GENERAL PROJECTIVE RICCATI EQUATIONS METHOD AND EXACT SOLUTIONS FOR A CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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General Projective Riccati equations method and exact solutions for a class of nonlinear partial differential equations

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ABSTRACT

By using a simple transformation technique, we have shown that the Hamiltonian amplitude equation, the nonlinear wave equation, the coupled Klein-Gordon-Zakharov (CKGZ) equations, the generalized Davey Sterwatson (DS) equations, the DS equations, the generalized Zakharov equations can be reduced to the same elliptic-like equation. Then, by using the generally projective Riccati equation method, many kinds of exact solutions of the above mentioned equations are obtained in a unified way. These solutions include new solitary wave, periodic and rational solutions.

1 INTRODUCTION

In recent years, seeking exact solutions of nonlinear partial differential equations (NLPDEs) is of great significance as it appears that these NLPDEs are mathematical models of complex physics phenomena arising in physics, mechanics, biology, chemistry and engineers. In order to help engineers and physicists to better understand the mechanism that governs these physical models or to better provide knowledge to the physical problem and possible applications, a vast variety of the powerful and direct methods have been derived. Among these are the inverse scattering method [1], Hirota's method[2], Bäcklund transformation [3,4] tanh-function method [5-8], extended tanh-function method [9-12].

In the literature [13], Conte et al. presented a general ansatz to seek more new solitary wave solutions of some NLPDEs that can be expressed as a polynomial in two elementary functions which satisfy a projective Riccati equations [14]. More recently, Yan developed conte's method and presented the general projective Riccati equation method [15]. Several authors used the Yan's technique to solve many NLPDEs [16-19].

In this paper, we will apply the general projective Riccati equation method to solve some class of NLPDEs The key idea of this method is to introduce a new projective Riccati equation and use its solutions to replace the elementary functions in the projective Riccati equation method [13], which simply proceeds as follows [15-19]

Step 1. For a given NLPDE in the unknown u(x,y,z,...,t) which are solutions of the ordinary differential equation (ODE) $E(u, u', u'', ...) = 0$ obtained by the travelling wave reduction $u(x, y, z, ..., t) \rightarrow u(\xi = \lambda_1 x + \lambda_2 y + \lambda_3 z + ... + \lambda_n t)$.

Step 2. We introduce two new variables $\sigma(\xi)$, $\tau(\xi)$. Then we seek solutions of $u(\xi)$ in the following form

Type I when R$\neq 0$

$$u(\xi) = \sum_{i=1}^{n} \sigma^{i-1}(\xi)[A_i\sigma(\xi) + B_i\tau(\xi)] + A_0, \qquad (\mathbf{1})$$

where A_i and B_i are constants to be determined later. $\sigma(\xi)$, $\tau(\xi)$ are solutions of the following new projective Riccati equation

$$\sigma'(\xi) = \epsilon\sigma(\xi)\tau(\xi), \;\; \tau'(\xi) = R+\epsilon\tau^2(\xi)-m\sigma(\xi), \;\; \epsilon = \pm 1. \qquad (2.a)$$

where m, R are constants. It is easy to see that Eq.(2.a) admits the first integral with $R \neq 0$

$$\tau^2(\xi) = -\epsilon \left[R - 2m\ \sigma(\xi) + \tfrac{m^2+r_i}{R} \sigma^2(\xi) \right],\ \ r_i = \pm 1 \qquad (2.b)$$

or we seek solutions of $u(\xi)$ in the following form

Type II when R=m=0,

$$u(\xi) = \sum_{i=1}^{n} A_i \tau^i(\xi) + A_0, \qquad (\mathbf{3.a})$$

where τ satisfies

$$\tau'(\xi) = \tau^2(\xi) \qquad (\mathbf{3.b})$$

The parameter n in (1) and (3.a) can be determined by balancing the highest-order derivative term in the $E(u, u', u'', ...) = 0$.

Step 3. Substituting system (1) along with condition (2.a) and (2.b) (or (3.a) along with (3.b)) into the $E(u, u', u'', ...) = 0$ and setting the coefficient of $\sigma^i \tau^j$ (j=0,1, i=0,1,2,3,...) (τ^l, l=0, 1,...) to zero yields a set of over-determined algebraic equations, from which the constants A_i, B_i, R, m, and λ_i (A_i, and λ_i) can be found explicitly.

Step 4. We know that (2.a) and (2.a) admit the following solutions

Case 1 when $\epsilon = -1$, $r_i = -1$, $R \neq 0$

$$\sigma_1(\xi) = \tfrac{R\ sech(\sqrt{R}\xi)}{m\ sech(\sqrt{R}\xi)+1}, \qquad \tau_1(\xi) = \tfrac{\sqrt{R}\ \tanh(\sqrt{R}\xi)}{m\ sech(\sqrt{R}\xi)+1}, \qquad (4)$$

Case 2 when $\epsilon = -1$, $r_i = 1$, $R \neq 0$

$$\sigma_2(\xi) = \tfrac{R\ csch(\sqrt{R}\xi)}{m\ csch(\sqrt{R}\xi)+1}, \qquad \tau_2(\xi) = \tfrac{\sqrt{R}\ \coth(\sqrt{R}\xi)}{m\ csch(\sqrt{R}\xi)+1}, \qquad (5)$$

Case 3 when $\epsilon = 1$, $r_i = -1$, $R \neq 0$

$$\sigma_3(\xi) = \tfrac{R\ \sec(\sqrt{R}\xi)}{m\ \sec(\sqrt{R}\xi)+1}, \qquad \tau_3(\xi) = \tfrac{\sqrt{R}\ \tan(\sqrt{R}\xi)}{m\ \sec(\sqrt{R}\xi)+1}, \qquad (6)$$

$$\sigma_4(\xi) = \tfrac{R\ \csc(\sqrt{R}\xi)}{m\ \csc(\sqrt{R}\xi)+1}, \qquad \tau_4(\xi) = -\tfrac{\sqrt{R}\ \cot(\sqrt{R}\xi)}{m\ \csc(\sqrt{R}\xi)+1}. \qquad (7)$$

Case 4 when $R = m = 0$,

$$\sigma_5(\xi) = \tfrac{C}{\xi} = C\epsilon\tau_5(\xi), \qquad \tau_5(\xi) = \tfrac{1}{\epsilon\xi}. \tag{8}$$

C is a constant.
Substituting the constants A_i, B_i, R, m and λ_i (A_i and λ_i) obtained in step 2 into (1) along with (4)-(7) (into (3.a) along with (8)) to obtain soliton and periodic (rational) solutions of the NLPDE in concern. We will continue by showing that we can use the above described method to present exact solutions to some class of NLPDEs which can be transformed to the same family of elliptic-like equation.

2 Exact solutions of some class of NLPDEs

2.1 A new Hamiltonian amplitude equation

A new Hamiltonian amplitude equation

$$iu_x + u_{tt} + 2\eta|u|^2u - \beta u_{xt} = 0, \tag{9}$$

where $\eta = \pm 1$, $\beta << 1$ was recently introduced by Wadati et al [20]. This equation governs certain instabilities of modulated wave trains, and the addition of the term $-\beta u_{xt}$ overcomes the ill-posedness of the unstable nonlinear Schrȯdinger equation.

Let

$$u(x,t) = \phi(\xi)e^{i(Kx-\Omega t)}, \quad \xi = px - \omega t. \tag{10}$$

Substituting Eq.(10) into Eq.(9), we have

$$(\omega^2+p\beta\omega)\phi^{''(\xi)}+i(p+2\omega\Omega+K\beta\omega+p\beta\Omega)\phi'(\xi)-(K+\Omega^2+pK\beta\Omega)\phi(\xi)+2\eta\phi^3(\xi) = 0. \tag{11}$$

If we take

$$\omega = -\frac{p(1+\beta\Omega)}{2\Omega + K\beta}. \tag{12}$$

Eq.(11) is transformed into the following one

$$\phi^{''}(\xi) + k_1\phi(\xi) + k_3\phi^3(\xi) = 0, \qquad (\mathbf{13.a})$$

where

$$k_1 = -\tfrac{K+\Omega^2+\beta K\Omega}{\omega^2+p\beta\omega}, \quad k_3 = \tfrac{2\eta}{\omega^2+p\beta\omega}. \qquad (\mathbf{13.b})$$

Then the solutions of (9) are given by $u(x,t) = \phi(\xi)e^{i(Kx-\Omega t)}$, where $\phi(\xi)$ is defined by (54)-(69) and the other parameters are determined by $\xi = px - \omega t, \quad \omega = -\frac{p(1+\beta\Omega)}{2\Omega+K\beta}, \quad k_1 = -\frac{K+\Omega^2+\beta K\Omega}{\omega^2+p\beta\omega}, \quad k_3 = \frac{2\eta}{\omega^2+p\beta\omega}$.

2.2 a nonlinear wave equation

Consider the nonlinear wave equation in Rev. [21]

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0, \qquad (\mathbf{14})$$

where $\alpha, \ \beta$ and γ are constants. Eq.(14) contains some particular important equations such as Duffing, Klein-Gordon and Landau=Ginzburg-Higgs equation. We assume that Eq.(14) has exact solution in the form

$$u(x,t) = \phi(\xi), \quad \xi = px - \omega t. \qquad (\mathbf{15})$$

Substituting Eq.(15) into Eq.(14), we have

$$(\omega^2 + \alpha p^2)\phi^{''}(\xi) + \beta\phi(\xi) + \gamma\phi^3(\xi) = 0. \qquad (\mathbf{16})$$

Then Eq.(16) can be written as

$$\phi^{''}(\xi) + k_1\phi(\xi) + k_3\phi^3(\xi) = 0, \qquad (\mathbf{17.a})$$

where

$$k_1 = \tfrac{\beta}{\omega^2+\alpha p^2}, \quad k_3 = \tfrac{\gamma}{\omega^2+\alpha p^2}. \qquad (\mathbf{17.b})$$

Then the solutions of (14) are given by $u(x,t) = \phi(\xi)$, where $\phi(\xi)$ is defined by (54)-(69), and the other parameters are expressed by $\xi = px - \omega t$, $k_1 = \frac{\beta}{\omega^2+\alpha p^2}, \ \ k_3 = \frac{\gamma}{\omega^2+\alpha p^2}$.

2.3 Coupled Klein-Gordon-Zakharov equations

The coupled nonlinear Klein-Gordon-Zakarov equations [1] read

$$u_{tt} - c_0^2\nabla^2 u + f_0^2 u + \delta uv = 0,$$
$$v_{tt} - c_0^2\nabla^2 v - \beta\nabla^2|u|^2 = 0. \qquad \mathbf{(18)}$$

We seek its following wave packet solution

$$u(x,y,z,t) = \phi(\xi)e^{i(kx+ly+nz-\Omega t)},\ v(x,y,z,t) = v(\xi),$$

$$xi = px + qy + rz - \omega t, \qquad \mathbf{(19)}$$

where both $\phi(\xi)$ and $v(\xi)$ are real function. Substituting Eq.(19) into Eq.(18) yields

$(\omega^2 - c_0^2\mathbf{P^2})\boldsymbol{\phi}^{''}(\boldsymbol{\xi}) + \mathbf{2i}(\boldsymbol{\omega\Omega} - \mathbf{c_0^2KP})\boldsymbol{\phi}^{'}(\boldsymbol{\xi}) - (\boldsymbol{\omega}^\mathbf{2} - \mathbf{c_0^2K^2} - \mathbf{f_0^2})\boldsymbol{\phi}(\boldsymbol{\xi}) + \boldsymbol{\delta}\mathbf{v}\boldsymbol{\phi}(\boldsymbol{\xi}) = \mathbf{0}$

$$(\omega^2 - c_0^2\mathbf{P^2})\mathbf{v}^{''}(\boldsymbol{\xi}) - \boldsymbol{\beta}\mathbf{P^2}(\boldsymbol{\phi}^\mathbf{2}(\boldsymbol{\xi}))^{''} = \mathbf{0}, \qquad \mathbf{(20)}$$

where
$\mathbf{K} =(k,l,n)$, $\mathbf{K^2} = k^2 + l^2 + n^2$, $\mathbf{P}=(p,q,r)$,
$\mathbf{P^2}=p^2 + q^2 + r^2$; $\mathbf{K}.\mathbf{P}=kp + lq + nr.$ **(21)**

If we take

$$\omega\Omega = c_0^2\mathbf{K}.\mathbf{P}, \qquad \mathbf{(22)}$$

then (20) is reduced to

$$(\omega^2 - \mathbf{P^2c_0^2})\boldsymbol{\phi}^{''}(\boldsymbol{\xi}) - (\boldsymbol{\omega}^\mathbf{2} - \mathbf{K^2c_0^2} - \mathbf{f_0^2})\boldsymbol{\phi}(\boldsymbol{\xi}) + \boldsymbol{\delta}\mathbf{v}\boldsymbol{\phi}(\boldsymbol{\xi}) = \mathbf{0}, \qquad \mathbf{(23.a)}$$
$$(\omega^2 - \mathbf{P^2c_0^2})\mathbf{v}^{''}(\boldsymbol{\xi}) - \boldsymbol{\beta}\mathbf{P^2}(\boldsymbol{\phi}^\mathbf{2}(\boldsymbol{\xi}))^{''} = \mathbf{0}. \qquad \mathbf{(23.b)}$$

Integrating (23.b) once with respect to ξ, we get

$$(\omega^2 - \mathbf{P^2 c_0^2})\mathbf{v}'(\boldsymbol{\xi}) - \boldsymbol{\beta}\mathbf{P^2}\boldsymbol{\phi}^{\mathbf{2}'}(\boldsymbol{\xi}) = \tilde{\mathbf{c}}. \tag{24}$$

where $\tilde{c}$ is integration constant. Because we find the special form of exact solutions for simplicity purpose, we take $\tilde{c} = 0$ and integrating this formula once again, we have

$$v(\xi) = \frac{C}{\omega^2 - c_0^2\mathbf{P^2}} + \frac{\beta\mathbf{P^2}}{\omega^2 - c_0^2\mathbf{P^2}}\phi^2(\xi), \tag{25}$$

where C is an integration constant. Substituting (25) into (23.a) yields

$$(\omega^2 - c_0^2\mathbf{P^2})^2\boldsymbol{\phi}''(\boldsymbol{\xi}) + [(\boldsymbol{\omega^2} - \mathbf{c_0^2P^2})(-\boldsymbol{\omega^2} + \mathbf{c_0^2K^2c_0^2} + \mathbf{f_0^2}) + \boldsymbol{\delta}\mathbf{C}]\boldsymbol{\phi}(\boldsymbol{\xi}) + \boldsymbol{\delta\beta}\mathbf{P^2}\boldsymbol{\phi}^{\mathbf{3}}(\boldsymbol{\xi}) = \mathbf{0}, \tag{26}$$

Eq.(26) can be expressed as

$$\phi''(\xi) + k_1\phi(\xi) + k_3\phi^3(\xi) = 0, \tag{27.a}$$

where

$$k_1 = \frac{[(\omega^2 - c_0^2\mathbf{P^2})(-\boldsymbol{\omega^2} + \mathbf{c_0^2K^2} + \mathbf{f_0^2}) + \boldsymbol{\delta}\mathbf{C}]}{(\omega^2 - c_0^2\mathbf{P^2})^{\mathbf{2}}}, \quad k_3 = \frac{\delta\beta\mathbf{P^2}}{(\omega^2 - c_0^2\mathbf{P^2})^{\mathbf{2}}}. \tag{27.b}$$

Then the solutions of CKGZ equations are given by $u(x, y, z, t) = \phi(\xi)$ $e^{i(kx+ly+nz-\Omega t)}$, $v(x, y, z, t) = \frac{C}{\omega^2 - c_0^2\mathbf{P^2}} + \frac{\beta\mathbf{P^2}}{\omega^2 - c_0^2\mathbf{P^2}}\phi^2(\xi)$, where $\phi(\xi)$ intervening in these solutions is given by (54)-(69), the other parameters are defined by $\Omega = \frac{c_0^2\mathbf{K.P}}{\omega}$, $\xi = px + qy + rz - \omega t$, $k_1 = \frac{[(\omega^2 - c_0^2\mathbf{P^2})(-\boldsymbol{\omega^2} + \mathbf{c_0^2K^2} + \mathbf{f_0^2}) + \boldsymbol{\delta}\mathbf{C}]}{(\omega^2 - c_0^2\mathbf{P^2})^{\mathbf{2}}}$, $k_3 = \frac{\delta\beta\mathbf{P^2}}{(\omega^2 - c_0^2\mathbf{P^2})^{\mathbf{2}}}$.

2.4 A class of NLPDEs with constant coefficients

We consider a class of NLPDEs with constant coefficients [22]

$$iu_t + \mu(u_{xx} + D_1u_{yy}) + E_1|u|^2u + C_1un = 0, \tag{28.a}$$

$$D_2n_{tt} + (n_{xx} - E_2u_{yy}) + C_2(|u|^2)_{xx} = 0, \tag{28.b}$$

where $\mu,\ D_i,\ E_i,\ C_i$ (i=1,2) are real constants and $\mu \neq 0,\ D_1 \neq 0,\ C_1 \neq 0,\ C_2 \neq 0.$
Eqs.(28.a), (28.b) are a class of physically important equation. In fact, if one takes

$$\mu = \frac{1}{2}\kappa^2, \quad D_1 = 2\mu, \quad E_1 = \alpha, \quad C_1 = -1,$$
$$D_2 = 0, \quad E_2 = D_1, \quad C_2 = -2\alpha, \quad \kappa^2 = \pm 1 \qquad (\mathbf{29})$$

Then Eqs.(28.a), (28.b) represent the Davey-Sterwatson (DS) equations [23]

$$iu_t + \frac{1}{2}\kappa^2(u_{xx} + \kappa^2 u_{yy}) + \alpha|u|^2 u - un = 0, \qquad (\mathbf{30}.\mathbf{a})$$

$$n_{xx} - \kappa^2 n_{yy} - 2\alpha(|u|^2)_{xx} = 0. \qquad (\mathbf{30}.\mathbf{b})$$

If one takes
$n = n(x,t)$ *i.e.* $n_y = 0,\ \mu = 1,\ D_1 = 0,\ E_1 = -2\lambda,$
$E_2 = -1, \quad C_2 = -1, \quad C_1 = 2.$ $\qquad (\mathbf{31})$

Then Eqs.(28a) and (28.b) become generalized Zakharov (GZ) equations [24]

$$iu_t + u_{xx} - 2\lambda|u|^2 u + 2un = 0, \qquad (\mathbf{32}.\mathbf{a})$$

$$n_{tt} - n_{xx} + (|u|^2)_{xx} = 0. \qquad (\mathbf{32}.\mathbf{b})$$

Since u is a complex function, we assume that

$$u(x,y,t) = \phi(\xi)e^{i(kx+ly-\Omega t)}, \quad v(x,y,t) = v(\xi), \ \ \xi = px+qy-\omega t \qquad (\mathbf{33})$$

where both $\phi(\xi)$ and $v(\xi)$ are real functions, k, l, p, q, Ω and ω are constants to be determined later. Substituting Eq.(33) into Eqs.(28.a) and (28.b), we have the following ODE for $\phi(\xi)$ and $v(\xi)$

$$\mu(p^2 + D_1 q^2)\phi''(\xi) + [\Omega - \mu(k^2 + D_1 l^2)]\phi(\xi) + E_1\phi^3(\xi) + i[-\omega + 2\mu(kp + D_1 lq)]\phi'(\xi) + C_1\phi(\xi)v(\xi) = 0, \qquad (\mathbf{34}.\mathbf{a})$$

$$(D_2\omega^2 + p^2 - E_2 q^2)v''(\xi) + C_2 p^2(\phi^2(\xi))'' = 0. \qquad (\mathbf{34}.\mathbf{b})$$

Setting

$$\omega = 2\mu(kp + D_1 lq). \qquad (\mathbf{35})$$

Then (34.a) and (34.b) reduce to

$$\mu(p^2+D_1q^2)\phi''(\xi)+[\Omega-\mu(k^2+D_1l^2)]\phi(\xi)+E_1\phi^3(\xi)+C_1\phi(\xi)v(\xi) = 0, \qquad (\mathbf{36}.\mathbf{a})$$

$$(D_2\omega^2+p^2-E_2q^2)v^{''}(\xi)+C_2p^2(\phi^2(\xi))^{''}=0. \qquad (\mathbf{36}.\mathbf{b})$$

Integrating (36.b) once, we get

$$(D_2\omega^2+p^2-E_2q^2)v'(\xi)+C_2p^2(\phi^2(\xi))^{'}=\tilde{C}, \qquad (\mathbf{37})$$

where $\tilde{C}$ is integration constant, then we take $\tilde{C}=0$ and integrating the formula once again, we have

$$v(\xi)=\frac{C}{D_2\omega^2+p^2-E_2q^2}-\frac{C_2p^2}{D_2\omega^2+p^2-E_2q^2}\phi^2(\xi). \qquad (\mathbf{38})$$

Substituting (38) into (36.a) yields

$$\mu(p^2+D_1q^2)(D_2\omega^2+p^2-E_2q^2)\phi^{''}(\xi)+[C_1C-(D_2\omega^2+p^2-E_2q^2)(\Omega-\mu(k^2+D_1l^2))]\phi(\xi)+[E_1(D_2\omega^2+p^2-E_2q^2)-C_1C_2p^2]\phi^3(\xi)=0, \qquad (\mathbf{39})$$

Eq.(39) can be written as

$$\phi^{''}(\xi)+k_1\phi(\xi)+k_3\phi^3(\xi)=0, \qquad (\mathbf{40}.\mathbf{a})$$

where

$$k_1=\frac{C_1C-(D_2\omega^2+p^2-E_2q^2)(\Omega-\mu(k^2+D_1l^2))}{\mu(p^2+D_1q^2)(D_2\omega^2+p^2-E_2q^2)},\ \ k_3=\frac{E_1(D_2\omega^2+p^2-E_2q^2)-C_1C_2p^2}{\mu(p^2+D_1q^2)(D_2\omega^2+p^2-E_2q^2)}. \qquad (\mathbf{40}.\mathbf{b})$$

Then the solutions of the GDS equations can be written as $u(x,y,t)=\phi(\xi)e^{i(kx+ly-\Omega t)}$, $v(x,y,t)=\frac{C}{D_2\omega^2+p^2-E_2q^2}-\frac{C_2p^2}{D_2\omega^2+p^2-E_2q^2}\phi^2(\xi)$, $\phi(\xi)$ intervening in these solutions are given by (54)-(69), the other parameters are defined by $\xi=px+qy-\omega t$, $\omega=2\mu(kp+D_1lq)$ and k_1 and k_3 satisfy (40.b)

We may obtain from (30) that

$$u(x,y,t)=\phi(\xi)e^{i(kx+ly-\Omega t)},$$
$$v(x,y,t)=\frac{C}{p^2-\kappa^2q^2}+\frac{2\alpha p^2}{p^2-\kappa^2q^2}\phi^2(\xi), \qquad (\mathbf{41}')$$

$$\xi=px+qy-\omega t, \qquad \omega=\kappa^2(kp+\kappa^2lq),$$

$$\phi^{''}(\xi)+k_1\phi(\xi)+k_3\phi^3(\xi)=0, \qquad (\mathbf{41}.\mathbf{a})$$

where

$$k_1 = \frac{2C+(p^2-\kappa^2 q^2)(-2\Omega+\kappa^2(k^2+\kappa^2 l^2))}{\kappa^2(p^2+\kappa^2 q^2)(\kappa^2 q^2-p^2)}, \quad k_3 = \frac{2\alpha}{\kappa^2(\kappa^2 q^2-p^2)}. \tag{41.b}$$

Then from (32) we have that

$$\begin{aligned} u(x,t) &= \phi(\xi)e^{i(kx-\Omega t)}, \\ v(x,t) &= \frac{C}{p^2-\omega^2} + \frac{p^2}{p^2-\omega^2}\phi^2(\xi), \end{aligned} \tag{42'}$$

$\xi = px - \omega t, \qquad \omega = 2kp,$

$$\phi''(\xi) + k_1\phi(\xi) + k_3\phi^3(\xi) = 0, \tag{42.a}$$

where

$$k_1 = \frac{2C-(p^2-\omega^2)(\Omega-k^2))}{p^2(p^2-\omega^2)}, \quad k_3 = \frac{2(p^2-\lambda(p^2-\omega^2))}{p^2(p^2-\omega^2)}. \tag{42.b}$$

Where for the solutions of the DS equations and the GZ equations defined above the $\phi(\xi)$ appearing in them is determined by (54)-(69).

2.5 Exact solutions of the elliptic-like equation

Now let us return back to Eqs.(13), (17), (27), (40), (41) and (42). According to step 2 in section 1, by balancing the higher-order derivative term ϕ'' with the nonlinear term ϕ^3, w get n=1 in (1). Therefore we suppose that Eqs.(13), (17), (27), (40), (41) and (42) have the following formal solutions with $R \neq 0$

$$\phi(\xi) = A_0 + A_1\sigma(\xi) + B_1\tau(\xi), \tag{43}$$

where A_0, A_1 and B_1 are constants to be determined later. $\sigma(\xi)$ and $\tau(\xi)$ satisfy (2.a) and (2.b). According to the step 3, we substitute (43) into the elliptic-like equations (13), (17), (27), (40), (41) and (42) along with (2.a) and (2.b). With the aid of Mathematica, collecting all terms with the same power in $\sigma^j(\xi)\tau^i(\xi)$, $\quad j = 0,\ 1,\ 2,\ 3,\ 4;\quad i = 0,\ 1$ and setting the coefficients of these terms $\sigma^j(\xi)\tau^i(\xi)$ to zero yields a set of over-determined algebraic equations with respect to $A_0,\ A_1,\ B_1,\ \ R,$ and m.

$$Const: \qquad k_1A_0+k_3[A_0^3-3\epsilon RA_0B_1^2]=0, \tag{44}$$

$$\sigma(\xi): \ k_1A_1+k_3[3A_0^2A_1-3\epsilon RA_1B_1^2+6\epsilon A_0B_1^2m]-\epsilon A_1R=0, \tag{45}$$

$$\tau(\xi): \qquad k_1B_1+k_3[3A_0^2B_1-\epsilon RB_1^3]=0, \tag{46}$$

$$\sigma(\xi)\tau(\xi): \qquad k_3[6A_0A_1B_1+2\epsilon mB_1^3]+\epsilon mB_1=0, \tag{47}$$

$$\sigma^2(\xi): \ k_3[3A_0^2A_1+6\epsilon mA_1B_1^2-\tfrac{3\epsilon}{R}A_0B_1^2(m^2+r_i)]+3\epsilon mA_1=0, \tag{48}$$

$$\sigma^2(\xi)\tau(\xi): k_3[3A_1^2B_1-\epsilon\tfrac{B_1^3}{R}(m^2+r_i)]-2\epsilon\tfrac{m^2+r_i}{R}B_1=0, \tag{49}$$

$$\sigma^3(\xi): k_3[A_1^3-3\epsilon\tfrac{m^2+r_i}{R}A_1B_1^2]-2\epsilon\tfrac{m^2+r_i}{R}A_1=0. \tag{50}$$

From which using Mathematica and Wu elimination Method [25,26], we get the following results

$$(I): \quad A_0=A_1=m=0, \quad R=\tfrac{k_1}{\epsilon}, \quad A_1^2=\tfrac{2r_i}{k_1k_3}, \ \epsilon=\pm1, \quad r_i=\pm1, \quad \mathbf{(51)}$$

$$(II): \quad A_0=A_1=m=0, \quad R=\tfrac{-k_1}{2\epsilon}, \quad B_1^2=\tfrac{-2}{k_3}, \ \epsilon=\pm1, \quad r_i=\pm1, \quad \mathbf{(52)}$$

$$(III): \quad A_0=0, \quad R=\tfrac{-2k_1}{\epsilon}, \quad A_1^2=-\tfrac{m^2+r_i}{4k_1k_3} \quad B_1^2=\tfrac{-1}{2k_3},$$
$$\epsilon=\pm1, \quad r_i=\pm1, \qquad \mathbf{(53)}$$

Therefore, from (4)-(7) and (51)-(53), we obtain fifteen kinds of exact travelling wave solutions of Eqs.(13), (17), (27), (40), (41) and (42).

Family 1. Dark soliton solutions

$$\phi_1(\xi)=\pm\sqrt{-\frac{k_1}{k_3}}\tanh(\sqrt{\frac{k_1}{2}}\ \xi), \qquad \mathbf{(54)}$$

Family 2. Singular dark soliton solutions

$$\phi_2(\xi)=\pm\sqrt{-\frac{k_1}{k_3}}\coth(\sqrt{\frac{k_1}{2}}\ \xi), \qquad \mathbf{(55)}$$

Family 3. Bright soliton solutions

$$\phi_3(\xi) = \pm\sqrt{-\frac{2}{k_3}}sech(\sqrt{-k_1}\ \xi), \tag{56}$$

Family 4. Singular soliton solutions

$$\phi_4(\xi) = \pm\sqrt{\frac{2}{k_3}}csch(\sqrt{-k_1}\ \xi), \tag{57}$$

Family 5. Periodic wave solutions

$$\phi_5(\xi) = \pm\sqrt{\frac{k_1}{k_3}}\tan(\sqrt{-\frac{k_1}{2}}\ \xi), \tag{58}$$

Family 6. Periodic wave solutions

$$\phi_6(\xi) = \pm\sqrt{\frac{k_1}{k_3}}\cot(\sqrt{-\frac{k_1}{2}}\ \xi), \tag{59}$$

Family 7. Periodic wave solutions

$$\phi_7(\xi) = \pm\sqrt{-\frac{2}{k_3}}\sec(\sqrt{k_1}\ \xi), \tag{60}$$

Family 8. Periodic wave solutions

$$\phi_8(\xi) = \pm\sqrt{-\frac{2}{k_3}}csc(\sqrt{k_1}\ \xi), \tag{61}$$

Family 9. combined formal soliton-like solutions

$$\phi_9(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\left[\frac{\sqrt{-1+m^2}sech(\sqrt{2k_1}\xi)}{m\ sech(\sqrt{2k_1}\xi)+1} + \frac{\epsilon\tanh(\sqrt{2k_1}\xi)}{m\ sech(\sqrt{2k_1}\xi)+1}\right], \tag{62}$$

Family 10. combined formal soliton-like solutions

$$\phi_{10}(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\left[\frac{\sqrt{1+m^2}csch(\sqrt{2k_1}\xi)}{m\ csch(\sqrt{2k_1}\xi)+1} + \frac{\epsilon\coth(\sqrt{2k_1}\xi)}{m\ csch(\sqrt{2k_1}\xi)+1}\right], \tag{63}$$

Family 11. combined formal periodic wave-like solutions

$$\phi_{11}(\xi) = \pm\sqrt{\frac{k_1}{k_3}}\left[\frac{\sqrt{1-m^2}\sec(\sqrt{-2k_1}\xi)}{m\ \sec(\sqrt{-2k_1}\xi)+1} + \frac{\epsilon\tan(\sqrt{-2k_1}\xi)}{m\ \sec(\sqrt{-2k_1}\xi)+1}\right], \qquad (\mathbf{64})$$

Family 12. combined formal periodic wave-like solutions

$$\phi_{12}(\xi) = \pm\sqrt{\frac{k_1}{k_3}}\left[\frac{\sqrt{1-m^2}\csc(\sqrt{-2k_1}\xi)}{m\ \csc(\sqrt{-2k_1}\xi)+1} + \frac{\epsilon\cot(\sqrt{-2k_1}\xi)}{m\ \csc(\sqrt{-2k_1}\xi)+1}\right], \qquad (\mathbf{65})$$

When $m = \pm 1$ then we have the following solutions
Family 13. new soliton solutions

$$\phi_{13}(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\frac{\tanh(\sqrt{2k_1}\xi)}{\epsilon\ sech(\sqrt{2k_1}\xi)+1}, \qquad (\mathbf{66})$$

Family 14. new periodic wave solutions

$$\phi_{14}(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\frac{\tan(\sqrt{-2k_1}\xi)}{\epsilon\ \sec(\sqrt{-2k_1}\xi)+1}, \qquad (\mathbf{67})$$

Family 15. new periodic wave solutions

$$\phi_{15}(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\frac{\cot(\sqrt{-2k_1}\xi)}{\epsilon\ \csc(\sqrt{-2k_1}\xi)+1}. \qquad (\mathbf{68})$$

According to the above mentioned method in section 1 and (3.a) and (3.b), we assume that Eqs. (13), (17), (27), (40), (41), (42) have the solutions in the form $\phi(\xi) = A_0 + A_1\tau(\xi)$, when R=m=0, then we obtain the rational solutions.

Family 16. Rational solutions:

$$\phi_{16}(\xi) = \pm\sqrt{-\frac{2}{k_3}\frac{1}{\xi}}, \qquad (\mathbf{69})$$

where $k_1 = 0, A_0 = 0$.

3 Conclusion

In this paper, by using a more general transformation (1) and the general projective Riccati equations method, we have been able to obtain in a unified way, by the help of symbolic computation system-Mathematica, many kinds of exact solutions to a class of NLPDEs. This class of NLPDEs is characterized by the fact that it can be reduced through a simple transformation to the elliptic-like equation $\phi^{''}(\xi) + k_1\phi(\xi) + k_3\phi(\xi)^3 = 0$. It is obvious that by using this simple transformation, the computation quantity evolved in solving nonlinear equations is greatly reduced. This method has proved its efficiency to the Hamiltonian amplitude equation, the nonlinear wave equation, the coupled Klein-Gordon-Zakharov (CKGZ) equations, the generalized Davey Sterwatson (GDS) equations, the DS equations, the generalized Zakharov (GZ) equations.
To our knowledge, it's the first time that the general projective equations method is used for solving a system of coupled equations.

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