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By

A. Bendali

and

Y. Boubendir

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: http://www.ima.umn.edu

NON-OVERLAPPING DOMAIN DECOMPOSITION METHOD FOR A NODAL FINITE ELEMENT METHOD

A. BENDALI* AND Y. BOUBENDIR †

Abstract. A new approach is proposed to deal with degrees of freedom shared by more than two subdomains in the context of a non-overlapping domain decomposition procedure for solving a system related to a nodal finite element method. The main feature of the method is to preserve the continuity requirements on the unknowns and the finite element equations at these degrees of freedom. We prove that the corresponding algorithm can be seen as a converging iterative method to solve the finite element system and that it cannot break down. Each iteration is obtained by solving uncoupled finite element systems posed in each subdomain and, in contrast to a strict domain decomposition method, is completed by solving a linear system whose unknowns are the above special degrees of freedom.

Key words. Domain decomposition methods, nodal finite element method, cross-points, stability, convergence.

1. Introduction. A difficulty facing of non-overlapping domain decomposition methods as those initiated by P.-L. Lions [16] for static problems and adapted to the time-harmonic case by B. Després [10, 11, 12], is the problem of dealing with points shared by more than two subdomains. Hereafter such points will be called "cross-points" for convenience. Indeed, for discretizations based on a typical nodal finite element method, a cross-point may support one or several degrees of freedom, each shared by more than two subdomains. The usual equivalent writing of the continuity conditions, on which the Lions, Després and some discrete FETI methods [9, 14] are based, does not remain valid at these special points.

Two approaches have been used to overcome this difficulty. The first one was introduced by Després [10] and used later by Collino, Ghanemi and Joly [8]. The basic principle is to solve the local problems posed in each subdomain by means of a mixed finite element method. For such methods, the degrees of freedom are located at the interior of either the elements or the interfaces separating two of them. So none of these degrees of freedom is supported by the cross-points, being thus shared by more than two subdomains. This cleaver approach may however complicate the solving process. The size of the problem can be increased by a factor two or three even in the two-dimensional case. One is led to solve saddle point problems instead of the usual one field equations (see, for instance, [6]). Another drawback of the mixed finite element approach is that it is not easy to use in the context of certain systems, such as those in elasticity and electromagnetism. In the second approach, mainly met in the context of FETI methods (see e.g., [9, 13, 14]), a cross-point is treated like any other point on the interface, roughly speaking. As a result, the continuity conditions are written in terms of an

* Laboratoire MIP, UMR 5640, INSA-CNRS-UPS, INSA (GMM), 135 avenue de Rangueil, 31057 Toulouse CEDEX 4, France CEDEX 1, France, bendali@insa-toulouse.fr.

† School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St., §. E. Minneapolis, MN 55455, boubendi@math.umn.edu.

overdetermined system which is solved using a Krylov method [9, 14]. We shall give some details on this subject below. The procedure is thus a purely algebraic process with no a priori guarantee of stability and convergence.

The approach adopted in this paper consists in keeping unchanged each unknown and each equation in the finite element system corresponding to a cross-point. Only those degrees of freedom shared by two subdomains, and the matching conditions related to them, are written in the relaxed form, making use of the algorithm of Lions [16] and Després [11]. The procedure we propose applies either to coercive or non coercive formulations, although the respective treatments are slightly different. Really, the distinction is between static problems describing the state of a physical system which is not varying with time or obtained by a discretization process in time in one hand and time-harmonic problems for which an implicit sinusoidal time dependence is assumed in the other hand. Our domain decomposition method applies also to systems. However, for the sake of simplicity, we consider scalar second order boundary-value problems only. These two classes of problems will be enough to get a good insight into the convergence and the stability properties of the method. Several numerical experiments validating this approach can be found in [5, 1, 4]. Parts of these results have been announced without proofs in [3].

The outline of the paper is as follows. In section 2, we introduce the two types of boundary-value problems to be considered. We recall the main properties of their variational formulation, as well as their approximation by the lowest order Lagrange finite method on triangles or tetrahedrons. We next describe the domain decomposition procedure, which can be interpreted as an iterative procedure for solving the system obtained by the finite element discretization. In section 3, we prove some stability and convergence properties of the iterative procedure. We end this paper by some concluding remarks concerning the scalability of the algorithm, that is, its convergence at a rate independent of the mesh, and some issues which are not yet covered by the present theoretical study.

2. The domain decomposition method.

2.1. Two types of boundary-value problems. Let Ω be a bounded domain of $\mathbb{R}^N$ ($N = 2$ or 3). We assume that Ω is a polygonal domain in the two-dimensional case (see for instance, figure 1), and a polyhedral domain for the three-dimensional case. We denote by Γ the boundary of Ω and by $\mathbf{n}$ the unit normal to Γ, directed outward from Ω.

Let there also be given $\{a_{ij}\}_{1\leq i,j\leq N}$, a_0 in $L^\infty(\Omega)$ satisfying

$$\left.\begin{array}{l}\exists\alpha>0 : \Re\left(\sum\limits_{i,j=1}^{N} a_{ij}(x)\xi_j\overline{\xi_i}\right) \geq \alpha\left|\xi\right|^2, \\ \Im\left(\sum\limits_{i,j=1}^{N} a_{ij}(x)\xi_j\overline{\xi_i}\right) \leq 0,\end{array}\right. \quad \forall\xi\in\mathbb{C}^N, \text{ a.e. for } x\in\Omega, \tag{2.1}$$

where $\Re z$ and $\Im z$ are respectively the real and the imaginary part of the complex

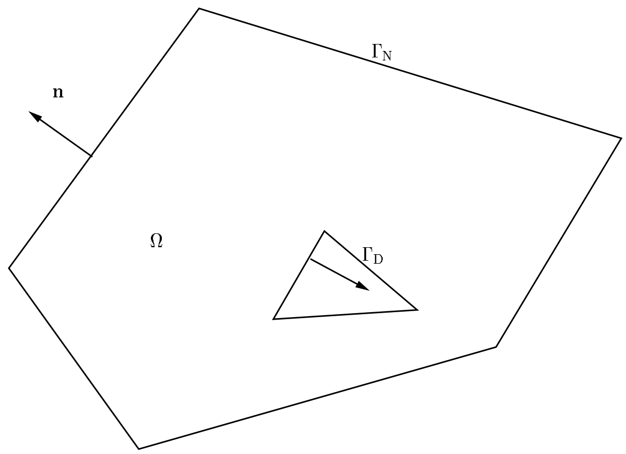

FIG. 1. *Example of 2D initial domain.*

number z. Standard notation from the theory of partial differential equation will be used without further comment (see, e.g., [19]). The consideration of function a_0 makes it possible to include in this framework problems resulting from a semi-discretization in time by means of an implicit method. We therefore assume that

$$\Re\left(a_0(x)\right) \geq 0,\ \Im\left(a_0(x)\right) \leq 0, \quad \text{a.e. for } x \in \Omega. \tag{2.2}$$

To deal with the two kinds of usual boundary conditions, we furthermore consider a non-overlapping partition of Γ into Γ_D and Γ_N as well as a function $\lambda \in L^\infty(\Gamma_N)$ satisfying

$$\Re\left(\lambda(x)\right) \geq 0,\ \Im\left(\lambda(x)\right) \leq 0, \quad \text{a.e. for } x \in \Gamma_N. \tag{2.3}$$

In time-harmonic problems, the hypotheses on the imaginary part of the data correspond to an assumed time dependence on $e^{-i\omega t}$, where $\omega > 0$ is the pulsation of the wave. For the sake of simplicity, we restrict the discussion to homogeneous boundary conditions; the extension to more general cases can be dealt with similarly. In the same way, we suppose that the right-hand side of the interior partial differential equation is a given function $f \in L^2(\Omega)$, but the consideration of more general data does not cause any further difficulty.

Coercive problems. The boundary-value problems of coercive type can be stated in the following common setting:

$$\begin{cases} u \in H^1(\Omega), \\ -\sum\limits_{i,j=1}^{N} \partial_{x_i} a_{ij} \partial_{x_j} u + a_0 u = f \text{ in } \mathcal{D}'(\Omega), \\ \sum\limits_{i,j=1}^{N} a_{ij} \partial_{x_j} u n_i + \lambda u = 0 \text{ on } \Gamma_N,\ u = 0 \text{ on } \Gamma_D. \end{cases} \tag{2.4}$$

To be solved by a finite element method, the above problem must first to be put into the following variational formulation:

$$\left\{ \begin{array}{l} u \in X, \quad \forall v \in X, \\ a(u,v) = Lv \end{array} \right. \tag{2.5}$$

where

$$X := \left\{ v \in H^1(\Omega);\ v|_{\Gamma_D} = 0 \right\} \tag{2.6}$$

$$a(u,v) := \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij} \partial_{x_j} u \partial_{x_i} v + a_0 uv \right) d\Omega + \int_{\Gamma_N} \lambda uv \, d\Gamma \tag{2.7}$$

$$Lv := \int_\Omega fv \, d\Omega. \tag{2.8}$$

The conditions (2.1,2.2,2.3) together with the usual Poincaré inequality, imply that problems (2.5), and, consequently, (2.4) admit one and only one solution.

Non coercive problems. Now we turn our attention to the second kind of boundary-value problem. Additional data and properties are required. To follow the usual setup of time-harmonic problems, we suppose that $b_0 \in L^\infty(\Omega)$ satisfies

$$\Re\left(b_0(x)\right) \leq 0, \quad \Im\left(b_0(x)\right) \leq 0, \quad \text{a.e. for } x \in \Omega, \tag{2.9}$$

and that, in addition to (2.3), λ verifies the strict dissipative condition

$$\exists \lambda_0 > 0 : -\Im\left(\lambda(x)\right) \geq \lambda_0, \quad \text{a.e. for } x \in \Gamma_N. \tag{2.10}$$

The statement of the boundary-value problem is obtained by replacing a_0 by b_0 in (2.4). However, since $a_0 = 0$ is not an excluded case, we can deal with the non coercive problem simply by adding to the bilinear form $a(\cdot,\cdot)$ the term

$$b(u,v) = \int_\Omega b_0 uv \, d\Omega \tag{2.11}$$

and by assuming that λ satisfies the more restrictive condition (2.10). This makes it possible to have the same framework for the coercive and the non coercive case. The variational formulation of the boundary-value problem of time-harmonic type can thus be written as follows

$$\left\{ \begin{array}{l} u \in X, \quad \forall v \in X, \\ a(u,v) + b(u,v) = Lv. \end{array} \right. \tag{2.12}$$

The existence and uniqueness of a solution to (2.12) are more complicated than in the coercive case. Indeed, the uniqueness now ensures the existence of a solution through the Fredholm alternative (see for instance, [19]). The uniqueness is a consequence of the following fact: by (2.10), any solution u corresponding to $L = 0$ can be extended by zero across Γ_N. One then has to resort to a kind of analytic continuation principle, requiring a bit more smoothness for the coefficients a_{ij}, a_0 and b_0 than $L^\infty(\Omega)$. For instance, assuming that these functions are piecewise smooth is enough (see, [15] for a precise statement and details). Henceforth, we assume that such regularity conditions hold. Also, additional difficulties of the non coercive case will be considered when we analyze the stability and the convergence of the domain decomposition algorithm.

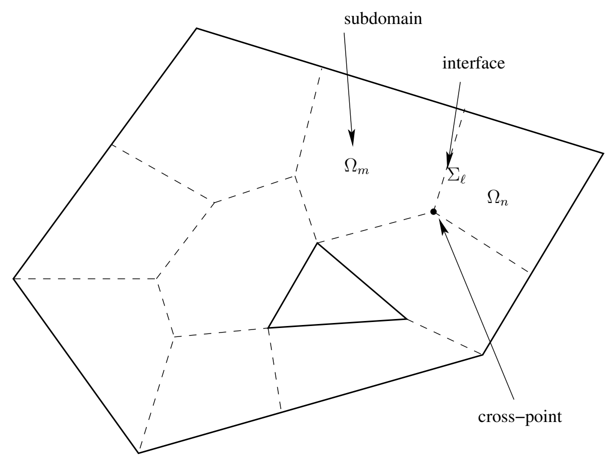

FIG. 2. *Non-overlapping decomposition of the domain Ω depicted in the figure 1.*

2.2. The finite element discretization. We intend to couple a non-overlapping domain decomposition and a finite element method for solving problems (2.5) and (2.12). Thus, we start from a non-overlapping decomposition $\{\Omega_m\}_{m=1}^{m=N_d}$ of Ω into polygonal or polyhedral subdomains (figure 2).

We then consider a regular mesh $\mathcal{T}^h$ of Ω, in triangles or tetrahedrons according to its dimension, compatible with the decomposition of Γ in Γ_D and Γ_N as usual. We assume that $\mathcal{T}^h$ is also compatible with the non-overlapping decomposition of Ω, in the sens that it induces a mesh $\mathcal{T}_m^h$ on each subdomain Ω_m ($m = 1, \ldots, N_d$). As usual, h stands for the mesh size, that is, $h := \max h_T$, where T is a generic element of the mesh and h_T is the longest side of T.

Using the lowest-order Lagrange finite element method (but higher-order methods can be considered as well), we define

$$X^h := \left\{v^h \in \mathcal{C}^0(\overline{\Omega});\ v^h|_T \in \mathbb{P}_1,\ \forall T \in \mathcal{T}^h,\ v^h|_{\Gamma_D} = 0\right\}$$

where $\mathbb{P}_1$ is the space of polynomials of degree less or equal to 1 in N independent variables.

We may now state the discrete versions of problems (2.5) and (2.12) as, respectively,

$$\begin{cases} u^h \in X^h, \quad \forall v^h \in X^h, \\ a(u^h, v^h) = Lv^h, \end{cases} \tag{2.13}$$

and

$$\begin{cases} u^h \in X^h, \quad \forall v^h \in X^h, \\ a(u^h, v^h) + b(u^h, v^h) = Lv^h. \end{cases} \tag{2.14}$$

Here again, it is precisely the coercivity that implies that problem (2.13) is well-posed. For problem (2.14), such a conclusion is guaranteed only for $0 < h \leq h_0$; in general, h_0 is not known explicitly.

2.3. Equivalent writing of the discrete problem. The domain decomposition procedure we consider can be seen as an efficient iterative method to solve discrete problem (2.13) or (2.14). It is based on a relaxation process of the continuity constraints at the interfaces. These interfaces can be defined in a precise way by considering all the sets

$$\overline{\Sigma}_\ell = \overline{\Omega}_{m_1^\ell} \cap \overline{\Omega}_{m_2^\ell} \quad (\ell = 1, \ldots, N_f), \tag{2.15}$$

whose respective interior Σ_ℓ (relative to the induced topology on the boundary) are non-empty. The numbers m_j^ℓ ($j = 1, 2;\ \ell = 1, \ldots, N_f$) play the role of a connectivity table for the interfaces. They will be helpful in the description of the domain decomposition algorithm below. Setting

$$u_m^h := u^h|_{\Omega_m} \quad (m = 1, \ldots, N_d), \tag{2.16}$$

the continuity constraints are either expressed in a strong way on the unknowns by

$$u^h_{m_1^\ell}|_{\overline{\Sigma}_\ell} = u^h_{m_2^\ell}|_{\overline{\Sigma}_\ell} \tag{2.17}$$

or in a weak form on the normal derivatives by means of the same conditions on the testing functions

$$v^h_{m_1^\ell}|_{\overline{\Sigma}_\ell} = v^h_{m_2^\ell}|_{\overline{\Sigma}_\ell}. \tag{2.18}$$

The space spanned by v_m^h will be called X_m^h.

Let us denote by $\mathcal{N}_c$ the set of all of the vertices (nodes) lying on the boundary $\partial\Sigma_\ell := \overline{\Sigma}_\ell \backslash \Sigma_\ell$ of some $\overline{\Sigma}_\ell$. The space

$$X_{\mathrm{B}}^h := \big\{ v^h \in L^2(\Omega);\ v_m^h \in X_m^h \\ \text{and } v^h \text{ is continuous at every node in } \mathcal{N}_c \big\} \tag{2.19}$$

expresses that the continuity constraints at the interfaces are maintained at the cross-points only. The index B indicates that X_{B}^h is the “broken” version of the space X^h. Strictly speaking, not every point in $\mathcal{N}_c$ is a cross-point. A point on the boundary of Ω and shared by only two subdomains is unduly counted as a cross-point. However, taking this into account would lead to unnecessarily complications.

Every function v^h in X_{B}^h can be decomposed in a single way via its nodal values in the following form:

$$v^h = \sum_{m=1}^{N_d} v_{m,0}^h + \sum_{\ell=1}^{N_f} (v_{1,\ell}^h + v_{2,\ell}^h) + v_c^h \tag{2.20}$$

where

- $v_{m,0}^h \in X^h$ is equal to 0 outside Ω_m and hence can be considered to belong to X_m^h,
- $v_{j,\ell}^h \in X_{m_j^\ell}^h$ has all its nodal values equal to 0, except for those located at the interior Σ_ℓ of the interface $\overline{\Sigma}_\ell$,
- v_c^h is a function in X^h whose all nodal values are equal to 0, except for those corresponding to a node in $\mathcal{N}_c$.

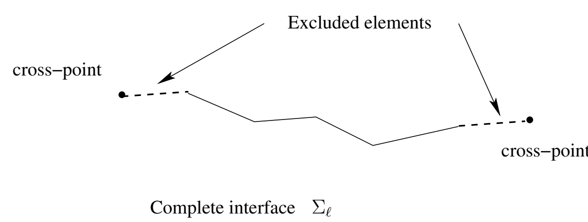

FIG. 3. *Part Σ_ℓ^h of a 2D interface $\Sigma\ell$.*

We now re-express the conditions for a function in X_{B}^h to be in X^h. To this end, we first remark that the mesh $\mathcal{T}^h$ induces a mesh $\mathcal{T}_{\Sigma_\ell}^h$ (in segments for $N = 2$ and in triangles for $N = 3$) on each interface Σ_ℓ. Let us then define Σ_ℓ^h to be the part of Σ_ℓ obtained by excluding from $\overline{\Sigma}_\ell$ any $T' \in \mathcal{T}_{\Sigma_\ell}^h$ having one or more of its vertices on $\partial\Sigma_\ell$ (see figure 3). The following bilinear form $c_\ell^h(\cdot,\cdot)$ yields the scalar product of $L^2(\Sigma_\ell^h)$

$$c_\ell^h(u,v) = \int_{\Sigma_\ell^h} uv \, d\Sigma_\ell^h, \quad u, v \in L^2(\Sigma_\ell^h). \tag{2.21}$$

Finally, we also need the following space

$$Y_\ell^h := \left\{ y^h \in L^2(\Sigma_\ell^h);\ \exists v^h \in X^h \text{ such that } y^h = v^h|_{\Sigma_\ell^h} \right\}. \tag{2.22}$$

Now, denoting by $\widetilde{a}_m(\cdot,\cdot)$ the restriction of either the bilinear form $a(\cdot,\cdot)$ or $(a+b)(\cdot,\cdot)$ to Ω_m, according to the case being considered, and by L_m that of L, both problems (2.13) and (2.14) can be written in the same form:

$$\left\{\begin{array}{l} u^h \in X_{\mathrm{B}}^h, \quad \forall v^h \in X_{\mathrm{B}}^h, \\ \widetilde{a}_m(u_m^h, v_{m,0}^h) = L_m v_{m,0}^h \; (m = 1, \dots, N_d), \\ \left.\begin{array}{r} \widetilde{a}_{m_1^\ell}(u_{m_1^\ell}^h, v_{1,\ell}^h) + \widetilde{a}_{m_2^\ell}(u_{m_2^\ell,0}^h, v_{2,\ell}^h) \\ = L_{m_1^\ell} v_{1,\ell}^h + L_{m_2^\ell} v_{2,\ell}^h \end{array}\right. \quad (\ell = 1, \dots, N_f), \\ \sum_{m=1}^{N_d} \widetilde{a}_m(u_m^h, v_c^h) = L v_c^h, \end{array}\right. \tag{2.23}$$

where u^h and v^h must satisfy the following equivalent matching conditions

$$c_\ell^h(u_{m_1^\ell}^h, y_\ell^h) = c_\ell^h(u_{m_2^\ell}^h, y_\ell^h), \quad \forall y_\ell^h \in Y_\ell^h, \; (\ell = 1, \dots, N_f), \tag{2.24}$$

$$v_{1,\ell}^h|_{\Sigma_\ell^h} = v_{2,\ell}^h|_{\Sigma_\ell^h} \; (\ell = 1, \dots, N_f). \tag{2.25}$$

Finally, let us define

$$Y^h = \prod_{\ell=1}^{N_f} Y_\ell^h \times Y_\ell^h.$$

A generic element $y^h \in Y^h$ will be written in the following form, compatible with notation (2.17):

$$y^h = \left\{(y_{1,\ell}^h, y_{2,\ell}^h\right\}_{\ell=1}^{\ell=N_f}. \tag{2.26}$$

The domain decomposition procedure is based on the following equivalent formulation of the finite element systems (2.13) or (2.14), making use of the interface unknowns $y^h \in Y^h$.

PROPOSITION 2.1. *Let $\beta \neq 0$ be given a complex number. Then, $u^h \in X_{\mathrm{B}}^h$ satisfies system (2.23), (2.24) with testing functions v^h satisfying (2.25), if and only if $(u^h, y^h) \in X_{\mathrm{B}}^h \times Y^h$ is a solution to the system*

$$\left\{\begin{array}{l} u^h \in X_{\mathrm{B}}^h, \quad \forall v^h \in X_{\mathrm{B}}^h, \\ \sum_{m=1}^{N_d} \widetilde{a}_m(u_m^h, v_m^h) + \beta c_m^h(u_m^h, v_m^h) = \sum_{m=1}^{N_d} (c_m^h(y_m^h, v_m^h) + L_m v_m^h), \end{array}\right. \tag{2.27}$$

$$\left\{\begin{array}{l} y_{1,\ell}^h = -y_{2,\ell}^h + 2\beta u_{m_2^\ell}^h|_{\Sigma_\ell^h}, \\ y_{2,\ell}^h = -y_{1,\ell}^h + 2\beta u_{m_1^\ell}^h|_{\Sigma_\ell^h}, \end{array}\right. \quad (\ell = 1, \dots, N_f). \tag{2.28}$$

Here, we have used the convenient abuses of notation

$$c_m^h(y_m^h, v_m^h) := \sum_{\ell \in \Lambda_m} c_\ell^h(y_{i_m^\ell,\ell}^h, v_m^h) \tag{2.29}$$

in which Λ_m stands for the set of indices ℓ such that $\overline{\Sigma}_\ell$ is an interface separating Ω_m from another subdomain Ω_n, and the table i_m^ℓ is defined for $\ell \in \Lambda_m$ by $m = m_j^\ell$ and $j = i_m^\ell$.

Proof. First, observe that (2.27) is a just a condensed way to write the following system:

$$\begin{cases} u^h \in X_{\mathrm{B}}^h, \quad \forall v^h \in X_{\mathrm{B}}^h, \\ \widetilde{a}_m(u_{m,}^h, v_{m,0}^h) = L_m v_{m,0}^h \quad (m = 1, \ldots, N_d)\,, \\ \begin{cases} \widetilde{a}_{m_1^\ell}(u_{m_1^\ell}^h, v_{1,\ell}^h) + \beta c_\ell^h(u_{m_1^\ell}^h, v_{1,\ell}^h) = c_\ell^h(y_{1,\ell}^h, v_{1,\ell}^h) + L_{m_1^\ell} v_{1,\ell}^h \\ \widetilde{a}_{m_2^\ell}(u_{m_2^\ell}^h, v_{2,\ell}^h) + \beta c_\ell^h(u_{m_2^\ell}^h, v_{2,\ell}^h) = c_\ell^h(y_{2,\ell}^h, v_{2,\ell}^h) + L_{m_2^\ell} v_{2,\ell}^h \end{cases} \\ \qquad\qquad (\ell = 1, \ldots, N_f), \\ \sum_{m=1}^{N_d} \widetilde{a}_m(u_m^h, v_c^h) = L v_c^h. \end{cases}$$

The proof can then be completed in a straightforward way by noting that conditions (2.28) are precisely equivalent to (2.24) and

$$y_{1,\ell}^h + y_{2,\ell}^h = \beta(u_{m_1^\ell}^h|_{\Sigma_\ell^h} + u_{m_2^\ell}^h|_{\Sigma_\ell^h}).$$

□

2.4. The domain decomposition algorithm. The domain decomposition algorithm is based on the following observation. For a given $y^h \in Y^h$, the substructured problem (2.27) can be solved yielding $S(L, y^h) \in Y^h$ with

$$\left\{S(L, y^h)\right\}_{j,\ell} = -y_{j,\ell}^h + 2\beta u_{m_j^\ell}^h|_{\Sigma_\ell^h} \quad (j = 1, 2;\ \ell = 1, \ldots, N_f). \tag{2.30}$$

We shall prove below that this system is well-posed always in the coercive case, and at least if h is sufficiently small in the non coercive one. In addition, its solution can be obtained by solving a block quasi-diagonal system of the following form:

$$\begin{bmatrix} A_{11} & & & A_{1C} \\ & \ddots & & \vdots \\ & & A_{N_dN_d} & A_{N_dC} \\ A_{C1} & \cdots & A_{CN_d} & A_{CC} \end{bmatrix} \begin{bmatrix} U_1 \\ \vdots \\ U_{N_d} \\ U_C \end{bmatrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_{N_d} \\ B_C \end{bmatrix} \tag{2.31}$$

where U_m are the nodal values of u_m^h, omitting those corresponding to cross-point, and U_C is a column-vector whose components are the nodal values corresponding to the points belonging to $\mathcal{N}_c$. We shall prove below that this system can be solved through a Schur complement technique reducing the coupling at cross-points to a simple post-processing of uncoupled finite element solutions in each subdomain.

As a result, the linear operator $(L, y^h) \rightarrow S(L, y^h)$ is well-defined. The linear operator $\Pi : Y^h \rightarrow Y^h$, which exchanges data between subdomains at the interfaces, is defined as follows

$$\left(\Pi y^h\right)_{1,\ell} := y_{2,\ell}^h, \quad \left(\Pi y^h\right)_{2,\ell} := y_{1,\ell}^h, \text{ for } \ell = 1, \ldots, N_f. \tag{2.32}$$

The domain decomposition algorithm is just the solution of the fixed point system

$$y^h = (1 - r)y^h + r\Pi S(L, y^h) \tag{2.33}$$

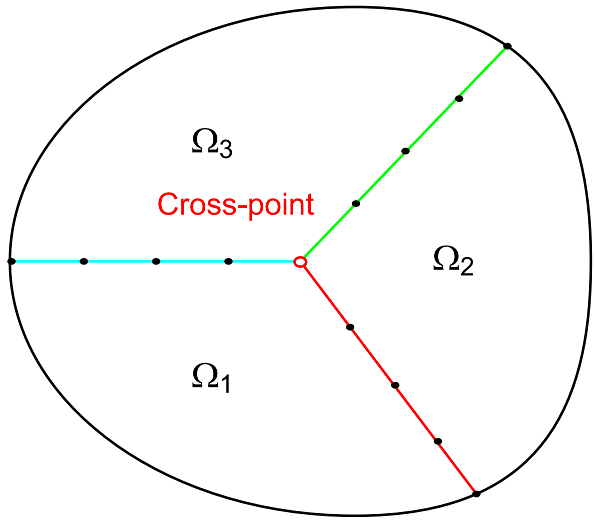

FIG. 4. *Typical domain decomposition involving a cross-point.*

by means of a successive approximations procedure. The parameter $r \neq 0$ is introduced to enhance its rate of convergence. We shall give later the complete description of the algorithm.

2.5. Main differences with a standard FETI method. Consider the case of domain decomposition depicted by figure 4. Let us denote the interface unknowns by y_{ij}^h $(i, j = 1, 2, 3)$. Each y_{ij}^h is defined only at the nodes of the interfaces $\Sigma^{i,j}$ respectively separating Ω_i from Ω_j, including its endpoints. For both FETI and the new method, instead of solving the actual FE problem satisfied by u^h, one solves a system satisfied by an unknown u_m^h, aimed to be, at least when convergence is reached, the restriction of u to the domain Ω_m.

In the conventional FETI method, the nodal values at the cross-point $\left(u^h\right)_{\mathrm{c}}$ are not assumed a priori to be equal. The interface equations are stated as follows at the cross-point

$$
\begin{array}{ll}
\text{Interface } \Sigma^{1,2} & \left\{ \begin{array}{ll} (\mathrm{Eq}^{1,2}) & \left(y_{12}^h\right)_{\mathrm{c}} + \left(y_{21}^h\right)_{\mathrm{c}} - 2\beta(u_1^h)_{\mathrm{c}}(y_{12}^h, y_{13}^h, L^1) = 0 \\ (\mathrm{Eq}^{2,1}) & \left(y_{21}^h\right)_{\mathrm{c}} + \left(y_{12}^h\right)_{\mathrm{c}} - 2\beta(u_2^h)_{\mathrm{c}}(y_{21}^h, y_{23}^h, L^2) = 0 \end{array} \right. \\
\text{Interface } \Sigma^{1,3} & \left\{ \begin{array}{ll} (\mathrm{Eq}^{1,3}) & \left(y_{13}^h\right)_{\mathrm{c}} + \left(y_{31}^h\right)_{\mathrm{c}} - 2\beta(u_1^h)_{\mathrm{c}}(y_{12}^h, y_{13}^h, L^1) = 0 \\ (\mathrm{Eq}^{3,1}) & \left(y_{31}^h\right)_{\mathrm{c}} + \left(y_{13}^h\right)_{\mathrm{c}} - 2\beta(u_3^h)_{\mathrm{c}}(y_{31}^h, y_{32}^h, L^3) = 0 \end{array} \right. \\
\text{Interface } \Sigma^{2,3} & \left\{ \begin{array}{ll} (\mathrm{Eq}^{2,3}) & \left(y_{23}^h\right)_{\mathrm{c}} + \left(y_{32}^h\right)_{\mathrm{c}} - 2\beta(u_2^h)_{\mathrm{c}}(y_{21}^h, y_{23}^h, L^2) = 0 \\ (\mathrm{Eq}^{3,2}) & \left(y_{32}^h\right)_{\mathrm{c}} + \left(y_{23}^h\right)_{\mathrm{c}} - 2\beta(u_3^h)_{\mathrm{c}}(y_{31}^h, y_{32}^h, L^3) = 0, \end{array} \right.
\end{array}
$$

where L^m are in some meaning the restriction of the data of the actual problem to domain Ω_m, and u_m^h are obtained as the solution to a system of FE equations set in domain Ω_m and can be seen as linear function of their arguments.

Clearly, each group of equations respectively leads to the system

$$\left\{\begin{array}{l}(u_1^h)_{\mathrm{c}}(y_{12}^h, y_{13}^h, L^1) = (u_2^h)_{\mathrm{c}}(y_{21}^h, y_{23}^h, L^2)\\(u_1^h)_{\mathrm{c}}(y_{12}^h, y_{13}^h, L^1) = (u_3^h)_{\mathrm{c}}(y_{31}^h, y_{32}^h, L^3)\\(u_2^h)_{\mathrm{c}}(y_{21}^h, y_{23}^h, L^2) = (u_3^h)_{\mathrm{c}}(y_{31}^h, y_{32}^h, L^3)\end{array}\right.$$

which is, in an obvious way, an overdetermined system of equations.

Moreover, every solution to the interface equations necessarily belongs to the subspace defined through the following linear relationships

$$\left\{\begin{array}{l}\left(\left(y_{12}^h\right)_{\mathrm{c}} + \left(y_{21}^h\right)_{\mathrm{c}}\right) - \left(\left(y_{13}^h\right)_{\mathrm{c}} + \left(y_{31}^h\right)_{\mathrm{c}}\right) = 0\\\left(\left(y_{21}^h\right)_{\mathrm{c}} + \left(y_{12}^h\right)_{\mathrm{c}}\right) - \left(\left(y_{23}^h\right)_{\mathrm{c}} + \left(y_{32}^h\right)_{\mathrm{c}}\right) = 0\\\left(\left(y_{31}^h\right)_{\mathrm{c}} + \left(y_{13}^h\right)_{\mathrm{c}}\right) - \left(\left(y_{32}^h\right)_{\mathrm{c}} + \left(y_{23}^h\right)_{\mathrm{c}}\right) = 0\end{array}\right.$$

This clearly indicates that the interface system has fewer independent equations than unknowns and is thus underdetermined.

By the construction process itself, this system has a solution. This is why it can be nevertheless solved by a Krylov method at least when using exact precision computations. For finite precision floating point computations, there is no theoretical guarantee of convergence of the iterative process.

By contrast, the newly introduced method keeps a strong coupling of the unknowns at the crosspoint

$$(u_1^h)_{\mathrm{c}}(y_{12}^h, y_{13}^h, L^1) = (u_2^h)_{\mathrm{c}}(y_{21}^h, y_{23}^h, L^2) = (u_3^h)_{\mathrm{c}}(y_{31}^h, y_{32}^h, L^3)$$

This makes it possible to not introduce any interface unknown at the crosspoint. In this way, we shall be able to prove that the successive approximation process converges. In other words, we shall prove that the eigenvalues of the matrix relative to the interfaces equations are located in a disk centered at the point 1 and with a radius < 1. As is well-known, this induces much better convergence properties than a solution process based on a Krylov method.

3. Stability and convergence.

3.1. Analysis of the domain decomposition algorithm. The domain decomposition algorithm breaks down if, given L and y^h, problem (2.27) is not solvable in u^h. Indeed, even when the solution exists, the algorithm is not reliable if u^h does not have some stability properties as $h \to 0$. The main reason that the algorithm is stable is that the coupling at the cross-points "disappears " at the limit $h = 0$. The main tool to establish this property is provided by the following lemma.

LEMMA 3.1. *Let Σ_ℓ be an interface separating Ω_m from another subdomain Ω_n. Then, the space of functions in $H^1(\Omega_m)$ that are equal to 0 in a neighborhood of $\partial\Sigma_\ell$ is dense in $H^1(\Omega_m)$.*

Proof. This is a well-known result, although it seems to be hard to give an accurate reference for its proof. For example, the statement is claimed without proof in [19] when $\partial\Sigma_\ell$ is a smooth submanifold of Ω_m of codimension $N-2$.

Since this property is the cornerstone in the theoretical justification of the method, and for the sake of completeness, we give its proof here.

First, we prove that the space of functions which vanish in a neighborhood of a point in $\overline{\Omega}_m$ is dense in $H^1(\Omega_m)$. Extending any function in $H^1(\Omega_m)$, it suffices to prove the density result for functions in $H^1(\mathbb{R}^N)$ vanishing in a neighborhood of 0. Clearly, it is equivalent to prove that a distribution $T \neq 0$ in $H^{-1}(\mathbb{R}^N)$ cannot have $\{0\}$ as support. But, from a classical Schwartz's theorem [18], such a distribution T is necessarily of the form

$$T = \sum_{|\alpha| \leq n} a_\alpha \partial^\alpha \delta$$

with δ the Dirac mass at 0. Since δ cannot belong to $H^s(\mathbb{R}^N)$ for $s \geq -N/2$, this proves the lemma for $N = 2$ and reduces the proof to the case where $\partial\Sigma_\ell$ is a segment of $\mathbb{R}^3$. In the same way, it suffices to prove that a distribution T with the same regularity cannot also have a segment as support. Applying another Schwartz's theorem [18], we need only to verify that the distribution $v(x_3)\delta_{x_1,x_2}$, even if $v \in \mathcal{D}(\mathbb{R})$ cannot belong to $H^{-1}(\mathbb{R}^3)$ for $v \neq 0$. For this purpose, we adapt an argument in [7]. The distribution $v(x_3)\delta_{x_1,x_2}$ is in $H^t(\mathbb{R}^3)$ if and only if the Fourier transform $\widehat{v}$ of v satisfies

$$\int_{\mathbb{R}} |\widehat{v}(\xi_3)|^2 \left\{ \int_{\mathbb{R}^2} (1 + \xi_1^2 + \xi_2^2 + \xi_3^2)^t d\xi_1 d\xi_2 \right\} d\xi_3 < +\infty.$$

But the variable change $\xi_1 = (1 + \xi_3^2)^{1/2}\eta_1$, $\xi_2 = (1 + \xi_3^2)^{1/2}\eta_2$ puts the above integral into the form

$$I(t) \int_{\mathbb{R}} |\widehat{v}(\xi_3)|^2 (1 + \xi_3^2)^{t+1} d\xi_3$$

with

$$I(t) = \int_{\mathbb{R}^2} (1 + \eta_1^2 + \eta_2^2)^t d\eta_1 d\eta_2.$$

To complete the proof, we have just to remark that the integral $I(t)$ is finite if and only if $t < -1$. □

An immediate consequence of this lemma is the following proposition establishing the above mentioned property.

PROPOSITION 3.1. *The finite dimensional space X_{B}^h, regarded as a subspace of the Hilbert space*

$$X_{\mathrm{B}} := \left\{ v \in L^2(\Omega);\ v_m := v|_{\Omega_m} \in H^1(\Omega_m),\ v_m|_{\Gamma_D \cap \partial\Omega_m} = 0\ (m = 1, \ldots, N_d) \right\}, \tag{3.1}$$

can be endowed with the natural norm

$$\|v\|_{X_{\mathrm{B}}} := \left\{ \sum_{m=1}^{Nd} \|v_m\|_{1,\Omega_m}^2 \right\}^{1/2} \tag{3.2}$$

where $\|v_m\|_{1,\Omega_m}$ *is the usual norm in* $H^1(\Omega_m)$*. Moreover, for every* $v \in X_{\mathrm{B}}$*, there exists a sequence* $\left\{v^h\right\}_{h>0}$ *such that* $v^h \in X_{\mathrm{B}}^h$ *for each* h *and*

$$\lim_{h\to 0} v^h = v \text{ strongly in } X_{\mathrm{B}}. \tag{3.3}$$

Proof. The first part of the proposition is obvious. For the second part, it is sufficient to approximate the restriction $v_m := v|_{\Omega_m}$ of $v \in X_{\mathrm{B}}$ in each Ω_m by a sufficiently smooth function w_m vanishing near the respective boundaries $\partial\Sigma_\ell$ of the interfaces Σ_ℓ. The function v^h can then be obtained by interpolating w_m in each Ω_m by a function in X_m^h. □

REMARK 3.1. *The above proposition establishes that* $\left\{X_{\mathrm{B}}^h\right\}_{h>0}$ *actually approaches* X_{B} *in the sense of Mosco [17], that is, it satisfies (3.3) and if*

$$\left\|v^h\right\|_{X_{\mathrm{B}}} \leq C \tag{3.4}$$

where C *is a constant independent of* h*. Then, possibly passing to a subsequence, one has*

$$\lim_{h\to 0} v^h = v \text{ weakly in } X_{\mathrm{B}}. \tag{3.5}$$

By the Rellich compactness theorem, it then follows that

$$\lim_{h\to 0} v^h = v \text{ strongly in } L^2(\Omega). \tag{3.6}$$

This property will be used several times below.

We can now prove the main stability property of the domain decomposition method. In particular, this ensures that the algorithm does not break down.

THEOREM 3.1. *Assume that*

$$\Re\beta = \beta_0 > 0 \text{ and } \Im\beta \leq 0 \tag{3.7}$$

in the coercive case and that

$$\Re\beta \geq 0 \text{ and } \Im\beta = -\beta_0 < 0 \tag{3.8}$$

in the non coercive one. Then there exists $h_0 > 0$ *such that the following property holds. Let there be given* $\left\{\chi_m^h\right\}_{m=1}^{m=N_d}$ *such that* $\chi_m^h \in \left\{H^1(\Omega_m)\right\}'$ $(m = 1, \ldots, N_d)$*. Then the problem*

$$\left\{ \begin{array}{l} u^h \in X_{\mathrm{B}}^h, \quad \forall v^h \in X_{\mathrm{B}}^h, \\ \sum_{m=1}^{N_d} \left(\widetilde{a}_m(u_m^h, v_m^h) + \beta c_m^h(u_m^h, v_m^h)\right) = \sum_{m=1}^{N_D} \chi_m^h v_m^h, \end{array} \right. \quad (m = 1, \ldots, N_d) \tag{3.9}$$

admits a unique solution, with no further condition in the coercive case, and for $0 < h \leq h_0$ *in the non coercive one. Furthermore, in both cases the solution* u^h *satisfies the bound*

$$\left\|u^h\right\|_{X_{\mathrm{B}}} \leq C \left\{ \sum_{m=1}^{N_d} \left\|\chi_m^h\right\|_{\{H^1(\Omega)\}'}^2 \right\}^{1/2} \tag{3.10}$$

where C is a constant independent of h, and, as usual,

$$\left\|\chi_m^h\right\|_{\{H^1(\Omega_m)\}'} = \sup_{\|v\|_{1,\Omega_m}\leq 1} \left|\chi_m^h v\right|. \tag{3.11}$$

Proof. To shorten the discussion, we focus on the non coercive problem. A simple adaptation yields the proof in the coercive one.

Clearly, it is sufficient to obtain the estimate. Recall that $\widetilde{a}_m(\cdot,\cdot) = (a_m + b_m)(\cdot,\cdot)$ can be decomposed into an almost coercive part $a_m(\cdot,\cdot)$ and a continuous bilinear form $b_m(\cdot,\cdot)$ on $L^2(\Omega_m)$. The conditions (2.1,2.2 ,2.3,3.8) lead to the following left estimate

$$\left|\sum_{m=1}^{Nd}\left(a_m(u_m,\overline{u}_m) + \beta c_m^h(u_m,\overline{u}_m)\right)\right|^2 \geq \\ \sum_{m=1}^{N_d}\left(\Re a_m(u_m,\overline{u}_m)\right)^2 + \left(\beta_0 c_m^h(u_m,\overline{u}_m)\right)^2.$$

Then the elementary inequality $|ab| \leq (a^2+b^2)/2$ yields

$$\left|\sum_{m=1}^{N_d}\left(a_m(u_m,\overline{u}_m) + \beta c_m^h(u_m,\overline{u}_m)\right)\right| \geq \\ \left(\sum_{m=1}^{N_d}\left(\Re a_m(u_m,\overline{u}_m) + \beta_0 c_m^h(u_m,\overline{u}_m)\right)\right)/\sqrt{2}.$$

For each ℓ ($\ell = 1,\ldots,N_f$), let there be given a non-empty open subset ω_ℓ of Σ_ℓ such that $\overline{\omega_\ell} \subset \Sigma_\ell^h$. The usual estimates of Poincaré type give

$$\exists\gamma_m > 0 : \Re a_m(u_m,\overline{u}_m) + \beta_0 c_m^h(u_m,\overline{u}_m) \geq \\ \Re a_m(u_m,\overline{u}_m) + \beta_0 \textstyle\sum_{\ell\in\Lambda_m} \displaystyle\int_{\omega_\ell} |u_m|^2\, d\omega_\ell \geq \gamma_m \|u_m\|_{1,\Omega_m}^2.$$

Gathering the above estimates, we immediately obtain the following left bound:

$$\left|\sum_{m=1}^{N_d} a_m(u_m,\overline{u}_m) + \beta c_m^h(u_m,\overline{u}_m)\right| \geq \alpha \|u\|_{X_{\mathrm{B}}}^2 \tag{3.12}$$

with a constant $\alpha > 0$ independent of $u \in X_{\mathrm{B}}$ and of h. This coercivity estimate makes it possible to define the following operator $T_h : X_{\mathrm{B}}^h \to X_{\mathrm{B}}^h$ as follows

$$\left\{\begin{array}{l} T_h u^h \in X_{\mathrm{B}}^h, \quad \forall v^h \in X_{\mathrm{B}}^h, \\ \sum_{m=1}^{N_d} a_m(\left(T_h u^h\right)_m, v_m^h) + \beta c_m^h(\left(T_h u^h\right)_m, v_m^h) = b(u^h, v^h). \end{array}\right. \tag{3.13}$$

The estimate is the consequence of the following left bound on T_h: there exist $h_0 > 0$ and $\gamma > 0$ such that if $0 < h \leq h_0$

$$\left\|u^h + T_h u^h\right\|_{X_{\mathrm{B}}} \geq \gamma \left\|u^h\right\|_{X_{\mathrm{B}}}, \quad \forall u^h \in X_{\mathrm{B}}^h \tag{3.14}$$

which is proved by contradiction. Suppose that it is false. Then there exists a sequence $\left\{u^h\right\}_{h>0}$ such that

$$\left\|u^h\right\|_{X_{\mathrm{B}}} = 1, \tag{3.15}$$

$$\lim_{h\to 0} \left\|u^h + T_h u^h\right\|_{X_{\mathrm{B}}} = 0. \tag{3.16}$$

By remark 3.1, $\lim_{h\to 0} u^h = u^*$ weakly in X_{B} and strongly in $L^2(\Omega)$. On the other hand, by the definition of $T_h u^h$, we can write

$$\sum_{m=1}^{N_d} \left(a_m(\left(u^h + T_h u^h\right)_m, v_m^h) + \beta c_m^h(\left(u^h + T_h u^h\right)_m, v_m^h)\right) =$$
$$\sum_{m=1}^{N_d} \left(\widetilde{a}_m(u_m^h, v_m^h) + \beta c_m^h(u_m^h, v_m^h)\right).$$

Proposition 3.1, condition (3.16), and the fact that $H^{1/2}(\partial\Omega_m)$ is compactly imbedded in $L^2(\partial\Omega_m)$ readily gives that

$$\sum_{m=1}^{N_d} \widetilde{a}_m(u_m^*, v_m) + \beta c_m(u_m^*, v_m) = 0, \quad \forall v \in X_{\mathrm{B}} \tag{3.17}$$

where

$$c_m(u_m^*, v_m) = \sum_{\ell\in\Lambda_m} \int_{\Sigma_\ell} u_m^* v_m \, d\Sigma_\ell.$$

No matching condition is imposed to functions in X_{B} at the interfaces. Therefore, the variational equation (3.17) actually consists of N_d uncoupled problems of the same type than (2.12)

$$\widetilde{a}_m(u_m^*, v_m) + \beta c_m(u_m^*, v_m) = 0, \quad \forall v_m \in X_m \tag{3.18}$$

where

$$X_m := \left\{v \in H^1(\Omega_m);\ \exists w \in X,\ v = w|_{\Omega_m}\right\}.$$

Problem (3.18) has exactly the same properties as (2.12), so its only solution is $u_m^* = 0$. As a result, $\lim_{h\to 0} u^h = 0$ strongly in $L^2(\Omega)$. Therefore, using the definition of $T_h u^h$ and estimate (3.12), we readily obtain $\lim_{h\to 0} T_h u^h = 0$ strongly in X_{B}. In turn, this limit, together with (3.16) gives

$$\lim_{h\to 0} u^h = \lim_{h\to 0}(u^h + T_h u^h) - \lim_{h\to 0} T_h u^h = 0 \text{ strongly in } X_{\mathrm{B}}.$$

which contradicts (3.15). This establish (3.14) and completes the proof of the theorem. □

We next prove that the algorithm is really a domain decomposition algorithm, that is, the main computation at each iteration consists of solving local problems posed in each subdomain. In other words, we must prove that system (2.31) can be solved through a Schur's complement procedure, based on the elimination of the unknowns corresponding to the blocks diagonal submatrix.

THEOREM 3.2. *Define $X_{m,c}^h$ to be the space spanned by $v_m^h \in X_m^h$ vanishing at all nodes belonging to $\mathcal{N}_c$. Under the general conditions of Theorem 3.1, there exist h_0 and a constant C independent of $0 < h \leq h_0$, such that for every $\chi_m^h \in \left\{H^1(\Omega_m)\right\}'$, the problem*

$$\left\{\begin{array}{l} u_m^h \in X_{m,c}^h, \quad \forall v_m^h \in X_{m,c}^h, \\ \widetilde{a}_m(u_m^h, v_m^h) + \beta c_m^h(u_m^h, v_m^h) = \chi_m^h v_m^h, \end{array}\right. \tag{3.19}$$

has a unique solution satisfying

$$\left\|u_m^h\right\|_{1,\Omega_m} \leq C \left\|\chi_m^h\right\|_{\{H^1(\Omega_m)\}'}. \tag{3.20}$$

Proof. The proof follows exactly the same lines as the previous one and is thus omitted. □

3.2. Convergence of the domain decomposition algorithm. In view of the stability estimate (3.10), we are left to prove that the mapping $y^h \mapsto (1-r)y^h + r\Pi S(L, y^h)$ from Y^h into itself satisfies

$$\left\|(1-r)y^h + r\Pi S(L, y^h) - ((1-r)z^h + r\Pi S(L, z^h))\right\|_{Y^h} \leq \kappa_h \left\|y^h - z^h\right\|_{Y^h}$$

for all y^h and z^h in Y^h with a constant $\kappa_h < 1$. This will imply that the domain decomposition algorithm converges. We say in this case that this mapping is a contraction (see for instance, [19]). Indeed, since Π is clearly a linear mapping from Y^h into itself which preserves the norm and S is a linear map from $\left\{H^1(\Omega)\right\}' \times Y^h$, it is sufficient to prove that the linear map from Y^h into itself defined by $y^h \mapsto (1-r)y^h + r\Pi S(0, y^h)$ has a norm < 1. The key tool to establish this estimate is provided by the following lemma.

LEMMA 3.2. *Let Y be an Hilbert space with norm $\|\cdot\|_Y$ and scalar product $(\cdot,\cdot)_Y$. Let Θ be a linear mapping from Y into itself such that*

$$\|\Theta y\|_Y \leq \|y\|_Y, \quad \forall y \in Y. \tag{3.21}$$

Then, for any $0 < r < 1$, Θ satisfies

$$\|(1-r)y + r\Theta y\|_Y^2 \leq \|y\|_Y^2 - r(1-r)\|y - \Theta y\|_Y^2, \quad \forall y \in Y. \tag{3.22}$$

Therefore, if the left bound

$$\exists \gamma > 0 : \|y - \Theta y\|_Y \geq \gamma \|y\|_Y, \quad \forall y \in Y, \tag{3.23}$$

holds, then the mapping $y \mapsto (1-r)y + r\Theta y$ verifies the following bound

$$\|(1-r)y + r\Theta y\|_Y \leq \sqrt{1-\gamma^2 r(1-r)}\,\|y\|_Y\,, \quad \forall y \in Y, \tag{3.24}$$

and hence is a contraction from Y into itself.

Proof. Actually, this result was established in [8], in an examination of the convergence of an algorithm of those considered here. We find it convenient to state and prove it in this general framework which is more adapted to the present context.

Expanding $\|(1-r)y + r\Theta y\|_Y^2$, we obtain

$$\|(1-r)y + r\Theta y\|_Y^2 = (1-r)^2\,\|y\|_Y^2 + 2r(1-r)\Re(\Theta y, y)_Y + r^2\,\|\Theta y\|_Y^2\,.$$

Since $\|y - \Theta y\|_Y^2 = \|y\|_Y^2 + \|\Theta y\|_Y^2 - 2\Re(\Theta y, y)_Y$, the previous expression can be written as

$$\|(1-r)y + r\Theta y\|_Y^2 = (1-r)\,\|y\|_Y^2 + r\,\|\Theta y\|_Y^2 - r(1-r)\,\|y - \Theta y\|_Y^2\,.$$

Using (3.21) and (3.23), we obtain(3.24), proving the lemma. □

REMARK 3.2. *The previous lemma deserves the following comment. If Θ is a contraction, that is,*

$$\|\Theta y\|_Y \leq K\,\|y\|_Y\,, \quad \forall y \in Y, \tag{3.25}$$

with $K < 1$, then the Neumann series expansion of $(I-\Theta)^{-1}$ implies that condition (3.23) is automatically satisfied with $\gamma = (1-K)$. Indeed, if Y and Θ depend on some parameter, then the contraction constant

$$\kappa := \sqrt{1-(1-K)^2 r(1-r)}$$

can be assumed to be independent of this parameter as long as estimate (3.25) is uniform relatively to this parameter. For the problem under consideration, this parameter is obviously the mesh size h. Unfortunately, it can be established using a compactness argument that such a desirable estimate is false in this context (cf. [1] for a proof for the continuous problem).
Actually, since Y^h is finite dimensional, verifying that

$$\left\|\Pi S(0, y^h)\right\|_{Y^h} \leq 1, \text{for all } y^h \text{ such that } \left\|y^h\right\|_{Y^h} = 1, \tag{3.26}$$

and that the only y^h satisfying $\Pi S(0, y^h) = y^h$ is $y^h = 0$, will suffices to prove that $y^h \mapsto (1-r)y^h + r\Pi S(0, y^h)$ is a contraction with a constant $\kappa_h < 1$.

Now we come to the proof of the convergence of the domain decomposition algorithm.

THEOREM 3.3. *Let $\left(u^h\right)^{(n)}$ be recursively defined as the solution of problem (2.27) with $y^h := (y^h)^{(n)}$, $(y^h)^{(0)} := 0$ and $(y^h)^{(n+1)} := (1-r)(y^h)^{(n)} +$*

$r\Pi S(L, \left(y^h\right)^{(n)})$. Then $\left(u^h\right)^{(n)}$ converges to the solution u^h of either problem (2.13) or (2.14) with the following error bound

$$\left\| \left(u^h\right)^{(n)} - u^h \right\|_{X_{\mathrm{B}}} \leq C\kappa_h^n \left\| \left(u^h\right)^{(0)} - u^h \right\|_{X_{\mathrm{B}}} \tag{3.27}$$

where C is a constant independent of h and κ_h is a constant < 1.

Proof. By Theorem 3.1, Lemma 3.2, and the fact that Π is an unitary operator of Y^h, we need only prove that the operator A from Y^h into itself defined by $Ay^h = S(0, y^h)$ satisfies

$$\left\|Ay^h\right\|_{Y^h} \leq 1, \quad \forall y^h \in Y^h \text{ such that } \left\|y^h\right\|_{Y^h} \leq 1 \tag{3.28}$$

and that

$$\Pi A y^h = y^h \text{ if and only if } y^h = 0 \tag{3.29}$$

according to the fact that Y^h is finite dimensional.

To prove (3.29), we have only to remark that for such y^h, setting $L = 0$ in (2.27) yields a solution u^h of (2.27) satisfying (2.28). This solution must equal zero, by the uniqueness of the solutions of (2.13) or (2.14).

The next part of the proof utilizes the techniques devised by Després [11, 10, 12] in the analysis of a similar algorithm for the Helmholtz equation. To obtain (3.28), we first expand $\left\|Ay^h\right\|_{Y^h}^2$ as

$$\left\|Ay^h\right\|_{Y^h}^2 = \sum_{m=1}^{N_d} \sum_{\ell \in \Lambda_m} \left\| -y^h_{i^\ell_m,\ell} + 2\beta u^h_m \right\|_{0,\Sigma_\ell}^2$$

where u^h is the solution of problem (2.27) with $L = 0$. Now

$$\left\| -y^h_{i^\ell_m,\ell} + 2\beta u^h_m \right\|_{0,\Sigma^h_\ell}^2 = \left\| y^h_{i^\ell_m,\ell} \right\|_{0,\Sigma^h_\ell}^2 - 4\Re\left(\overline{\beta} c^h_\ell (y^h_{i^\ell_m,\ell}, \overline{u^h_m})\right) + 4\left|\beta\right|^2 \left\|u^h_m\right\|_{0,\Sigma_\ell}^2$$

and by variational formulation (2.27), we readily obtain

$$\left\|Ay^h\right\|_{Y^h}^2 = \left\|y^h\right\|_{Y^h}^2 - 4\Re\left(\overline{\beta} \sum_{m=1}^{N_d} \widetilde{a}_m(u^h_m, \overline{u^h_m})\right).$$

We now consider separately the two types of boundary-value problems.

Time-harmonic problems: $\Re\beta = 0, \quad -\Im\beta = \beta_0 > 0$. Since

$$\overline{\beta} \sum_{m=1}^{N_d} \widetilde{a}_m(u^h_m, \overline{u^h_m}) = -\beta_0 \sum_{m=1}^{N_d} \left(\Im\widetilde{a}_m(u^h_m, \overline{u^h_m}) - i\Re\widetilde{a}_m(u^h_m, \overline{u^h_m})\right).$$

By conditions (2.1), (2.2), (2.9) and (2.10), we obtain

$$\Re\left(\overline{\beta} \sum_{m=1}^{N_d} \widetilde{a}_m(u^h_m, \overline{u^h_m})\right) = -\beta_0 \sum_{m=1}^{N_d} \Im\widetilde{a}_m(u^h_m, \overline{u^h_m}) \geq \beta_0\lambda_0 \int_{\Gamma_N} \left|u^h\right|^2 d\Gamma_N \geq 0. \tag{3.30}$$

Coercive problems: $\Re\beta = \beta_0 > 0$ and $\Im\beta = 0$. Now, we have

$$\overline{\beta}\sum_{m=1}^{N_d}\widetilde{a}_m(u_m^h,\overline{u_m^h}) = \beta_0\sum_{m=1}^{N_d}\left(\Re a_m(u_m^h,\overline{u_m^h}) + i\Im a_m(u_m^h,\overline{u_m^h})\right).$$

According to (2.1), (2.2) and (2.3), we get

$$\Re\left(\overline{\beta}\sum_{m=1}^{N_d}a_m(u_m^h,\overline{u_m^h})\right) = \beta_0\sum_{m=1}^{N_d}\Re a_m(u_m^h,\overline{u_m^h}) \geq 0. \tag{3.31}$$

Condition (3.28) thus holds in both cases. This completes the proof of the theorem. □

4. Concluding remarks. According to the Remark 3.2, the only way to ensure that the constant κ_h can be bounded by some constant $\kappa < 1$, independent of h, is to find a uniform lower bound for the constant γ_h. Such an estimate is unattainable with the local operators associated to the bilinear forms c_ℓ^h used here. If the domain decomposition does not lead to any cross-point, then Σ_ℓ^h coincides with Σ_ℓ. In this case, taking c_ℓ^h for scalar product of $H^{1/2}(\Sigma_\ell)$ will ensure this uniform bound for γ_h (cf. [1, 8]). However, it appears from several numerical experiments [5, 1] that using a constant β with $\Re\beta > 0$ heuristically to damp the evanescent part of the error yields better convergence rates than those provided by a non-local operator on Σ_ℓ, at least for meshes having typical density of nodes.

Theorem 3.3 is false in general for β such that $\Re\beta > 0$ in the case of non coercive problems (cf. [2, 1]). However, these studies show that even if the simple algorithm obtained through successive approximations as described above does not converge, the problem can be efficiently solved by applying a Krylov method [1] to the fixed point problem (2.33). In the case of the continuous problem, this phenomenon can be explained partially by a spectral decomposition in a spherical geometry, and in a wave guide case [1, 2], since then only a very few eigenvalues of the operator in the right-hand side of (2.33) have magnitude > 1. A challenging issue is to give the same theoretical explanation in the general case and for the discrete problem.

REFERENCES

[1] Y. Boubendir, Techniques de Décomposition de Domaine et Méthodes d'Equations Intégrales. INSA-Toulouse, France, 2002.

[2] Y. Boubendir and A. Bendali, Domain decomposition methods for solving scattering problems by a boundary element method. Domain decomposition methods in Science and Engineering (2002), eds., N. Debit and al., pp. 321–328.

[3] A. Bendali, Y.Boubendir, Méthode de décomposition de domaine et éléments finis nodaux pour la résolution de l'équation d'Helmholtz, C. R. Acad. Sci, Paris, Ser. I **339**, (2004) 229–234.

[4] A. Bendali, Y. Boubendir and M'B. Fares, A FETI-like domain decomposition method for coupling FEM and BEM in large-size problems of acoustic scattering. To be submitted.

[5] Y. Boubendir, A. Bendali and M'B. Fares, Non-overlapping domain decomposition method and nodal finite element method. Submitted.

[6] F. Brezzi and M. Fortin, *ixed and Hybrid Finite Element Methods*. Springer-Verlag, New York and al. 15 (1991).

[7] J. Chazarain and A. Piriou, *Introduction to the theory of partial differential equations*. North-Holland Publishing Co., Studies in Mathematics and Applications, Amsterdam-New York, 14 (1982).

[8] F. Collino, S. Ghanemi, and P. Joly, Domain Decomposition Method for Harmonic Wave Propagation: a General Presentation. Computer Methods in Applied Mechanics and Engineering, **184** (2000), pp. 171–211.

[9] A. de la Bourdonnaye, C. Farhat, A. Macedo, F. Magoules, and F.-X. Roux, A Non-Overlapping Domain Decomposition Method for the Exterior Helmholtz Problem. Contemporary Mathematics, **218**, January (1998), pp. 42–66.

[10] B. Després, Domain decomposition method and the Helmholtz problem, Mathematical and numerical aspects of wave propagation phenomena, Strasbourg, (1991), SIAM Philadelphia, PA, pp. 44–52.

[11] B. Després, Méthodes de décomposition de domaine pour les problèmes de propagation d'ondes en regime harmonique. Le théorème de Borg pour l'équation de Hill vectorielle. University Paris VI, 1991.

[12] B. Després, Domain decomposition method and the Helmholtz problem.II. Second International Conference on Mathematical and Numerical Aspects of Wave Propagation Newark, DE, (1993), SIAM Philadelphia, PA, pp. 197–206.

[13] C. Farhat and J. Mandel, The two-level FETI for static and dynamic plate problems. Part I: An optimal iterative solver for biharmonic systems. Computer methods in applied mechanics and engineering, **155** (1998), pp. 129–151.

[14] C. Farhat, J. Mandel, and F.X. Roux, Optimal convergence properties of the FETI domain decomposition method. Computer methods in applied mechanics and engineering, **115** (1994), pp. 365–385.

[15] R. Leis, *Initial boundary value problems in mathematical physics*. J. Wiley and sons, Chichester et al., 1986.

[16] P.-L. Lions, On the Schwarz alternating method III". Third international Symposium on domain decomposition methods for partial differential equations, 1989 July/August, 22, No.4, SIAM.

[17] U. Mosco, Convergence of convex sets and of solutions for a class of variational inequalities. Adv. Math. **3** (1969), pp. 510–585.

[18] L. Schwartz, *Théorie des distributions*. Hermann, Paris, 1984, c1996.

[19] M. Taylor, *Partial Differential Equations I, Basic Theory*. Springer-Verlag, New-York, 1996.