O(1)$ terms for $\varepsilon\downarrow 0$ on most of $\mathcal{L}_\varepsilon$, the relaxed gradient of the vector field given by $(b,\zeta)\mapsto(\,\varepsilon-|\zeta|\,)\,\mathbf{w}_{\mathcal{S}}(b)$ inside $\mathcal{L}_\varepsilon$ and vanishing outside it. The remodelling force, when evaluated on test fields having the structure (27), in the limit for $\varepsilon\downarrow 0$ yields a pure surface working:

$$\lim_{\varepsilon\downarrow 0}\int_{\mathcal{B}}\mathbb{A}\cdot\mathbb{V} \;=\; -\sum_{\mathcal{S}\in\mathfrak{C}_{\mathrm{F}}}\int_{\mathcal{S}}(\,[\![\,\mathbb{A}\,]\!]_{\mathcal{S}}\mathbf{m}_{\mathcal{S}})\cdot\mathbf{w}_{\mathcal{S}}\,, \tag{28}$$

in agreement with (23) and (18), where the bulk integrals appearing in the last sum asymptotically vanish. Eqs. (27) and (28) explain how the remodelling *couples* dominating bulk growth induce configurational *forces*—in the sense of Gurtin—in the limit theory of surface growth (cf. (25)).

I now integrate the reduced dissipation inequality (14) over $\mathcal{L}_\varepsilon$ and take the limit for $\varepsilon\downarrow 0$, assuming that the brute dissipation per unit volume $\overset{+}{\mathbb{S}}\cdot\dot{\mathrm{F}}$ is $O(1)$. I thus obtain the **surface dissipation inequality**

$$([\![\,\overset{+}{\mathbb{C}}\,]\!]_{\mathcal{S}}\mathbf{m}_{\mathcal{S}})\cdot\mathbf{w}_{\mathcal{S}} \;=\; \boldsymbol{g}^{\mathcal{S}}\cdot\mathbf{w}_{\mathcal{S}} \;\leq\; 0 \tag{29}$$

(recall (25_1)). Now, if *invariance under reparametrization* of the interface $\mathcal{S}$ is required, implying that the working depends only on the (scalar) *normal velocity* of the interface $V_{\mathcal{S}} := \mathbf{w}_{\mathcal{S}}\cdot\mathbf{m}_{\mathcal{S}}$, then the internal configurational force on the interface is necessarily *normal*:

$$\boldsymbol{g}^{\mathcal{S}} = g^{\mathcal{S}}\,\mathbf{m}_{\mathcal{S}}\,, \tag{30}$$

with $g^{\mathcal{S}}$ scalar-valued (cf. (11–11) on page 69 of Gurtin (2000)), and (29) coincides with the *interfacial dissipation inequality* (11–21) on page 71 of Gurtin (2000): $g^{\mathcal{S}}\,V_{\mathcal{S}} \leq 0$.

5 Concluding remarks

Lack of space prevents me from treating theories with interfacial structure, intersecting interfaces (junctions), or growing cracks (fracture). To allow for interfacial structure, the integral representation of the working (5) should be extended to include *measures* concentrated on interfaces, both for brute and remodelling forces;

in (18), this would bring out an extra sum over $\mathfrak{C}_{\mathsf{F}}$ of area integrals over interfaces. Junctions and cracks require concentrations on subsets of higher *codimension*: the intersection of two or more interfaces, or the crack tip.

As a closing remark, let me quote the introduction to Gurtin (2000): "Indetermination arises in the configurational system whenever there is no change in material structure." This is exactly why the theory of configurational forces, applied to changes in material structure confined to meagre subsets of the body manifold, is fraught with indeterminacy. The *bulk* balance (19a) (coincident with (5–10) on page 37 of Gurtin (2000)), while much weaker than (7), is altogether superfluous in *surface* growth, being identically satisfied by a *reactive* internal body force $\boldsymbol{g}$ (introduced in (20)), which is the reaction to the constraint of no structural change in the bulk. This does not help in advocating the independence and usefulness of the configurational force system. However, bulk growth gives us our revenge.

Acknowledgments

I am grateful to the colloquium organisers, to the discussants at the colloquium and to the anonymous referee for prompting me, in various ways, to organize my ideas and to write them down. My work was supported by GNFM-INdAM (the Italian Group for Mathematical Physics), and by MIUR (the Italian Ministry of Education, University and Research) through the Project "Mathematical Models for Materials Science."

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